Acoustic liner - mean flow interaction
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Acoustic liner - mean flow interaction

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## Nomenclature

### Coordinate systems

| $x, y, z$ | Cartesian coordinates |
| $x, r, \theta$ | Cylindrical coordinates |

### Operators

| $\text{Re}$ | Real part |
| $\text{Im}$ | Imaginary part |
| $e^u$ | Exponential function |
| $\delta$ | Dirac delta distribution |
| $W$ | Wronskian |

### Greek symbols

| $\alpha$ | Radial wavenumber |
| $\gamma$ | Euler constant |
| $\zeta$ | Scaled impedance |
| $\lambda$ | Wavelength (Ch. 1, 2, 3, 4, 5) or generic variable (Ch. 6 and 7) |
| $\rho$ | Density |
| $\sigma$ | Reduced wavenumber (Ch. 5) or shear rate (Ch. 7) |
| $\omega$ | Frequency |
| $\Omega$ | Doppler shifted frequency |
# Nomenclature

## Latin symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_0$</td>
<td>speed of sound</td>
</tr>
<tr>
<td>$C_p$</td>
<td>heat capacity at const. pressure</td>
</tr>
<tr>
<td>$C_V$</td>
<td>heat capacity at const. volume</td>
</tr>
<tr>
<td>$D$</td>
<td>dispersion relation</td>
</tr>
<tr>
<td>$E$</td>
<td>modified exponential integral</td>
</tr>
<tr>
<td>$F_k$</td>
<td>inverse Fourier integr. contour</td>
</tr>
<tr>
<td>$G$</td>
<td>Green’s function</td>
</tr>
<tr>
<td>$h$</td>
<td>boundary layer thickness</td>
</tr>
<tr>
<td>$h_c$</td>
<td>critical boundary layer thickness</td>
</tr>
<tr>
<td>$I_m$</td>
<td>modified Bessel function (first kind)</td>
</tr>
<tr>
<td>$J_m$</td>
<td>Bessel function (first kind)</td>
</tr>
<tr>
<td>$k$</td>
<td>axial wavenumber</td>
</tr>
<tr>
<td>$K$</td>
<td>stiffness (mass-spring-damper)</td>
</tr>
<tr>
<td>$L_{\omega}$</td>
<td>inverse Laplace integr. contour</td>
</tr>
<tr>
<td>$m$</td>
<td>inertance (mass-spring-damper) (Ch. 2, 3, 5)</td>
</tr>
<tr>
<td>$m$</td>
<td>azimuthal wavenumber (Ch. 6)</td>
</tr>
<tr>
<td>$L$</td>
<td>cavity depth (Helmholtz resonator)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>Mach number</td>
</tr>
<tr>
<td>$n$</td>
<td>unit vector normal into the wall</td>
</tr>
<tr>
<td>$p$</td>
<td>pressure</td>
</tr>
<tr>
<td>$Q$</td>
<td>source term</td>
</tr>
<tr>
<td>$R$</td>
<td>resistance (impedance)</td>
</tr>
<tr>
<td>$R_{\text{spec}}$</td>
<td>specific gas constant</td>
</tr>
<tr>
<td>$s$</td>
<td>entropy</td>
</tr>
<tr>
<td>$S$</td>
<td>Strouhal number (Ch. 5) or source strength (Ch. 7 and App. B)</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$T$</td>
<td>temperature</td>
</tr>
<tr>
<td>$u$</td>
<td>axial velocity</td>
</tr>
<tr>
<td>$U$</td>
<td>mean flow velocity</td>
</tr>
<tr>
<td>$v$</td>
<td>radial or vertical velocity</td>
</tr>
<tr>
<td>$v$</td>
<td>velocity vector</td>
</tr>
<tr>
<td>$w$</td>
<td>velocity in azimuthal direction</td>
</tr>
<tr>
<td>$Y_m$</td>
<td>Bessel function (second kind)</td>
</tr>
<tr>
<td>$Z$</td>
<td>impedance (dimensional)</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

1.1 Aircraft noise

There is a global tendency towards air traffic because of the time efficiency of aircraft on long distances and because of their route flexibility, due to the fact that the only infrastructure needed are airports. However, there is a serious limitation to increasing civil air traffic, namely the associated growth of noise emission that affects residential areas around the airport. Aircraft noise is carefully regularized by the International Civil Aviation Organization (ICAO) and its regional offices in Annex 16, Volume I to the Chicago convention [4]. The increasingly stringent admitted noise levels before 1977, between 1977 and 2006, and after 2006 are contained in Chapter 2, 3 and 4 of the Annex, respectively.

These noise regulations have a twofold purpose: to protect the population from adverse sound level exposure, and to stimulate research and development towards better performances. So, basically the quieter an aircraft, the more economically competitive, and more likely to survive longer on the market.
There were significant advances in the past fifty years in designing quieter aircraft, with the first important step marked by the introduction of the bypass turbofan engine (see Figure 1.1) in the late 1960s which utilizes the Tyler and Sofrin selection rule for the number of rotor blades and stator vanes [108], thus making sure that the first harmonic of the engine duct is cut-off. Further noise reduction was achieved by increasing the engine by-pass ratio, by developing the technology of acoustic liners, and by other engineering changes.

Figure 1.2 [1] shows the downward trend in the EPN\(^1\) levels of various aircraft, the current quietest in use being variants of the Boeing 777 and the Airbus 380 [2] with predictions for the 787 model of Boeing following the same line. Suggested noise management includes reducing it at the source, improving maneuvers and operational procedures (for example, reducing the time the aircraft is within the region where it affects population and also reducing the noise level there by increasing the angle at landing, or having a rather continuous landing procedure, in contrast to a step-wise approach), and land-use planning.

It should be noted that the gradual flattening of the noise reduction curve has a very important economical consequence, since the noise load of airport areas is invariably measured in terms of the (logarithm of the) number of flight motions (starts and landings) plus the noise per aircraft (SPL\(^2\)). Therefore, economic

\(^{1}\)effective perceived noise, see [5] ch. 19
\(^{2}\)sound pressure level
1.1 Aircraft noise

Figure 1.2: Noise certification of various jet aircraft. Chapter 4 regulations effective as of 2006.

Figure 1.3: Significant aircraft noise sources at approach and take-off.
growth, which is proportional to the increase of the number of flight move-
mements, is only possible if the noise per aircraft is reduced, especially since we
need to remain below the legal noise load.

The successful reduction of aircraft noise during the past five decades has re-
sulted in the fact that many sources of noise are nowadays of equal importance.
We refer to Figure 1.3 for a comprehensive illustration, observing that the en-
gines are the dominant noise source at take-off, with airframe noise becoming
comparable during landing procedures. All in all, the aim for significant noise
reduction is calling for a revolutionary technology. This is, nonetheless, a long-
term investment since the certification of new technology takes a considerable
amount of time, not to mention the financial aspect of adapting airports to this.
An alternative, especially short-term effective way, is improving existing tech-
nology. This is in itself a broad topic with many aspects, one of which is the
further development of the acoustic liners that are applied in different regions
of the engine and engine duct in order to absorb produced sound (see Figures
1.5 and 1.1). To introduce liners, let us first have a look into sources of noise in
a turbofan jet-engine.

To understand the working of an engine, note that air is first sucked into the inlet
duct by the fan, then split into two different paths: the by-pass and the core duct
(see Figure 1.4). In the core duct, the air is further compressed, mixed with fuel
and burnt in the combustion chamber, then sent downstream into a turbine and
1.2 Problem description

exhausted. So there is [99]

- *interaction noise* produced by the interaction between the wakes of the rotor or fan, and the stator immediately downstream, representing an important fraction of the total engine noise both at take-off and at approach [3];

- *multiple pure tone* or “buzzsaw” *noise* when the rotor-tip speed exceeds the speed of sound (this rises above the interaction noise);

- *combustion and turbine noise*;

- *jet noise*, generated by turbulence of the core and by-pass jet.

Some of the noise is broadband and is attenuated by specific liners, for example bulk absorbers. To attenuate the discrete tones, walls are lined (or coated) with sound-absorbing material, especially in the inlet and the bypass-duct. These liners are often built of Helmholtz resonators, assembled between two metallic sheets in a way similar to a honeycomb and optimized for certain frequencies (see Figure 1.5). Depending on the space available and the noise to be attenuated, there can be a single- or double-layer of honeycombs, and a perforated or a closely woven facesheet.

Due to required precision in the interconnection of two liner-cells, as well as the complexity of having a bidirectional duct curvature, the production cost for honeycomb-liners is relatively high, making physical experiments expensive and mathematical models attractive. However, to move the greater part of flight simulations to the virtual domain requires not only significant computational power, but also reliable computer codes and mathematical models that can faithfully simulate and predict the sources of noise, and sound propagation into the surrounding air. We address in this thesis aspects of parts of the mathematical model used in duct acoustics, focusing on flow - acoustic liner couplings, and discuss reliability and accuracy in this context.

1.2 Problem description

To introduce our topic, let us first make some remarks on the modeling procedure itself. A model translates real-life aspects of a complex problem to
Figure 1.5: Jet engine lining (courtesy of Rolls-Royce [5], used with permission).
1.2 Problem description

mathematical language, in order to make use of the power of the latter towards a better understanding and explanation of the former. This process involves a balance between a necessary simplification in order to have a solvable mathematical problem, and a faithful account of the relevant physical features under study. So there is, on the one hand, a strong stimulus for developing the mathematical theories and tools, while on the other hand we put effort into refining and optimizing existing models.

One of the problems in the state-of-the-art models of sound propagation in lined ducts with non-uniform mean flow, is the role of the boundary layer. From practical perspective, this boundary layer is very thin compared to any characteristic wavelength (especially in the inlet), so it is reasonable to assume a model with a vanishing boundary layer. However, this simplification leads in time-domain to an ill-posed problem. Furthermore, in a finite boundary layer (in the bypass duct) we have theoretically various singularities in the equations, in particular the so-called critical layer, which is not yet fully understood, and for a part, represents an interaction between the acoustic waves and the hydrodynamic mean flow. We explore these two aspects in this thesis, and for a preliminary understanding we shall next briefly introduce the context of each of the two problems.

Typical engine ducts consist of a more or less cylindrical inlet and an annular exhaust (see Figure 1.4 and 1.5), both acoustically lined to suppress radiated noise. The flow through the ducts can cover a wide range of subsonic velocities, depending on the operating conditions of the engine, and is practically nonuniform. This non-uniformity is, however, weak in the inlet where we can accurately consider the flow to be parallel and constant except for a small boundary layer along the duct wall (this approximation is also sometimes used for other parts of the duct). Moreover, if the boundary layer is much thinner than any characteristic wavelength, its acoustic effects are equivalent to that of a vanishing boundary layer, and it is reasonable to consider this limit, leading to a reformulated boundary condition. This is known in the literature as the Ingard-Myers condition, acknowledging U. Ingard for being the first to correctly formulate it for plane surfaces, and M.K. Myers for generalizing it for curved surfaces.

This approximation works well for problems described in frequency domain.
Nevertheless, with the rise of the otherwise advantageous time-domain solvers, this led to unexplainable instabilities. The possibility of instabilities was suspected since the 1970s (see [46, 106]), but only recently understood as a manifestation of ill-posedness in time-domain (see [20]), thus prompting for research towards improving the model and including the missing aspects that would regularize it. First steps in this direction were to include viscosity [21], a fact which rendered better results in frequency domain, but still leaving the problem ill-posed in time domain. We aim for a regularization by including a finite, non-zero boundary layer thickness such that the model retains the stability properties of the finite boundary layer.

The second aspect we address, that is the critical layer, is a byproduct of the inviscid assumption in the mathematical model we work with. It arises as a singularity of the inviscid Fourier-transformed equations, that is often regularized in other branches of fluid mechanics by simply adding some viscosity in the problem. In the acoustics-related shear-flow theory, it is mostly neglected and rarely fully understood. Our purpose is to study it in a simple context in order to understand the subtleties related to it, and its contribution to the total field.

1.3 Thesis outline

We present the general model in Chapter 2, together with corresponding simplifications commonly used in acoustics. We further introduce the Fourier and the Laplace transforms used in the next chapters, as well as the impedance boundary condition for acoustically lined walls. We briefly introduce the Ingard-Myers condition for flows of vanishing boundary layers, and surface waves in this limit.

Chapter 3 is a summary of hydrodynamic stability results primarily used in Chapter 5. We introduce linear instability concepts such as absolute and convective instabilities and explain the Briggs-Bers criterion for distinguishing between them. We conclude the chapter with limitations on the applicability of this method, and with a typical example on how the criterion works.

In Chapter 4 we introduce the Pridmore-Brown equation in 2D ducts and discuss some aspects regarding solutions of the corresponding boundary value
problem. We remark that a symmetric mean flow imposes a certain symmetry on the acoustic pressure field and proceed with a comparison between the continuous linear-then-constant and the continuously differentiable parabolic-then-constant mean flow profiles.

In Chapter 5 we study the stability of a linear-then-constant mean flow over an impedance wall as a function of liner and mean flow properties, and finally propose a regularization of the Ingard-Myers boundary condition. We start with a justification of the chosen model, semi-infinite linear-then-constant incompressible mean flow, and argue that if there is a critical thickness where the flow turns absolutely unstable, it should be a function of the liner and the flow parameters and it does not scale on the wavelength. We study the stability of the flow in the aforementioned set-up, and find that for values of the problem parameters typical in aeronautical applications, this critical boundary layer thickness is smaller than any practical value, which may be one of the possible reasons why the instability was not observed in practice. We re-derive a regularized Ingard condition by including first-order terms with respect to the boundary layer thickness, and show that the resulting approximate dispersion relation follows the stability behavior of the original one. More than that, we give an estimate of the critical thickness for a relevant range of problem parameters and study the behavior of the surface waves with this regularization. We compare our approximate dispersion relation with the one proposed by E.J. Brambley [19] for compressible smooth profiles, and argue that both are unaffected by compressibility effects, which are of higher order in the small parameter. We also discuss some smoothness effects and conclude with some limitations and restrictions of our results, as well as further possible developments and recommendations.

Chapter 6 (a result of a close collaboration with dr. E.J. Brambley) continues with further aspects of the intertwined acoustics and hydrodynamics, and discusses the critical layer in ducted shear flows over impedance walls. We start by proposing a relation between the parameters in the problem that leads to the existence of such a singularity, and then choose a linear-then constant mean flow in a cylindrical duct, as being one of the simplest configurations able to reproduce it. We write the solutions of the Pridmore-Brown equation for this case in terms of Bessel functions in the constant flow part, and Frobenius expansions
in the linear shear region, and construct the Green’s function of the problem for a point mass source. The critical layer arises as a logarithmic singularity in the Fourier transformed pressure solution, and is not negligible once the source is in the boundary layer as it is closely linked with an important contribution of the source. We find another particularity in a downstream pole that leaks into the critical layer for small frequencies and argue that, for the accuracy of the model, we always have to consider or neglect this pole together with the rest of the critical layer.

Chapter 7 emerged as a subproblem of Chapter 6, and represents a close-up of the trailing vorticity of a point mass source in linear shear flow. We find that it is possible to find explicit solutions for pressure and velocity in an infinite geometry via integrals of generalized functions. We extend these results for semi-infinite shears along impedance walls, and arrive at an explicit solution in terms of logarithms and exponential integrals with customized branch cuts.
Chapter 2

The general model

2.1 Modeling equations

Consider the conservation equations (mass, momentum and energy) governing the motion of a fluid

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \tag{2.1}
\]

\[
\frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho vv) = - \nabla p + \nabla \cdot \tau \tag{2.2}
\]

\[
\frac{\partial}{\partial t} \left( e + \frac{1}{2}v^2 \right) + \nabla \cdot \left( \rho v(e + \frac{1}{2}v^2) \right) = -\nabla \cdot q - \nabla \cdot (pv) + \nabla \cdot (\tau \cdot v) \tag{2.3}
\]

written in terms of pressure \( p \), density \( \rho \), velocity vector \( v \) and internal energy per unit mass, \( e \). Here \( v = |v| \) and \( q \) the heat flux vector due to heat conduction.

The stress tensor is given by \(-pI + \tau\), while \( \nabla p \) arises from the isotropic part and is given by normal stresses, the anisotropic part being \( \nabla \cdot \tau \) representing viscous and shear effects (\( \tau \) is called deviatoric stress tensor). For a more compact
notation we introduce the material derivative that follows the fluid motion along a particle moving with velocity \( v \)

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla.
\]

The energy equation can be rewritten in terms of entropy \( s \) and temperature \( T \)

\[
\rho T \frac{ds}{dt} = -\nabla \cdot q + \nabla \cdot (\tau \cdot v) - v \cdot (\nabla \cdot \tau) \quad (2.4)
\]

via the fundamental thermodynamic relation (resulting from the first two fundamental laws of thermodynamics)

\[
T ds = de + p d\rho^{-1}. \quad (2.5)
\]

In order to be able to solve equations (2.1), (2.2), (2.4), we need constitutive equations. We assume here

\[
p = \rho R_{\text{spec}} T \quad \text{ideal gas law} \quad (2.6)
\]

\[
q = -\kappa \nabla T \quad \text{Fourier’s law} \quad (2.7)
\]

\[
\tau = \mu (\nabla v + (\nabla v)^T) - \frac{2}{3} \mu (\nabla \cdot v) I \quad \text{Newtonian fluid} \quad (2.8)
\]

where \( \kappa \) is the thermal conductivity, \( \mu \) the dynamic viscosity, \( R_{\text{spec}} = C_p - C_V \) is the specific gas constant, and \( C_p, C_V \) are the heat capacities at constant pressure and volume

\[
C_p = \left( \frac{\partial i}{\partial T} \right)_{\rho}, \quad C_V = \left( \frac{\partial e}{\partial T} \right)_{V},
\]

with \( i = e + \frac{p}{\rho} \) the enthalpy. Since acoustics is in its essence inviscid and non-heat-conducting, we will not make use of the Newtonian fluid and Fourier’s law, therefore we focus in the next steps on the ideal gas assumption. For an ideal gas the internal energy is only a function of temperature and \( de = C_V dT \).

Thus from (2.5) and \( p = \rho R_{\text{spec}} T \) we have

\[
ds = \frac{de}{T} + \rho R_{\text{spec}} d\rho^{-1}
\]

\[
= C_V \frac{dp}{p} - C_V \frac{d\rho}{\rho} - R_{\text{spec}} \frac{d\rho}{\rho} = C_V \frac{dp}{p} - C_p \frac{d\rho}{\rho}.
\]
Moreover, for a perfect gas the heat capacities are constant and we can integrate obtaining
\[
s = C_V \log p - C_P \log \rho + s_{\text{init}}. \tag{2.9}
\]

In acoustics we say that information transfer happens so fast that viscous or turbulent stresses will play a role only in the source region, while any perturbation is not affected by thermal conduction. In order to quantify this, we nondimensionalize the equations on characteristic scales, \( \rho_0, v_0, \ell \) (for example \( \ell = \lambda \), the wavelength) [94], remarking that the Reynolds number is \( Re = \rho_0 v_0 \ell / \mu \) (\( \mu \), the dynamic viscosity of air, in this case), the Peclet number \( Pe = \rho_0 C_P v_0 \ell / \kappa \), the Eckert number \( Ec = v_0^2 / C_P \Delta T \) (\( \Delta T \), the temperature difference due to compression - decompression of air). With these, conservation of mass, momentum and energy amount to
\[
\frac{d}{dt} \rho = -\rho \nabla \cdot v \tag{2.10}
\]
\[
\rho \frac{d}{dt} v = -\nabla p + \frac{1}{Re} \nabla \cdot \tau \tag{2.11}
\]
\[
\rho T \frac{d}{dt} s = -\frac{1}{Pe} \nabla \cdot q + \frac{Ec}{Re} (\nabla \cdot (\tau \cdot v) - v \cdot (\nabla \cdot \tau)) \tag{2.12}
\]

For the applications considered here the Reynolds number is high, and we can neglect viscosity. In these conditions the Peclet number will be of the same order, since they are related via the Prandtl number \( Pr = Pe/Re \), which is 0.7–0.8 for most gases (and therefore of order 1). The adiabatic Eckert number is for the interesting Prandtl numbers (for air and other gases) between 2 and 2.4 [98] and we can conclude then from (2.12) that neglecting viscous effects and heat transfer, implies isentropic perturbations. Thus, (2.10)-(2.12) reduce to the Euler equations in fluid dynamics
\[
\frac{d}{dt} \rho = -\rho \nabla \cdot v \tag{2.13}
\]
\[
\rho \frac{d}{dt} v = -\nabla p \tag{2.14}
\]
\[
\rho T \frac{d}{dt} s = 0. \tag{2.15}
\]

Moreover, (2.15) can be written (using (2.9), for example) as
\[
\frac{d}{dt} p = c^2 \frac{d}{dt} \rho, \tag{2.16}
\]
with the speed of sound, $c$, given by $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$. If the flow is homentropic ($s$ is uniform and constant, in contrast to isentropic flow where $s$ of each particle does not change with time, but may vary from particle to particle) then from (2.9),

$$p \propto \rho^{C_p/C_V}.$$ 

Since sound waves are small perturbations of a stationary base flow, $(p_0, \rho_0, v_0)$, we can represent pressure, density and velocity as

$$p = p_0 + p', \quad \rho = \rho_0 + \rho', \quad v = v_0 + v',$$

where $(p', \rho', v')$ are in general time-dependent and, in almost all applications, of extremely small amplitude (see also the first part of Chapter 3). Starting from the Euler equations (2.13), (2.14), (2.16) satisfied by $(p, \rho, v)$, and neglecting nonlinear terms involving the perturbations, we obtain for $(p', \rho', v')$ the linearized Euler equations

$$\frac{d\rho'}{dt} + v' \nabla \rho_0 + \rho' \nabla \cdot v_0 + \rho_0 \nabla \cdot v' = 0 \quad (2.17)$$

$$\rho_0 \left(\frac{dv'}{dt} + v' \nabla v_0\right) + \rho' v_0 \cdot \nabla v_0 = -\nabla p' \quad (2.18)$$

$$\frac{dp'}{dt} (p' - c_0^2 \rho') = 0. \quad (2.19)$$

We work throughout the thesis only with the acoustic pressure $p'$, density $\rho'$ and velocity vector $v'$, and never consider the total field as a whole. We therefore drop the $t$, keeping in mind that each time we refer to “pressure”, we mean “pressure perturbation”.

A common approach in acoustics is to treat the problem in frequency domain and stationary mean flow conditions, since there is usually a dominant frequency of interest. There are two ways to do that: either consider a full Fourier transform of the signal, or directly assume time harmonic behavior. In many applications, the approaches are equivalent, due to the fact that there are dissipative effects in the model\(^1\) so the frequencies associated with the transient

\(^{1}\text{this is the case in our present study, due to the sound-absorbing liner}\)
2.1 Modeling equations

effects of the switching-on of the source, and any perturbations in general, die out, and we are left only with the frequency that is maintained by the source. Not being interested in those initial effects, we can simplify the problem by taking the frequency fixed and equal to the frequency of the source. If, however, the system is unstable, this single frequency assumption cannot be maintained, as the instability wave will grow and eventually dominate. In such cases, when all frequencies are of potential interest (we do not know the instability wave a priori) we need to consider the full Fourier transform.

In the following chapters we will have several configurations and geometries, comprising 2D semi-infinite domains as well as 2D and 3D duct geometries considered in cartesian and cylindrical coordinates, respectively. For all the aforementioned, \( x \) will be the unconstrained direction and we will assume time-harmonic behavior with the convention \( e^{i\omega t} \) or, exploiting the full (inverse) transforms

\[
\hat{p}(x, y; \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(x, y, t) e^{-i\omega t} dt, \quad \hat{\hat{p}}(y; k, \omega) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{p}(y; k, \omega) e^{i\omega t - ikx} dkd\omega,
\]

for the 2D case, or in a cylindrical geometry,

\[
p(x, r, \theta, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{p}(x, r, \theta; \omega) e^{i\omega t} d\omega
\]

\[
= \frac{1}{4\pi^2} \sum_{-\infty}^{\infty} e^{-im\theta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\tilde{p}}(r; k, \omega, m) e^{i\omega t - ikx} dkd\omega \quad (2.21)
\]

with

\[
\tilde{\tilde{p}}(x, y; \omega) = \int_{-\infty}^{\infty} p(x, y, t) e^{-i\omega t} dt, \quad \tilde{\tilde{p}}(y; k, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y, t) e^{-i\omega t + ikx} dxdt,
\]

and similar in cylindrical coordinates. Let us note that the physical pressure field is given by the real part of (2.21); moreover, in the above, \( \omega \) is the wave

---

\(^2\)By introducing the Fourier transform of a function \( f \), we implicitly assume that \( f \) is continuous - except for at most a finite number of points - and \( L^1 \) and \( L^2 \) integrable, i.e. \( \int_{-\infty}^{\infty} |f(x)| dx < \infty \) and \( \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \)
frequency, and $k$ and $m$ the axial and the azimuthal wavenumbers, respectively.

Throughout the main part of this work, mean flow pressure $p_0$ and density $\rho_0$ are assumed constant and the base flow velocity of the type $\mathbf{v}_0 = (U, 0, 0)$ (in case of a 3D geometry, or $\mathbf{v}_0 = (U, 0)$ for the 2D case) with $U$ varying only in the vertical (or radial, in case of a cylindrical geometry) direction. Under these assumptions, (2.17)-(2.19) with pressure and velocity fields of the form (2.20) or (2.21)$^3$, can be reduced to a second order differential equation for the Fourier transformed pressure [81].

### 2.2 Impedance boundary condition

To complete the harmonic linearized Euler equations, boundary conditions are imposed at the walls, accounting for an acoustically absorbing lining modeled by a wall impedance. The impedance is defined in frequency domain as the ratio of acoustic pressure and acoustic velocity normal to the surface

$$Z(x; \omega) = \frac{\hat{p}(x; \omega)}{\hat{\mathbf{v}}(x; \omega) \cdot \mathbf{n}_S(x)},$$

with $x$ a point on a surface $S$, and unit normal vector $\mathbf{n}_S$ pointing into the surface. $Z$ is in general complex and a function of frequency, position and angle of incidence.

In the cases encountered here, the normal component of the particle velocity at any wall element depends only on the sound pressure at that element and not on the pressure at neighboring elements. Such walls are called locally reacting. For example a bulk absorber, such as foam or fiber glass, is non-locally reacting, but the local reaction of the various surface elements of the arrangement can be brought about by rigid partitions which obstruct the air space in any lateral direction and prevent sound propagation parallel to the surface.

By definition, impedance refers to sound of a single frequency and its value for any given boundary will be different at different frequencies. On the other hand, within the linear response range, $Z$ is independent of the surface pressure amplitude. Typical applications of sound absorbing surfaces are in a large enclosure

---

$^3$I.e. harmonic linearized Euler equations in 2D or 3D
2.2 Impedance boundary condition

or room, to reduce the reverberant noise level, or in a duct, to absorb sound as it propagates along the duct. In the first case, the sound field can be treated as diffuse, and the most important acoustical characteristic of the surface is its random-incidence absorption coefficient. If the surface is locally reacting and waves and walls are plane, its most absorbing impedance is purely resistive. In the second case, however, the optimum is not necessarily real as we are usually not dealing with a diffusive sound field - in particular if the dimensions of the duct cross-section are not large compared with the wavelength. The optimum wall impedance for maximum attenuation, in a given length, depends on the duct dimensions, the frequency, and the spatial characteristics of the sound to be attenuated [71].

2.2.1 Impedance condition in time domain

In Computational Aeroacoustics (CAA), time-domain solvers have sometimes numerical advantage [9, 36] over frequency domain ones for broadband problems, and are indispensable for investigations of nonlinear interactions and for transient wave simulation. This brought up the need of an impedance condition formulated in time-domain, obtained by an inverse Fourier transform, normally leading to a convolution as

$$p(x; t) = \int_{-\infty}^{\infty} z(\tau)v(x, t - \tau) d\tau,$$

with $z(t)$, the impedance function in time-domain, defined by

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(\omega) e^{i\omega t} d\omega,$$

and $v$ the component of the velocity vector normal to the wall. In order to be physical, $Z$ has to be the Fourier transform of a causal, real function, and it must satisfy the passivity condition.

Causality mainly states that there should be no effects prior to the cause; a function is therefore causal if it is zero for negative time [77] (pg. 13), [94] (section C.1.1). In an initial value problem (or in a system with a source switched on
at a certain moment in time), causality simply means we have zero field before the initial conditions (or before the source begins to act).

In order to obtain a causal solution of a problem defined by boundary conditions expressed in terms of an impedance, this should have a particular form: since the pressure at the wall $p_w(t)$ should only depend on the values of $v_w(t)$ of the past, the inverse Fourier transform $z(t)$ of $Z(\omega)$ has to be zero for $t < 0$, in other words

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(\omega) e^{i\omega t} d\omega = 0 \quad \text{for} \quad t < 0.$$  

The same applies for the admittance $1/Z(\omega)$, when we express $v_w(t)$ in terms of $p_w(t)$, requiring (assuming convergence for $|\omega| \to \infty$) [89]

$$Z(\omega) \quad \text{and} \quad 1/Z(\omega) \quad \text{analytic in} \quad \text{Im}(\omega) < 0.$$  

Furthermore, since both $p_w$ and $v_w$ are real, $z$ has to be real too, meaning $z^*(t) = z(t)$, with $^*$ denoting the complex conjugate. So

$$z^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z^*(\omega) e^{-i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z^*(-\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(\omega) e^{i\omega t} d\omega = z(t).$$  

This is true for all $t > 0$ if

$$Z^*(\omega) = Z(-\omega), \quad \text{for all} \quad \omega \in \mathbb{R}.$$  

Finally, since we want to model walls that absorb and/or reflect, but never produce sound, we impose on $Z$ the passivity condition stating that the acoustic intensity into the wall

$$I = \frac{1}{2} \text{Re}(\bar{p}\bar{v}^*) = \frac{1}{2} \text{Re}(Z|\bar{v}|^2)$$  

is positive. This reduces to

$$\text{Re}(Z(\omega)) > 0 \quad \text{for all} \quad \omega \in \mathbb{R}.$$  

A particular case is the hard-wall impedance, where we have in the limit $|Z| \to \infty$. 
2.2 Impedance boundary condition

The mass-spring-damper system

One of the simplest locally reacting wall models satisfying all of the above conditions is the mass-spring-damper. It is a low frequency limit [84] of a Helmholtz resonator system (see Figure 2.1 and the next subsection) with the assumptions that each independent cell system has mass density or inertance $m$ representing the amount of air in the neck of the opening, spring constant or stiffness $K$ characterizing the compressibility of the air in the cavity, and damping $R$ due to viscous and turbulent friction of the air at, and near, the holes of the top porous facesheet.

To model this, suppose the wall consists of infinitesimally small, independently movable, small elements of surface $dA$ ($m^2$) and mass density $m$ (kg/m$^2$), kept in vertical position $y = \eta(x, t)$ by a spring of constant $K$ (kg/m$^2s^2$) and a damper of resistance $R$ (kg/m$^2s$). When a (downward) force $-p(x, \eta, t)dA$ is applied to the wall (at some location $x$), the wall element will respond according to

$$\left( m \frac{\partial^2 \eta}{\partial t^2} + R \frac{\partial \eta}{\partial t} + K \eta \right) dA = -p(x, \eta, t)dA$$

where $dA$ can be divided out. If the displacement is small enough for linearization, this simplifies to

$$m \frac{\partial^2 \eta}{\partial t^2} + R \frac{\partial \eta}{\partial t} + K \eta = -p(x, 0, t).$$

The motion of the air at the wall follows the wall, so

$$v(x, \eta, t) = v(x, 0, t) = \frac{\partial \eta}{\partial t}$$
and we have at \( y = 0 \) after differentiation to time

\[
m \frac{\partial^2 v}{\partial t^2} + R \frac{\partial v}{\partial t} + K v = - \frac{\partial p}{\partial t}.
\]

(2.22)

For time harmonic motion (where \( v = \bar{v} e^{i\omega t} \) and \( p = \bar{p} e^{i\omega t} \)), when initial effects are damped out, this becomes

\[
(-\omega^2 m + i \omega R + K)\bar{v} = -i \omega \bar{p}
\]

and the impedance is indeed independent of the applied field:

\[
\frac{\bar{p}}{-\bar{v}} = i \omega m + R + \frac{K}{i \omega} = Z(\omega).
\]

(2.23)

**Helmholtz resonators**

The Helmholtz resonator is described [85, 87, 94] by

\[
Z(\omega) = R + i \omega \tilde{m} - i \rho_0 c_0 \cot(\omega L/c_0),
\]

Figure 2.2: Honeycomb liner

\( R, \tilde{m}, L > 0 \), with uniform sound speed \( c_0 \). It models resonators (see Figure 2.2) with face-sheet resistance \( R \), cavity reactance \( \cot(\omega L/c_0) \) and cell depth \( L \), and satisfies the causality, reality and passivity conditions [89].

For small \( \omega L/c_0 \) (which holds in the relevant frequency range) we can approximate the Helmholtz resonator by a mass-spring-damper with spring constant \( \rho_0 c_0^2 / L \) and mass density \( m + \frac{1}{3} \rho_0 L \).

If we take into account damping in the fluid cavity (in an ad-hoc way represented by \( \epsilon > 0 \)) and a varying cavity reactance (by introducing a parameter \( b > 0 \) in front of the cot), we have the extended Helmholtz resonator [89]

\[
Z(\omega) = R + i \omega m - i b \cot(\omega L/c_0 - i \frac{1}{2} \epsilon),
\]

which was proved to perform well in broadband applications in time-domain [85].
2.2 Impedance boundary condition

2.2.2 The Ingard-Myers limit

In numerical computations there are situations when considering the full problem may turn out cumbersome, especially if one needs different resolutions for different length scales. A situation of this sort arises in the acoustics of turbofan engine inlets due to the fact that the boundary layer thickness ($h$) of the flow in the inlet is small compared to other length scales, such as the duct radius, or the acoustic wave length, and is in the order of millimeters. It is therefore extremely convenient numerically to neglect it, and correct for the main physical effects by an appropriate slip condition for the acoustic field. Such a model was developed in 1959 by Ingard [56] for plane boundaries and then extended in 1980 by Myers [74] for curved surfaces, making use of the fact that if $h \ll \lambda$ (the wavelength) the sound waves do not see any difference between a finite boundary layer and a vortex sheet, and the limit $h \to 0$ can be taken [44, 56, 74, 106].

The work of Ingard is based on a plane wave assumption and the study of the relative motion between boundary and fluid on the reflection and absorption coefficients. It was later confirmed by a more systematic asymptotic analysis for general mean flow profiles by Eversman and Beckemeyer [44]. The proposed boundary condition, in the notation and convention used here, writes

$$i\omega (\bar{v} \cdot \mathbf{n}) = \left[ i\omega + U \frac{\partial}{\partial x} \right] \left( \frac{\hat{p}}{Z} \right).$$

In [74] we have a surface $S(t)$, which is acoustically deformed in response to an incident sound field. If $S_0$ is its unperturbed position and we fix here a coordinate system, we have to leading order, by imposing the particles to follow the surface,

$$i\omega (\vec{v} \cdot \mathbf{n}) = [i\omega + \mathbf{V} \cdot \nabla - \mathbf{n} \cdot (\mathbf{n} \cdot \nabla \mathbf{V})] \left( \frac{\hat{p}}{Z} \right).$$

The Ingard-Myers boundary condition is advantageous due to its compact, fairly simple form. Moreover, it has been confirmed by comparison with experiments to be in most circumstances a valid approximation of the full problem. However, it was recently proved by Brambley [20] that this boundary condition leads to an ill-posed problem in time domain, a fact which we will discuss more
in Chapter 5. (Illposedness is implied by the unbounded exponential growth in time, which is equivalent to Hadamard’s definition; for a proof see [61].)

2.3 Surface waves

The wavenumber spectrum for a straight cylindrical duct with impedance walls and a non-uniform subsonic axial mean flow can be divided into a discrete and a continuous part. The latter is related to the critical layer, and we will discuss it in detail in Chapter 6. The discrete spectrum is formed by acoustic duct modes resulting from the finiteness of the geometry, although some are confined to a thin layer in the neighborhood of the wall, and are therefore called surface waves.

As the name already suggests, surface waves [8, 27, 40, 41, 88] are solutions propagating horizontally at a phase velocity less than the velocity of sound in air, and decaying exponentially with height above the surface. Some exist only in the presence of a mean flow, and are hence called “hydrodynamic”, others exist also without the mean flow and could be therefore called “acoustic”, but differ from the acoustic waves in that they propagate subsonically.

Take the steady state 2D Helmholtz equation in pressure (pressure with time factor $e^{i\omega t}$)

$$\nabla^2 p + k^2 p = 0, \quad k = \omega/c_0,$$

with an impedance condition at the wall ($y = 0$)

$$ikp(x, 0) = Z\frac{\partial}{\partial y}p(x, 0),$$

where $Z$ is the impedance made dimensionless on $\rho_0 c_0$. Looking for waves of the type

$$p(x, y) = A e^{-i\alpha x - ikyy}, \quad \gamma(\alpha) = \sqrt{1 - \alpha^2}, \quad \text{Im} \gamma \leq 0,$$

that decay away from the wall, we find that they exist only for $\text{Im} Z < 0$, and that there are at most 2 with axial wave number given [28] by

$$\alpha = \pm \sqrt{1 - Z^{-2}}.$$
2.4 The Fourier transform of a causal function

In the uniform flow case with Mach number \( M \), Ingard-Myers boundary condition at the wall and the field varying in time as \( e^{i\omega t} \), we have at most 4 surface waves [88] given by

\[
\beta(1 - Ma)^2 + Z \sqrt{1 - (M + \beta^2 \alpha)^2} = 0,
\]

with \( \beta = \sqrt{1 - M^2} \). They appear or vanish out of the physical domain as a function of \( Z \) via the branch cuts of \( \gamma \). For hard walls (\( |Z| \to \infty \)) we recognize 2 acoustic and 2 hydrodynamic modes which behave in this limit like

- \( \alpha_{SR}, \alpha_{SL} \approx \pm \frac{1 \pm M}{\beta^2} \pm \frac{(1 \pm M)^4}{2Z^2\beta^8} \) (acoustic)
- \( \alpha_{HI}, \alpha_{HS} \approx \pm \frac{1}{M^2} \pm \frac{1 + \beta^2}{M\beta^2} \) (hydrodynamic).

In the following chapters we focus on the study of hydrodynamic surface waves and their possibly unstable behavior.

2.4 The Fourier transform of a causal function and the Laplace transform

We have so far used the Fourier transform for both the time and the space variables. However, requirements for all flow fields (pressure, density, velocities) include causality which we defined, as in [77], with the reference point \( t = 0 \).

The Fourier transform of a causal pressure field, for example, is

\[
\hat{\rho}(\omega) = \int_{0}^{\infty} p(t) e^{-i\omega t} \, dt, \quad (2.24)
\]

wherein we neglected in \( p \) the dependence on anything except time.

The Laplace transform of the causal, integrable function \( p \) is given by

\[
(\mathcal{L}p)(a) = \int_{0}^{\infty} p(t) e^{-at} \, dt, \quad (2.25)
\]
for complex $a = a_r + ia_i$. Its region of convergence is of the type $\text{Re} \, a > \alpha$, $\alpha \in \mathbb{R}$ (see Figure 2.3). If the imaginary axis is in the region of convergence of $(\mathcal{L}p)(a)$ then the Fourier transform of a causal function (2.24) is a special case of (2.25) with $a = i\omega$, and

$$\hat{\rho}(\omega) = (\mathcal{L}p)(i\omega).$$

This fact motivates an interchangeable use of the Laplace and the causal Fourier transform for the time variable.
Chapter 3

Hydrodynamic instabilities

In the study of sound in lined ducts we have to account for hydrodynamic effects that are in general coupled with the acoustics, adding up to the complexity of the problem. Moreover, it was found [11, 26] that this interaction can lead to sound production instead of the expected sound absorption, due to an excited instability mode driven by the available large amount of mean flow energy. In this section we recall concepts of hydrodynamic stability and give the outlines of the method developed first in plasma physics by Bers and Briggs [17] and then later adapted and applied to fluid mechanics and, in particular, shear flows [52, 53, 101, 102].

3.1 Stability concepts

We start with a basic state $U_0 = (p_0, \rho_0, \nu_0)$ (see previous chapter), which is said to be stable in the sense of Lyapunov, if for any positive $\varepsilon$ there is a positive
number $\mu(\varepsilon)$ such that

\[
\|U(x, 0) - U_0(x, 0)\| \leq \mu(\varepsilon), \quad \text{then for any } t \geq 0 \quad \|U(x, t) - U_0(x, t)\| \leq \varepsilon,
\]

for a perturbed state $U$, where the norm may be defined in the physical domain $V$ as

\[
\|U(x, t)\| = \max\{|U(x, t)|, x \in V\}
\]

(see also [54], and [43] for stability in other norms). Hence, how close a system remains to the basic state, depends on the size of the initial perturbations.

In the context of linear stability theory, perturbations are restricted to be infinitesimal, which is the case in the study of sound since acoustic perturbations are indeed very small compared to the base flow. For example, the sound pressure of traffic on a busy roadway 10m away from the measuring device, is around 0.5 Pa (corresponding to 88dB); very loud noise of 120dB corresponds to only 20 Pa, while typical values in an engine inlet are around 130 dB - and this is to be compared with $10^5$ Pa, the ambient atmospheric pressure.

Based on the Hartman-Grobman theorem [49] we can say that the linearization procedure faithfully approximates the dynamics of the nonlinear system in the vicinity of the base state [54]. Consequently, we will hereafter refer to linear stability (i.e. the stability of the linearized system, see [43, 80] for a complete and rigorous theory) simply as stability, explicitly mentioning the type only if used otherwise.

For the simplicity of the exposition we will consider here the two-dimensional case, in cartesian coordinates, infinite in $x$, with boundary conditions in $y$, giving rise to a dispersion relation dispersion relation

\[
D(k, \omega) = 0,
\]

for waves $\sim e^{i\alpha t - ikx}$.

A thorough presentation of linear stability concepts is beyond the purpose of this thesis, and we rather state directly criteria useful in the subsequent Chapters, referring to [43, 62] for a rigorous presentation of definitions and results.
3.1 Stability concepts

In the theory of hydrodynamic stability we have both temporal and spatial characteristics\(^1\) (first recognized by Twiss [107], Landau and Lifshitz [62]). We follow the lines in [29, 54] and consider the flow stable if \(\omega_i > 0\) for all \(k \in \mathbb{R}\) with \(D(\omega, k) = 0\), \(\omega = \omega_r + i \omega_i\). In other words, the flow is stable if all the spatially periodic disturbances (because \(k \in \mathbb{R}\)) decay in time. Conversely, a flow is unstable if there is a real wavenumber \(k\) for which at least one of the corresponding frequencies \(\omega\) given by the dispersion relation has negative imaginary part.

Further on, we can make a distinction between temporal (or spatio-temporal) and (pure) spatial instabilities. A flow is absolutely unstable, if disturbances grow in time at every point in the domain. If the disturbance grows for a co-moving observer, but decays eventually at each point in space, it corresponds to a convective instability. Note the spatial character of this type of instability, a fact which already indicates that its appropriate characteristic will be the growth in space rather than in time. A good graphical example of the two types of instabilities is found in [16] Figure 3.2.1 which, adapted to our notations, is Figure 3.1 here: in the case of a convective instability a fixed observer sees eventually an attenuation of the signal in time, and the actual growth is observed only if traveling with the wave’s propagation speed; an absolute instability is perceived as amplifying both temporally and spatially. Consequently, the notions defined here are not Galilean invariants: if the reference frame is changed (for instance from a stationary one to one moving with a certain velocity) a convective instability may become an absolute one [53].

---

\(^1\)temporal analysis is for \(k \in \mathbb{R}\), while the purely spatial analysis is when \(\omega \in \mathbb{R}\).
3.2 The Briggs-Bers criterion for instabilities

The method we use here to study the stability of the flow, was developed by R.J. Briggs and A. Bers for plasma applications in 1960s. Their reviews [16, 29] contain both proofs and rigorous motivations for the steps to be made, and a large palette of examples. We will restrict ourselves here to merely mentioning the results we will use in the coming chapters and then illustrating them by two examples. The reasoning is done in the framework of open flows (in $x$) and does not consider amplifications or damping due to reflections at the boundary of the domain, and therefore states the presence/absence of inherent instabilities of the flow.

In order to have all modes excited, and to be able to distinguish between the left- and the right-running ones, we will consider the pulse-response of the system given by a Green’s function represented in Fourier-Laplace form

$$\Psi(x, y, t) = \frac{1}{(2\pi)^2} \int_{L_\omega} \int_{F_k} \frac{\varphi(y)}{D(k, \omega)} e^{i\omega t - ikx} dk d\omega. \quad (3.2)$$

$F_k$ is the Fourier contour chosen along the real $k$ axis, and - since we are interested in causal responses - $L_\omega$ the Laplace contour set below all singularities in the $\omega$-plane (see Figure 3.2). However we have freedom in choosing these contours anywhere in the domains of absolute convergence of the integrals which is

- for the $\omega$-integral, anywhere in the half-plane below the poles $\omega_j(k)$ given by $D(k, \omega) = 0$, where $k \in F_k$; this is due to causality that requires $\Psi = 0$
for $t < 0$ and the $e^{i\omega t}$-factor that imposes the closure of the integration contour for $t < 0$ downwards (as in Figure 3.2).

- for the $k$-integral, in a strip along the real axis between the left- and right-running poles, $k^-(\omega)$ and $k^+(\omega)$, given by $D(k, \omega) = 0$, for $\omega \in L_\omega$.

Following the definition, if there is any pole $\omega(k), k \in \mathbb{R}$ with negative imaginary part, then the flow is unstable. Let us first assume that the singularities in the integrand in (3.2) are given by zeros of the dispersion relation only. According to [16, 29], in order to investigate the spatio-temporal nature of this instability, we can start lifting the contour in the $\omega$-plane towards the real axis. Because of the $\omega - k$ interdependence, poles in the $k$-plane will also move and $F_k$ will need to be then deformed such that the separation between left- and right-running modes is well-preserved (see Figure 3.3). If there is no restriction in lifting the $L_\omega$ contour to the real $\omega$ axis, the instability is a spatial one (convective instability). But if the $F_k$ contour is pinched by two $k$ poles - one right and one left-running, as in Figure 3.3 - then that mode formed by the collision of these two is an absolute instability, as there is both a temporal and a spatial amplification to it.

The pinch singularity in the $k$ plane is a double root of the dispersion relation, and corresponds to a cusp singularity in the $\omega$ plane, so we could search for the absolute instability in either of the two planes.

To summarize, in order to study the stability of a flow we follow the steps below:

1. solve $D(k, \omega) = 0$ for $\omega = \omega_r + i\omega_i \in \mathbb{C}$ with $k \in \mathbb{R}$ known; $\omega_i(k)$ should be bounded by a minimum, the absolute value of which is the maximum temporal amplification rate;
   - if $\min \omega_i > 0$, the flow is stable and the analysis ends here;
   - if $\min \omega_i = -\sigma_0 < 0$, the flow is unstable, we proceed to investigate whether this instability is absolute or convective;

2. look for a solution $(k^*, \omega^*)$ of the system

\[
\begin{align*}
D(k, \omega) &= 0 \\
\partial_k D(k, \omega) &= 0
\end{align*}
\]
and check whether it comes from the collision of a right- and a left-running mode (otherwise it does not pinch the integration contour in the $k$ plane); if for the pinch point, $\text{Im} \omega^* > 0$, it is a convective instability, otherwise if $\text{Im} \omega^* < 0$, it is an absolute instability.

The first step is actually a temporal instability analysis, as we consider solutions that are periodic in space ($k$ is real, signifying no decay or amplification in the $x$ direction). For convective instabilities it is suitable to perform a spatial analysis, that is to investigate the behavior of $\text{Im} k(\omega)$, $\omega \in \mathbb{R}$, representing the purely spatial response. Another significant quantity is the propagation velocity

$$V_0 = \left( \frac{\partial \omega_r}{\partial k} \right)_{k=k_0},$$

where $(k_0, \omega_{r_0} - i\sigma_0)$ is the most amplified wave in time ($D(k_0, \omega_{r_0} - i\sigma_0) = 0$, see also step 1 before). An observer moving with $V_0$ will see the disturbance increase as $e^{\sigma_0 t}$.

If in addition to poles, (3.2) has also branch points and branch cuts, extra care is needed in the way these move as the contours are deformed; note that integration contours can never cross a branch cut without changing the integral. An example of such a case is discussed in Chapter 6, so we do not insist on the details here.

In the study of convectively unstable flows we often have harmonic sources ($\sim e^{i\omega_0 t}$) driving the system. The frequency of the source will be the frequency
at which the flow amplifies the disturbance [20,29], as no perturbations of other frequencies are amplified in time.

The flow can support one or more absolute instabilities as well as convective instabilities. However, it will be dominated by the most amplified mode.

3.3 Applicability and conclusions

The method described in this chapter is not applicable if $\omega_i(k)$, the imaginary part of the frequency, is unbounded from below for real wavenumbers (see step 2 above). In fact in such a situation the Laplace transform representation is not applicable and there is no maximum amplification rate in the system, making it ill-posed [20, 97]. As was shown by Brambley, the Ingard-Myers boundary condition leads to ill-posed problems [20], making it improper for time-domain calculations.

3.4 An illustrative example

As an example let us consider the dispersion relation

$$d_\omega(\omega - \omega^*) + d_{kk}(k - k^*)^2 = 0 \quad (3.4)$$

with $d_\omega, \omega^*, d_{kk}, k^* \in \mathbb{C}$ parameters. It is prototypical for well-posed systems, in the sense that the dispersion relation is of this form in the neighborhood of a saddle point in $k$ (see Appendix A). In the present case it is the local approximation of equation (5.11) to be studied in Chapter 5.

We can rewrite (3.4) as

$$\omega = \omega^* - \frac{d_{kk}(k - k^*)^2}{d_\omega}$$
or equivalently, explicit in the real and imaginary part for \( \omega = \omega_r + i \omega_i \),

\[
\begin{align*}
\omega_r &= \omega_r^* - [(k - k_r^*)^2 - k_i^2] \Re \left( \frac{d_{kk}}{d_\omega} \right) - 2k_i^*(k - k_r^*) \Im \left( \frac{d_{kk}}{d_\omega} \right) \\
\omega_i &= \omega_i^* - [(k - k_r^*)^2 - k_i^2] \Im \left( \frac{d_{kk}}{d_\omega} \right) + 2k_i^*(k - k_r^*) \Re \left( \frac{d_{kk}}{d_\omega} \right).
\end{align*}
\]

As a function of \( k \in \mathbb{R} \), \( \omega_i \) has a minimum at

\[
k_0 = k_r^* + k_i^* \Re \left( \frac{d_{kk}}{d_\omega} \right) / \Im \left( \frac{d_{kk}}{d_\omega} \right),
\]

if \( \Im \left( \frac{d_{kk}}{d_\omega} \right) < 0 \), and that minimum is equal to

\[
\omega_{i,\text{min}} = \left| \frac{d_{kk}}{d_\omega} \right|^2 k_i^2 / \Im \left( \frac{d_{kk}}{d_\omega} \right) + \omega_i^*.
\]

For an almost incompressible flow along a weakly resistive wall, dimensionless parameters\(^2\) may be in the order of

\[
\begin{align*}
d_\omega &= 8.5 + 15i \\
\omega^* &= 0.1 - 0.02i \\
d_{kk} &= -1.2 - 8.3i \\
k^* &= 0.2 + 0.3i.
\end{align*}
\]

The maximum amplification rate is in this case \( \sigma_0 = 0.13 \) and we find a saddle point at \((\omega^*, k^*)\) representing an absolutely unstable mode (because \( \omega_i^* < 0 \)).

---

\(^2\)these parameters correspond to \( R = 100 \text{ kg/m}^2\text{s}, m = 0.1215 \text{ kg/m}^2, K = 8166 \text{ kg/m}^2\text{s}^2, U_\infty = 82 \text{ m/s}, h = 1 \text{ cm} \) in the notation of section 5.1.3 (as in Figure 5.2); velocities made dimensionless on \( U_\infty \), lengths on \( h \), and impedance on \( U_\infty \rho_0 \).
3.4 An illustrative example

Figure 3.4: The $k$-plane for $\omega \in \mathbb{R}$. It’s visible that there is spatial growth in the down-stream direction, because there are wavenumbers with positive imaginary parts.

If we now change the parameters to\(^3\)

\[
\begin{align*}
    d_\omega &= 12.9 + 12.53i \\
    \omega^* &= 0.08 + 0.002i \\
    d_{kk} &= -6.2 - 8.3i \\
    k^* &= 0.14 + 0.29i,
\end{align*}
\]

the maximum temporal amplification rate changes to $\sigma_0 = 0.3$. The most amplified mode is $(\omega^*, k^*)$, but since $\text{Im}\, \omega^* > 0$, this saddle corresponds to a convective instability. To calculate the propagation speed we need

\[
\omega'(k) = -2k_i^* \text{Im} \frac{d_{kk}}{d_\omega} - 2(k - k_r^*) \text{Re} \frac{d_{kk}}{d_\omega},
\]

and then according to (3.3)

\[
V_0 = -2 \text{Im} \frac{d_{kk}}{d_\omega} k_i^* - 2(k_0 - k_r^*) \text{Re} \frac{d_{kk}}{d_\omega} = 2.13.
\]

In order to investigate the spatial amplification rate, we plot $k(\omega)$ for $\omega \in \mathbb{R}$. From Figure 3.4 we can see that the growth is downstream, and the maximum spatial amplification rate is approximatively 0.267.

\(^3\text{corresponding to a change of resistance to } R = 150 \text{ kg/m}^2\text{s; otherwise parameters as in the previous case}\)
Modeling sound propagation in turbofan engines has motivated a series of studies of different geometries ranging from rectangular, cylindrical axisymmetric, annular to varying-cross-section ducts, and different types of mean flow profiles starting with the simple uniform flow up to more realistic smooth boundary layers. The idea is to simplify the problem in various ways and observe emerging characteristics. In the inlet, the mean flow is generally taken to be uniform except for a small region in the neighborhood of the walls where we have a boundary layer, the equations at the center admitting exact solutions in terms of exponentials (or sines and cosines), in 2D cases, and Bessel functions, for 3D geometries.

This chapter is meant to present some general results on boundary layer mean flows. We summarize some properties and remarks concerning shear flows along impedance walls and end with a comparison between continuous linear-then-constant mean flows and continuously differentiable parabolic boundary layers, avoiding the singularity at the critical layer, a problem which will be amply treated in Chapter 6.
4.1 The Pridmore-Brown equation

Take the linearized Euler equations (2.17)-(2.19) in a 2D straight, infinitely long duct of width $2R_0$, axial direction taken along the $x$ axis. Scaling lengths on the duct radius $R_0$, velocities on the speed of sound, $c_0$, densities on the mean flow density, $\rho_0$, pressure on mean flow pressure $p_0 = \rho_0 c_0^2$, we have in non-dimensional form

\[
\frac{\partial p}{\partial t} + U(y) \frac{\partial p}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.1}
\]

\[
\frac{\partial u}{\partial t} + U(y) \frac{\partial u}{\partial x} + \frac{dU}{dy} v + \frac{\partial p}{\partial x} = 0 \tag{4.2}
\]

\[
\frac{\partial v}{\partial t} + U(y) \frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = 0, \tag{4.3}
\]

where $U$ is the dimensionless mean flow velocity, and we have made use of the pressure - density relation of isentropic perturbations, $p = c_0^2 \rho$.

Allowing for unstable, causal solutions (see section 2.4), let us take $p$, $u$ and $v$ of the form

\[
p(x, y, t) = \frac{1}{4\pi^2} \int_{L_\omega} \int_{-\infty}^{\infty} \hat{p}(y; k, \omega) e^{i\omega t - ikx} \, dk \, d\omega
\]

\[
u(x, y, t) = \frac{1}{4\pi^2} \int_{L_\omega} \int_{-\infty}^{\infty} \hat{v}(y; k, \omega) e^{i\omega t - ikk} \, dk \, d\omega
\]

where $L_\omega$ is the inverse Laplace integration contour taken in the $\omega$-plane below all the singularities of the integrand. For ease of notation we will consider only the $y$-dependence in the Fourier transforms $\hat{p}$, $\hat{u}$, $\hat{v}$, the $k$- and $\omega$-dependencies being implicitly assumed. Writing $p$, $u$ and $v$ as above, we come to the harmo-
4.2 Some general remarks

Pridmore-Brown equations in 2D or 3D forms were extensively studied in the past decades for different mean flow, density and sound speed profiles and various boundary conditions [39, 44, 48, 57, 66, 72, 73, 81, 105, 106]. The equation was initially solved in 2D for hard walls using asymptotic methods for constant gradient and fractional power boundary layers [81]. Further development included numerical solutions for hard- and finite-admittance-walls [72].
with analytical solutions available only for uniform [66] and linear profiles.

In this chapter we will focus on soft-wall boundary conditions, excluding also the totally admissive wall, \( Z = 0 \), and the hard-wall, \( |Z| \to \infty \), cases.

We avoid here the singularity at the critical layer (to be treated in Chapter 6), and assume that \( \omega - kU(y) \neq 0 \) for all \( y \in [-1, 1] \).

The trivial solution to (4.7) with boundary conditions (4.8) is \( \hat{p} \equiv 0 \). We are however interested in modes, which are non-zero solutions, possible for special values of \((\omega, k)\). For these solutions, \( \hat{p}(\pm 1) \) and \( \hat{p}'(\pm 1) \) are different from zero\(^1\).

This can be motivated via the Picard-Lindel"of theorem\(^2\):

- if we would have for example \( \hat{p}(1) = 0 \), then the boundary condition (4.8) at \( y = 1 \) implies \( \hat{p}'(1) = 0 \);

- the Pridmore-Brown equation is linear in \( \hat{p} \) and its coefficients continuous (provided the mean flow is at least \( C^1 \)), and we have from the Picard-Lindel"of theorem that (4.7) with \( \hat{p}(1) = \hat{p}'(1) = 0 \) has a unique solution, \( \hat{p} \equiv 0 \), which contradicts our initial assumption;

- if the mean flow is only continuous, then the coefficients in (4.7) are discontinuous; still due to the fact that we impose continuity of pressure and normal velocity\(^3\) at the points where \( U \) is non-smooth, we can apply the uniqueness theorem piecewise on intervals and get to the same conclusion.

So non-zero solutions of (4.7) must have non-zero values and derivatives at the walls.

To show the importance of having all the conditions in the Picard-Lindelöf theorem verified, take for example the Bessel equation\(^4\)

\[
\frac{d^2 \hat{p}}{dy^2} + \frac{1}{y} \frac{d \hat{p}}{dy} + \left( \alpha^2 - \frac{m^2}{y^2} \right) \hat{p} = 0.
\]

\(^1\)with \( \hat{p}' \) denoting the derivative to \( y \)

\(^2\)we need Picard-Lindelöf’s theorem as the Pridmore-Brown equation (or the harmonic linearized Euler equations) is equivalent to a non-autonomous problem of the form \( y'(t) = F(t, y(t)) \), \( y(0) = y_0 \); for such cases the theorem assures existence and uniqueness of solution iff \( F \) is Lipschitz continuous in \( y \) and continuous in \( t \)

\(^3\)equivalent to continuity of particle displacement due to the continuity of the mean flow

\(^4\)obtained by solving the 3D problem with uniform flow in cylindrical coordinates (keeping the notation of the cartesian geometry, \( y \), instead of the radial variable \( r \) with azimuthal wave-number \( m \) and radial wavenumber \( \alpha \)
For each \( m \leq 2 \), \( \hat{p}(y) = CJ_m(\alpha y) \) is a solution with \( \hat{p}(0) = \hat{p}'(0) = 0 \), \( C \in \mathbb{R} \), the non-uniqueness being due to the singularity of the Bessel equation in 0.

### 4.2.1 Symmetric and antisymmetric solutions

Mean flows in ducts are generally speaking axisymmetric due to the symmetric geometry, thus \( U(y) = U(-y) \). For uniform flows \( U \equiv M \), with \( M \) the constant Mach number in the center of the duct) we can write the general solution as

\[
\hat{p}(y) = C_1 \cos(\alpha y) + C_2 \sin(\alpha y), \quad \alpha^2 = (\omega - kM)^2 - k^2. \tag{4.9}
\]

So there are [66] odd modes if \( C_1 = 0 \) and even ones for \( C_2 = 0 \). An alternative form of (4.9), more convenient when we are interested in surface waves, is in terms of exponentials

\[
\hat{p}(y) = D_1 e^{i\alpha y} + D_2 e^{-i\alpha y}, \tag{4.10}
\]

since the decay with the distance from the wall is directly visible.

For a symmetric mean flow profile with a boundary layer of thickness \( h \) on either walls, the nonzero modes of (4.7) are either odd or even [13, 24]. To find the antisymmetric modes, \( \hat{p}(-y) = -\hat{p}(y) \), we restrict the problem to \( y \in [-1, 0] \) with \( \hat{p}(0) = 0 \), and match at the top of the boundary layer, \( y = -1 + h \), the solution for the constant flow (4.9) with the solution in the shear region, by imposing continuity of pressure and normal velocity\(^5\). From the boundary and the continuity conditions we can write the dispersion relation governing the frequency - wavenumber interdependence for antisymmetric modes

\[
\alpha \cot(\alpha(-1 + h)) - \frac{\hat{p}'(-1 + h)}{\hat{p}(-1 + h)} = 0. \tag{4.11}
\]

In a similar way, we have the dispersion relation for symmetric solutions, where \( \hat{p}'(0) = 0 \),

\[
\alpha \tan(\alpha(-1 + h)) + \frac{\hat{p}'(-1 + h)}{\hat{p}(-1 + h)} = 0. \tag{4.12}
\]

\(^5\) equivalent, for continuous mean flows, to continuity of particle displacement
U(y) [48, 92]. A particularly interesting case is the linear-then-constant mean flow, a common simplifying assumption in the planar case [57], with the advantage that the differential equations reduce to solvable forms. We present in the next section a comparison between solutions for this and solutions for a more realistic continuously differentiable mean flow.

4.3 The parabolic boundary layer - a particular case

Let us take a linear-then-constant mean flow with boundary layer thickness $h_l$ and centerline Mach number $M$,

$$U_l(y) = \begin{cases} 
  \frac{M}{h_l} + M/|h_l|, & y \in [-1, -1 + h_l] \\
  M, & y \in [-1 + h_l, 1 - h_l] \\
  -\frac{M}{h_l} + M/|h_l|, & y \in [1 - h_l, 1] 
\end{cases} \quad (4.13)$$

and a continuously differentiable profile with a parabolic boundary layer of thickness $h_p$, outside which the mean flow is constant and equal to $M$

$$U_p(y) = \begin{cases} 
  -\frac{My^2}{h_p^2} - 2M(1 - h_p)y/h_p^2 - M(1 - 2h_p)/h_p^2, & y \in [-1, -1 + h_p] \\
  M, & y \in [-1 + h_p, 1 - h_p] \\
  -\frac{My^2}{h_p^2} + 2M(1 - h_p)y/h_p^2 - M(1 - 2h_p)/h_p^2, & y \in [1 - h_p, 1]. 
\end{cases} \quad (4.14)$$

In order to compare the two cases, let us choose the two boundary layer thicknesses such as to have the same mass flux through a segment

$$\int_{-1}^{1} \rho_0 U_l(y)dy = \int_{-1}^{1} \rho_0 U_p(y)dy.$$

This amounts to

$$h_p = \frac{3}{2} h_l. \quad (4.15)$$
An equivalent approach is via the displacement thickness, which is for a velocity profile \( U \) and a free stream Mach number \( M \)

\[
h^* = \int_{-1}^{1} \left( 1 - \frac{U(y)}{M} \right) dy;
\]
equation (4.15) can be obtained by fixing the displacement thickness of the two profiles \( U_l \) and \( U_p \).

To compare the two profiles (see also Figure 4.1 a), we look at their effect on the plane waves for hard and soft walls. To start with, plane waves (of normal incidence) in a 2D geometry with hard walls and uniform flow \( U \equiv M \) [94], are given by

\[
\hat{p}''(y) + \alpha^2 \hat{p} = 0, \quad \alpha^2 = (\omega - kM)^2 - k^2,
\]

with \( \hat{p}' \) denoting derivatives with respect to \( y \), and boundary condition at \( y = \pm 1 \)

\[
\hat{p}'(\pm 1) = 0.
\]

For a non-zero \( \hat{p} \), this amounts to \( \alpha = 0 \), or

\[
k = \pm \frac{\omega}{1 \pm M}.
\]
The results for the linear and the parabolic mean flows for hard walls and soft walls of impedance $Z = 2 + i$, are given in Table 4.1 and 4.2, respectively. We can see that, for the hard wall case, the difference with the uniform flow results is mostly in the order of the boundary thickness. The difference between the parabolic and the linear case is for both hard and soft-walls in the same order. Such being the case we will work in the following chapters with linear profiles, due to significant simplifications in the differential equations.

Table 4.1: Plane waves for thick ($h_l = 5\%$) and thin ($h_l = 0.5\%$) linear (l) and parabolic (p) boundary layers; $h_p$ is calculated to keep a constant mass flow. Results are for hard walls, and are to be compared to the uniform flow plane waves ($u$).

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$h_l$</th>
<th>$u$</th>
<th>$l$</th>
<th>$p$</th>
<th>$u$</th>
<th>$l$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>5%</td>
<td>-1</td>
<td>-0.9470</td>
<td>-0.9739</td>
<td>0.3828</td>
<td>0.3919</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.5%</td>
<td>-1</td>
<td>-1.0054</td>
<td>-1.0079</td>
<td>0.3523</td>
<td>0.3215</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5%</td>
<td>-4</td>
<td>-4.1200</td>
<td>-3.8950</td>
<td>1.1987</td>
<td>1.2256</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.5%</td>
<td>-4</td>
<td>-3.9900</td>
<td>-4.0011</td>
<td>1.3777</td>
<td>1.1279</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.5%</td>
<td>-40</td>
<td>-39.9594</td>
<td>-40.0017</td>
<td>13.3553</td>
<td>13.3368</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2: Plane waves for thick ($h_l = 5\%$) and thin ($h_l = 0.5\%$) linear (l) and parabolic (p) boundary layers; $h_p$ is calculated to keep a constant mass flow. Results are for hard walls, and are to be compared to the uniform flow plane waves ($u$).

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$h_l$</th>
<th>profile</th>
<th>left-running pl. wv.</th>
<th>right-running pl. wv</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega = \frac{1}{2}$</td>
<td>1%</td>
<td>lin.</td>
<td>$-0.59169 + 0.51531i$</td>
<td>$0.32174 - 0.1049i$</td>
</tr>
<tr>
<td></td>
<td>1%</td>
<td>parab.</td>
<td>$-0.59153 + 0.51547i$</td>
<td>$0.32175 - 0.10491i$</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>lin.</td>
<td>$-0.59143 + 0.49148i$</td>
<td>$0.32435 - 0.10657i$</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>parab.</td>
<td>$-0.59017 + 0.49305i$</td>
<td>$0.32446 - 0.10672i$</td>
</tr>
<tr>
<td>$\omega = 2$</td>
<td>1%</td>
<td>lin.</td>
<td>$-3.58383 + 0.20992i$</td>
<td>$1.28678 - 0.08607i$</td>
</tr>
<tr>
<td></td>
<td>1%</td>
<td>parab.</td>
<td>$-3.58388 + 0.21046i$</td>
<td>$1.28683 - 0.08611i$</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>lin.</td>
<td>$-3.53267 + 0.16629i$</td>
<td>$1.29712 - 0.08854i$</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>parab.</td>
<td>$-3.5333 + 0.17003i$</td>
<td>$1.29768 - 0.08877i$</td>
</tr>
<tr>
<td>$\omega = 20$</td>
<td>1%</td>
<td>lin.</td>
<td>$-39.93961 + 0.00182i$</td>
<td>$13.29004 - 0.02108i$</td>
</tr>
<tr>
<td></td>
<td>1%</td>
<td>parab.</td>
<td>$-39.93959 + 0.00187i$</td>
<td>$13.29003 - 0.02125i$</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>lin.</td>
<td>$-39.93036 + 0.00002i$</td>
<td>$13.24696 - 0.03683i$</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>parab.</td>
<td>$-39.92861 + 0.00002i$</td>
<td>$13.24471 - 0.03648i$</td>
</tr>
</tbody>
</table>

Similar results with those presented in Tables 4.1, 4.2 are obtained if we com-
pare the linear and the parabolic boundary layers by fixing the momentum flux

\[ \int_{-1}^{1} \rho_0 U_l(y)(M - U_l(y))dy = \int_{-1}^{1} \rho_0 U_p(y)(M - U_p(y))dy. \]

In this case, the relation between the two boundary thicknesses amounts to (see Figure 4.1 b.)

\[ h_p = \frac{5}{4} h_l. \quad (4.16) \]

For different free stream Mach numbers, \( M_l \) and \( M_p \), the above relation becomes

\[ h_p M_p^2 = \frac{5}{4} h_l M_l^2. \]
Chapter 5

Mean flow boundary layer effects of the hydrodynamic instability along an impedance wall

As we argued in the Introduction, in order to proceed with research and development in the acoustics of engine ducts, we need a reliable model to start with. The uniform mean flow case is of particular interest, since all the complexity of boundary layer effects are concentrated at the boundary. A suitable boundary condition for incident plane waves on a vortex sheet [56], was developed by recognizing that the correct continuity conditions to be employed, were those of pressure and of particle displacement [56, 70, 83]. This was later shown [44, 106] to be indeed a limit case of flows of vanishing boundary thickness $h$, where $h \ll \lambda$, the typical wave length. For mean flows, $U$, along a
straight wall in (say) $x$-direction, this results into the Ingard condition

$$i \omega (\hat{v} \cdot n) = \left[ i \omega + U \frac{\partial}{\partial x} \right] \left( \frac{\hat{p}}{Z} \right),$$

or generalizing for curved surfaces [74], the later developed Myers condition

$$i \omega (\hat{v} \cdot n) = \left[ i \omega + V_0 \cdot \nabla - n \cdot (n \cdot \nabla V_0) \right] \left( \frac{\hat{p}}{Z} \right).$$

However, analyzing the flow along an impedance wall, revealed an instability mode [105] that could be held responsible for the howling of some liners [14, 46], and that bares some similarities with the Kelvin-Helmholtz instability of a free vortex sheet [90]. However, the problem was not thoroughly considered until fairly recently when encountered with CAA solvers in time-domain [36, 76, 85, 103]. Further experimental investigations revealed a sound amplification along liners in some special cases [10, 11, 26, 68] while analytical efforts pointed out the existence of an unstable surface wave for flows along impedance walls [88, 96]. Nevertheless, an important step towards the fundamental understanding of the problem, was the observation of Brambley [20] that the Ingard-Myers limit leads to an ill-posed problem in time-domain causing the field to grow at an arbitrary exponential rate.

Although there is little doubt that the limit $h \to 0$ of vanishing boundary layer thickness, is correct, there must be something wrong in our modeling assumptions. In particular, there must be a very small length scale in the problem, other than $\lambda$, on which $h$ scales at the onset of instability. This is what we consider and study here.

We choose a simple model of a semi-infinite linear-then-constant mean flow along an impedance wall in the incompressible limit, and show in the first part that the flow changes stability when varying $h$ beyond a critical value, $h_c$. Moreover, the flow is absolutely unstable for $h < h_c$, and convectively unstable otherwise. We argue that for practical liner and flow parameters, $h_c$ is smaller than any available boundary layer thickness. We propose a regularization to the available Ingard-Myers boundary condition, by extending it to include $O(h)$ terms, and study the emerging approximation in comparison with the original problem. In addition to that, we derive an asymptotic formula for $h_c$, comple-
5.1 Incompressible linear-then-constant shear flow

mented by a contourplot for the yet uncovered parameter ranges. We also adapt the surface-waves relation for the new boundary condition. We compare our regularization with that of Bramley [23] arguing that they are both unaffected by compressibility, and show that smoothness effects are for practical aeronautical applications also of negligible order. We conclude with some possible limitations and extensions of the here presented results.

5.1 Incompressible linear-then-constant shear flow

We start with the Pridmore-Brown equation for an inviscid 2D parallel mean flow $U(y)$ (Figure 5.1), with uniform mean pressure $p_0$ and density $\rho_0$, and small isentropic perturbations (see also Chapter 4, section 4.1) in dimensional form

$$\frac{d^2 \hat{p}}{dy^2} + \frac{2k}{\omega - kU} \frac{dp}{dy} + \left( \frac{(\omega - kU)^2}{c_0^2} - k^2 \right) \hat{p} = 0. \quad (5.1)$$

At $y = 0$ we have a lined wall, giving the impedance boundary condition

$$- \frac{\hat{p}(0)}{\hat{v}(0)} = Z(\omega), \quad (5.2)$$

with normal velocity given by

$$\hat{v} = \frac{i}{\rho_0(\omega - kU)} \frac{d\hat{p}}{dy}. \quad (5.3)$$

At this point we conjecture that the instability in the model is due to the interaction between the impedance wall, and the mean flow, and hence select solutions of surface wave type by assuming exponential decay for $y \to \infty$.

The mean flow is typically uniform everywhere, except for a thin boundary layer of thickness $h$. We look for frequency ($\omega$) and wavenumber ($k$) combinations that allow a solution. The stability of this solution is investigated as a function of the problem parameters. In particular, we are interested in the critical thickness $h = h_c$ below which the flow becomes absolutely unstable.
5.1.1 Dimension analysis and scaling

Since the frequency and wave number at which the absolute instability first appears are part of the problem, $h_c$ does not depend on $\omega$ or $k$. Consequently, the Ingard-Myers limit taking $h \to 0$, based on $h/\lambda \ll 1$ ($\lambda$ a typical acoustic wavelength) is not applicable to the instability problem. Furthermore, since the associated surface wave (see [88] equations (12)-(13)) is of hydrodynamic nature and inherently incompressible, $h_c$ is only weakly depending on sound speed $c_0$, with $p_0$ also playing no role anymore. As there are no other length scales in the fluid, $h_c$ must scale on an inherent length scale of the liner. If we take a liner of mass-spring-damper type (see Chapter 2, section 2.2.1)

$$Z(\omega) = R + i\omega m - iK/\omega,$$

we eventually have 6 parameters ($h_c, \rho_0, U_\infty, R, m, K$) and 3 dimensions\(^1\) (m, kg, s), and it follows from Buckingham’s theorem that our problem has three dimensionless numbers, for example

$$\frac{R}{\rho_0 U_\infty^2}, \frac{mK}{\rho_0 U_\infty^2}, \frac{Kh_c}{\rho_0 U_\infty^2}.$$

(5.5)

We see in section 5.1.5 that a proper reference length scale for $h_c$, i.e. one that preserves its order of magnitude, is a more complicated combination of these parameters, namely we find that we can write, for a function $H = O(1)$,

$$h_c = \left(\frac{\rho_0 U_\infty}{R}\right)^2 U_\infty \sqrt{\frac{m}{K}} H \left(\frac{R}{\rho_0 U_\infty}, \frac{\sqrt{mK}}{\rho_0 U_\infty}\right).$$

(5.6)

However, since a relation between the different parameters is part of our problem, it is not useful at this point to nondimensionalize on arbitrary scales, and we therefore keep our problem in dimensional form.

\(^1\)since the parameters of the mass-spring-damper impedance are in dimensional units $[m] = \text{kg}/\text{m}^2$, $[K] = \text{kg}/\text{m}^2\text{s}^2$ and $[R] = \text{kg}/\text{m}^2\text{s}$, see also Chapter 2 section 2.2.1
5.1 Incompressible linear-then-constant shear flow

5.1.2 The dispersion relation

As the stability problem is essentially incompressible, we consider the incompressible limit, where \(\omega/k\) and \(U\) are negligible compared to \(c_0\). Then the Pridmore-Brown equation reduces to

\[
d\frac{d^2\hat{p}}{dy^2} + \frac{2k\frac{d\hat{p}}{dy}}{\omega - kU} - k^2\hat{p} = 0. \tag{5.7}
\]

If we assume a linear-then-constant velocity profile of thickness \(h\)

\[
U(y) = \begin{cases} 
yU_\infty & \text{for } 0 \leqslant y \leqslant h \\
h & \text{for } h \leqslant y < \infty \\
U_\infty & \text{for } h \leqslant y < \infty
\end{cases} \tag{5.8}
\]

we have an exact solution for our problem.

For \(y \geqslant h\) we have

\[
\hat{p} = A e^{-|k|y}, \quad \text{where } |k| = \text{sign}(\text{Re}k)k. \tag{5.9}
\]

Other representations of \(|k|\) are \(\sqrt{k^2}\) or \(\sqrt{1/k} \sqrt{-1}k\) with principal square roots assumed for \(\sqrt{-}\) in all cases. \(|k|\) is the generalization of the real absolute value function which is analytic in the right and in the left complex half-plane. It has discontinuities along \((-\infty, 0)\) and \((0, \infty)\), which correspond with the branch cuts of the square roots. The notation \(|k|\) is very common in this kind of problems [65, 78, 79], but of course should not be confused with the complex modulus of \(k\).

In the shear layer region \((0, h)\) we have

\[
\hat{p}(y) = C_1 e^{ky}(h\omega - kyU_\infty + U_\infty) + C_2 e^{-ky}(h\omega - kyU_\infty - U_\infty) \tag{5.10a}
\]

\[
\hat{v}(y) = \frac{kh}{\rho_0}(C_1 e^{ky} + C_2 e^{-ky}) \tag{5.10b}
\]

\[
\hat{v}(y) = \frac{ikh}{\rho_0}(C_1 e^{ky} - C_2 e^{-ky}). \tag{5.10c}
\]

This last solution is originally due to Rayleigh [43], but has been used in a similar context of stability of flow along a flexible wall by [65].
When we apply continuity of pressure and particle displacement at the interface \( y = h \), and the impedance boundary condition at \( y = 0 \), we eliminate \( C_1 \) and \( C_2 \), and obtain the dispersion relation of possible values for \( \omega \) and \( k \)

\[
0 = D(k, \omega) = Z(\omega) + \frac{i \rho_0 (h \omega - U_\infty) (kh \Omega + |k|(h \Omega + U_\infty)) e^{kh} + (h \omega + U_\infty) (kh \Omega - |k|(h \Omega - U_\infty)) e^{-kh}}{kh (kh \Omega + |k|(h \Omega + U_\infty)) e^{kh} - (kh \Omega - |k|(h \Omega - U_\infty)) e^{-kh}}.
\]

(5.11)

where

\[
\Omega = \omega - k U_\infty.
\]

(5.12)

5.1.3 Stability analysis

We consider a careful stability analysis for our problem once we transit from frequency-wavenumber to the time-space domain, and consequently evaluate the inverse Laplace-Fourier transform, generically of the form

\[
p(x, y, t) = \frac{1}{(2\pi)^2} \int_{I_o} \int_{F_x} \frac{\varphi(y)}{D(k, \omega)} e^{i \omega t - ikx} dk \omega.
\]

(5.13)

It is of foremost interest to see whether there is a physically genuine instability, as may be the case in \([10, 11, 26, 68]\), and to come to an explanation as to why this is very rare for aeronautical applications \([14, 59]\).

We follow the method described in Chapter 3, and check first if there exists a minimum imaginary part of the possible \( \omega_j \):

\[
\omega_{\text{min}} = \min_{k \in \mathbb{R}} \left[ \text{Im} \, \omega_j(k) \right].
\]

(5.14)

This is relatively easy for a mass-spring-damper impedance, because the dispersion relation is equivalent to a third order polynomial in \( \omega \) with just 3 solutions, which can be traced without difficulty. See Figure 5.2 for a typical case (note that we have to consider only \( \text{Re}(k) > 0 \) because of the symmetry of \( D \)). There is always a minimum imaginary part, so Briggs-Bers’ method is applicable, and since \( \omega_{\text{min}} < 0 \), the flow is unstable.
5.1 Incompressible linear-then-constant shear flow

Figure 5.2: Plots of $\text{Im}(\omega_j(k))$ for $k \in \mathbb{R}$. All have a minimum imaginary part so Briggs-Bers’ method is applicable. ($\rho_0 = 1.22, U_\infty = 82, h = 0.01, R = 100, m = 0.1215, K = 8166$.)

We consider in the $k$-plane right- and left-running poles, $k^+$ and $k^-$, and plot $k^\pm(\omega)$-images of the line $\text{Im}(\omega) = c \geq \omega_{\text{min}}$; note that while $c$ is increased, The Fourier-contour $F_k$ has to be deformed in order not to cross the poles, and this always via the origin because of the branch cuts along the imaginary axis. As $c$ is increased, $k^+$ and $k^-$ approach each other until they collide for $\omega = \omega^*$ into $k = k^*$, a pinch point of the $F_k$-integration contour. The imaginary part of the frequency, $c$, cannot be increased anymore since $F_k$ cannot be further deformed; see Figure 5.3 for a typical case here. If $\text{Im}(\omega^*) < 0$, resp. $> 0$, then $(\omega^*, k^*)$ corresponds to an absolute, resp. convective instability.

A typical example from aeronautical applications

As a typical aeronautical example we consider a low Mach number mean flow $U_\infty = 60$ m/s, $\rho_0 = 1.225$ kg/m$^3$ and $c_0 = 340$ m/s, with an impedance of Helmholtz resonator type [89]

$$Z(\omega) = R + i \omega \tilde{m} - i \rho_0 c_0 \cot \left( \frac{\omega L}{c_0} \right) \approx R + i \omega \left( \tilde{m} + \frac{1}{3} \rho_0 L \right) - i \rho_0 c_0^2 \frac{\omega^2}{\omega L}. \quad (5.15)$$

which is chosen such that $R = 2 \rho_0 c_0 = 833$ kg/m$^2$s, $L = 3.5$ cm and $\tilde{m}/\rho_0 = 20$ mm, leading to $K = 4.0 \cdot 10^6$ kg/m$^2$s$^2$ and $m = 0.039$ kg/m$^2$.

When we vary the boundary layer thickness $h$, and plot the imaginary part (= minus growth rate) of the found frequency $\omega^*$, we see that once $h$ is small
enough, the instability becomes absolute (see Figure 5.4). We call the value of $h$ where $\text{Im}(\omega^*) = 0$ the critical thickness $h_c$, because for any $h < h_c$ the instability is absolute. Note that $\text{Im}(\omega^*) \to -\infty$ for $h \downarrow 0$ so the growth rate becomes unbounded for $h = 0$, which confirms the ill-posedness of the Ingard-Myers limit, as observed by [20]. For the present example, the critical thickness $h_c$ appears to be extremely small, namely

$$h_c = 10.5 \cdot 10^{-6} \text{m} = 10.5 \mu\text{m},$$

with $\omega^* = 11023.4 \text{ s}^{-1}$, $k^* = 364.887 + i4188.99 \text{ m}^{-1}$. (5.16)

For other industrially relevant liner top plate porosities and thicknesses (leading to other values of $\tilde{m}$), we find similar values, namely $h_c = 8.5 \mu\text{m}$ for $\tilde{m}/\rho_0 = 10 \text{ mm}$, and $h_c = 13.6 \mu\text{m}$ for $\tilde{m}/\rho_0 = 40 \text{ mm}$.

It is clear that these values are smaller than any practical boundary layer thickness, so a real flow will not be absolutely unstable, in contrast to any model that adopts the Ingard-Myers limit, even though this is at first sight a very reasonable assumption if the boundary layer is only a fraction of any relevant acoustic wave length.
5.1 Incompressible linear-then-constant shear flow

5.1.4 A regularized boundary condition

Approximations for vanishing $kh$

If we carefully consider the third order approximations of the exponentials for $kh \to 0$, i.e. $e^{\pm kh} \approx 1 \pm kh + \frac{1}{2}(kh)^2 \pm \frac{1}{6}(kh)^3$, of both the numerator and denominator of the dispersion relation $D(k, \omega) = 0$, then collect powers of $kh$ up to $O(kh)$, with $\omega h/U_\infty = O(kh)$, and ignore higher order terms, we find

$$Z(\omega) \approx \frac{\rho_0}{i} \frac{\Omega^2 + |k|(\omega \Omega + \frac{1}{2}U_\infty^2 k^2)h}{|k|\omega + k^2 \Omega h}$$

(5.17)

where $\Omega = \omega - kU_\infty$. This expansion is obviously not unique. We can multiply numerator and denominator by any suitable function of $kh$, re-expand, and obtain a different, but asymptotically equivalent form. For example, we can multiply by $e^{-|k|h\theta} / e^{-|k|h\theta}$ and obtain after re-expanding numerator and deno-
We see in section 5.3.1 that the value $\theta = -1$ corresponds to a direct systematic approximation of pressure and velocity, $\hat{p}$ and $\hat{v}$. Indeed, it is not immediately clear if there is a practically preferable choice of $\theta$, but a particularly pleasing result seems to be obtained by $\theta = \frac{1}{3}$. For this choice the coefficient of the highest power of $k$ in the numerator is set to 0 and the approximate solutions are remarkably close to the “exact” ones, at least in the industrial example considered here, as will shown below (section 5.1.4, Figure 5.5; see for a comparison Figure 5.10 illustrating the $\theta = -1$ case). So in the following we will continue with the approximation

$$Z(\omega) \approx \frac{\rho_0}{i} \frac{\Omega^2 + |k|(1 - \theta)\omega^2 - (1 - 2\theta)\omega k U_\infty + (\frac{1}{3} - \theta)k^2 U_\infty^2 h}{|k|\omega + k^2(\Omega - \theta\omega)h} \quad (5.18)$$

recast in a form convenient later.

**A modified Ingard-Myers boundary condition**

Although the approximation is for vanishing $kh$, it should be noted that the behavior for large real $k$ is such that the solutions of *this* approximate dispersion relation have exactly the same stability behavior as the solutions of the original $D(k, \omega) = 0$ (see below). Not only are all modes $\omega_j(k)$ bounded from below when $k \in \mathbb{R}$, but also is the found $h$ as a function of the problem parameters very similar to the “exact” one for the practical cases considered above. It therefore makes sense to consider an equivalent boundary condition that exactly produces this approximate dispersion relation and hence replaces the effect of the boundary layer (just like the Ingard-Myers limit) but now with a nonzero $h$. For such a choice of $h$, the ill-posedness and associated absolute instability can
be avoided. Most importantly, this is without sacrificing the physics but, on the contrary, by restoring some of the inadvertently neglected physics.

If we identify at $y = 0$

$$-ik \hat{p} \sim \frac{\partial}{\partial x} \hat{p}, \quad -\frac{|k|}{i \Omega \rho_0} \hat{p} = (\hat{v} \cdot \hat{n}), \quad |k|(\hat{v} \cdot \hat{n}) \sim \frac{\partial}{\partial n} (\hat{v} \cdot \hat{n}), \quad (5.20)$$

for the normal vector $\hat{n}$ pointing into the surface, then we have a “corrected” or “regularized” Ingard-Myers boundary condition

$$Z(\omega) = \left( i \omega + U_\infty \frac{\partial}{\partial x} \right) \hat{p} - h \rho_0 i \omega \left( \frac{3}{2} i \omega + \frac{1}{2} U_\infty \frac{\partial}{\partial x} \right) (\hat{v} \cdot \hat{n}) \left( i \omega (\hat{v} \cdot \hat{n}) + h \frac{\partial^2}{\partial x^2} \hat{p} - \frac{1}{2} hi \omega \frac{\partial}{\partial n} (\hat{v} \cdot \hat{n}) \right), \quad (5.21)$$

which indeed reduces for $h = 0$ to the Ingard approximation, but now has the physically correct stability behavior. (Note that the Myers generalization for curved surfaces is more complicated.)

**Stability behavior of the modified dispersion relation**

A way to study the well-posedness of the problem with the regularized boundary condition for a mass-spring-damper impedance is to verify the lower boundedness of $\text{Im}(\omega)$ as a function of real $k$. Since $\omega$ is continuous in $k$ and finite everywhere, it is enough to consider the asymptotic behavior to large real $k$ while keeping the other length scales fixed. Equation (5.19) leads to a third order polynomial in $\omega$. Using perturbation techniques for small $1/k$ we find that two of the roots behave to leading order as

$$\omega = i \frac{R}{2m} \pm i \frac{1}{2m} \sqrt{R^2 - 4km + O(1/k)},$$

while the third one is given by

$$\omega = \frac{3}{2} k U_\infty \pm \text{sign}(k) \left( \frac{9}{4} + \frac{\rho_0 h}{m} \right) \frac{U_\infty}{h} + O(1/k).$$

So for two of the three solutions, the imaginary part of $\omega$ tends to some constant
values, while the third is $O(1/k)$ and so approaches zero (see Figure 5.5 for an example). Such being the case, the Briggs-Bers’ method is applicable.

If, for the proposed boundary condition (5.21), we vary again $h$ and plot, as in Figure 5.4, the imaginary part of the frequency $\omega^*$, with $\text{Im}(\omega^*) = 0$, we find very similar results as for the “exact value” (see Figure 5.6). Also the value $h_c$ for which the flow turns from convectively unstable to absolutely unstable is very close to the “exact” value. Note that rather good agreement was also found for the approximation that corresponds with $\theta = 0$ (equation 5.17) but the present high accuracy is definitely due to the particular choice of $\theta = \frac{1}{3}$ (equation 5.19). See for example Figure 5.5 where exact results are compared with the approximations for $\theta = 0$ and $\theta = \frac{1}{3}$. A similar comparison in Figure 5.6 is not useful because the typical difference of $O(10^2)$ would be both too small for the large graph and too big for the zoom-in to be visible.

From these results we think it is reasonable to assume that the stability behavior of the regularized Ingard-Myers boundary condition is for the industrially relevant cases the same as for the finite boundary layer model studied here, at least for $h$ small enough such that $kh$ is small for all wavenumbers $k$ considered.

5.1.5 Asymptotic relation for the critical boundary layer thickness

Insight is gained into the functional relationship between $h_c$ and the other problem parameters by considering relevant asymptotic behavior [92]. If the wall
has a high “hydrodynamic” resistance, i.e. \( r = R/\rho_0 U_\infty \gg 1 \) and a high quality factor of the resonator, i.e. \( \sqrt{mK}/R = O(\tau) \), then the inherent scalings for \( h_c \) appear to be \( m/\rho_0 h_c = O(\tau^4) \), \( kh_c = O(\tau^{-1}) \) and \( \omega h_c/U_\infty = O(\tau^{-2}) \), such that we get to leading order from \( D(k, \omega) = 0 \) and \( D_k(k, \omega) = 0 \)

\[
\begin{align*}
\begin{split}
i \left( m\omega - \frac{K}{\omega} \right) + \left( R + i\rho_0 U_\infty \frac{kh_c}{U_\infty} \right) \frac{kh_c}{U_\infty} - k^2 h_c^2 & + \cdots = 0, \\
\left( \frac{i}{\omega h_c U_\infty} - k^2 h_c^2 \right) \frac{2ik^2 h_c^2}{\omega h_c U_\infty - k^2 h_c^2} & + \cdots = 0
\end{split}
\end{align*}
\tag{5.22}
\]

With the condition that \( \omega \) is real, we have

\[
\omega \approx \sqrt{\frac{K}{m}}, \quad \frac{\omega h_c}{U_\infty} + (kh_c)^2 \approx 0, \quad \frac{R}{\rho_0 U_\infty} - \frac{i}{2kh_c} \approx 0, \tag{5.23}
\]

resulting into the approximate relation

\[
h_c \approx \frac{1}{4} \left( \frac{\rho_0 U_\infty}{R} \right)^2 U_\infty \sqrt{\frac{m}{K}}. \tag{5.24}
\]
This is confirmed by the numerical results given in Figures 5.7 and 5.8, where we plot the dimensionless quantity

\[ \frac{h_c R^2 \sqrt{K/m}}{\rho_0 U_{\infty}^2} = H \left( \frac{R}{\rho_0 U_{\infty}}, \frac{\sqrt{mK}}{\rho_0 U_{\infty}} \right) \] (5.25)

(the function \( H \) of equation 5.6) as a function of dimensionless parameters \( R/\rho_0 U_{\infty} \) and \( \sqrt{mK}/\rho_0 U_{\infty} \). In Figure 5.7 one parameter is varied while the other is held fixed at the conditions of the example in section 5.1.3, and vice versa. An even more comprehensive result is given in Figure 5.8 where a contourplot of \( H \) is given [91]. From (5.24) we know that \( H \) becomes asymptotically equal to 0.25. We see indeed that for a rather large parameter range - including the above example (indicated by a dot) - \( H \) is found between 0.2 and 0.25. So expression (5.24) appears to be an good estimate of \( h_c \) for \( R, K \) and \( m \) not too close to zero.

### 5.2 Surface waves with the new boundary condition

Changing the Ingard-Myers boundary condition, naturally affects the location and, as we see here, the number of surface waves. This was considered in [22] for the approximate dispersion proposed in [23]. We study here the surface waves for a 2D compressible uniform mean flow along an impedance wall, and follow the lines of [86] and [88]. We work here with dimensionless quantities, with density scaled on the mean density, velocities on \( c_0 \) and lengths on some
5.2 Surface waves with the new boundary condition

length in the problem, and start with the wave equation for the perturbation potential

\[ \nabla^2 \phi - \left( i \omega + M \frac{\partial}{\partial x} \right) \phi = 0, \tag{5.26} \]

with \( M \), the mean flow Mach number. To avoid an overload of notation, we kept in this section \( \omega \) and \( k \) as the symbols for the dimensionless frequency and wavenumber. In this case pressure and normal velocity are given by

\[ \bar{p} = -\left( i \omega + M \frac{\partial}{\partial x} \right) \phi, \quad \bar{v} = \frac{\partial}{\partial y} \phi. \]

We consider solutions of the form

\[ \phi = A e^{-ikx-ix}, \quad \alpha^2 = (\omega - Mk)^2 - k^2, \]

that verify the (correspondingly nondimensionalized) boundary condition (5.21) at the wall \( y = 0 \), and that decay exponentially with the distance to the wall. To satisfy the latter, we define the branch cuts of \( \alpha \) such that \( \text{Im} \alpha \leq 0 \). Applying

Figure 5.8: Contour plot of \( hR^2 \sqrt{K/m}/(\rho_0^2 U_\infty^3) \) as a function of \( R/\rho_0 U_\infty \) and \( \sqrt{mK}/\rho_0 U_\infty \). The dashed lines correspond to Figure 5.7. The dot corresponds with the conditions of example 5.1.3.
Figure 5.9: The solutions $\sigma$ of equation (5.27) as a function of $\zeta$ values. Contours showing lines of constant $\text{Re}\zeta \in [0, 50]$ (gray) and constant $\text{Im}\zeta \in [-50, 50]$ (red) in equation (5.27) for $M = 0.5$ and different boundary layer thicknesses $h$. 

(a) $h = 0.1$  
(b) $h = 0.05$  
(c) $h = 0.02$  
(d) $h = 0.005$
5.3 Comparison with Brambley’s proposed boundary condition

the modified boundary condition at the wall, we arrive at the dispersion relation

$$\zeta = \frac{-(\omega - Mk)^2 - i \omega \alpha (\frac{\omega}{3} \omega - \frac{1}{3} Mk)}{\omega \alpha - i h k^2 (\omega - Mk) - \frac{i}{3} h \omega \alpha^2}.$$  

$\zeta$ the dimensionless impedance ($Z = \rho_0 c_0 \zeta$). With the Lorentz or Prandtl-Glauert transformation

$$\omega = \beta \sigma, \quad \beta = \sqrt{1 - M^2}, \quad \sigma = \beta k + M \sigma,$$

$\sigma^2 = \sigma^2 - \sigma^2$ and the dispersion relation becomes

$$\zeta = \frac{-(\sigma - M \sigma)^2 + \frac{i}{3} h \beta^2 \sigma \alpha ((1 + \beta^2) \sigma - M \sigma)}{\beta^3 \sigma \alpha - \frac{i}{\beta} (\sigma - M \sigma)^3 - \frac{i}{3} \beta h^3 \sigma \alpha^2}. \quad (5.27)$$

Rearranging terms, and squaring in order to eliminate the $\alpha$ square root, we end up with a polynomial of 6th order in $\sigma$, the reduced wavenumber. This is in line with the findings in [23]. Figure 5.9 represents contours in the $\sigma$-plane of constant Re $\zeta$ (gray) and Im $\zeta$ (red) for various boundary thicknesses $h$, and is to be compared with Figure 4 of [88]. We can see that the “egg-shape” from the before-mentioned pictured is recovered for $h$ sufficiently small, however, the $\sigma$-far-field is to be taken with care, as it is outside the region of accuracy of the boundary condition, since $\sigma h$ is not small anymore.

5.3 Comparison with Brambley’s proposed boundary condition

A regularization similar to 5.21 was proposed in [23], with analysis and discussion done on a smooth velocity profile of a compressible flow in a cylindrical geometry. The derivation follows the lines of Eversman & Beckemeier [44], and considers an approximate solution of matched expansion type for thin mean flow boundary layers. A similar approach is taken in [60] for Galbrun’s equation of a semi-infinite smooth compressible mean flow.\footnote{Galbrun’s equation is equivalent to the linearized Euler equations; it describes in a mixed Eulerian-Lagrangian frame the propagation of linear waves in inhomogeneous moving fluids}
We take a semi-infinite sufficiently smooth incompressible mean flow, and, following the ideas of Brambley [23] and Eversman & Beckemeier, systematically approximate pressure and velocity for small boundary layer - wavelength ratio, thus re-deriving the regularized boundary condition. We show this way, that the difference to [23] comes only from the cylindrical geometry (in contrast to the 2D frame here), compressibility effects playing yet no role, as they appear only in higher order \( h \) approximations. Moreover, we find the same expressions for the regularized impedance condition as with the alternative model of above. However, the solution is, by construction, not valid for all \( k \), so it is not certain that causality analysis is applicable if the results are inadvertently diverging or otherwise singular for large \( k \).

5.3.1 Approximate dispersion relations for small boundary layer thicknesses: a matched asymptotic expansions approach

We start again with the Pridmore-Brown equation 5.7 governing a 2D incompressible shear flow along a lined wall located at \( y = 0 \), where we have an impedance boundary 5.2. We have exponential decay away from the wall,

\[
\lim_{y \to \infty} \hat{p}(y) = 0,
\]

and assume for the present analysis that

- there is a length scale \( l \), comparable to \( 1/\|k\| \), and \( l \gg h \); hence note that the analysis is not valid for wavenumbers outside this range\(^3\);
- \( \text{Re}(k) > 0 \);
- \( U(y) = U_\infty \varphi(y/h) \) where \( \varphi(z) = O(1) \) is a sufficiently smooth function with \( \varphi(0) = 0 \) and
  \[
  \lim_{z \to \infty} \varphi(z) = 1 \text{ exponentially fast};
  \]
- \( \varepsilon = h/l \to 0 \), such that \( kh = O(\varepsilon) \), \( kl = O(1) \), while \( \omega = (U_\infty/l)O(1) \).

\(^3\)note that the norm here, \( \| \cdot \| \), is to be understood as the absolute value of the argument
5.3 Comparison with Brambley’s proposed boundary condition

We redefine \( y = lY \), \( Y \) the outer variable, introducing the dimensionless \( kl = \kappa \), \( \omega l/U_\infty = S \) (Strouhal number) and \( \tilde{p}(y) = \tilde{p}(Y) \), such that

\[
\frac{d^2 \tilde{p}}{dY^2} + \frac{2\kappa}{S - \kappa \varphi} \frac{d\tilde{p}}{dY} - \kappa^2 \tilde{p} = 0 \quad \text{and} \quad i(S - \kappa \varphi)\tilde{v} + \frac{d\tilde{p}}{dY} = 0
\]

with the boundary conditions

\[
\tilde{p}(0) = -Z\tilde{v}(0) = \frac{Z}{iS\rho_0 U_\infty} \frac{d}{dY} \tilde{p}(0); \quad \lim_{Y \to \infty} \tilde{p}(Y) = 0.
\]

For the method of matched asymptotic expansions consider two regions: one outside the boundary layer where the mean flow variation is very small and can be approximated by \( U_\infty \), and an inner region defined by the boundary layer, where a different scaling is needed in order to capture the correct behavior of the solution.

Assume the expansion for \( Y = O(1) \) (i.e. outside the boundary layer where \( \varphi' \) is exponentially small)

\[
\tilde{p}(Y; \varepsilon) = p_0(Y) + \varepsilon p_1(Y) + \varepsilon^2 p_2(Y) + \ldots
\]

we get

\[
\frac{d^2 p_0}{dY^2} + \varepsilon \frac{d^2 p_1}{dY^2} + \varepsilon^2 \frac{d^2 p_2}{dY^2} - \kappa^2 p_0 - \varepsilon \kappa^2 p_1 - \varepsilon^2 \kappa^2 p_2 = 0 + \ldots.
\]

with the outer solution (satisfying the boundary condition at \( \infty \))

\[
p_0(Y) = A_0 e^{-\kappa Y}, \quad p_1(Y) = A_1 e^{-\kappa Y}, \quad p_2(Y) = A_2 e^{-\kappa Y}
\]

There is no restriction against taking \( A_1 = A_2 = 0 \). We expand, for later use, the outer solution for small \( Y \)

\[
p(Y; \varepsilon) = A_0 e^{-\kappa Y} = A_0 - \kappa YA_0 + \frac{1}{2}(\kappa Y)^2 A_0 + \ldots \quad (5.28)
\]

In the boundary layer, we rescale \( Y = \varepsilon z \) and denote \( \tilde{p}(Y; \varepsilon) = P(z; \varepsilon) \) to obtain

\[
\frac{d^2 P}{dz^2} + \frac{2\kappa \varphi'(z)}{S - \kappa \varphi(z)} \frac{dP}{dz} - (\varepsilon k)^2 P = 0, \quad i\varepsilon S\rho_0 U_\infty P(0) = Z \frac{d}{dy} p(0). \quad (5.29)
\]
We determine $P$ by matching it to the outer solution $\tilde{p}(Y;\varepsilon)$ for $z \to \infty$. So, we first expand

$$P(z;\varepsilon) = P_0(z) + \varepsilon P_1(z) + \varepsilon^2 P_2(z) + \ldots$$

to get from (5.29)

$$
\frac{d^2 P_0}{dz^2} + \varepsilon \frac{d^2 P_1}{dz^2} + \varepsilon^2 \frac{d^2 P_2}{dz^2} + \frac{2\kappa \psi'(z)}{S - \kappa \psi(z)} \frac{dP_0}{dz} + \varepsilon \frac{2\kappa \psi'(z)}{S - \kappa \psi(z)} \frac{dP_1}{dz} + \varepsilon^2 \frac{2\kappa \psi'(z)}{S - \kappa \psi(z)} \frac{dP_2}{dz} - (\varepsilon \kappa)^2 P_0 = 0 + \ldots
$$

Thus $P_0$ satisfies

$$
\frac{d^2 P_0}{dz^2} + \frac{2\kappa \psi'(z)}{S - \kappa \psi(z)} \frac{dP_0}{dz} = 0
$$

with

$$P_0(z) = C_1 + C_0 \int_0^z (S - \kappa \psi(\zeta))^2 \, d\zeta.$$

The first matching step requires that $\lim_{z \to \infty} P_0(z) = A_0$, and since, for $z \to \infty$, $P_0(z) \sim C_0(S - k)^2z$, we need to take $C_0 = 0$ and $C_1 = A_0$.

There’s a similar differential equation for $P_1$, rendering

$$P_1(z) = D_1 + D_0 \int_0^z [(S - \kappa \psi(\zeta))^2 - (S - k)^2] \, d\zeta + D_0(S - k)^2z.$$

Since for large $z$

$$P_1(z) \approx D_1 + D_0 \int_0^\infty [(S - \kappa \psi(\zeta))^2 - (S - k)^2] \, d\zeta + D_0(S - k)^2z \equiv -\kappa z A_0,$$

we have

$$D_0 = -\kappa A_0/(S - k)^2, \quad D_1 = \kappa A_0 I_0, \quad \text{with} \ I_0 = \int_0^\infty \left( \frac{(S - \kappa \psi(\zeta))^2}{(S - k)^2} - 1 \right) \, d\zeta.$$

For $P_2$ we have

$$
\frac{d^2 P_2}{dz^2} + \frac{2\kappa \psi'(z)}{S - \kappa \psi(z)} \frac{dP_2}{dz} = \kappa^2 A_0
$$
5.3 Comparison with Brambley’s proposed boundary condition

with

\[ P_2(z) = E_1 + E_0 \int_0^z (S - \kappa \varphi(\zeta))^2 \, d\zeta + \kappa^2 A_0 \int_0^z \int_0^z \left( \frac{S - \kappa \varphi(\zeta)}{S - \kappa \varphi(\eta)} \right)^2 \, d\eta \, d\zeta. \]

For large \( z \), we set apart the parts of the integrals that converge to constants, the parts that contain terms that are linear in \( z \) and the ones with terms quadratic in \( z \). This leads, after a little rearrangement, to

\[ P_2(z) = \frac{1}{2} A_0 \kappa^2 z^2 + E_0 (S - \kappa)^2 z + A_0 \kappa^2 I_1 z + E_1 + \]

\[ A_0 \kappa^2 \int_0^z \left[ \int_0^z \left( \frac{(S - \kappa)^2}{(S - \kappa \varphi(\eta))^2} - 1 \right) \, d\eta \, d\zeta + E_0 \int_0^z [(S - \kappa \varphi(\zeta))^2 - (S - \kappa)^2] \, d\zeta \right. \]

\[ + \left. A_0 \kappa^2 \int_0^z \left[ \left( \frac{S - \kappa \varphi(\zeta)}{S - \kappa} \right)^2 - 1 \right] \int_0^z \left( \frac{S - \kappa}{S - \kappa \varphi(\eta)} \right)^2 \, d\eta \, d\zeta, \]

with

\[ I_1 = \int_0^\infty \left( \frac{(S - \kappa)^2}{(S - \kappa \varphi(\zeta))^2} - 1 \right) \, d\zeta. \]

The quadratic term matches perfectly the quadratic term in the outer solution. Now matching for the other terms for \( z \to \infty \):

\[ E_0 = -\frac{A_0 \kappa^2 I_1}{(S - \kappa)^2}, \]

\[ E_1 = -A_0 \kappa^2 \int_0^\infty \left[ \int_0^z \left( \frac{(S - \kappa)^2}{(S - \kappa \varphi(\eta))^2} - 1 \right) \, d\eta \, d\zeta - I_1 \right] \, d\zeta. \]

To derive a regularized approximate dispersion relation, we retain in both \( \hat{p}(0) \) and \( \hat{v}(0) \), terms up to \( O(\varepsilon) \)

\[ \hat{p}(0) = P_0(0) + \varepsilon P_1(0) = A_0 (1 + \varepsilon \kappa I_0) \]

\[ \hat{v}(0) = \frac{1}{S} (P'_0(0) + \varepsilon P'_1(0)) = -\frac{i A_0 S}{(S - \kappa)^2} (\kappa + \varepsilon \kappa^2 I_1). \]
5.3.6 Boundary layer thickness effects

Figure 5.10: Plots of \( \text{Im}(\omega_j(k)) \) for \( k \in \mathbb{R} \) for the exact (solid) and the approximate (dashed) dispersion relation (5.31). \( (\rho_0 = 1.22, U_\infty = 82, h = 0.01, R = 100, m = 0.1215, K = 8166). \)

The above agrees with [23] up to a term proportional to \( m \) accounting for the cylindrical geometry, with the remark that the Bessel functions there, are to be taken here in correspondence to exponentials (see equation (5.28)).

For a linear-then-constant velocity profile, \( I_0 \) and \( I_1 \) can be explicitly computed, and we arrive to the following approximate dispersion relation

\[
Z(\omega) + \rho_0 U_\infty \frac{1 + \varepsilon \frac{k}{(S-k)^3}(-\frac{2}{3}k^2 + kS)}{\frac{i k}{(S-k)^3}(-S + \varepsilon k^2)} = 0, \quad (5.31)
\]

which corresponds to \( \theta = -1 \) in (5.18). A comparison to the exact dispersion relation for this case is shown in Figure 5.10a for the temporal growth rate. However, as we already mentioned in section 5.1, the best results, with respect to predicting the absolute instability, seem to be obtained for \( \theta = 1/3 \).

5.4 Smoothness effects on the critical boundary layer thickness

Consider now a family of smooth mean flow profiles

\[
u_n(y) = U_\infty \left( \tanh \frac{x_n}{R} \right)^{1/n}. \quad (5.32)
\]

A comparison for various \( n \) is given in Figure 5.11.
5.5 Summarizing remarks

Figure 5.11: Comparison between the linear-then-constant profile (black) and (5.32) for $n = 1$ (blue) and $n = 3$ (red); $U_\infty = 1, h = 1$.

Taking a semi-infinite incompressible medium, and an impedance wall at $y = 0$, we compare this to the results of section 5.1.3, and find that the growth rate, as well as the corresponding pinch frequency follow the same curve as in the linear-then-constant case (see Figure 5.12). However, the $k$ values of the pinch point seem to be more sensitive, and its behavior is still to be explained.

Since we used a central difference method to solve (5.1) for $U = u_3$, we expect accuracy in the regions encircled in Figure 5.13.

5.5 Summarizing remarks

We propose in this chapter a regularization of the Ingard-Myers boundary condition by using a semi-infinite incompressible linear-then-constant mean flow, and argue that, in its valid range of wavenumbers and frequencies, compressibility and smoothness effects are negligible. However, care should be taken for values of parameters outside the asymptotically valid ranges, as results could be seriously disturbed by errors.

A recent study [67] shows that the finiteness of the duct, also possibly viscosity [12, 21], can play an important role for the prediction of the unstable mode. Another important result is the estimate of the critical boundary layer thickness, which gives an indication as to when to expect genuine absolute instabilities in our model. An interesting further investigation, would be an experimental validation of the prediction of the critical thickness (either (5.24) or Figure 5.8).
Figure 5.12: Comparison between the linear-then-constant profile (black) and (5.32) for $n = 3$ (red); $U_\infty = 60$: growth rate $\text{Im}(\omega^*)$ against $h$ of potential absolute instability at vanishing group velocity (pinch point) is plotted together with the corresponding complex frequency $\omega^*$ and wave number $k^*$. Parameters as in Figure 5.6.

Figure 5.13: Comparison between the linear-then-constant profile (black) and (5.32) for $n = 3$ (red); $U_\infty = 60$. Parameters as in Figure 5.6; plots as in Figure 5.12. The reliable range of $h$, $\omega^*$ and $k^*$ is encircled.
Chapter 6

Critical layer singularities

This study is the result of a close collaboration with dr. E.J. Brambley. We will focus here on some parts of the analysis, acknowledging among other things his insight in separating the various contributions in the Green’s function. We mention that the figures in section 6.6 are courtesy of dr. Brambley and have been included here for the clarity and logical build-up of the exposition, referring for further details to the joint paper [25].

Critical layers arise for waves of the form $f(r)e^{i\omega t - ikx}$ in inviscid shear flows as a mathematical singularity at points where the axial phase velocity, $\omega/k$, is equal to the local fluid velocity, and can give rise to a logarithmic singularity in the spatial Fourier transform of the solution. This can be smoothed out by taking into account additional viscous or nonlinear terms in the neighborhood of the singular point (see [51, 55, 69]). However, one can avoid adding complexity to the problem by defining a proper branch cut for the complex logarithm [33].

1. We see in section 6.2.1 that for planar shear, the critical layer though present, does not lead to a log-singularity in pressure; therefore we distinguish between “critical layers” and “critical layer singularities”
based on causality arguments, with the restriction that the spatial Fourier inversion contour in the wavenumber plane should not cross the branch cut.

We investigate the effects of the critical layer for a simple model having as few parameters as possible, and understand the phenomena linked to it as well as its various contributions to the total field.

We study the field generated by a time-harmonic point mass source in a circular duct with a constant-then-linear mean flow and a constant density profile. This choice is justified in section 6.2.1 as one of the simplest scenarios where a critical layer singularity occurs. Linearizing the compressible Euler equations and Fourier transforming in the axial coordinate results in a single second order differential equation for pressure in radial coordinate. Solutions are given in terms of Bessel functions for the constant flow and Frobenius series for the linear shear [31], thus having the necessary tools for constructing the Green’s function for a point mass source. By analyzing the contributions of the poles as well as the effects of branch cuts, we show that when the source is in the uniform flow part the critical layer’s contribution may well be neglected. This is certainly not the case for sources in the boundary layer, because neglecting the critical layer in this case, implies neglecting an important contribution of the source. We discuss this in section 6.4.2 and illustrate it in section 6.6. Another particularity we find is a downstream pole closely related to the critical layer, a pole that should always be taken together with the critical layer, leading otherwise to serious inaccuracies in the total field. To complete the study we conclude with a section of numerical results and findings.

Throughout this chapter we will consider differential equations and solutions both in the classical and distributional sense without an explicit announcement of the framework we situate ourselves in. We do so, in order to keep a natural flow of ideas and arguments, as the sound field is everywhere smooth except at the source. For a rigorous justification we refer to Appendix C.

### 6.1 State of the art

In an analogy with spectral theory, the discrete wavenumbers that satisfy the dispersion equation for a fixed frequency are associated to the discrete spec-
6.1 State of the art

trum of the wave operator considered [100, 111] whereas the critical layer singularities are put in correspondence with the continuous spectrum [34, 82]. In fact, since critical layers are singularities in the differential equations where \( \omega - kU = 0 \), they correspond in the \( k \)-wavenumber plane to a continuum of logarithmic branch points \((\omega/U_{\text{max}}, \omega/U_{\text{min}})\), for mean flow velocity \( U \) with values between \( U_{\text{min}} \) and \( U_{\text{max}} \). Their contribution has so far been proved to have an algebraic rather than exponential decay or growth rate along the duct, as is the case for swirling flows which in general exhibit algebraic growth [32, 50, 104].

The reference for critical layers in a duct carrying sheared flow was thus far the work of Swinbanks (1975) [100]. Considering the sound field in such a case, and arbitrary Dirichlet-Neumann boundary conditions, he found that the eigenfunction representation of the pressure field generated by a mass source broke down at the critical layer. Thus, the normal modes no longer formed a complete basis, and one had to add also the integrals around the log branch cuts. For sheared flows, this latter part is only present downstream. In the case of a hard-walled duct with constant velocity profile except for a thin boundary layer, this contribution appears, in the worst case scenario when the source is at the critical layer, as a singularity consisting of a simple pole and a logarithmic branch point. Inverting the Fourier transform, the prediction in [100] was algebraic decay of \( O(1/x^3) \) for a point mass source and of \( O(1/x^{1/2}) \) for a source of distributed nature.

Recently, numerical computations showed [45] an \( O(1/x) \) decay for a point source in a two-dimensional hard-walled duct with a parabolic mean flow profile, suggesting that further analysis is necessary.

When it comes to determining the sound field in a lined duct, the critical layer is either explicitly (as for example in [30]) or tacitly (as in [75]) ignored in the majority of cases, assuming its contribution to the total field to be insignificant. Numerical methods detect the critical layer branch points as a set of spurious eigenvalues clustered on the positive real wavenumber axis [18, 110], their number depending on the spatial discretization considered.
6.2 Problem formulation

Consider an inviscid compressible parallel sheared flow within a cylinder, with mean flow velocity $U(r)$ and density $\rho_0(r)$. Time harmonic acoustic perturbations to this flow of frequency $\omega$ can be found by Fourier series expansion in the circumferential coordinate $\theta$ and Fourier transformation to the axial coordinate $x$ with wave number $k$. For suitable solutions $\hat{p}(r; k, m, \omega)$, the physical pressure field $p(x, r, \theta, t)$ is given by (the real part of) the sum over Fourier integrals (see also section 2.1)

$$p(x, r, \theta, t) = \text{Re} \left( \frac{e^{i\omega t}}{2\pi} \sum_{m=-\infty}^{\infty} e^{-im\theta} \int_{-\infty}^{\infty} \hat{p}(r; k, m, \omega) e^{-ikx} dk \right).$$

Well-chosen indentations of the $k$-inversion contour are understood when singularities of any kind along the real axis are to be avoided. In this section we will be interested in a single $m$-mode, represented by the $k$-integral only.

The resulting equation to be solved is of Pridmore-Brown type [81], given in dimensional inhomogeneous form as (see Appendix B)

$$\dddot{p} + \left( \frac{2kU'}{\omega - kU} + \frac{1}{r} - \frac{\rho'}{\rho_0} \right) \ddot{p} + \left( \frac{(\omega - kU)^2}{c_0^2} - k^2 - \frac{m^2}{r^2} \right) \dot{p} = -i(\omega - kU)\hat{Q}, \quad (6.1)$$

with $c_0$ the constant speed of sound and forcing term $\hat{Q}(r; k, m)$, the Fourier transform and Fourier series coefficient of a mass source. This equation contains a regular singularity at $r = r_c$, where $\omega - kU(r_c) = 0$, referred to as a critical layer (located at $r = r_c$).

For simplicity, we will skip the $\omega$- and $m$-dependence in $\hat{p}$ (and later in $\hat{G}$), retaining occasionally the $k$-dependence when needed.

6.2.1 Frobenius method

A way to obtain two linearly independent solutions to the homogeneous Pridmore-Brown equation, expanded about $r_c$, is via the method of Frobenius. We look
6.2 Problem formulation

for solutions to the differential equation (6.1) of the form

\[ \hat{p}(r) = \sum_{n=0}^{\infty} a_n (r - r_c)^{n+3}, \quad a_0 \neq 0. \]  \hfill (6.2)

Substituting this expansion into the Pridmore-Brown equation and assuming that \( a_0 \) is nonzero, we obtain for the coefficient of \( (r - r_c)^1 \) an equation in \( \lambda \),

\[ \lambda(\lambda - 3) = 0, \]  \hfill (6.3)

also called indicial equation, with roots \( \lambda_1 = 3 \) and \( \lambda_2 = 0 \). There is always at least one solution in the form of a Frobenius series (see [15] pg. 71), in our case normalized by taking \( a_0 = 1 \),

\[ \hat{p}_1(r) = (r - r_c)^3 + \sum_{n=1}^{\infty} a_n (r - r_c)^{n+3}. \]  \hfill (6.4)

The coefficients \( a_n \) for \( n \geq 1 \) are determined by requiring \( p_1 \) to satisfy the homogeneous correspondent of the differential equation (6.1), arriving at the recursion relation

\[ [(\lambda + n)^2 + (q_{1,0} - 1)(\lambda + n) + q_{2,0}]a_n = -\sum_{i=0}^{n-1} [(\lambda + i)q_{1,n-i} + q_{2,n-i}]a_i, \quad n = 1, 2, \ldots \]  \hfill (6.5)

where \( \lambda = \lambda_1 \) and \( q_{1,n}, q_{2,n} \) are defined by the (assumed convergent) expansions

\[ 2kU' + \frac{\omega - kU}{r} - (\omega - kU)\frac{\rho_0'}{\rho_0} = \sum_{n=0}^{\infty} q_{1,n}(r - r_c)^n \]
\[ (\omega - kU)^2 \left( \frac{(\omega - kU)^2}{c_0^2} - k^2 - \frac{m^2}{r_c^2} \right) = \sum_{n=0}^{\infty} q_{2,n}(r - r_c)^n. \]

Repeating the procedure for \( \lambda_2 = 0 \), we have for a general solution of the form (6.2) with coefficients\(^2\) \( b_n \), that equation (6.5) reduces for \( n = 3 \) to

\[ 0 = b_0 k^2 U''(r_c)^2 \left( k^2 + \frac{m^2}{r_c^2} \right) \left( \frac{U''(r_c)}{U'(r_c)} + \frac{\rho_0'(r_c)}{\rho_0(r_c)} - \frac{1}{r_c} \right) - b_0 k^2 U'(r_c) \frac{2m^2}{3r_c^3}. \]  \hfill (6.6)

\(^2\)using \( b_n \) for \( \lambda_2 \), in contrast to \( a_n \) for \( \lambda_1 \)
If this is true, there are two linearly independent solutions in Frobenius form, with coefficients given by (6.5) for $\lambda = \lambda_{1,2}$. If, however, (6.6) does not hold, as is generally the case, we find that there is a second linearly independent solution of the form (see [15] pg. 74-75)

$$\hat{p}_2(r) = \Lambda \hat{p}_1(r) \log(r - r_c) + \sum_{n=0}^{\infty} b_n(r - r_c)^n,$$

with $b_0$, $b_3$ arbitrary, and a given relation between $\Lambda$ and $b_0$, $b_1$, $b_2$. The log-singularity is removed when coefficient $\Lambda$ is zero which is equivalent to (6.6). In general, $\Lambda$ will be nonzero, even for planar shear (where the $1/r_c$ term in (6.6) is not present). The notable cases where (6.6) holds are the uniform flow and the linear planar shear of a uniform density fluid. A relation similar to (6.6) regarding the existence of the critical layer is mentioned in [109] (for uniform density).

We take the simplest possible problem exhibiting a critical layer singularity which, following (6.6), turns out to be linear-then-constant mean flow profile with constant density in a cylindrical duct (see Figure 6.1). Here $\Lambda$ is nonzero owing to the non-planar shear due to the cylindrical geometry.
6.2 Problem formulation

6.2.2 Solutions to the Pridmore-Brown equation

We aim at constructing Green’s function solutions for equation (6.1) with boundary conditions at $r = 0$ and $r = 1$, as for a singular Sturm-Liouville problem. Thus, we proceed to solve the homogeneous Pridmore-Brown equation for a linear-then-constant shear with constant density in a cylindrical duct. We scale distances on the duct radius, velocities on the sound speed $c_0$, density on the mean density $\rho_0$, pressure on $\rho_0 c_0^2$ and impedance on $\rho_0 c_0$, thus having the duct wall at $r = 1$ and

$$U(r) = \begin{cases} M, & 0 \leq r \leq 1 - h, \\ M(1 - r)/h, & 1 - h \leq r \leq 1, \end{cases}$$

with $M$ the mean flow Mach number. Since for the rest of the chapter we will work with the non-dimensional quantities, we will keep, for the sake of simplicity, the same notations as for the dimensional case.

The homogeneous, dimensionless form of (6.1) is then

$$\hat{p}'' + \left( \frac{2kU'}{\omega - kU} + \frac{1}{r} \right) \hat{p}' + \left( \omega - kU \right)^2 - k^2 - \frac{m^2}{r^2} \hat{p} = 0,$$

subject to the boundary conditions that $\hat{p}$ is non-singular at $r = 0$ and that

$$iZ\hat{p}' - \omega \hat{p} = 0 \quad \text{at} \quad r = 1,$$

where $Z$ is the dimensionless impedance of the lined duct walls. Since $U(1) = 0$ (no slip at the wall), there is no need to resort to the Ingard-Myers boundary condition [56, 74].

In order to solve (6.8), we consider separately the solutions within the constant-flow region $r < 1 - h$ and within the sheared region $r > 1 - h$, and then connect them (in section 6.3, in the Green’s function) by requiring the pressure to be continuously differentiable in $r \in [0, 1]$. 
Solution within the constant-flow region

The solutions to (6.8) are given within the uniform-flow section \( r < 1 - h \) by

\[
\hat{p} = A J_n(\alpha r) + B Y_n(\alpha r),
\]

where \( \alpha^2 = (\omega - kM)^2 - k^2 \). It is not necessary to choose branch cuts for \( \alpha \) in the present case, since \( \hat{p} \) appears in any instance to be a function of \( \alpha^2 \).

Frobenius expansion for constant shear

For \( 1 - h < r < 1 \), the mean flow becomes \( U(r) = M(1 - r)/h \). The Pridmore-Brown equation is singular at \( r = r_c \), where 

\[
r_c = 1 - \omega h / (kM).
\]

Note that in the physical domain \( r_c \) is real, and the singularity in the Pridmore-Brown equation occurs for real \( k \) and \( \omega \). Nevertheless in the complex \( r \)-plane, \( r_c \) is complex for complex \( k \). We expand about this singularity in a similar way as in section 6.2.1, and choose two linearly independent solutions

\[
\hat{p}_1(r) = \sum_{n=0}^{\infty} a_n (r - r_c)^{n+3},
\]

\[
\hat{p}_2(r) = \frac{1}{3r_c} \left( k^2 - \frac{m^2}{r_c^2} \right) \hat{p}_1(r) \log(r - r_c) + \sum_{n=0}^{\infty} b_n (r - r_c)^n,
\]

\[
a_n = \frac{1}{n(n+3)} \left[ k^2 a_{n-2} - \eta^2 a_{n-4} - \sum_{j=0}^{n-1} a_{n-1-j}(-1)^j(n + 2 + (m^2 - 1)j)/r_c^{j+1} \right],
\]

\[
b_n = \frac{1}{n(n-3)} \left[ k^2 b_{n-2} - \eta^2 b_{n-4} - \sum_{j=0}^{n-1} b_{n-1-j}(-1)^j(n - 1 + (m^2 - 1)j)/r_c^{j+1} \right. \\
\left. - \frac{1}{3r_c} \left( k^2 - \frac{m^2}{r_c^2} \right) \left( 2n - 3 \right) a_{n-3} + \sum_{j=0}^{n-4} a_{n-4-j}(-1)^j/r_c^{j+1} \right],
\]

\[
a_n = b_n = 0, \text{ for } n < 0; \quad a_0 = b_0 = b_3 = 1; \quad \eta = \frac{Mk}{h}. \]
6.3 The Green’s function solution

For the above choice of \( a_0, b_0 \) and \( b_3 \) we have from (6.5) at \( \lambda = 0 \) and \( n = 3 \)

\[
\Lambda = \frac{1}{3r_c} \left( k^2 - \frac{m^2}{r_c^2} \right).
\]

The log term in (6.10b) leads for \( r \in [1-h, 1] \) to branch points in the \( k \)-plane on the real axis, namely in \([\omega/M, \infty)\). For convenience, we will choose the branch cuts ranging each from the branch point to \( \infty \). For \( k \) below the branch cut we have \( \text{Im}(r_c) < 0 \), and for \( k \) above the branch cut we have \( \text{Im}(r_c) > 0 \). For a fixed \( r \), the change in \( p_2(r) \) when \( k \) crossing a log-branch cut from below is therefore

\[
\Delta p_2(r) = -\frac{2\pi i}{3r_c} \left( k^2 - \frac{m^2}{r_c^2} \right) p_1(r) H(r_c - r)
\]

(6.11)

where \( H(r) \) is the Heaviside step function. The solution \( p_1(r) \) remains continuous.

6.3 The Green’s function solution

Green’s function solutions capture all possible physics of the problem, in the sense that solutions for general driving sources can easily be generated in terms of the associated Green’s function. In order to draw physical conclusions and interpretations, we are concerned in this chapter only with the point mass source type of Green’s function; nonetheless, the results we obtain should be seen as general, in the sense mentioned before.

The field generated by a point mass source of unit strength located at \((x, \theta, r) = (0, 0, r_0), 0 < r_0 < 1\), is given by\(^3\)

\[
\hat{G}'' + \left( \frac{1}{r} + \frac{2kU'}{\omega - Uk} \right) \hat{G}'' + \left( (\omega - Uk)^2 - k^2 - \frac{m^2}{r^2} \right) \hat{G} = -i(\omega - U(r_0)k) \frac{\delta(r - r_0)}{2\pi r_0},
\]

with the Green’s function solution

\[
\hat{G} = \frac{-i(\omega - U(r_0)k)}{2\pi r_0 W(r_0; \psi_1, \psi_2)} \psi_1(r_0) \psi_2(r_0), \quad (6.12)
\]

\(^3\)see last section in Appendix C
where $\bar{W} = \psi_1 \psi'_1 - \psi'_1 \psi_2$ is the Wronskian of $\psi_1$ and $\psi_2$, $r_c = \min(r, r_0)$, and $r_2 = \max(r, r_0)$. The function $\psi_1$ is the solution to the homogeneous Pridmore-Brown equation (6.8) satisfying $\psi_1(0) = 0$ for $m \neq 0$ and $\psi'_1(0) = 0$ for $m = 0$. The function $\psi_2$ is a solution to the same equation, satisfying the impedance boundary condition (6.9). Both $\psi_1$ and $\psi_2$ are required to be continuously differentiable at $r = 1 - h$. We take

$$
\psi_1 = \begin{cases} J_m(ar) & r \leq 1 - h, \\ C_1 \dot{p}_1(r) + D_1 \dot{p}_2(r) & r > 1 - h, \end{cases} \quad (6.13a)
$$

$$
\psi_2 = \begin{cases} A_2 J_m(ar) + B_2 Y_m(ar) & r \leq 1 - h, \\ C_2 \dot{p}_1(r) + D_2 \dot{p}_2(r) & r > 1 - h, \end{cases} \quad (6.13b)
$$

where $A_2, B_2, C_1$ and $D_1$ are chosen to give $C^1$ continuity at $r = 1 - h$, and $C_2$ and $D_2$ are chosen to satisfy the boundary condition (6.9) at $r = 1$, which for definiteness we take to be $\psi_2(1) = 1$ and $\psi'_2(1) = -i\omega / Z$. This eventually leads to

$$
C_1 = W^{-1}J_m(ar) \dot{p}_2' - \alpha J'_m(ar) \dot{p}_2 \bigg|_{r=1-h} \quad C_2 = W^{-1} \left( \dot{p}_2' + \frac{i\omega}{Z} \dot{p}_2 \right) \bigg|_{r=1} \quad (6.14a)
$$

$$
D_1 = -W^{-1}J_m(ar) \dot{p}_1' - \alpha J'_m(ar) \dot{p}_1 \bigg|_{r=1-h} \quad D_2 = -W^{-1} \left( \dot{p}_1' + \frac{i\omega}{Z} \dot{p}_1 \right) \bigg|_{r=1} \quad (6.14b)
$$

$$
\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \frac{\pi(1-h)}{2W(1)} \begin{pmatrix} \alpha Y'_m - Y_m \\ -\alpha Y'_m - \dot{p}_1 \dot{p}_2 \\ \dot{p}_1' \dot{p}_2' \end{pmatrix} \begin{pmatrix} \dot{p}_2 - \dot{p}_2 \\ \dot{p}_2' \dot{p}_1' \end{pmatrix} \bigg|_{r=1-h} \begin{pmatrix} 1 \\ -i\omega / Z \end{pmatrix}, \quad (6.14c)
$$

where $W(r) = \dot{p}_1(r) \dot{p}_2'(r) - \dot{p}_1'(r) \dot{p}_2(r)$, and we have used for the final line the identity $W(J_m(ar), Y_m(ar)) = 2/(\pi r)$ (see [7]). The function $W(r)$ may be calculated directly by substituting into (6.8), to give

$$
W' + \left( \frac{1}{r} + \frac{2kU'}{\omega - U_k} \right) W = 0, \quad \Rightarrow \quad W = -\frac{3}{r} \frac{r_c}{r - r_c} (r - r_c)^2, \quad (6.15)
$$

where the multiplicative constant in $W$ has been determined by considering the limit $r \to r_c$ for the normalization of $p_1$ and $p_2$ used in (6.10a,6.10b).
We now consider separately the two cases of a source outside \(r_0 > 1 - h\) and within \(r_0 < 1 - h\) the boundary layer.

### 6.3.1 Source in the constant flow region

In this case \(r_0 < 1 - h\), so that \(\psi_1(r_0)\) and \(\psi_2(r_0)\) are given in terms of Bessel functions by (6.13a) and (6.13b). Expanding \(\hat{W}(r_0; \psi_1, \psi_2)\) in (6.12) in this case and making further use of Bessel function identities from [7], we finally arrive at

\[
\hat{G} = \frac{-i(\omega - Mk)W(1)}{2\pi(1 - h)Q} \psi_1(r_\omega)\psi_2(r_\omega),
\]

where \(Q = W(1 - h)W(1)(C_1D_2 - C_2D_1)\), so that

\[
Q = (J_m(\alpha \hat{r}_1') - \alpha J'_m(\alpha \hat{r}_1')) \biggl|_{r = 1 - h} \left( \hat{r}_2' + \frac{i\omega}{Z} \hat{r}_2 \right) \biggl|_{r = 1} - (J_m(\alpha \hat{r}_2') - \alpha J'_m(\alpha \hat{r}_2')) \biggl|_{r = 1 - h} \left( \hat{r}_1' + \frac{i\omega}{Z} \hat{r}_1 \right) \biggl|_{r = 1}.
\]

### Comparison with the uniform flow case

The Green’s function for a point mass source in a circular duct containing uniform mean flow was studied in [95] and writes in our notation

\[
\hat{G} = \frac{1}{8\pi} J_m(\alpha r_\omega) \left( \frac{\omega Z \alpha Y'_m(\alpha)}{\omega Z \alpha Y_m(\alpha)} \right) \frac{1}{J_m(\alpha(1 - h)) - Z_1 \alpha J'_m(\alpha(1 - h))}.
\]

The only \(h\)-dependent term is

\[
\frac{Y_m(\alpha(1 - h)) - Z_1 \alpha Y'_m(\alpha(1 - h))}{J_m(\alpha(1 - h)) - Z_1 \alpha J'_m(\alpha(1 - h))}.
\]
In the limit \( h \to 0 \) we have \( Z_1 \to \frac{i \omega}{\Omega^2} \), and we recover the Green’s function for uniform flow, with the difference that in (6.18) we have a \( \frac{1}{hr} \) factor, while in (6.19) the factor is \( \frac{-i \Omega}{4} \). This is because here we considered a point mass source, i.e. a point source in the mass conservation equation (see Appendix B), while for the uniform-flow-Green’s-function we took the wave equation with \( \delta(r - r_0)/r_0 \) as a right-hand-side (see [95] eq. (2)).

6.3.2 Source in the boundary layer

If \( r_0 > 1 - h \) the mass source is located in the boundary layer; in this case, the Green’s function is given by (6.12), with \( \bar{W}(r_0; \psi_1, \psi_2) = (C_1 D_2 - C_2 D_1) W(r_0) \). Using (6.14a), (6.14b) gives

\[
\hat{G} = \frac{-i(\omega - U(r_0)k) W(1) W(1 - h)}{2\pi r_0 W(r_0) Q} \psi_1(r_0) \psi_2(r_0),
\]

with \( Q \) as defined in (6.17).

6.4 Evaluating the Green’s function

As described in the previous section and illustrated schematically in Figure 6.6, the Green’s function solution for a fixed \( m \), after Fourier inversion

\[
G(x, r, t; r_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(r; r_0, k, \omega) dk,
\]

will consist of a sum of residues of poles and an integral around the critical layer branch cuts. The contributions to this integral come from acoustic and hydrodynamic modes, as well as from the possible log-branch cuts (the poles and branch cuts are positioned in a typical situation as in Figure 6.2).

To calculate \( G \), we choose a Fourier inversion contour in the \( k \)-plane satisfying causality via the Briggs-Bers criterion. For an \( \omega \) with a large, negative imaginary part, \( \hat{G} \) is analytic in \( k \) in a strip containing the real \( k \)-axis and we are free to choose the inversion contour anywhere inside this domain (see Chapter 3). Further on, we can smoothly increase \( \text{Im}(\omega) \) to zero (because for any practical
values of $Z$ and $h$, there are no absolute instabilities, as discussed in Chapter 5) and deform accordingly the $k$-contour maintaining analyticity, arriving thus to a contour as shown in Figure 6.6. We close it for $x < 0$ in the upper half plane, having in (6.21) only contributions from acoustic poles. For $x > 0$ the contour must be taken to enclose the branch cut\(^4\) and the down-stream poles. The Green’s function will hence consist of a sum of residues of poles and an integral along the branch cut.

### 6.4.1 Modes

We note from the expression of the Green’s function inside and outside the boundary layer (equations (6.20) and (6.16)) that since $ψ_1$ and $ψ_2$ have no poles, these come in in $\hat{G}$ as the zeros of $Q$, or as the zeros of $W(r_0)$. We will treat the singularity rendered by the zeros of $W(r_0)$ separately, remarking though that it exists only in (6.20), i.e. when the source is in the boundary layer. The zeros of $Q$ provide the acoustic and the hydrodynamic modes, including a convective instability when the boundary layer is thin enough (see also section 6.6).

\(^4\)the critical layer contributing downstream is in line with the statements of [100]
6.4.2 A pole on the branch cut

For \( r_0 \in (1 - h, 1) \) the Green’s function \( \hat{G} \) will have, besides the poles coming from the zeros of \( Q \), another pole when \( W(r_0) = 0 \). This happens according to (6.15) when \( r_0 = r_c \), i.e. when the source is at the critical layer. We can rewrite (6.20) representing the Green’s function in the boundary layer, as

\[
\hat{G}(r; r_0, k) = -\frac{i}{6\pi} \frac{\psi_1(r_<)\psi_2(r_>)}{C_1D_2 - C_2D_1} \frac{1}{hr_c(1 - r_0)k - k_0},
\]

where

\[
\frac{\psi_1(r_<)\psi_2(r_>)}{C_1D_2 - C_2D_1} = O(1), \quad \text{for } k \rightarrow k_0,
\]

and \( k_0 = \omega h / ((1 - r_0)M) \), with the \( k_0 \)-pole located on the branch cut \( (k_0 > \omega / M) \), giving a contribution of constant amplitude (see [64]) which is not straightforwardly determined. As we show in Chapter 7, it represents vorticity trailing from the source, and exists for a point source within sheared flow, being determined solely by the position of the source.

Remark that we have in (6.22) an \( r_c \) which is also a function of \( k \). However, since \( r_c \) is given by \( \omega - kU(r_c) = 0 \), it can never be zero, and therefore does not influence the number of poles in \( \hat{G} \). In order to avoid any possible confusions,
6.4 Evaluating the Green’s function

we did not write \( r_c \) explicitly as a function of \( k \) here.

To calculate the residue of \( k_0 \) we need to evaluate \( \log(\omega/M - k_0) \) (present in \( C_1 \)) and, for \( 1 - h < r < r_0 < 1, \log(\frac{\omega h}{(1-h)M} - k_0) \) (or equivalently \( \log(r - r_0) \), in \( \psi_1(r) \)). The branch cuts for these logarithms can be chosen anywhere as long as causality and analyticity are preserved, therefore anywhere in the lower \( k \)-half-plane (see also [100] Figure 2). Hence we calculate the contribution of \( k_0 \) as if approached from above these branch cuts, and have

\[
P_+ = -\frac{\omega k_0 e^{-ik_0 x}}{6\pi r_0(1 - r_0)^2} \Psi_0^+, \tag{6.23}
\]

where

\[
\Psi_0^+ = \begin{cases}
\frac{1}{C_1^+D_2-D_1^+} \bigg|_{k=k_0} & r < 1 - h < r_0, \\
\frac{(C_1\hat{p}_1(r)+D_1\hat{p}_2^+(r))D_2}{C_1^+D_2-D_1^+} \bigg|_{k=k_0} & 1 - h < r < r_0, \\
\frac{(C_2\hat{p}_1(r)+D_2\hat{p}_2^+(r))D_1}{C_1^+D_2-D_1^+} \bigg|_{k=k_0} & 1 - h < r_0 < r,
\end{cases}
\]

with \( C_1^+ \) calculated by substituting the singular Frobenius expansion, \( \hat{p}_2^+ \), with \( \hat{p}_2^+ \).

6.4.3 The critical layer

We choose all logarithmic branch cuts as in section 6.2.2, aligned on top of each other and ranging to +\( \infty \). Integrating along \( [\omega/M, \infty) \) for continuously changing frequencies \( \omega \) gives at a certain instant a discontinuity in the order of magnitude of the resulting field (see Figure 6.4.3). Numerical analysis reveals a pole that is located below \( (\omega/M, \infty) \) for large frequencies, which leaks into a branch cut for \( \omega \) smaller than a critical value (see also Figure 6.5). Our observations from numerical integrations were that this leaking mode has to be always taken together with the branch cut, otherwise leading to significant errors in the total field (see also Figure 5. in [25]). We analyze next the importance in the downstream far-field of the contributions coming from the logarithmic branch cuts linked with the critical layer, together with any possible \( k_\) poles.
Critical layer singularities

Figure 6.4: Integral around the branch cut without and with the residue of $k_-$. Results are for $\omega = 9.5$, $m = 2$, $M = 0.5$, $h = 0.05$, $Z = 2 + i$.

Figure 6.5: The pole $k_-$ for $\omega$ between 10.7 and 9.33 (arrow pointing the direction of change of the location $k_-); m = 2$, $r_0 = 0.5$, $M = 0.5$, $h = 0.05$, $Z = 2 + i$. Note that the branch point is at $\omega/M$, such that $k_-$ is in all cases under the branch cut.

6.5 Critical layer asymptotics for large $x$

Before proceeding with our analysis, we mention that the large-$x$ limit was also discussed in [100] based on the singularities of the Green’s function linked to the critical layer. Swinbanks explains that these are of the form $(k - k_r)^n \log(k - k_r)$ with $n > 0$, with an additional simple pole when the source location and the critical layer location coincide - this being the situation when the critical layer contributes most seriously. An eventual inverse Fourier transform of this Green’s function will therefore consist of a constant amplitude term (due to the presence of the pole) and terms behaving like $x^{-(n+1)}$ (due to the logarithms in $\hat{G}$). In the case when the source is at the critical layer, the logarithmic singula-
rity is present as \((k - k_0)^2 \log(k - k_0)\) and leads to an \(O(x^{-3})\) decay.

We approach this limit differently, and carefully analyze the integral around the log-branch cut(s), \(I_{CL}\), which reduces in most cases to the integral of the jump \(
\Delta \hat{G} = \hat{G}_+ - \hat{G}_-\) of the Green’s function across the branch cuts. Depending on the source and the observer’s location, we have different situations concerning the number and the location of the branch points:

| Case i. | \(r, r_0 < 1 - h\), & br. pt. at \(\omega/M\) |
| Case ii. | \(r_0 < 1 - h < r\), & br. pts. at \(\omega/M\) and \(k_r\) |
| Case iii. | \(r < 1 - h < r_0\), & pole and br. pt. at \(k_0\); br. pt. at \(\omega/M\) |
| Case iv. | \(r, r_0 > 1 - h\), & pole and br. pt. at \(k_0\); br. pts. at \(\omega/M\) and \(k_r\) |

with \(k_r = \frac{\omega h}{(1-r)M}\) for fixed \(r\) (see also Figure 6.6). We analyze each of these situations separately, and find different decay rates for source locations in the uniform or in the shear flow region.

For the compactness of the formulas in this section, we define

\[
c_{1,2} = (J''_{\nu}(ar)\hat{p}'_{1,2} - \alpha J'_{\nu}(ar)\hat{p}_{1,2}(r))\bigg|_{r=1-h} \quad d_{1,2} = iZ\hat{p}'_{1,2}(1) - \omega\hat{p}_{1,2}(1).
\]

In determining the large \(x\) asymptotics we make use of Watson’s lemma [6].

**Lemma. (Watson’s lemma)**

Consider the integral

\[
I(x) = \int_0^\infty f(s) e^{-sx} \, ds.
\]

Suppose that \(f(s)\) is integrable in \((0, \infty)\), with \(|f(s)| \leq M e^{cs}\), for some constants \(c\) and \(M\), and that it has the asymptotic series expansion

\[
f(s) \sim s^\alpha \sum_{n=0}^{\infty} a_n s^{\beta n}, \quad s \to 0^+; \quad \alpha > -1, \quad \beta > 0.
\]

Then

\[
I(x) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + \beta n + 1)}{x^{\alpha + \beta n + 1}}, \quad x \to \infty.
\]
Figure 6.6: Schematic in the $k$-plane of the critical layer branch cut integral.
6.5 Critical layer asymptotics for large $x$

6.5.1 $r$ and $r_0$ in the mean flow

This is the only case when the critical layer consists of only one branch point, namely $\omega/M$. We integrate around the branch cut at $(\omega/M, \infty)$ and collapse the integration contour as sketched in the first part of Figure 6.6 (a) to have

$$I_{CL} = \frac{1}{2\pi} \int_{\omega/M}^{\infty} (\hat{G}_+ - \hat{G}_-) e^{-ikx} \, dk,$$

where $\hat{G}_\pm = \lim_{\epsilon \to 0^\pm} \hat{G}(r; k + i\epsilon)$. In order to determine the asymptotic behavior of $I_{CL}$ we deform to the steepest descent contour (see the second part of Figure 6.6 (b)) including now also the residue of the poles $k_-$ of $G_-$ located below the branch cut:

$$I_{CL} = \frac{e^{-i\omega x/M}}{2\pi i} \int_0^{\infty} \left( \hat{G}_+(r; \omega/M - i\xi) - \hat{G}_-(r; \omega/M - i\xi) \right) e^{-ix\xi} \, d\xi. \quad (6.24)$$

For $r, r_0 < 1 - h$ we use the expression of $\hat{G}$ from (6.18) and have

$$\hat{G}_+(r; k) - \hat{G}_-(r; k) = -\frac{i\Omega}{2\pi(1-h)} \frac{(Z_1^1 - Z_1^0) I_m(\alpha r) J_m(\alpha r_0)}{2(1-h)^2 - m^2 M^2 \omega^2},$$

where, to avoid confusion, we take the total derivative (we keep this convention throughout this section). Moreover, for $k = \omega/M - i\xi$ and $\xi \to 0^+$ we have $\alpha \to -i\omega/M$ and $J_m(\alpha r) \to (-1)^m J_m(-\omega/M r)$, and we have to leading order in $\xi$, for a fixed $r$,

$$\hat{G}_+(r; \omega/M - i\xi) - \hat{G}_-(r; \omega/M - i\xi) = \mathcal{A}_1(r) \xi^3 + O(\xi^4),$$

with

$$\mathcal{A}_1(r) = -ih^2 M \frac{I_m(-\omega/M r) I_m(-\omega/M r_0)}{I_m^2(-\omega/M(1-h))} \left( \frac{1}{(1-h)^2} - \frac{m^2 M^2}{\omega^2(1-h)^4} \right).$$
Applying Watson’s lemma to obtain the large $x$ behavior of $I_{sd}$ renders

$$I_{CL} \sim \frac{\mathcal{A}_1(r)\Gamma(4)e^{-i\omega x/M}}{2\pi i x^4} + O(1/x^5) \quad \text{for } x \to \infty,$$

indicating that the critical layers decays far downstream like $x^{-4}$.

### 6.5.2 $r_0$ in the mean flow; $r$ in the boundary layer

In this case the Green’s function is

$$\hat{G}(r; k) = \frac{i(\omega - kM) W(1) J_m(\alpha r_0)(d_2 \hat{p}_1(r) - d_1 \hat{p}_2(r))}{2\pi(1-h) c_1 d_2 - c_2 d_1},$$

with $d_2$ containing a log($k - \omega/M$), and $\hat{p}_2(r)$ a log($k - k_r$), $k_r = \frac{\omega h}{(1-r)M}$ giving rise to two overlapping branch cuts as in the first part of Figure 6.6 (b).

To consider, as before, a contour collapsed onto the real $k$-axis, and then deform to a steepest descent path is no longer straightforward. Nevertheless, the log branch cuts can be chosen - due to causality and for the convergence of the integrals - anywhere in the lower half-plane and we find it convenient to use the redefined

$$\text{Log}(z) = -\frac{i\pi}{2} + \log(z e^{\frac{i\pi}{2}}), \quad (6.26)$$

where log is the principal value logarithm. The branch cuts in $\hat{G}$ will now be as in the last part of Figure 6.6 (b) (including any possible $k_-$ pole), and the Fourier inversion contour can be collapsed onto these two branch cuts by applying Jordan’s lemma, with $I_{CL}$ given in this case by

$$I_{CL} = I_1 + I_2,$$

$$I_1 = \frac{e^{-i\omega x/M}}{2\pi i} \int_0^\infty \left( \hat{G}_+(r; \omega/M - i\xi) - \hat{G}_-(r; \omega/M - i\xi) \right) e^{-x\xi} d\xi, \quad (6.27)$$

$$I_2 = \frac{e^{-ik_r x}}{2\pi i} \int_0^\infty \left( \hat{G}_+(r; k_r - i\xi) - \hat{G}_-(r; k_r - i\xi) \right) e^{-x\xi} d\xi,$$

where $\hat{G}_\pm(k) = \lim_{\varepsilon \to 0^\pm} \hat{G}(k + \varepsilon)$, for $k$ along a branch cut of $\hat{G}$. For $I_1$ we have
6.5 Critical layer asymptotics for large $x$

for small $\xi$

$$\hat{G}_+(r; \omega/M - i\xi) - \hat{G}_-(r; \omega/M - i\xi) = \mathcal{A}_{21}(r)\xi^3 + O(\xi^4),$$

with

$$\mathcal{A}_{21}(r) = \frac{-9M^3}{\omega^2(3iZ - \omega h)^2} \left( \frac{J_m(-\omega h)}{I_m(-\omega h)} \right) \left( d_2\hat{p}_1(r) - d_1\hat{p}_2(r) \right) \xi = 0.$$

For $I_2$ we have for small $\xi$

$$\hat{G}_+(r; k_r - i\xi) - \hat{G}_-(r; k_r - i\xi) = \mathcal{A}_{22}(r)\xi^3 + O(\xi^4),$$

with

$$\mathcal{A}_{22}(r) = \frac{3M^3(1 - r)^8r}{\omega^2h^2(1 - h)} \left( \frac{J_m(\alpha r_0)\Lambda d_1(\omega - kM)}{c_1d_2 - c_2d_1} \right)_{k = k_r}.$$

Note that for $k = k_r$, $\alpha$ is purely imaginary and $J_m$ behaves like $I_m$ as in the previous subsection. Following Watson’s lemma

$$I_{CL} \sim \frac{(\mathcal{A}_{21}(r)e^{-i\omega x/M} + \mathcal{A}_{22}(r)e^{-ik_r x})\Gamma(4)}{2\pi ix^4} + O(1/x^5),$$

and the critical layer behaves for large $x$ as $x^{-4}$, as before.

6.5.3 $r_0$ in the boundary layer; $r$ in the mean flow

The Green’s function is in this case

$$\hat{G}(r, k) = -\frac{iMk^2W(1 - h)}{6\pi hr_0(1 - r_0)} \frac{1}{k - k_0} \frac{d_2\hat{p}_1(r_0) - d_1\hat{p}_2(r_0)}{c_1d_2 - c_2d_1} J_m(\alpha r)$$

$$= \frac{iMkW(1 - h)J_m(\alpha r)}{6\pi hr_0(c_1d_2 - c_2d_1)} \left( \frac{k^2\zeta_1}{1 - r_0} \frac{1}{k - k_0} + d_1b_2(1 - r_0)(k - k_0) - d_1\Lambda \frac{(1 - r)^2}{k} (k - k_0)^2 \log(r_0 - r_c) + \ldots \right)$$

(6.28)

and has in the $k$-plane, for fixed $r$, a logarithmic branch point at $\omega/M$ and a pole and logarithmic branch point at $k_0$, see also Figure 6.6 (c). To decouple the contributions from the two logarithmic singularities, we take in the above
\( \hat{G} \) a logarithm as defined in (6.26), thus shifting the branch cuts in the \( k \)-plane, as in the last part of Figure 6.6 (c). We split again the critical layer integral in two, \( I_{CL} = I_1 + I_2 \), with \( I_1 \) given by (6.27).

For \( I_1 \) we have for small \( \xi \),

\[
\hat{G}_+(r; \omega/M - i\xi) - \hat{G}_-(r; \omega/M - i\xi) = A_{31}(r)\xi^4 + O(\xi^5),
\]

\[
A_{31}(r) = \frac{-3hM^4I_m(-\frac{\omega}{M} r)I_m(-\frac{\omega}{M} (1-h))}{\omega^3(1 - r_0 - h)(3Z + i\omega h)I_m^2(-\frac{\omega}{M} (1-h))} \left( d_2 \hat{p}_1(r_0) - d_1 \hat{p}_2(r_0) \right)_{k=\frac{\omega}{M}},
\]

with Watson’s lemma giving

\[
I_1 \sim \frac{A_{31} e^{-i\omega x/M} \Gamma(5)}{2\pi i x^5} + O(1/x^6).
\]

Since we are interested only in the contribution of the branch cut integrals, we subtract the contribution of the pole \( k_0 \) (which is \( P_+ \)) and take

\[
I_2 = \frac{e^{-ik_0 x}}{2\pi i} \lim_{\varepsilon \to 0^+} \int_0^\infty \left( \hat{G}_+(r; k_0 - i\varepsilon - i\xi) - \hat{G}_-(r; k_0 - i\varepsilon - i\xi) \right) e^{-\lambda(\xi+\varepsilon)} d\xi.
\]

Since \( \hat{G}_+(r; k_0 - i\varepsilon - i\xi) - \hat{G}_-(r; k_0 - i\varepsilon - i\xi) = O(\varepsilon^3) + O(\varepsilon)^3 + O(1)(\varepsilon + \ldots) \), we can apply the dominated convergence theorem and interchange limit and integral, having for small \( \xi \)

\[
\lim_{\varepsilon \to 0^+} \hat{G}_+(r; k_0 - i\varepsilon - i\xi) - \hat{G}_-(r; k_0 - i\varepsilon - i\xi) = A_{32}(r)\xi^2 + O(\xi^3),
\]

with \( A_{32}(r) = \frac{(1 - r_0)\omega(1 - r_0 - h)}{(1-h)k_0} \left( \frac{\Lambda d_1 J_m(ar)}{c_1d_2 - c_2d_1} \right)_{\xi=0} \),

and this due to the fact that the factor \( \frac{1}{\xi-k_0} \sim \frac{1}{\xi} \) diminishes the decay in \( \hat{p}_1(r_0) \) by one order, while the other terms in \( \Delta \hat{G} \) remain \( O(1) \).

To leading order

\[
I_{CL} \sim \frac{A_{32}(r)\Gamma(3) e^{-ik_0 x}}{2\pi i x^3} + O(1/x^4),
\]

concluding, as in [100], that the critical layer contribution decays, apart from the constant amplitude term \( P_+ \), as \( x^{-3} \) for \( r < 1 - h < r_0 \).
6.5 Critical layer asymptotics for large $x$

6.5.4 $r_0, r$ in the boundary layer

In this case we have three logarithmic branch points, $\omega/M, k_0$ and $k_r$, located as in Figure 6.6 (d), or with $k_r$ between $\omega/M$ and $k_0$, depending on the relative position of $r_0$ and $r$. The Green's function is now

$$\hat{G}(r, k) = -\frac{iMk^2}{6\pi hr_c(1-r_0)} \frac{1}{k-k_0} \frac{d_2 \hat{p}_1(r_\rightarrow) - d_1 \hat{p}_2(r_\rightarrow)}{c_1d_2 - c_2d_1} \frac{c_2 \hat{p}_1(r_\leftarrow) - c_1 \hat{p}_2(r_\leftarrow)}{c_1d_2 - c_2d_1}.$$  \hspace{1em} (6.29)

With the logarithm as in (6.26), we have three branch cuts as in the last part of Figure 6.6 (d), and we split $I_{CL}$ in three integrals, around each branch cut, $I_{CL} = I_1 + I_2 + I_3$. The integrals $I_2$ and $I_3$ are around the branch cuts of Log$(k-k_0)$ and Log$(k-k_r)$, respectively, while $I_1$ is, as before, around the branch cut of Log$(k-k_r)$. We have for $k = \omega/M - i\xi$, and small $\xi$,

$$\hat{G}_+(r; \omega/M - i\xi) - \hat{G}_-(r; \omega/M - i\xi) = \mathcal{A}_{41}(r)\xi^4 + O(\xi^5),$$

$$\mathcal{A}_{41}(r) = \frac{1}{iM} \left( \frac{1}{\omega M(1-h)^2} \frac{1}{\omega^2} \right)^2 \left( \frac{M^2}{(1-h)^2} \left( \left( \frac{d_2 \hat{p}_1(r_\rightarrow) - d_1 \hat{p}_2(r_\rightarrow)}{d_2 \hat{p}_1(r_\leftarrow) - d_1 \hat{p}_2(r_\leftarrow)} \right) \xi = 0 \right) \right),$$  \hspace{1em} (6.30)

giving

$$I_1 \sim \frac{\mathcal{A}_{41} \Gamma(5)}{2\pi i x^5} + O(1/x^6).$$

We define next

$$I_2 = \frac{e^{-ik_0x}}{2\pi i} \lim_{\varepsilon \to 0^+} \int_0^\infty (\hat{G}_+(r; k_0 - i\varepsilon - i\xi) - \hat{G}_-(r; k_0 - i\varepsilon - i\xi)) e^{-x(\xi + \varepsilon)} d\xi,$$

and find, similarly to section 6.5.3, that

$$\lim_{\varepsilon \to 0^+} \hat{G}_+(r; k_0 - i\varepsilon - i\xi) - \hat{G}_-(r; k_0 - i\varepsilon - i\xi) = \mathcal{A}_{42}(r)\xi^2 + O(\xi^3),$$
\[ \mathcal{A}_{42}(r) = \frac{1}{3} \frac{M}{\hbar k_0} \left( \frac{\Lambda}{c_1 d_2 - c_2 d_1} \right) \left[ H(r_0 - r) \left( d_1 c_2 \hat{p}_1(r) - d_1 c_1 \hat{p}_2(r) \right) + H(r - r_0) \left( c_1 d_2 \hat{p}_1(r) - c_1 d_1 \hat{p}_2(r) \right) \right]_{\xi = 0}. \] (6.31)

The asymptotic behavior of \( I_2 \) is given by Watson's lemma:

\[ I_2 \sim \frac{\mathcal{A}_{42} \Gamma(3) e^{-ik_0 x}}{2\pi i x^3} + O(1/x^4). \]

Finally,

\[ I_3 = \frac{e^{-ik_0 x}}{2\pi i} \int_0^\infty (\hat{G}_+(r; k_r - i\xi) - \hat{G}_-(r; k_r - i\xi)) e^{-k_0 x} d\xi. \]

For small \( \xi \)

\[ \hat{G}_+(r; k_r - i\xi) - \hat{G}_-(r; k_r - i\xi) = \mathcal{A}_{43}(r) \xi^3 + O(\xi^4), \]

\[ \mathcal{A}_{43} = \frac{M}{3} \frac{(1 - r)^2 k_0}{\hbar k_r^3 (k_r - k_0)} \left( \frac{\Lambda}{c_1 d_2 - c_2 d_1} \right) \left[ H(r_0 - r) \left( c_1 d_2 \hat{p}_1(r_0) - c_1 d_1 \hat{p}_2(r_0) \right) + H(r - r_0) \left( d_1 c_2 \hat{p}_1(r_0) - d_1 c_1 \hat{p}_2(r_0) \right) \right]_{\xi = 0}. \] (6.32)

and we have for large \( x \) from Watson's lemma that

\[ I_3 \sim \frac{\mathcal{A}_{43} e^{-ik_0 x} \Gamma(4)}{2\pi i x^4} + O(1/x^5). \]

Hence,

\[ I_{CL} = I_1 + I_2 + I_3 \sim \frac{\mathcal{A}_{42} \Gamma(3) e^{-ik_0 x}}{2\pi i x^3} + O(1/x^4). \]

To finish this section, we note that when the source is in the mean flow, we find that the critical layer integral decays far downstream as \( 1/x^4 \), whereas for a source located in the boundary layer, we have, apart from a constant contribution of the \( k_0 \) pole, a decay as \( 1/x^3 \).
6.6 Summarizing remarks

To conclude, we reaffirm our present results with some numerical illustrations, courtesy of dr. E.J. Brambley.

We show in this chapter that the critical layer is negligible if we have a mass source in the uniform flow part, and we can see this well in Figure 6.7(a). Nonetheless, this holds only if we neglect the branch cut together with any leaking mode, as argued in section 6.4.3.

In contrast to this, the critical layer becomes important when we have sources in the boundary layer, on the one hand, due to the trailing vorticity of the source (see Figure 6.8(a)), and on the other, because this immediately excites any inherent (convective) instability of the flow (see Figure 6.8(b), (c) and Figure 6.7(b)); see also [25] for more numerical illustrations, and details regarding their computation.
Figure 6.8: Contour plot of the pressure field in \((x, r)\) plane: first column of figures represent the sum of residues of acoustic poles; second column adds the branch cut and the \(k_\perp\) contribution; third column adds to all this the instability mode. (Courtesy of dr. E.J. Bramley)
This study was motivated by the findings of Chapter 6, in particular by the presence of the $k_0$ pole in the Fourier transformed Green’s function for sources located in the shear region [93].

In order to analyze the non-modal\(^1\) contribution of $k_0$, we seek a basic configuration in which its effects can be reproduced, and consider thus a time-harmonic point mass source in an incompressible inviscid two-dimensional linear shear of infinite extent. This simple model has the advantage of admitting exact solutions - albeit, via introduction of generalized functions, since we have a logarithmic-like pressure field. We present and discuss the model and the inverse Fourier transformation of the pressure and velocities in the first section. Throughout the chapter we will keep the problem dimensional, since in an incompressible infinite linear shear the characteristic length is given by the posi-

\(^1\)since it depends on the source
The trailing vorticity field behind a point source

tion of the observer relative to the source, the velocity in the problem being the velocity at the source, and they are both difficult to use for scaling. The explicit analytic solutions obtained appear to be new in spite of this rather simple configuration, with the nearest known solution being the velocity field given in [39] for the initial value problem of an impulsive point source in a linear shear layer.

A first natural extension to the results in section 7.1 is an incompressible, linear, semi-infinite shear along an impedance wall - the subject of section 7.2. We find again exact solutions for pressure and velocities given in terms of logarithms and exponential integrals, and, in contrast with the infinite case, two more poles corresponding to two surface waves, similar to the surface waves with the Ingard-Myers boundary condition for uniform flow in the incompressible limit [88].

We conclude this chapter with the equations for vorticity, explaining thus in a different way, the origins of \( k_0 \). In linear theories of relatively simple mean flows, vortex shedding is a well known effect of an external force (see [94], [42]) but it is not common from mass sources. However, what we have here is in fact a vorticity redistribution [93], as a particle’s vorticity can only change by an external force.

Although the work presented here is primarily a ramification of Chapter 6, we believe it to also have importance on its own. It is yet again a case when simplifying and taking a crude model that concentrates on specific aspects, facilitates a more detailed analysis and provides exact solutions, which have their importance, as they potentially contribute to the fundamental understanding of similar problems. The advantage comes from having an explicit Green’s function in which we can recognize effects of the source, effects of the wall, and couplings between the two.

From a mathematical point of view, our case study provided an interesting challenge to properly customize branch cuts, such that solutions are in line with practical requirements. We discuss this in section 7.2, in the impedance wall case, where the use of standardized complex functions would lead to discontinuous pressure and velocities.

The results presented here are part of a collaboration with dr. E.J. Brambley.
7.1 Infinite shear

7.1 A time-harmonic point mass source in infinite linear shear

Consider a time harmonic point-source of strength $S$ located at $x = y = 0$ in the two-dimensional incompressible inviscid linear mean flow of perturbations with time dependency $e^{i\omega t}$. The model is given by the linearized incompressible Euler equations

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = S \delta(x) \delta(y),
\rho_0 \left( i\omega + U \frac{\partial}{\partial x} \right) u + \rho_0 \frac{dU}{dy} v + \frac{\partial p}{\partial x} = 0, 
\rho_0 \left( i\omega + U \frac{\partial}{\partial x} \right) v + \frac{\partial p}{\partial y} = 0.
$$

(7.1)

with far field boundary conditions of vanishing velocity.

We assume pressure and velocities are Fourier transformable in $x$, at least in a generalized sense\(^2\) (see [58]). We eliminate $u$ and $v$ and obtain, after Fourier transformation in $x$, a Pridmore-Brown equation [81] in pressure

$$
\hat{p}'' + \frac{2k dU/dy}{\omega - kU} \hat{p}' - k^2 \hat{p} = -i \rho_0 S (\omega - kU(0)) \delta(y).
$$

(7.2)

This simplifies for a doubly-infinite linear shear flow, $U(y) = U_0 + \sigma y$, to

$$
\hat{p}'' + \frac{2k\sigma}{\Omega} \hat{p}' - k^2 \hat{p} = -i \rho_0 S (\omega - kU) \delta(y), \quad \Omega = \omega - kU, \quad \Omega_0 = \omega - kU_0
$$

(7.3)

subject to boundary conditions of a decaying field at infinity.

The homogeneous equation has two independent solutions (see also [43, 82]), $e^{\pm k\sigma \sqrt{-\Omega}}$. In order to construct the Green’s function, we choose two solutions, $\hat{p}_1$ and $\hat{p}_2$, such that $\hat{p}_1$ satisfies the boundary condition at $-\infty$, and $\hat{p}_2$ at $\infty$,

$$
\hat{p}_1(y) = e^{k\sqrt{-\Omega} \text{sign}(\text{Re} k) \sigma}, \quad \hat{p}_2(y) = e^{-k\sqrt{-\Omega} \text{sign}(\text{Re} k) \sigma},
$$

(7.4)

\(^2\)as we see later, the pressure admits an inverse Fourier transform only in the context of generalized functions
with
\[ |k| = \text{sign}(\text{Re} \, k) \sqrt{k^2}, \] (7.5)
where \( \sqrt{\cdot} \) denotes the principal value square root, and \( |k| \) has thus branch cuts along \((-i\infty, 0)\) and \((0, i\infty)\). The Wronskian of \( \hat{p}_1 \) and \( \hat{p}_2 \) is
\[ W(y; k) = -2|k|\Omega^2, \] (7.6)
and the Fourier transformed solution (or Green’s function) for (7.1) is thus
\[ \hat{p}(y, k) = \frac{i\rho_0 S}{|k|\Omega_0} e^{-|k|y}(\Omega\Omega_0 - \sigma^2|ky| - \sigma^2). \] (7.7)

We obtain the physical field in the \( x, y \)-domain as the (real part of the) inverse Fourier transform
\[ p(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}(y, k) e^{-ikx} \, dk = \frac{i\rho_0 S}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-|k|y}}{|k|\Omega_0} (\Omega\Omega_0 - \sigma^2|ky| - \sigma^2) \, dk, \] (7.8)
which has singularities at \( k = 0 \) (if \( \omega^2 \neq \sigma^2 \)) and at \( k = k_0 = \omega/U_0 \) (where \( \Omega_0 = -U_0(k - k_0) = 0 \)). Note that \( k_0 \) is completely depending on the source, and contributes thus downstream. Folding the contour around the branch cuts of \( |k| \) (upwards for \( x < 0 \) and downwards for \( x > 0 \)) and around the \( k_0 \) pole, we obtain
\[ p(x, y) = \frac{\rho_0 S}{2\omega} \sigma^2 H(x)(1 + k_0|y|) e^{-ik_0x - k_0|y|} \]
\[ + \frac{i\rho_0 S}{2\pi} \int_0^{\infty} \frac{e^{-|\lambda|y}}{\lambda\Omega^2_0} \left[ (\Omega^\pm\Omega^\pm_0 - \sigma^2\cos\lambda y - \sigma^2\lambda y \sin\lambda y \right] \, d\lambda \] (7.9)
where \( \Omega^\pm = \omega \pm i\lambda U \), and \( H(x) \) is Heaviside’s step function.

The singularity at \( k = 0 \) is caused by the fact that \( p \) is not Fourier transformable. As mentioned before, if \( \omega^2 \neq \sigma^2 \), \( p \) diverges for \( x^2 + y^2 \to \infty \) as \( \sim \log(x^2 + y^2) \) and is hence not integrable. This is an artefact of the point source model in an incompressible medium. When we consider the incompressible problem as an inner problem of a larger compressible problem, this diverging behavior disappears as it changes in the far field into an outward radiating, and hence decaying, acoustic wave [38, 63].
The inverse Fourier integral, however, can be found if the singular integral is interpreted in the generalized sense, and the singular part is split off. Following [58], p. 105, we identify the semi-infinite integral by a doubly infinite one by replacing $1/\lambda$ by the generalized function

$$
\lambda^{-1}H(\lambda) = \frac{d}{d\lambda}H(\lambda) \log |\lambda|.
$$

After integration by parts we obtain the convergent integrals

$$
p(x, y) = \frac{i\rho_0 S}{2\pi} \sigma^2 \int_0^\infty \log \lambda \frac{d}{d\lambda} \left[ e^{-|\lambda|} \frac{\cos \lambda y + \lambda y \sin \lambda y}{\Omega_0^x} \right] d\lambda
$$

$$
+ \frac{\rho_0 S}{2\omega} \sigma^2 H(\chi)(1 + k_0 |y|) e^{-ik_0 x - k_0 |y|} - \frac{i\rho_0 S}{2\pi} \int_0^\infty \log \lambda \frac{d}{d\lambda} \left[ e^{-|\lambda|} \Omega^x \cos \lambda y \right] d\lambda
$$

(7.10)

Each one can be integrated as follows

$$
\int_0^\infty \log \lambda \frac{d}{d\lambda} \left[ e^{-|\lambda|} \Omega^x \cos \lambda y \right] d\lambda = \omega \gamma + \frac{1}{2} \omega \log(x^2 + y^2) - iU \frac{x}{x^2 + y^2}
$$

$$
\omega \int_0^\infty \log \lambda \frac{d}{d\lambda} \left[ e^{-|\lambda|} \frac{\cos \lambda y + \lambda y \sin \lambda y}{\Omega_0^x} \right] d\lambda = \gamma + \frac{1}{2} \log(x^2 + y^2)
$$

$$
+ \frac{1}{2}(1 - k_0 |y|) E(k_0, z) + \frac{1}{2}(1 + k_0 |y|) E(k_0, \bar{z})
$$

where $z = x + iy$, $\gamma = 0.5772156649 \ldots$ is Euler’s constant and $E(k_0, z) = e^{-ik_0 z} E_1(-ik_0 z)$, with $E_1$ the exponential integral with the standard branch cut along the negative real axis of its argument. This results for $E_1(-ik_0 z)$ in a branch cut along the line $x = 0, y > 0$ and for $E_1(-i k_0 \bar{z})$ in a branch cut along the line $x = 0, y < 0$. Altogether, we have

$$
p(x, y) = -\rho_0 S \left( U_0 + \sigma y \right) \frac{x}{x^2 + y^2} + \frac{i\rho_0 S}{2\pi \omega} \left( \sigma^2 - \omega^2 \right) \left[ \gamma + \frac{1}{2} \log(x^2 + y^2) \right]
$$

$$
+ \frac{i\rho_0 S \sigma^2}{4\pi \omega} \left( 1 - k_0 |y| \right) E(k_0, z) + \left( 1 + k_0 |y| \right) E(k_0, \bar{z})
$$

$$
- 2\pi i H(x)(1 + k_0 |y|) e^{-ik_0 x - k_0 |y|}.
$$

(7.11)

A seemingly different result would have been obtained had we scaled $\lambda$ by a positive factor. The above regularization of the divergent integral would have
produced, via the logarithm, a result that differs by a constant. This, however, is not a problem because the pressure is only defined up to a constant in the first place. Indeed, the term proportional to $\gamma$ in (7.11) is also not relevant and can be discarded$^3$.

As opposed to $p$, the integrals for $v$ or $u$ are convergent and can be found without resorting to generalized functions, since

$$v(y, k) = \frac{1}{2}S \ e^{-|ky|} \left( \text{sign}(y) + \text{sign}(\text{Re} k) \frac{\sigma}{\Omega_0} \right),$$
$$u(y, k) = \frac{1}{2}iS \ e^{-|ky|} \left( \text{sign}(\text{Re} k) + \text{sign}(y) \frac{\sigma}{\Omega_0} \right).$$

(7.12)

We have

$$v(x, y) = \frac{S}{2\pi} \frac{y}{x^2 + y^2} - \frac{S \sigma}{4\pi U_0} \left[ E(k_0, z) + E(k_0, \bar{z}) - 2\pi i H(x) e^{-ik_0|x-k_0|} \right]$$
$$u(x, y) = \frac{S}{2\pi} \frac{x}{x^2 + y^2} + \frac{iS \sigma}{4\pi U_0} \left[ E(k_0, z) - E(k_0, \bar{z}) + 2\pi i H(x) \text{sign}(y) e^{-ik_0|x-k_0|} \right].$$

(7.13)

The branch cuts of the exponential integrals cancel the jumps due to the $H(x)$-terms, to produce continuous $p$ and $v$ fields. Only $u$ has a tangential discontinuity along $y = 0$, $x > 0$, but this is due to the $\text{sign}(y)$ term$^4$.

A graphical example of this solution, in the form of snap shots in time of the pressure and velocity fields (i.e. including the factor $e^{i\omega t}$), is given in figure 7.1. The figures are determined by the hydrodynamic wavelength $2\pi/k_0$ (which is the wavelength of the vortices) and the location of the source, the other parameters contributing only as a strength factor. In order to remove the effect of the undetermined constant, the plot-domain averaged value of $p$ is subtracted from $p$. The axial velocity $u$ is discontinuous across $y = 0$, $x > 0$, whereas $v$ and $p$ are continuous everywhere (outside the source). Since $\omega \neq \sigma$, the pressure diverges logarithmically.

$^3$the fact that there is an undetermined additive constant explains the dimensional argument of $\log(x^2 + y^2)$

$^4$this discontinuity corresponds to the $\delta(y)$-function behavior of the vorticity given in (7.32)
7.1 Infinite shear

![Figure 7.1: Snapshots in time, free field. Source located at \( x = y = 0; k_0 = 2.67 \).](image)

7.1.1 Interpretation for compressible duct flow

For a comparison with the three-dimensional acoustic problem of a cylindrical duct of radius \( a \), mean flow of Mach number \( M \) and boundary layer thickness \( ah \) (as considered by [25]), we note that in the shear layer we have (in dimensionless form)

\[
U(r) = Mh^{-1}(1 - r) = Mh^{-1}(1 - r_0) + Mh^{-1}(r_0 - r)
\]

which is equivalent to the 2D problem if we identify \( y = a(r_0 - r), U_0 = c_0M(1 - r_0)/h, \sigma = c_0M/ah \) and \( \omega := \omega c_0/a \), such that the dimensionless duct equivalent of \( k_0 \) is

\[
k_0 := k_0 a = \frac{a\omega}{U_0} = \frac{\omega h}{M(1 - r_0)}.
\]

Exactly the same trailing vorticity wave number \( k_0 \) is found in the acoustic duct problem as in the present 2D incompressible problem.

This analogy extends to the configuration where the source is positioned near an impedance wall, as we show in the next section.
7.2 A time-harmonic point mass source in linear shear over an impedance wall

7.2.1 The soft wall

We consider again (7.2), but now in \([0, \infty)\) with a source at \(y = y_0\) (while \(x = 0\), as before), and a wall of impedance \(Z_w = \rho_0 \zeta\) at \(y = 0\), where \(U\) vanishes. With \(U_0 = \sigma y_0\) the velocity at the source, we take now

\[
U(y) = U_0 + \sigma(y - y_0) = \sigma y, \quad \text{with} \quad \Omega = \omega - kU, \quad \Omega_0 = \omega - kU_0, \quad k_0 = \omega \frac{U}{U_0}.
\]

This leads to the same incompressible Pridmore-Brown equation (7.3) with same far-field condition and impedance boundary condition at \(y = 0\)

\[i \omega \hat{p}(0) = \zeta \hat{p}'(0).\]

Note that \(\zeta\) has the dimension of velocity.

We construct the Fourier-transformed solution in a similar way to the free-field one, to obtain

\[
\hat{p} = \frac{i \rho_0 S}{2 |k| \Omega_0} e^{-|k|y_0} \left( \Omega - \text{sign}(\text{Re} k) \sigma \right) (\Omega - \text{sign}(\text{Re} k) \sigma) + \frac{i \rho_0 S}{2 |k| \Omega_0} e^{-|k|y_0} \left( \Omega - \text{sign}(\text{Re} k) \sigma \right) (\Omega - \text{sign}(\text{Re} k) \sigma) \frac{ik \zeta + \sigma + \text{sign}(\text{Re} k) \omega}{ik \zeta + \sigma - \text{sign}(\text{Re} k) \omega}
\]

where \(y_0 = \min(y, y_0), y_+ = \max(y, y_0)\) and \(\Omega_{<} = \Omega_{<=}(y_+).\) For convenience, we split \(\hat{p}\) into an incident and a reflected part\(^5\)

\[
\hat{p} = \hat{p}_\text{in} + \hat{p}_\text{ref}
\]

\[
\hat{p}_\text{in} = \frac{i \rho_0 S}{2 |k| \Omega_0} e^{-|k|y_0} \left( \Omega \Omega_0 - \sigma^2 |k|y - y_0| - \sigma^2 \right)
\]

\[
\hat{p}_\text{ref} = \frac{i \rho_0 S}{2 |k| \Omega_0} e^{-|k|(y+y_0)} \left( \Omega \Omega_0 - |k| \sigma (\Omega - \Omega_0) + \sigma^2 \right) \frac{ik \zeta + \sigma + \text{sign}(\text{Re} k) \omega}{ik \zeta + \sigma - \text{sign}(\text{Re} k) \omega}
\]

\(^5\) the incident part being analogous to the free field (7.7), while the reflected part is due to the wall and has \(\zeta\)-dependent terms.
7.2 Linear shear over an impedance wall  

Figure 7.2: Complex $k$-plane with possible positions of poles, branch cuts of $|k|$, and original (---) and deformed (----) Fourier inversion contours for $x < 0$ and $x > 0$.

and have the physical field given by inverse Fourier transformation

$$p(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}(y, k) e^{-ikx} \, dk = p_{in} + p_{ref}.$$  

An important remark is that $\hat{p}$ is no longer singular at $k = 0$, as the singular term in $\hat{p}_{ref}$ cancels the one in $\hat{p}_{in}$ and we have

$$\lim_{k \to 0} \hat{p} = S \rho_0 (\zeta + i\omega_y).$$  

So even though both $\hat{p}_{in}$ and $\hat{p}_{ref}$ are singular at $k = 0$, the singularities cancel each other in $\hat{p}$ (at least when $\zeta$ is finite) and we have a non-diverging pressure for large $x^2 + y^2$.

Regarding poles, both $\hat{p}_{in}$ and $\hat{p}_{ref}$ have one at $k = k_0$ (the vorticity shed from the source), while $\hat{p}_{ref}$ has possibly another one or two simple poles given by the dispersion relation for surface wave-like modes

$$k_y = i\zeta^{-1} (\sigma - \text{sign}(\text{Re} \, k_y) \omega).$$  \hfill (7.14)

In particular, assuming $\sigma > 0$ and $\omega > 0$, and noting that $\text{Re} \, \zeta > 0$ on physical grounds, we may distinguish the following cases (see also Figure 7.2)
Case i. $\text{Im} \zeta > 0, \sigma > \omega$: $k_s = k_1 \in 1\text{st quadrant}$

Case ii. $\text{Im} \zeta > 0, \sigma \leq \omega$: no $k_s$ present

Case iii. $\text{Im} \zeta < 0, \sigma \geq \omega$: $k_s = k_2 \in 2\text{nd quadrant}$

Case iv. $\text{Im} \zeta < 0, \sigma < \omega$: $k_s = k_1 \in 4\text{th quadrant}$

$k_s = k_2 \in 2\text{nd quadrant}$

where

$$k_1 = i\zeta^{-1}(\sigma - \omega), \quad k_2 = i\zeta^{-1}(\sigma + \omega).$$

If $\text{Im}(\zeta) = 0$, the $k_s$ poles are exactly on the imaginary axis, i.e. on the branch cut of $|k|$, and we have to take the limit $\text{Im}(\zeta) \to 0$ from above or below, with either limit giving the same result. If $\omega = \sigma$, $k_1$ would be zero, but this is canceled out in $\hat{p}_{\text{ref}}$ and we have no contribution from it (moreover, in this case both $\hat{p}_{\text{in}}$ and $\hat{p}_{\text{ref}}$ are regular in 0).

In order to see any eventual relations between $k_s$ and the incompressible limit of surface waves obtained with the Ingard-Myers boundary condition, we rewrite equation (12) of [88] with the present notation, obtaining the dispersion relation

$$(k_\infty - k)^2 - i(\zeta/U_\infty)k_\infty |k| = 0,$$

where $k_\infty = \omega/U_\infty$ and $U_\infty$ is the uniform mean flow velocity. This equation has 0, 2 or 4 solutions (one in each quadrant) depending on $\text{Im} \zeta/U_\infty$ being $> 2$, between $-2$ and $2$, and $\leq -2$ respectively. This is to be compared with the 0, 1 or 2 solutions, depending on the signs of $\text{Im} \zeta$ and $\sigma - \omega$, for the present shear flow case. The subtleness of the correspondence is to be expected as we have a vanishing boundary layer thickness in one case, and an infinite shear in the other.

### 7.2.2 Stability analysis of the $k_s$ poles

Because of the presence of the mean flow, it is not immediately clear if the $k_s$-modes are stable. However, a Briggs–Bers stability analysis ([16, 29]) shows that any $k_s$ is a stable mode. In fact, we will show that, with $\zeta = \zeta(\omega)$, and $\omega = \omega(k)$ defined by dispersion relation (7.14), $\text{Im} \omega$ is bounded from below by
7.2 Linear shear over an impedance wall

zero as a function of real \( k \). Indeed, if we have \( k \in \mathbb{R} \), then

\[
\text{Im} \omega = |k| \text{Re} \zeta(\omega).
\]

For a passive liner with \( \text{Re} \zeta > 0 \) for real \( \omega \), this shows that \( \text{Im} \omega = 0 \) only if \( k = 0 \). Under reasonable assumptions of smoothness of \( \zeta(\omega) \), \( \text{Im} \omega(k) \) is continuous and hence can only change sign once, namely at \( k = 0 \). However, it does not change sign, for the following reason. When \( |\zeta(\omega)| \gtrsim O(\omega) \) for \( \omega \to \infty \) (a reasonable assumption if the impedance involves inertia effects), \( \zeta \) must vanish for large \( k \), and so \( \lim_{k \to \pm \infty} \zeta(\omega) = 0 \). Because of causality (\cite{[89]}), \( 1/\zeta \) must be analytic in \( \text{Im} \omega < 0 \) and so any zero of \( \zeta \) has a positive imaginary part. So \( \lim_{k \to \pm \infty} \text{Im} \omega(k) \) is always positive, and in particular \( \text{Im} \omega \) is positive on either side of \( k = 0 \) and therefore does not change sign. Hence, \( \min_{k \in \mathbb{R}} (\text{Im} \omega) = 0 \). Since this minimum is not negative, the modes are not unstable.

### 7.2.3 The solution

To determine \( p \), the solution for our problem, we note first that \( p_{\text{in}} \) is the same as for the free field, with \( y \) replaced by \( y - y_0 \); we denote this free field pressure by \( p_f \) (with similar notations for the velocities). The reflected field is a contribution of the \( k_0 \) pole, any \( k_s \) poles present, and the branch cut integrals. The contribution from \( k_0 \) is only present downstream (\( x > 0 \)). If we close the integral via \(-i\infty\), we capture the \( k_0 \) residue of \( p_{\text{ref}} \)

\[
-\frac{1}{2} \rho_0 S \sigma^2 \omega \left( 1 + k_0(y - y_0) \right) \frac{k_0 - k}{k_0 - k_1} e^{-ik_0 \bar{z}} H(x)
\]

(7.15)

(where \( z_\pm = x + i(y \pm y_0) \)) representing the image field of the shed vorticity. For the contributions from \( k_s \) poles we discuss different cases, according to their position and existence.

Folding the integration contour around the branch cuts (see figure 7.2), we ob-
tain integrals of the following type (the branch cut of $E$ still undetermined)

$$
\int_0^\infty \frac{e^{-\lambda z}}{\lambda(\lambda-iq_1)\lambda- iq_2} \, d\lambda = \frac{p_1p_2p_3}{q_1q_2} (y + \log z)
+ \frac{(p_1 + iq_1)(p_2 + iq_1)(p_3 + iq_1)}{q_1(q_1 - q_2)} E(q_1, z)
- \frac{(p_1 + iq_2)(p_2 + iq_2)(p_3 + iq_2)}{q_2(q_2 - q_1)} E(q_2, z). \tag{7.16}
$$

Altogether the pressure looks like

$$
p(x, y) = p_f(x, y - y_0) - \rho_0 S \frac{x}{\sigma y^2 + (y + y_0)^2}
- \frac{i\rho_0 S}{2\pi \omega} \left[ \sigma^2 \log(x^2 + (y + y_0)^2) \right]
- \frac{i\rho_0 S}{4\pi \omega} \left[ \left(1-k_0(y-y_0) \right) \frac{k_0 - k_1}{k_0 - k_2} E(k_0, z_+\right)
+ \frac{k_0}{k_0 - k_2} \left( E(k_0, z_+ - 2\pi i e^{-ik_0 z_+} H(x)) \right)\right]
+ \frac{\rho_0 S k_0}{2\pi \zeta} \left[ \frac{k_1(U_0 - i\zeta)}{k_0 - k_1} (\sigma y - i\zeta)(E(k_1, z_+ - C_1)
+ \frac{k_2(U_0 + i\zeta)}{k_0 - k_2} (\sigma y + i\zeta)(E(k_2, z_+) - C_2) \right] \tag{7.17}
$$

where $C_1$ and $C_2$ relate to the possible contributions of the poles $k_1$ and $k_2$.

As noted before, the pressure is not divergent for large $x^2 + y^2$, and has no undetermined constant, since the terms containing log and $\gamma$ appear in both $\hat{p}_{\text{in}}$ and $\hat{p}_{\text{ref}}$. Depending on the signs of $\text{Im} \, \zeta$ and $\sigma - \omega$, we are in one of the following cases (see also section 7.2.1)

**Case i** (1 pole). One $k_s = k_1$ present in the upper half plane and therefore contributing upstream. Thus

$$
C_1 = -2\pi i e^{-ik_1 z_+} H(-x), \quad C_2 = 0. \tag{7.18a}
$$

**Case ii** (no pole). No $k_s$ pole present, so

$$
C_1 = 0, \quad C_2 = 0. \tag{7.18b}
$$

**Case iii** (1 pole). One $k_s = k_2$ is found in the upper half plane and contributes
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upstream,

\[ C_1 = 0, \quad C_2 = 2\pi i e^{-ik_2z_+} H(-x). \quad (7.18c) \]

**Case iv** (2 poles). One pole equal to \( k_1 \) is now found in the lower half plane and contributes downstream, while the second pole is the same as in case iii above, \( k_3 \); these lead altogether to

\[ C_1 = 2\pi i e^{-ik_1z_+} H(x), \quad C_2 = 2\pi i e^{-ik_3z_+} H(-x). \quad (7.18d) \]

For \( \sigma = \omega \), (7.17) simplifies since the log- and the \( k_1 \)-terms vanish.

Using similar reasoning, we obtain from

\[ \hat{v}_{ref} = \frac{1}{2} S e^{-|k|(y+y_0)} \left(1 - \text{sign(Re } k)\right) \frac{ik\zeta + \sigma + \text{sign(Re } k)\omega}{\Omega_0} \frac{k_0 - k_1}{k_0 - k_2} \left(E(k_0, z_+) - 2\pi i e^{-ik_0z_+} H(x)\right) \]

\[ \hat{u}_{ref} = \frac{1}{2} i S e^{-|k|(y+y_0)} \left(\text{sign(Re } k)\right) \frac{ik\zeta + \sigma + \text{sign(Re } k)\omega}{\Omega_0} \frac{k_0 - k_1}{k_0 - k_2} \left(E(k_1, z_+ - C_1) + \frac{k_2(U_0 + i\zeta)}{k_0 - k_2} (E(k_2, z_+) - C_2)\right), \quad (7.19) \]

\[ u(x, y) = u_f(x, y - y_0) + \frac{S}{2\pi} \frac{x}{x^2 + (y - y_0)^2} \]

\[ v(x, y) = v_f(x, y - y_0) + \frac{S}{2\pi} \frac{y + y_0}{x^2 + (y + y_0)^2} \]

\[ - \frac{iS}{4\pi U_0} \left[ \frac{k_0 - k_1}{k_0 - k_2} E(k_0, z_+) - \frac{k_0 - k_2}{k_0 - k_1} E(k_0, z_+) - 2\pi i e^{-ik_0z_+} H(x)\right] \]

\[ - \frac{iS}{2\pi} \left[ \frac{k_1(U_0 - i\zeta)}{k_0 - k_1} (E(k_1, z_+ - C_1) - C_1) - \frac{k_2(U_0 + i\zeta)}{k_0 - k_2} (E(k_2, z_+) - C_2)\right]. \quad (7.20) \]

**On the exponential integral**

An important function in the analysis above is the function \( E(a, z) \), closely related to the exponential integral \( E_1 \) (see [7], equation 5.1.1). For \( a, z \in \mathbb{C} \) we

\[ E(a, z) = \int_0^z e^{a \zeta} \frac{d\zeta}{\zeta} \]
The trailing vorticity field behind a point source

\[
E(q, z) = e^{-i\alpha z} E_1(-i\alpha z),
\]

\[
E_1(z) = \int_{\xi}^{\infty} \frac{e^{-t}}{t} \, dt = -\gamma - \log z + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \zeta^k}{k!}.
\]  (7.21)

where \(\gamma = 0.5772156649 \ldots\) is Euler's constant. The variable \(a\) corresponds here to complex wave numbers \((k_0, k_s)\), while \(z = x + iy\) relates to the physical \((x, y)\)-space. As is clear from the series representation, \(E_1(z)\) has a logarithmic singularity, for which the standard definition is the principal value \(\log\), with \(\log(1) = 0\) and a branch cut along the negative real axis.

However, using this definition, we have for \(E(k_{21}, \bar{z})\) a branch cut from the standard \(E_1\) along the line

\[
y + y_0 = -\frac{\text{Im} \, \zeta}{\text{Re} \, \zeta} x \quad \text{or, equivalently,} \quad y + y_0 = -\frac{\text{Re} \, k_{21}}{\text{Im} \, k_{21}} x
\]

for \((k_1 - k_0) \text{ Re } \zeta x < (k_1 - k_0) \text{ Im } \zeta (y + y_0)\). For \(E(k_{22}, z)\), the principal value \(E_1\) leads to a branch cut on

\[
y + y_0 = \frac{\text{Im} \, \zeta}{\text{Re} \, \zeta} x \quad \text{or, equivalently,} \quad y + y_0 = \frac{\text{Re} \, k_{22}}{\text{Im} \, k_{22}} x
\]

for \(x < -\frac{\text{Im} \, \zeta}{\text{Re} \, \zeta} (y + y_0)\) (see figure 7.3). So with the principal value \(E\) we would have discontinuities in \(p, u\) and \(v\).

There are two ways to correct for these unphysical discontinuities in \(E(a, z)\): either shift the contribution of the corresponding pole, and move it from \(x = 0\) to the line of discontinuity, or redefine \(E(a, z)\) such that its branch cut is always on \(\text{Re } z = 0\). For the latter case, the poles are included as usual, as right- or left-running, so we choose this option believing it to be the most intuitive one.

We make our choice such that if \(\text{Re } a > 0\), the branch cut of \(E(a, z)\) is along the line \(x = 0, y < 0\), and thus for \(E(a, \bar{z})\) along the line \(x = 0, y > 0\). If \(\text{Re } a < 0\) it is the opposite: the branch cut of \(E(a, z)\) is then taken along the line \(x = 0, y > 0\). This is most easily obtained by the logarithm

\[
\text{Log}(-i\alpha z) \overset{\text{def}}{=} \log\left(-iz\frac{a}{|a|}\right) + \log(|a|)
\]

with the principal value \(\log\) and \(|a|\) as defined in (7.5). So we define our function
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Figure 7.3: Branch cuts (−) of the standard $E_1(-ik_{21}z)$ and $E_1(-ik_{22}z)$ together with their support lines (--). Red indicates branch cuts in the physical field.
E with this \( \text{Log} \) and with the standard exponential integral, \( E_1 \), as

\[
E(a, z) \overset{\text{def}}{=} e^{-iaz} \left( E_1(-iaz) + \text{Log}(-iaz) - \text{Log} \left( \frac{a}{|a|} \right) - \text{Log}(|a|) \right). \tag{7.22}
\]

This definition only differs from the standard one if \( a \) is not both real and positive, and therefore agrees with \( E(k_0, z) \) in (7.11).

With (7.22), the branch cuts of the \( E \) functions now compensate (as in the free field problem) for the unphysical jumps\(^6\) of the Heaviside functions. The resulting fields are therefore smooth and continuous apart from the discontinuity in \( u \) due to the \( \text{sign}(y) \) term in the free field solution \( u_f \) mentioned previously.

### 7.2.4 The hard wall limit

A special case of interest is the hard wall limit, i.e. \( \zeta \to \infty \). This straightforwardly found for the velocities, and they are symmetric about \( y = 0 \)

\[
v_{\text{HW}}(x, y) = \frac{S}{2\pi} \frac{y - y_0}{z_- \bar{z}_-} - \frac{S \sigma}{4\pi U_0} [E(k_0, z_-) + E(k_0, \bar{z}_-) - 2\pi i H(x) e^{-ik_0 x - k_0 |y - y_0|}]
- \left( \frac{S}{2\pi} \frac{-(y + y_0)}{z_+ \bar{z}_+} - \frac{S \sigma}{4\pi U_0} [E(k_0, \bar{z}_+) + E(k_0, z_+) - 2\pi i H(x) e^{-ik_0 x - k_0 (y + y_0)}] \right), \tag{7.23}
\]

\[
u_{\text{HW}}(x, y) = \frac{S}{2\pi} \frac{x}{z_- \bar{z}_-} + \frac{S}{2\pi} \frac{x}{z_+ \bar{z}_+}
+ \frac{i S \sigma}{4\pi U_0} [E(k_0, z_-) - E(k_0, \bar{z}_-) + 2\pi i H(x) \text{sign}(y - y_0) e^{-ik_0 x - k_0 (y - y_0)}]
+ \frac{i S \sigma}{4\pi U_0} [E(k_0, \bar{z}_+) - E(k_0, z_+) + 2\pi i H(x) \text{sign}(-(y + y_0)) e^{-ik_0 x - k_0 (y + y_0)}]. \tag{7.24}
\]

As a result, the expressions can be written in terms of the free field velocities as follows.

\[
v_{\text{HW}}(x, y) = v_f(x, y - y_0) - v_f(x, -(y + y_0)), \tag{7.25}
\]

\[
u_{\text{HW}}(x, y) = u_f(x, y - y_0) + u_f(x, -(y + y_0)).
\]

\(^6\)these jumps are artifacts of the contour being closed via the lower (if \( x > 0 \)) or upper (if \( x < 0 \)) complex half plane
7.2 Linear shear over an impedance wall

The hard wall limit for pressure, on the other hand, is more subtle than may be expected, because there is inherent undetermined additional constant. So any hard-wall limit will not (immediately) agree with a corresponding case of finite but large $\zeta$.

If we take directly the limit of (7.17) and ignore the diverging log $\zeta$-terms, we find

$$p_{HW}(x,y) = p_f(x,y-y_0) - p_f(x,-(y+y_0)) - \frac{i\rho_0 S \sigma}{2\pi} [E(k_0,z_+) - E(k_0,\bar{z}_+)] + 2\pi i H(x) e^{-ik_0y_0} + (1 + \omega/\sigma) \log(-iz_+) - (1 - \omega/\sigma) \log(i\bar{z}_+)] \quad (7.26)$$

where log denotes the principal value logarithm (see eq. 7.22). The hard-wall pressure is, remarkably, not similar to the corresponding expressions (7.25) for the velocities.

7.2.5 Examples

The cases are useful in order to know how to correctly calculate the pressure and velocities. Apart from that, there is no fundamental difference. Important characteristics in field are again $k_0$ and $y_0$, giving the position and the wavelength of the trailing vortices, with $\text{Im} \ \zeta$ and $\omega/\sigma$ having some intrinsic importance not immediately visible in the plots. For a typical case, see Figure 7.4, where the pressure and velocities are plotted together with their hard-wall limits. The effect of the undetermined constant in the hard-wall pressure is removed by subtracting its plot-domain averaged value. Since the hydrodynamic wave length $2\pi/k_0 = 3.93$ is large compared to the distance $y_0 = 0.5$ to the wall, we have a strong interaction of the shed vorticity field with the wall, seen especially in the velocity plots.

7.2.6 Interpretation for acoustic duct modes with lined walls

In order to compare with the acoustic problem of a lined flow duct, we note that $\rho_0 \xi = \rho_0 c_0 Z$ such that the dimensionless duct equivalents of $k_x$ are

$$k_x := k_x a = \frac{i}{Z} \left( \frac{M}{h} \pm \omega \right).$$
Figure 7.4: Soft- and hard-wall pressure and velocities with $y_0 = 0.5$ and $k_0 = 1.6$ computed for a case iii parameter choice.
These correspond indeed to two of the compressible surface modes, as is clearly seen in Figure 7.5. The limit $c_0 \to \infty$ is taken such that the Strouhal number is fixed and the impedance $Z = O(M)$. We use the notation and geometry of [25], i.e. a cylindrical duct with linear-then-constant mean flow

$$U(r) = \begin{cases} M, & 0 \leq r \leq 1 - h, \\ M(1 - r)/h, & 1 - h \leq r \leq 1, \end{cases}$$

with $M$ the mean flow Mach number.

### 7.3 Vorticity equation

To conclude this chapter, we analyze the effect of a point mass source in linear shear from a different perspective. Since we know that $k_0$ is the trailing vorticity of a source in sheared flow, we look directly at the vorticity equation for this case. To begin with, consider the equation for conservation of momentum in 3D compressible inviscid flow, together with the continuity equation with a mass
source \( Q \)

\[
\rho \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) v + \nabla p = 0
\]

\[
\left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \rho + \rho \nabla \cdot v = Q.
\]

Vorticity is a measure of the angular rotation of fluid particles [37] and is given as the curl of the velocity vector

\[
\omega = \text{curl} \, v.
\]

Analogously, we obtain the vorticity equation (in vector form) by taking the curl of the momentum equation for a barotropic fluid\(^7\) (see [37], [94], etc.)

\[
\left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \omega = \omega \cdot \nabla v - \omega \nabla \cdot v. 
\]

From the continuity equation

\[
\nabla \cdot v = -\frac{1}{\rho} \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \rho + \frac{Q}{\rho},
\]

and we have that vorticity for an inviscid, barotropic flow, with a mass source \( Q \) is given by

\[
\left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \left( \frac{\omega}{\rho} \right) = \left( \frac{\omega}{\rho} \cdot \nabla \right) v - \omega \frac{Q}{\rho}. \quad (7.27)
\]

When the source varies harmonically in time with small amplitude and we have a parallel mean flow given by \( v_0 = (U(y), 0, 0) \), with uniform pressure \( p_0 \), density \( \rho_0 \) and sound speed \( c_0 \), we linearize the problem by introducing small perturbations of the form

\[
\rho = \rho_0 + c_0^{-2} \tilde{p} e^{i\omega t}, \quad v = v_0 + \tilde{v} e^{i\omega t}, \quad p = p_0 + \tilde{p} e^{i\omega t}, \quad Q = q e^{i\omega t}, \quad \omega = \omega_0 + \tilde{\omega}.
\]

For a simple linear shear flow \( U(y) = U_0 + \sigma y \), \( \omega_0 = (0, 0, -\sigma) \) and the linearized form gives

\[
\left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \left( \frac{\omega}{\rho} \right) = \left( \frac{\omega}{\rho} \cdot \nabla \right) \tilde{v} - \tilde{\omega} \frac{Q}{\rho}. \quad (7.27)
\]

\( ^7 \)Barotropic fluids are characterized by the fact that density is only a function of pressure, and vice versa; examples include isentropic ideal gases.
zed form of (7.27) is

\[
\left( i\omega + U \frac{\partial}{\partial x} \right) \left( \tilde{\omega} - \omega_0 \frac{p}{\rho_0 c_0^2} \right) = -\omega_0 \frac{q}{\rho_0} + (\tilde{\omega} \cdot \nabla) v_0 - \sigma \frac{\partial v}{\partial z},
\]

(7.28)

or written out in coordinates for \( \tilde{\omega} = (\tilde{\omega}_x, \tilde{\omega}_y, \chi) \)

\[
\begin{align*}
\left( i\omega + U \frac{\partial}{\partial x} \right) \tilde{\omega}_x &= \sigma \tilde{\omega}_y - \sigma \frac{\partial \tilde{\dot{u}}}{\partial z} \quad (7.29a) \\
\left( i\omega + U \frac{\partial}{\partial x} \right) \tilde{\omega}_y &= -\sigma \frac{\partial \tilde{\dot{v}}}{\partial z} \quad (7.29b) \\
\left( i\omega + U \frac{\partial}{\partial x} \right) \left( \chi + \frac{\sigma}{\rho_0 c_0^2} \tilde{\ddot{p}} \right) &= \sigma \frac{q}{\rho_0} - \sigma \frac{\partial \tilde{\dot{w}}}{\partial z} \quad . \quad (7.29c)
\end{align*}
\]

In 2D we have \( \tilde{\dot{v}} = (\tilde{\dot{u}}, \tilde{\dot{v}}, 0) \), with the only nonzero vorticity component being \( \chi \) (implying that the first two of the above equations are null identities) given by

\[
\left( i\omega + U \frac{\partial}{\partial x} \right) \left( \chi + \frac{\sigma}{\rho_0 c_0^2} \tilde{\ddot{p}} \right) = \sigma \frac{q}{\rho_0} . \quad (7.30)
\]

For a monopole type point source \( q = \rho_0 S \delta(x) \delta(y) \), (7.30) has, under causal free-field conditions for \( U_0 > 0 \), the solution

\[
\chi + \frac{\sigma}{\rho_0 c_0^2} \tilde{\ddot{p}} = S \frac{\sigma}{U_0} H(x) e^{-ik_0 x} \delta(y), \quad k_0 = \frac{\omega}{U_0} .
\]

(7.31)

Noting that the pressure term above cannot be discontinuous, we have that a line source in sheared flow produces a semi-infinite sheet of vorticity, undulating with hydrodynamic wavenumber \( k_0 \).

Moreover, for an incompressible case, the above simplifies to an explicit solution for the vorticity

\[
\chi = S \frac{\sigma}{U_0} H(x) e^{-ik_0 x} \delta(y), \quad k_0 = \frac{\omega}{U_0}.
\]

(7.32)
Appendix A

The dispersion relation in the neighborhood of a saddle point

Since dispersion relations $D(\omega, k) = 0$ are in general complicated allowing little or no direct conclusions, it may be instructive to note that under rather general conditions their behavior near a critical point $(\omega^*, k^*)$ is always the same.

Expand $D(k, \omega)$ around $(k^*, \omega^*)$ as

$$D(k, \omega) \approx D(k^*, \omega^*) + D_k(k^*, \omega^*)(k - k^*) + D_\omega(k^*, \omega^*)(\omega - \omega^*) + \frac{1}{2}D_{kk}(k^*, \omega^*)(k - k^*)^2 + D_{k\omega}(k^*, \omega^*)(k - k^*)(\omega - \omega^*) + \frac{1}{2}D_{\omega\omega}(k^*, \omega^*)(\omega - \omega^*)^2,$$

(A.1)

where indexes represent partial derivatives with respect to that variable. If
The dispersion relation in the neighborhood of a saddle point

$(k^*, \omega^*)$ is a mode, and a saddle point in $k$, we are left with

$$D(k, \omega) = D_{\omega}(k^*, \omega^*)(\omega - \omega^*) + \frac{1}{2}D_{kk}(k^*, \omega^*)(k - k^*)^2 + D_{k\omega}(k^*, \omega^*)(k - k^*)(\omega - \omega^*) + \frac{1}{2}D_{\omega\omega}(k^*, \omega^*)(\omega - \omega^*)^2.$$  \hspace{1cm} (A.2)

Since we are in the neighborhood of the saddle point, both $\omega - \omega^*$ and $k - k^*$ are small, and we seek to determine a proper scaling between the two. Denote therefore

$$\omega - \omega^* = \epsilon X, \quad \text{and} \quad k - k^* = \delta Y,$$

with $X, Y \in O(1)$, and introduce the short-hand notation

$$d_\omega = D_{\omega}(k^*, \omega^*), \quad d_{kk} = \frac{1}{2}D_{kk}(k^*, \omega^*), \quad d_{k\omega} = D_{k\omega}(k^*, \omega^*), \quad d_{\omega\omega} = \frac{1}{2}D_{\omega\omega}(k^*, \omega^*).$$

Substituting in (A.2), for $D(k, \omega) = 0$,

$$\epsilon Xd_\omega + \delta^2 Y^2 d_{kk} + \epsilon \delta XY d_{k\omega} + \delta^2 Y^2 d_{\omega\omega} = 0$$

or

$$Xd_\omega + \frac{\delta^2}{\epsilon} Y^2 d_{kk} + \delta XY d_{k\omega} + \epsilon Y^2 d_{\omega\omega} = 0.$$  

The middle terms are both small, so the first and the last terms have to balance each other, resulting in

$$\frac{\delta^2}{\epsilon} = O(1).$$

Returning with this result in (A.2), we have that to leading order $D(k, \omega) = 0$ reduces to

$$d_\omega(\omega - \omega^*) + d_{kk}(k - k^*)^2 = 0.$$  \hspace{1cm} (A.3)
Appendix B

Pridmore-Brown equation in cylindrical coordinates

Consider the linearized Euler equations in cylindrical coordinates (see Chapter 2 equations (2.17)-(2.19) for $v_0 = (U(r), 0, 0)$; for convenience we have dropped the $r$ in the perturbation variables)

\[ \frac{1}{\rho_0 c_0^2} \left( \frac{\partial p}{\partial t} + U \frac{\partial p}{\partial x} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial w}{\partial \theta} \right) = Q \]  

(B.1)

\[ \frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + v \frac{\partial U}{\partial r} = -\frac{\partial p}{\partial x} \]  

(B.2)

\[ \frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial r} \]  

(B.3)

\[ \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} = -\frac{1}{r} \frac{\partial p}{\partial \theta}, \]  

(B.4)
perturbations given by the sum over Fourier integrals

\[ p(x, r, \theta, t) = \frac{e^{i\omega t}}{2\pi} \sum_{m=-\infty}^{\infty} e^{-im\theta} \int_{-\infty}^{\infty} \hat{p}(r, k, m) e^{-ikx} dk \]

with similar expressions for \( u, v \) and \( w \), and \( Q \) a time-harmonic point source of strength \( S \) located at \((0, r_0, 0)\), \( Q = S \delta(x)\delta(r - r_0)\delta(\theta) e^{i\omega t} \). We represent the \( \delta(\theta) \) by a generalized Fourier series in \( m \), and \( \delta(x) \) by a Fourier integral, amounting to:

\[ Q(x, r, \theta, t) = S \frac{\delta(r - r_0)}{2\pi r_0} \sum_{m=-\infty}^{\infty} e^{-im\theta} \int_{-\infty}^{\infty} e^{-ikx} dk. \]

With these assumptions we can write the velocities in (B.2-B.4) in terms of \( \hat{p} \)

\[ \hat{u} = \frac{k}{\rho_0\Omega} \hat{p} - \frac{dU}{dr} \frac{1}{\rho_0\Omega^2} \frac{d\hat{p}}{dr} \]  \hspace{1cm} (B.5)

\[ \hat{v} = \frac{i}{\rho_0\Omega} \frac{d\hat{p}}{dr} \]  \hspace{1cm} (B.6)

\[ \hat{w} = \frac{m}{\rho_0 r_0 \Omega} \hat{p}, \]  \hspace{1cm} (B.7)

\( \Omega = \omega - kU \), and substitute them in the continuity equation to find

\[ \frac{i}{\Omega} \frac{d^2 \hat{p}}{dr^2} + \left( \frac{i}{\rho_0\Omega} + \frac{ikdU/dr}{\Omega^2} \right) \frac{d\hat{p}}{dr} + i \left( \frac{\Omega}{\rho_0^2} - \frac{k^2}{\Omega} - \frac{m^2}{r^2\Omega} \right) \hat{p} = S \frac{\delta(r - r_0)}{2\pi r_0}. \]

Multiplying this equation with \((-i\Omega)\), to have the left side in a Pridmore-Brown form, we arrive to

\[ \frac{d^2 \hat{p}}{dr^2} + \left( \frac{1}{r} + \frac{kdU/dr}{\Omega} \right) \frac{d\hat{p}}{dr} + \left( \frac{\Omega^2}{\rho_0^2} - \frac{k^2}{\rho_0^2} - \frac{m^2}{r^2\Omega} \right) \hat{p} = -iS \frac{\delta(r - r_0)}{2\pi r_0}. \]  \hspace{1cm} (B.8)
Appendix C

Tempered distributions

In this section we present some mathematical tools extensively used in previous chapters. We mainly focus on the concept of tempered distributions and then link it to the generalized functions’ approach.

In our modeling attempts, we isolate features and simplify the problem such as to satisfactorily reproduce the set of characteristics we are interested to preserve. We pay off for this simplification by needing a more complex mathematical theory to solve our reduced problem. Take for example an impulse, which is the change of momentum resulting from a force acting over a period of time; we formally represent it as the integral of the force over the time interval. In many practical cases this period is negligible (but nonzero), and it is, hence, convenient to consider its limit to zero for a fixed change of momentum (i.e. keeping the value of the integral fixed). By doing so, we end up with something like Figure C.1 a., which is one of the representations of the delta function, $\delta(t)$. As we can already guess from the arguments above, there is no pointwise meaning attached to $\delta(t)$, and it therefore does not qualify as a func-
For practical reasons, the δ-function and other certain type of distributions were used in engineering sciences already in the nineteenth century, even though a rigorous mathematical framework appeared only half-way through the twentieth century, with the work of Laurent Schwartz. A rigorous treatment of the topic requires the abstract notion of Fréchet spaces [97] whose introduction is beyond our modeling scope. We therefore choose a more intuitive approach.

The space of tempered distributions is a suitable setting to look for solutions of differential equations since existence of derivatives, convolutions and applicability of Fourier and Laplace transforms are here immediate. A solution in the sense of distributions is in many cases equivalent to a classical solution \(^1\) by means of regularity theory. We briefly explain all these concepts below.

First consider the Schwartz space\(^2\) of rapidly decreasing functions, which is invariant under Fourier transformation [97]

\[
\mathcal{S}(\mathbb{R}) := \left\{ \phi \in C^\infty(\mathbb{R}) \mid \lim_{|x| \to \infty} |x^q \phi^{(r)}(x)| = 0 \text{ for all integers } q, r \geq 0 \right\}, \quad (C.1)
\]

\(^1\)differentiable in the classical sense

\(^2\)sometimes referred to as the space of good functions, see [58]
or in other words, a function is in the Schwartz space if it vanishes at infinity, together with all its derivatives, faster than any polynomial.

We define convergence in this space, by saying that a sequence \( \{\phi_n\} \subset \mathcal{S}(\mathbb{R}) \) converges to \( \phi \in \mathcal{S}(\mathbb{R}) \) if for any non-negative integer \( r \)

\[
\lim_{n \to \infty} \max_{x \in \mathbb{R}} |\phi_n^{(r)}(x) - \phi^{(r)}(x)| = 0.
\]

We can hence introduce continuous functionals on \( \mathcal{S}(\mathbb{R}) \).

We define the space of tempered distribution as the dual of the Schwartz space, \( \mathcal{S}'(\mathbb{R}) \), i.e. the space of linear and continuous functionals

\[
u : \mathcal{S}(\mathbb{R}) \to \mathbb{C}.
\]

The convergence in \( \mathcal{S}'(\mathbb{R}) \) is defined by

\[
u_n \to \nu \quad \text{in} \quad \mathcal{S}'(\mathbb{R}) \quad \text{if} \quad \lim_{n \to \infty} \langle u_n, \phi \rangle = \langle u, \phi \rangle
\]

for every test function \( \phi \in \mathcal{S}(\mathbb{R}) \), where \( \langle u, \phi \rangle = u(\phi) \).

A useful result is that we can represent every tempered distribution by sequences of smooth functions in the following way: given any \( u \in \mathcal{S}'(\mathbb{R}) \), there exists a sequence of smooth functions \( \{f_n\} \subset \mathcal{S}(\mathbb{R}) \), such that for every test function \( \phi \in \mathcal{S}(\mathbb{R}) \)

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x)\phi(x)dx = \langle u, \phi \rangle.
\]

This can be also taken as an alternative definition for \( \mathcal{S}'(\mathbb{R}) \), and justifies the terminology 'generalized function'. As an illustration, see Figure C.1 for two representations of the \( \delta \) function. Figure C.1(a) is given by

\[
\delta_n = \left(\frac{n}{\pi}\right)^{1/2} e^{-nx^2},
\]

and Figure C.1(b) by

\[
\delta_n = \frac{\sin nx}{\pi x} e^{-x^2/n^2},
\]

see also [94], Appendix C.
Operations on distributions, such as derivatives and integral transforms, are defined via the corresponding operations on test functions:

- \(\langle u', \phi \rangle = -\langle u, \phi' \rangle\), for every test function \(\phi \in \mathcal{S}^{}(\mathbb{R})\);
- for every \(f, \phi \in \mathcal{S}^{}(\mathbb{R})\), \(\langle fu, \phi \rangle = \langle u, f\phi \rangle\);
- the Fourier transform is defined by \(\langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle\), for every test function \(\phi \in \mathcal{S}^{}(\mathbb{R})\).

Any locally integrable function \(f\) with at most polynomial growth yields a continuous linear functional on \(\mathcal{S}^{}(\mathbb{R})\), \(u_f\) given by

\[
\langle u_f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx,
\]

for all test functions \(\phi \in \mathcal{S}^{}(\mathbb{R})\). In this case we identify \(u_f\) with \(f\), an identification implicitly used in the handling of Green’s functions. Moreover, we can say in this context that tempered distributions generalize the slowly growing locally integrable functions.

**Regularity in a nutshell**

Remarkable subspaces of \(\mathcal{S}'^{}(\mathbb{R})\) defined via Fourier transforms, are the Sobolev spaces \(H^s\), \(s \geq 0\), (see [35] pg. 209) of distributions with the first \(s\) derivatives in \(L^2\) (we have identified the distributional derivatives with functions). A delicate branch of analysis is the regularity theory, in particular, the so-called Morrey embedding theorems which allow us to prove that functions in higher order Sobolev spaces posses classical derivatives [47].

**Distributions in this thesis**

We mainly use distributions to describe point-sources in different geometries. We refer to tempered distributions only with respect to the axial direction \(x\), in which the domain is unbounded and non-periodic. This functional setting enables the use of the Fourier transform and its inverse, and gives meaning to a differential calculus on function which are not differentiable in a classical sense. However, in \(y\) and \(r\), due to the finiteness of the geometry, we need
to consider distributions acting on smooth test functions of compact support (see [35], [97]). In addition to this, we formally use in the cylindrical case, Fourier series in $\theta$. Fourier series are well defined in the context of periodic generalized functions, and we refer to [58] sections 5.5 and 5.6 for the theoretical context, since we are here interested in single $m$-modes, and do not explicitly compute the fields in the azimuthal direction.
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Summary

Acoustic liner - mean flow interaction

Since the introduction in the 1950’s of the jet engine in civil aviation, its noise has been a stringent problem. National laws and international regulations forbid selling noisy aircraft and limit the total yearly noise load of airports, this noise load depending on both the number of flight motions (starts and landings), and the noise emitted per aircraft. Since the number of flight motions increases in direct proportion to economic growth, the noise per aircraft has to decrease in compensation. As a result, year by year a world wide effort in aeroacoustic research remains necessary.

From elementary dimensional arguments, scaling is hardly possible, while full scale experiments are very expensive, thus making almost any mathematical model cheaper and better. We study in this thesis aspects of parts of the mathematical model used in duct acoustics, and focus on the interaction of acoustic liners with the mean flow, discussing reliability and accuracy in this context.

A more than 50 years old modeling problem was the correctness of a vanishing boundary layer along an acoustic liner. Arguing classically, the acoustic effects of a boundary layer that is much thinner than any characteristic wave length is the same as of a vanishing boundary layer. So for a numerically efficient and thrifty model without unnecessary parameters, it is reasonable to apply this limit yielding the so-called Ingard-Myers condition. This works well in frequency domain. Our research showed that in time domain, on the other hand, a boundary layer less than a certain (very small, but non-zero) thickness is absolutely unstable. This makes the model useless for any industrially relevant configuration. We propose here a corrected version of the limit that retains the
stability properties of the finite boundary layer. In addition to this, we give an estimate for the critical boundary layer thickness, beyond which the flow is absolutely unstable.

The last part of the thesis discusses the critical layer singularity arising in the mathematical model due to the inviscid assumption. Common treatment is to by-pass its contribution assuming it is negligible, without fully understanding its subtleties. We study this problem for linear-shear boundary layers over acoustic linings. We show that if the source is located in the boundary layer, neglecting the critical layer means neglecting also the trailing vorticity produced by source, which introduces significant errors. Moreover, extra care is needed for high frequencies, due to an existing leaking mode.
Mirela Dărău was born on the 13th of October 1984 in Arad, Romania. After finishing her pre-university education at the Moise Nicoară National College (Arad) in 2003, she started her Bachelor studies in Mathematics at the West University in Timișoara. She obtained her Master’s title in 2008 in the same university, with a specialization in Applied Informatics.

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