The development of proficiency in the fraction domain
affordances and constraints in the curriculum

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PROEFSCHRIFT

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Contents

1 General introduction 3
  1.1 Dutch debate on basic mathematical skills 3
  1.2 Research aims 4
    1.2.1 Focus 4
    1.2.2 Background 5
  1.3 Research strategy 6
    1.3.1 Student proficiency 6
    1.3.2 Textbook analysis 7
  1.4 Outline 7

I Student proficiency 9

2 Framework for the proficiency test 13
  2.1 Rationale 14
    2.1.1 Proficiency 14
    2.1.2 Delineation 15
  2.2 Big ideas 15
    2.2.1 Relative comparison 16
    2.2.2 Reification 18
    2.2.3 Equivalence 19
    2.2.4 From natural number to rational number system 19
    2.2.5 Relation division-multiplication 20
  2.3 Complexity factors 20
    2.3.1 General complexity factors 20
    2.3.2 Initial fraction concepts 22
    2.3.3 Basic operations 24
    2.3.4 Application of fraction knowledge 25
  2.4 Conclusion 26

3 Construction and evaluation of the proficiency test 31
  3.1 Test construction 31
    3.1.1 Design 32
    3.1.2 Construction and validation 32
    3.1.3 Data collection 33
  3.2 Creating a scale for proficiency in fractions 33
    3.2.1 Fit 35
    3.2.2 Rasch scale 35
### CONTENTS

3.3 Analysis of the development of fraction proficiency 37
   3.3.1 Item level analysis: addition of fractions 37
   3.3.2 Concept level analysis: development of fraction understanding 38
3.4 Discussion 41
   3.4.1 Test construction: uni-dimensionality, validity and reliability 41
   3.4.2 Describing the development of proficiency 41
   3.4.3 Diagnostic value of the test 42
   3.4.4 Further research 42

4 Progress of proficiency in secondary education 45
   4.1 Methodology 45
      4.1.1 Test construction and data collection 45
      4.1.2 Scale for proficiency in fractions 47
   4.2 Development of proficiency 47
      4.2.1 Cross-sectional analysis 47
      4.2.2 Longitudinal analysis 48
   4.3 Level of understanding 49
      4.3.1 Item level analysis 49
      4.3.2 Concept level analysis 54
   4.4 Discussion 56
      4.4.1 Development of proficiency 56
      4.4.2 Preparation for higher secondary education 57

II Textbook analysis 61

5 Incoherence as reflected in primary and secondary education textbooks 65
   5.1 Background 66
      5.1.1 Theoretical lens 66
      5.1.2 Meaning of artifacts 68
      5.1.3 Organization of the Dutch educational system 68
      5.1.4 Activity systems in Dutch education 69
      5.1.5 Focus of this study 70
   5.2 Dataset construction 70
      5.2.1 Selecting textbooks 70
      5.2.2 Units - text and inscription 71
   5.3 Coarse-grained analysis: common structures in the textbooks 71
   5.4 Fine-grained analysis: meaning of multiplication 73
      5.4.1 Description of analysis 73
      5.4.2 Comparison of primary and secondary education textbooks 75
      5.4.3 Findings 77
   5.5 Discussion 78
   5.6 Implications 79
CONTENTS

List of publications                               149
ESoE dissertation series                          151
Index                                             153
“Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room and it’s dark, really dark. One stumbles around bumping into the furniture. And gradually you learn where each piece of furniture is. And finally after six months or so you find the light switch, you turn it on and suddenly it is all illuminated. You can see exactly where you were.”

(Andrew Wiles)
In recent years, a growing concern about the proficiency level of Dutch students has led to a public debate about mathematics education. This controversy has been the context and starting point of this dissertation, in which we investigate the development of proficiency in the domain of fractions. In this section we sketch some characteristics of the Dutch debate, introduce our research aims, describe our research strategy and finally outline this dissertation.

1.1 Dutch debate on basic mathematical skills

In the last decade, there has been a heated debate on the way mathematics is taught in the Netherlands. The public debate has started with complaints from higher education that the proficiency level of new students did not meet expectations and was dropping. Since then the public debate has been polarized in two opposing camps; those who defend reform mathematics and those who advocate a return to traditional methods of mathematics teaching.

Main empirical input for the debate has been results from large scale assessments providing historical and/or international comparisons. These assessment projects however are not intended nor appropriate to search for causes of these problems and to find pointers for solving them. Some of these studies, like TIMMS and PPON, have even been used to substantiate arguments from both opposing camps. On the one hand it is claimed that the proficiency level of Dutch students did not decrease, while on the other hand the opposite is substantiated by using the same studies. The main problem with such studies is that the attainment targets for mathematics education have changed during the last decades, and, accordingly, the assessment of students’ basic mathematical skills. It is hard to say if an increase of skills in one domain makes up for a decline in other domains.

In the debate, Realistic Mathematics Education (RME), that has been widely adopted in primary education, is subject to fire. To it are, justly and unjustly, attributed all characteristics of current mathematics education. Judging by the many different causes that have been hypothesized, the problem appears to be complex and touches upon many educational issues. That is, the hypothesized causes range from primary to secondary education, from the use of contexts to the graphical calculator, from the role of the teacher to the “maintenance” of basic skills. There is however hardly any research to substantiate
General introduction

these hypotheses. The debate mirrors debates in many other countries regarding reform mathematics (e.g. Kilpatrick, 2001). Studies such as PISA and TIMMS have given rise to concern in many countries, especially in those countries with a relative low rating in the league tables (e.g. Anderson et al., 2010; Neumann et al., 2010). The debate touches upon some strong dichotomies such as procedural versus conceptual knowledge, drill and practice versus reform mathematics, skills versus understanding, a focus on daily live versus a theoretical focus on mathematics, and mechanistic-structuralistic versus empirical-realistic. In this sense, many parallels can be drawn with discussions in other countries.

This sketch of the Dutch debate illustrates two issues regarding the problem of the basic mathematics proficiency level. The first issue is that the problem is complex, it is not likely that there is just one cause. Rather, we expect a whole range of causes that together contribute to the change of proficiency level over the last decades. The second issue is that there is a lack of knowledge of the actual proficiency level of students. Without knowledge of these levels in the whole range from primary to tertiary education, there are no pointers in where to look for adverse factors in the educational setting. This knowledge of proficiency levels is essential for improving the curriculum. Without knowing why things have not worked out as intended, it is difficult to come to structural and efficient solutions. Without such knowledge there is a risk of just treating the symptoms instead of real improvement of the curriculum. This leads us to our research aims.

1.2 Research aims

The general aim of the studies described in this dissertation, is to analyse the development of proficiency in the domain of fractions and to link this development to the formal curriculum of textbooks. Although, this dissertation originated in the polarized Dutch debate, it is not on comparing reform mathematics and traditional ways of mathematics education. Rather, in the analysis, the current situation, following from pedagogical choices and the interplay with other aspects of the educational setting, is taken as is.

1.2.1 Focus

In this dissertation we focus on the development of proficiency over a number of grades in primary and secondary education. Since mathematics is a strongly interrelated and hierarchical domain, basic mathematical skills are interwoven and therefore subject to study over long periods in the curriculum. Since the mathematical basics are the fundament of more advanced mathematics, the mathematical basis that is laid in primary education thus also influences mathematics in higher education. Furthermore, some mathematical ideas, such as functions or fractions, are so wide-ranging that their learning path is distributed over a number of grades. Given the context, we focus on students preparing for higher education. In the orientation phase of this dissertation, we studied reports of the problems felt in higher education (e.g. Martens et al., 2006; Sterk and Perrenet, 2005; Tempelaar, 2006). This led us to conclude that some of the problems in the mathematics curriculum originate early in the learning process, especially in primary education and lower secondary education. These reports showed that among others, problems in the domain of fractions. This has led to our
choice to focus our study on the domain of fractions. In Dutch education, the fraction symbol is introduced in primary education. In the third year of secondary education of HAVO and VWO a formal understanding of fractions is required.

To conclude, we focus on HAVO and VWO, on proficiency development from primary to secondary education and on the domain of fractions.

1.2.2 Background

In Dutch primary education, and to some extent in secondary education, Realistic Mathematics Education (RME) has influenced mathematics teaching. In this section we will briefly sketch the basic principle of RME.

Realistic mathematics education

RME originated in the early 1970s. Keywords are “guided reinvention” and “progressive mathematization” (Freudenthal, 1991). Especially the ideas on the use of contexts and models in this tradition are important for our study. These ideas are reflected in the theory of emergent modeling that classifies mathematical activity into four levels, namely task setting, referential level, general level and formal level (Gravemeijer, 1999). The main idea is that students’ thinking shifts from reasoning about the context of a problem to reasoning about the mathematical relations involved. This process is supported by the introduction of proper models. Ideally, these models initially come to the fore as models of informal situated activity, and later gradually develop into models for more formal mathematical reasoning. This is in contrast with the nature of contexts and models and the use thereof in more “traditional” approaches to mathematics education as they are still found in secondary education. In a more traditional approach, contexts are used as an application of the learning process (Gravemeijer and Doorman, 1999), while models are rather used as mathematical models in problem solving and proving. These ways of using the words context and model are more akin with its use in academic mathematics.

Furthermore, in RME, mathematics is seen as a human activity. RME can be considered as reform mathematics in line with constructivist ideas. Reality is seen both as a source for learning mathematics as well as a field of applying mathematics (Freudenthal, 1968). Finally, intertwinement of mathematical topics is to be reflected in instruction (e.g. Streefland, 1988; Treffers et al., 1989).

Dutch educational system

In the Dutch system there is an early differentiation at the end of primary education (age 12). At the end of primary education, students choose for one of the three streams for secondary education (Figure 1.1). These streams are VWO (pre-university education), HAVO (general education or pre-higher vocational education), and VMBO (pre-vocational secondary education). The VWO stream, which represents about 15% of the population, typically prepares students for university. Gymnasium is a VWO stream that offers students additional courses in the classical languages Latin and Greek. The HAVO stream (25%) prepares students for higher vocational education. The VMBO stream is divided in four
General introduction

sub-streams, i.e. BBL (basic vocational programme), KBL (middle management vocational programme), GL (combined programme), and TL (theoretical programme). These streams represent 17 %, 16 %, 6 %, and 21 % of the total number of students in secondary education respectively. The majority (72 %) of the VMBO “certified” students continues their studies in MBO (post-secondary vocational education and training). Only a small number of VMBO students (approximately 6% from GL and 12% from TL), continues in the HAVO stream (Van Esch and Neuvel, 2007). The selection for these streams is based on the advice of the teacher of grade 6 and the results of a nation-wide test, the CITO-test (e.g. Cito, 2010).

Figure 1.1: Dutch school system.

1.3 Research strategy

In our analysis of proficiency we look at two sides of education. On the one hand we investigate the proficiency of students and on the other hand we analyse textbooks and their link to the ideal curriculum.

1.3.1 Student proficiency

In our study we aim to assess the level of proficiency of the students in grade 4 through 9. This requires an assessment instrument that allows for longitudinal study and for detailed
empirical information on students’ understanding of fractions. The data provided by such a test is to offer footholds for improving instruction and even inform theory building. This test needs to represent the whole domain, match the students’ level, facilitate comparison between grades, and provide efficient assessment of a large number of students. Since there was not a pre-existing test that met these requirements, we designed a new test. That is, although large scale studies provide an indication of students’ level in basic mathematical skills, they have not been designed to provide enough detail for improving instruction in a specific domain, such as for instance fractions. Domain-specific research, on the other hand, usually focuses on one aspect of a specific topic only and does not provide an overview of the whole domain.

The test was administered at schools for primary and secondary education and the assessment followed both cross-sectional and longitudinal aspects (Section 3.1.3). The Rasch model (Rasch, 1980) was chosen as a method to analyze our data. This model is a one parameter Item Response Model (IRT). Rasch analysis and other IRT models have proven to provide a good scale for student achievement in mathematics in large scale assessment projects such as TIMMS, PISA (e.g. Boone and Rogan, 2005) and PPON (Janssen et al., 2005). With this model a linear scale can be created on which both students and items can be arranged according to their ability and difficulty (e.g. Wright and Stone, 1999). The linearity of the scale implies that the items and students are not only ordered on the scale but that the relative distances between them also have meaning. The Rasch scale will be comprehensively described in Chapter 3.

1.3.2 Textbook analysis

The second line of analysis is related to the formal curriculum. For practical reasons we decided not to analyze activities in the classroom or the role of the teacher in this process. However, it is known that Dutch teachers, both in primary and secondary education, stay close to the content and pace of the textbooks. Even more so, in primary education teachers follow closely the guidelines of the teacher guides. Additionally, Dutch students work intensively with the textbooks. In secondary education, they even own the textbook, and bring it back home, to prepare homework (e.g. Hiebert et al., 2003). Thus, there are weighty reasons to believe that an analysis of the textbooks gives a representative picture of the main lines in classroom practice. Additionally we take the theoretical basis of the textbooks into account. That is, we study the relation between the prototypical work that was performed to elaborate the general ideas of RME in the domain of fractions, and the textbooks that were based on this work.

1.4 Outline

The research on the development of proficiency of Dutch students in the domain of fractions in grades 4 to 9 followed two main paths. Although these two lines of research have been interwoven in practice, for clarity we describe these lines in two separate parts of this dissertation. In the first part we describe the test design and the analysis of the students results. This part starts with a description of the theoretical framework (Chapter 2) and
the construction and validation (Chapter 3) of our test on proficiency. We continue with
the analysis of the progress in proficiency of students in lower secondary education (Chapter 4). The second part of this dissertation describes the analysis of textbooks. We describe
the construction of our database that is the basis for our textbook analysis (Chapter 5). We
start with a focus on the transition between primary and secondary education. Later we
focus on the multiplication of fractions in primary education (Chapter 6). These chapters
are based on articles that are submitted for publication. We conclude this dissertation with
a summary of the main findings of these two lines of analysis and a reflection on these
results in the broader perspective of basic mathematical skills as well as recommendations
for Dutch education (Chapter 7).
Part I

Student proficiency
“Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts.”

(David Hilbert)
Framework for the proficiency test

In the Netherlands, educators and policy makers are concerned about the basic mathematical skill level of students at the beginning of both secondary and tertiary education. These concerns, which parallel those in many other countries, often result in discussions about the mathematics curriculum. Input for these kind of curriculum-related discussions are among others the results of large scale international studies on basic mathematical skills such as PISA and TIMMS, particularly in countries with a relatively low rating in the summary reports (league-tables effect) (e.g. Anderson et al., 2010; Neumann et al., 2010). The Dutch discussion concentrates on the questions, in how far there is reduced mastery, what causes this problem, and what the direction of curriculum reform should be. Although large scale studies provide information about basic mathematical skills, they have not been designed to provide the detail needed for informing instruction in a specific domain, such as fractions. Instead, they target a broad segment of the mathematics curriculum and aim at correlating student achievement to factors that influence learning such as student background, perceptions and attitudes (e.g. Anderson et al., 2007). Also, international differences and gender effects (e.g. Goodchild and Grevholm, 2009) have been studied by analyzing data of these large scale assessments. Studies like PPON (Janssen et al., 2005) do provide more insight in the nature of students’ proficiency halfway and at the end of primary education, but do not cover secondary education. Domain-specific research, in contrast, usually focusses on one aspect of a specific topic only. More information on (hampering in) the development of basic mathematical skills is needed to find a way to solve such problems. This requires an assessment instrument that allows for longitudinal study and for a detailed analysis of students’ proficiency. In this chapter we report on the construction of an assessment instrument that meets these requirements for the domain of fractions.

Fractions are prominent in the curriculum of the final years in primary education and are seen as a prerequisite for further mathematical progress. The learning of fractions involves a great variety of interrelated concepts and a learning process stretching over several grades. Thus, the aim of this study is to construct a proficiency test on fractions that provides detailed information on students’ understanding of fractions in a longitudinal setting (grade 6 to 9).

The outline of this chapter is as follows. We start with a description of the rationale for the construction of a fraction proficiency test. Next, we identify five big ideas that describe the domain of fractions at the level of underlying concepts. These big ideas are to be used during the test construction to select items such that the test covers the domain evenly. In
the analysis, the big ideas will be used as guidelines for an analysis at the concept level. In Section 2.2 we describe these big ideas and how they connect fraction concepts and existing research. The exact appearance of the items is described in a list of so-called “complexity factors” (Section 2.3). These complexity factors are the external characteristics of tasks that theoretically influence its difficulty. We distinguish general complexity factors that apply for a large range of tasks, and specific complexity factors for each of the subdomains, concerning initial fraction concepts, basic mathematical operations on rational numbers and the application of fraction knowledge. In Chapter 3 we will illustrate that the framework and approach that we proposed in this chapter fulfills our aims.

2.1 Rationale

The aim of this study is to design a test that can enhance our understanding of how the fraction proficiency of students develops in grade 4 through 9. The objective is to gain detailed empirical data that can inform curriculum innovation. Since we expect substantial variations in the students’ proficiency, the design has to allow the efficient assessment of a large number of students. Therefore, we opt for a paper and pencil tests to assess students’ proficiency.

2.1.1 Proficiency

We unraveled proficiency in the ability to solve certain tasks and the understanding of the structure and the concepts underlying such tasks. On the one hand, students are to know the rules of arithmetic and its definitions and be able to use both these rules and definitions. On the other hand, students are to understand the underlying structure of the domain of rational numbers, its concepts and the interrelations between these concepts and adjacent mathematical domains. Consequently, the analysis of test results was also to comprise of these two levels. Therefore, the first level of analysis consists of identifying the tasks students either can or cannot perform. At the second level, the analysis will be oriented to probing students’ understanding of underlying fraction concepts. This level comprises of the interplay between tasks, that is, how students’ understanding can be analyzed by considering students’ responses to tasks that vary in certain detail.

In order to identify the aspects on which the test items should vary we developed a framework of big ideas and complexity factors. The big ideas are used as guidelines in selecting items to cover the underlying concepts of the domain evenly. The complexity factors represent the external characteristics of the items that, according to domain-specific pedagogical theory determine their difficulty or complexity. These are aspects on which tasks should vary. Such a systematic test construction also reduces the chances on one-sided testing by either overemphasizing or ignoring particular domain-specific pedagogical aspects.
2.1.2 Delineation

The domain of fractions encompasses a lot of topics and there is a considerable body of research on the learning of fractions and rational numbers. Such research typically focusses on specific questions and thus on small parts of the domain (Confrey et al., 2009; Pitkethly and Hunting, 1996). In this study we aim to synthesize this rich body of research. In our view this entails both the interrelations between subdomains of the fraction domain and the link between arithmetic and algebra. Still, we had to delineate the domain to some extent for practical reasons.

Since our research addresses the learning progress of students who prepare for higher education, we put emphasis on the transition to more formal reasoning with fractions as a preparation for algebra. Therefore, initial concepts from related domains such as proportion and percentages are not explicitly covered in our test. Instead we regarded these domains as application of fraction knowledge. Our focus on students in the age of 10 to 15 (grades 4 up to 9) implied that intuitive mechanisms (Pitkethly and Hunting, 1996) were not addressed in our test, unless they entail restrictions in fraction understanding in a later stage. Thus, we did not address pre-fraction concepts as described by (Steffe, 2002), Cobb and Olive and Steffe (2002).

2.2 Big ideas

Before we could describe the complexity factors to construct items for our test, we needed a description of the domain on the level of underlying concepts. For this purpose we looked for big ideas that come to the fore in the literature on fractions and that link the concepts underlying the domain.

Understanding fractions entails different aspects, such as ratio and rational number. It is important that education addresses these aspects, but for deeper understanding students have to understand how all these aspects relate, rather than just understand each of them separately (e.g. Kieren, 1976). To find the complexity factors that address this deeper understanding, we searched for big ideas that address the relations between these different aspects. We realized that the ambiguous nature of fractions would make it impossible to find big ideas that do not overlap. This ambiguity implies that each of the interpretations that can be given to the fraction symbol is inseparable from every other possible interpretation. This makes it so difficult to describe these interpretations separately. So, we did not aim at finding distinct constructs or latent variables. Rather, we decided that it would still be worth trying to find big ideas to cover and relate the most important aspects of fraction learning. These big ideas were to serve two goals. They were to guide our decisions on the choices which tasks to include in our test and to offer us guidelines for the analysis of the test results. That is, the selection of tasks in our test had to include sets of tasks that only differed in some detail, such that the analysis of the differences in students responses can reveal some of the students understanding with respect to the big ideas. In the analysis phase, these particular sets of tasks are to be analyzed regarding these big ideas. After an analysis of available research, the curriculum and the mathematical definition of the ra-
tional numbers, we formulated the following big ideas: relative comparison, equivalence, reification, from \( \mathbb{N} \) to \( \mathbb{Q} \), and relation division-multiplication (Table 2.1).

With this choice of big ideas we deviated from existing research that takes the so-called “subconstructs” as starting point. Subconstructs are the aspects of fractions that are generally referred to as part-whole, measure, quotient, ratio number and multiplicative operator. They have been distinguished in the work of for example Behr et al. (1984), Kieren (1980), Streefland (1983), and have been widely accepted as fundamental for fraction learning. That is, in the learning process, the understanding of fractions is to be enriched and deepened by familiarizing students with these different aspects of fractions. In the work of Streefland (1991), the five subconstructs come to the fore as the result of a phenomenological exploration and were employed as so-called “inroads” to fraction learning. That is, they would have to serve as contexts of exploring fraction concepts. In a later stage of learning, one of these contexts would serve as basis for generalization towards formal rules of arithmetic. Thus, the subconstructs are closely related to the context and external features of a task. On the other hand, they are used to describe the “meaning” that is addressed to the fraction symbol in a specific contexts. However, at a certain point of mastery, the meaning is not uniquely one of the subconstructs. Rather these subconstructs are inextricably bounded up.

To sum, although each of the subconstructs pertains to a particular and significant aspect of understanding rational numbers, they can not stand alone (Pitkethly and Hunting, 1996). Kieren (e.g. 1976) already expressed that a complete understanding of rational numbers requires an understanding of how the subconstructs interrelate, rather than only understanding each of them separately. Consequently, the subconstructs can be considered as distinguishable, but inseparable aspects of this mathematical construct, that constitutes its ambiguity. Key in understanding fractions is to realize how these aspects interrelate and that these are by principle inseparable.

Thus, given our focus on the relations between subconstructs we decided not to take the subconstructs as basis of the analysis of deeper understanding of fractions. Instead, they have been present in the background of the formulation of our big ideas. In the following we address how the big ideas of relative comparison, equivalence, reification, from natural to rational numbers and relation division-multiplication, interrelate and connect existing literature.

2.2.1 Relative comparison

From a mathematical perspective, the set of rational numbers can be considered as an extension of the set of integers to provide for a solution of \( b \times a = a \) for every \( a \) and \( b \in \mathbb{Z} \) and \( b \) nonzero; that is \( x = a \div b = \frac{a}{b} \). We may further note that in general the fraction notation is used to denote a division, even if it does not refer to a rational number, e.g. \( \frac{1}{\sqrt{2}} \).

With regard to division, in pedagogical research generally two types of division are distinguished, i.e. measurement division and partitive division. Both types play their own role in the fraction curriculum. In the introduction of fractions, partitive division or sharing is most frequently used (e.g. Empson et al., 2005). Partitioning has been identified as a constructive mechanisms for fractions operating across the subconstructs (Behr et al., 1983). Similar to the context of natural numbers, it is crucial that the whole is divided into equal
parts; this notion is usually referred to as fair sharing or equi-partitioning (e.g. Charles and Nason, 2000). In contrast with the setting of natural numbers, for fractional numbers also the natural unit of one object is subdivided. Fractions come into play when the division does not work out "neatly". The fraction symbol is used to name "fractional parts" of an object. In general, the first introduction of fractions is in the context of sharing of just one –usually a pizza-like– object. Later more objects have to be shared among a number of persons. Technically, in this case the dimension of the outcome is "pizza per person", although the "per person" is usually left out. In short, in the introduction of fractions, the fraction bar (vinculum) is associated with partitive division. Measurement division, on the other hand, is more in line with iterating a given unit and is usually connected to the division sign (\( \div \)) in fraction tasks (e.g. how many pieces of \( \frac{3}{4} \) meter can you cut from rope of \( 6 \frac{3}{4} \) meter). In such a context, division can be interpreted as repeated subtraction. The outcome is dimensionless.

However, both the "\( \div \)" sign and fraction bar can not be associated with either one of these types of division in all sorts of contexts. An example is speed, which is clearly related to fractions and division given its dimension \( \frac{\text{km}}{\text{h}} \). Speed can neither be regarded as partitive division nor as a measurement division. In relation to this, Freudenthal (1973) argues "It cannot be held that two kinds of division are less absurd than a few hundred. It is [...] a characteristic of mathematics that isomorphic procedures are reduced to one abstract scheme." Therefore, we propose a more general and unifying view on division, as an alternative. That is, we interpret division as relative comparison (of quantities), just as subtraction can be regarded as naturally following from absolute comparison of quantities. In this view, partitive division can be regarded as the relative comparison of a total number of units with the total number of shares, and measurement division can be regarded as the relative comparison of a total amount with the amount of each share. However, while partitioning and division are closely related for natural numbers, this connection is not always made evident towards the students when it concerns fractions. In regard to this big idea of relative comparison, we argue that two important insights are needed, to which we refer as ratio-rate and unit.

**Ratio-rate**

From a mathematical standpoint, the fraction \( \frac{a}{b} \) is the solution of \( bx = a \). Regardless the context this means that we can consider \( x \) as the factor that relates \( b \) to \( a \). This boils down to questions like "how many times does \( b \) fit into \( a \)" or "with which factor do I have to multiply \( b \) to get \( a \)". What makes the concept of fraction so difficult is that \( \frac{a}{b} \) is both the division \( a \div b \) and the factor between these two numbers, i.e. it expresses a relationship between two quantities and it defines a new quantity. This is a duality that is often referred to as ratio and rate (e.g. Behr et al., 1983) and which is visible in the mathematical construction: \( Q = (\mathbb{Z} \times (\mathbb{Z} \setminus 0))/\sim \). It is an example of what Sfard (1991) referred in terms of proces-object duality. In this case the fraction is both a process (dividing) and an object (the outcome of this division, a number or a factor).
Unit

It is not only important to recognize that $\frac{a}{b}$ is the factor between two numbers $a$ and $b$. It is also important to recognize the quantities $a$ and $b$, and how they are related to each other in the given context. The relative comparison of these quantities relates closely to the choice of unit. The nature of the unit changes when a student enters the domain of rational numbers. Lamon (1996) articulated unitizing as “the cognitive assignment of a unit of measurement to a given quantity; it refers to the size chunk one constructs in terms of which to think about a given commodity.” She found a close relationship between the students’ overall problem solving level and the choice of units in fraction and ratio problems (Lamon, 1993). In the domain of fractions, the choice of which amount is to be regarded as “one” (the unit) is not uniquely determined.

In the part-whole model for example the unit is usually one object (e.g. a cake or pizza) and thus easy recognizable (e.g. Clarke and Roche, 2009). However also one share ($\frac{1}{b}$) can be regarded as a unit, especially when $\frac{a}{b}$ is interpreted as $a$ pieces of $\frac{1}{b}$th. In the context of part-group models the natural units or objects stay intact and are not subdivided. However, in this case not one object is the unit to which standardization takes place, but rather the whole group of objects. Thus, the unit changes from context to context. In the context of rates, the third quantity (the rate) which reflects the relationship between two other quantities is itself also a new entity. The work of Lamon (1993) suggests that it is useful to also consider a ratio as a unit.

2.2.2 Reification

For a full understanding, it is important that students develop an operational and a structural conception of fractions (Sfard, 1991). That is, for the student the fraction has to develop from a process into an object that can be reasoned with. The fraction must become a rational number that, among other things, can be operated on.

Charles and Nason (2000) argue that if students do not construct conceptual mappings between the entities in the context on the one hand to the fraction name of each share (yths) and the number of yths in each share on the other hand, the outcomes of their partitioning activity may be the mere completion of the partitioning task rather than the building of the “partitive quotient fraction construct” (Behr et al., 1992).

A conception of a fraction as a number also requires fractions larger than 1. However, the step from the part-whole interpretation of proper fractions to fractions larger than 1, either in mixed number notation (e.g. $1\frac{1}{4}$) or as an improper fraction (e.g. $\frac{5}{4}$), is reported to be a non-trivial step in the development of fraction understanding (e.g. Hackenberg, 2007; Stafylidou and Vosniadou, 2004). It involves the transition from a part-whole interpretation of fractions to seeing them as measurements. For this transition the fraction has to develop from an amount to an abstract number, from $\frac{1}{4}$ of a given pizza via $\frac{4}{4}$ as a measure to eventually $\frac{3}{4}$ as a number in and of itself.

In addition, the conversion from mixed number representation to improper fraction notation or vice-versa appears to be difficult for students. Tasks that involve such a conversion in, for example, simplifying the answer, can lead to additional errors (Brueckner, 1928).
In addition, a quantitative notion of fractions is considered to be crucial for the development of other rational number concepts (Behr et al., 1984), such as the ordering of fractions and the use of the number line as a representation of fractions.

### 2.2.3 Equivalence

In the construction of \( \mathbb{Q} \), the equivalence relation is a basic element. The idea of equivalence classes may be experienced as difficult by students because it is inconsistent with the notion of a unique relation between number and symbolic representation that they developed on the basis of their experiences with natural numbers (Prediger, 2008). The concept of equivalence entails more than to reduce or complicate fractions by multiplying or dividing the numerator and denominator with the same number. The concept also includes the notion of an infinite number of representatives for the same rational number, and the recognition that the representative with smallest denominator has a special role.

### 2.2.4 From natural number to rational number system

In the construction of the set of rational numbers, explicit attention is given to how the natural numbers –or more generally integers– fit in \( \mathbb{Q} \). Consequently, for students it is important to understand that an integer can be thought of as \( a = \frac{a}{1} \). However, the extension from \( \mathbb{Z} \) (or \( \mathbb{N} \)) to \( \mathbb{Q} \) does not only encompass the numbers themselves. This extension also concerns operations, especially addition/subtraction and multiplication/division. That is, students will have to come to see these operations for natural numbers as essentially the same as for rational numbers. From a mathematical perspective, the transition from \( \mathbb{N} \) to \( \mathbb{Q} \) is an extension, which means that \( \mathbb{N} \) fits into \( \mathbb{Q} \). For the students it may create cognitive conflicts, as educational research has shown. Some ideas that children develop in the context of natural numbers, are in conflict with the new “reality” of rational numbers. These ideas create a contrast between natural numbers and fractional numbers (Prediger, 2008).

Notions such as “multiplication makes larger” and “division makes smaller” are often mentioned as causes of cognitive conflicts (e.g. Stafylidou and Vosniadou, 2004). One may argue, however, that some of these conflicts are a consequence of the asymmetry of multiplication and division that was created by education—an asymmetry that is due to a strict distinction between multiplier and multiplicand, which has not been resolved by the time students are introduced to fractions. A similar discontinuity with whole numbers exists in the ordering of fractions. Ordering of rational numbers is not directly supported by familiarity with the natural numbers’ sequence. Streefland (1991) speaks of “\( \mathbb{N} \)-distractor.” Another fundamental discontinuity concerns the concept of density (Prediger, 2008). For fractional numbers there is no unique successor like there is for natural numbers, and moreover there is an infinite number of numbers between each arbitrary pair of numbers.
2.2.5 Relation division-multiplication

The last big idea concerns the relation between division and multiplication. A fraction is in its essence a division. In the discussion of the big idea “relative comparison,” we related division both to the ratio between two numbers and the more abstract factor or rate that relates those numbers. However the division operation itself can also become an object. That is, division becomes part of a network of relations between operations (Van Hiele, 1986). This involves an understanding that division and multiplication are inverse operations and, grasping the notion of an inverse number \( \frac{a}{b} \times \frac{b}{a} = 1 \). These notions are conditional for understanding that “division is multiplication with the inverse” and to flexibly change the order of multiplication and division, e.g. \( 31 \times \frac{17}{4} = 17 \) because we first divide by 31, and then multiply the remaining 1 with 17. This flexibility also involves interpreting \( \frac{3}{4} \) as \( 3 \times \frac{1}{4} \), \( 3 \div 4 \), \( \frac{1}{4} \times 3 \), ... \( \div 4 \times 3 \), \( 1 \div \frac{1}{4} \), etc.

In addition, understanding the fraction bar symbol as denoting division relates to understanding that “the larger the denominator (thus the number of shares) the smaller the fraction”. This reciprocal effect of the denominator is counter-intuitive to concepts developed for whole numbers. In its turn this relates to notions such as that \( \frac{2}{3} \) is in the middle of \( \frac{3}{7} \) and \( \frac{5}{7} \) (average of numerators), but that \( \frac{1}{3} \) is not the average of \( \frac{1}{2} \) and \( \frac{1}{4} \).

2.3 Complexity factors

In the previous section we described the big ideas that link concepts that are known to underly the understanding of fractions. These big ideas were used to guide the construction of a test that allows for an analysis of students understanding at the concept level. That is, the combination of tasks is to provide more insight in students’ understanding of the underlying fraction concepts. Such an analysis can only be performed if the test is constructed systematically. In this section we discuss a framework of complexity factors—characteristics of tasks that can be varied and theoretically determine the difficulty of a task—that will serve as the basis of systematic test construction. In addition this framework of complexity factors is also to support the analysis at task level. That is, the separate tasks should represent required skills. In this section we distinguish between general complexity factors that apply to the whole domain and specific complexity factors for groups of tasks. Concerning initial fraction concepts we consider the subdomains of part-whole (in the remainder of this dissertation these task are denoted with P), order (O), reduce and complicate (E), mixed numbers and improper fractions (I), and the number line (N). Regarding the basic operations we distinguish on the subdomains of addition and subtraction (A), multiplication (M) and division (D). Finally we consider the application of fraction knowledge (T).

2.3.1 General complexity factors

General complexity factors relate to mathematics learning in general and fraction learning more specifically. We distinguished the familiarity of numbers, support of contexts and models, representation, reduction of the answer, form of fraction and direction of the task.
Familiarity of numbers

Learning about fractions generally starts with the partitioning of objects. The earliest contextual experiences consist usually of sharing objects or quantities among two, three or four people. In the context of circular models, only particular—usually small—numbers are appropriate for partitioning. More or less the same holds for rectangular models, for which again the denominator is usually small. Based on these partitioning experiences, students may form a network of number relations (Treffers et al., 1994). It is expected that reasoning within this network (and thus with familiar denominators) is less difficult than reasoning with larger denominators (Van Galen et al., 2005). Indeed it has been shown that smaller denominators are easier initially (Mack, 1995). The familiarity of the denominator can thus be regarded as one of the factors that influence the difficulty of tasks.

Support of contexts and models

Arguably, contexts and models can offer support for students’ reasoning. The use of contexts and models is common practice in the Dutch curriculum (Van den Bergh et al., 2006). The role of contexts and models has been theoretically elaborated in the theory of emergent modeling (Gravemeijer, 1999). We expect that only if students are familiar with a certain model or context this will help them in solving a task.

Representation

The contextual introduction of part-whole situations with concrete partible objects like pizza’s and cakes shifts gradually in a circular or rectangular part-whole representation of fractions when the students progress in the curriculum. The formal notation using a vinculum is usually introduced as counterpart of such models. Later, other representations, such as the number line and ratio (e.g. ‘1 : 100’ or ‘1 to 100’) are added to the list of representations. The students must not only be able to work with each of these representations, they also have to be able to flexibly switch between representations (Behr et al., 1984). The most commonly used models for representing fractions are geometric regions, sets of discrete objects and the number line (Behr et al., 1983). These models are not only to support students’ reasoning about fractions. They are also representations of fractions and thus goals of learning in and of themselves.

Reduction of the answer

Brueckner (1928) found that reduction of an answer can lead to many errors. In Dutch classrooms students are required to reduce their answer to lowest terms. In primary education, a mixed number notation is required. In secondary education the answer may also be given as an improper fraction (with smallest denominator).

Form of fraction

The introduction of fractions usually starts with unit fractions (e.g. $\frac{1}{2}$) and is soon expanded to proper fractions (e.g. $\frac{3}{5}$). We already discussed that the transition from proper fraction to
improper fraction or mixed number is difficult. This argument for distinguishing between these forms of fractions is strengthened by a more procedural argument. That is, informal strategies for the standard operations may not apply for all forms, and formal strategies may involve extra steps such as for fractions larger than 1. As a result, we distinguish between unit fractions, proper fractions, improper fractions, mixed numbers and whole numbers.

**Direction of the task**

There is a difference in difficulty between understanding fraction language or representations or using them oneself, i.e. a difference in passive and active use of fraction language. Accordingly, there is a difference, for example, between naming a location on the number line or the shading of a part-whole model and to represent a given fraction.

### 2.3.2 Initial fraction concepts

We distinguish several types of tasks that involve initial fraction concepts. These are part-whole, order, mixed & improper, number line and reduce & complicate.

**Part-whole**

Tasks in this subdomain involve the part-whole model. This model is related to fair sharing that can involve discrete or continuous quantities (e.g. Clarke and Roche, 2009). Continuous models like pizza’s or cakes are usually referred to as part-whole, whereas discrete models are referred to as part-group. As discussed previously, the unit is usually easy to recognize in continuous models in contrast with discrete models for which this is usually more difficult. Indeed, Behr et al. (1983) argue that the cognitive structures involved in rational problem solving referring to a discrete model differ from those referring to a continuous model. Differences between part-group and part-whole models were also found by Hunting and Korbosky (e.g. 1990); Novillis (e.g. 1976). The type of objects —whether they are easily partitioned— also influences the ease of sharing. (Charles et al., 1999) found that not only the shape of the objects (especially circular or rectangular) mattered, but also the contextual ease of partitioning (its ecological validity (Streefland, 1991)). The fact that the shape of the part-whole model (e.g. circular or rectangular) influences the performance was also found by others (e.g. Hunting and Korbosky, 1990). In addition, the partitioning of the unit is also a complexity factor. That is, whether there is no pre-partitioning, a ‘fair’ partitioning or if parts have unequal size.

**Order**

Tasks that ask students to order fractions according to their size refer to the quantitative notion of fractions. Ordering tasks may vary on various aspects. For the ordering of fractions the type of problem is the first factor that we distinguish. We expect a difference between plain ordering and tasks that can also involve equivalent fractions. Additionally we expect a difference between ordering multiple fractions and tasks that require to indicate the largest
of two fractions. Furthermore, the use of formal signs like $<, >$ and $=$ can complicate a task.

The complexity of such tasks is among others determined by the strategies that can be used for solving the task. The applicability of these strategies is number specific. A rather formal strategy that applies for all fractions is to “cross multiply”, e.g. $\frac{2}{7} < \frac{4}{13}$ because $2 \times 13 < 7 \times 4$. A more intuitive strategy is the use of a referent, often ‘$\frac{1}{2}$’ or ‘1’ (e.g. Clarke and Roche, 2009; Van Galen et al., 2005). To compare $\frac{7}{8}$ and $\frac{3}{4}$ for example, the difference of these fractions with ‘1’ –which yields $\frac{1}{8}$ and $\frac{1}{4}$ respectively– can be used to reason on the size of these fractions. A strategy that is promoted in the work of Treffers et al. (1994) is to find an appropriate “sub-unit” (“passende ondermaat”). This entails for example using a chocolate bar with 12 pieces to compare $\frac{3}{4}$ and $\frac{7}{8}$. Thus $\frac{3}{4}$ is larger than $\frac{7}{8}$ because 9 pieces of a bar of 12 are more than 8 pieces of that bar. A common error in comparison tasks is the so-called whole number dominance, i.e. the fraction with the largest numbers is assumed to be the largest fraction (e.g. Behr et al., 1984; Streefland, 1991). In this case item characteristics that allow certain strategies or are prone to certain errors, can be considered as complexity factors. We named these the relation between fractions. ‘Referent $\frac{1}{2}$’ strategies for example can only be used if one of the fractions to be compared is smaller and the other larger than $\frac{1}{2}$.

To test for whole number dominance fractions must be chosen such that the largest fraction has the smallest denominator and numerator (e.g. $\frac{4}{6}$ and $\frac{7}{13}$). Thus, the type of strategies that can be applied are determined by the relation between fractions, Behr et al. (e.g. 1984) for instance distinguish between tasks with fractions with (1) the same numerators, (2) the same denominators and (3) different numerators and denominators.

For equivalent fractions the relation between denominators and numerators is of importance, in other words the factor that relates the numerators. We expected that a factor of 2 or 3 (e.g. $\frac{3}{4}$ and $\frac{7}{8}$) is much less complex than a non integer factor (e.g. $\frac{3}{5}$ and $\frac{4}{7}$). In Dutch such fractions are named “gelijksoortige breuken”.

**Mixed numbers & improper fractions**

For this subdomain we expect differences between the direction of the task. That is, from improper to mixed is an activity that is performed more often than vice versa. Especially since in Dutch primary education, splitting is promoted as strategy for multiplying mixed numbers. That is, students are to multiply $4 \frac{1}{2} \times \frac{1}{2}$ via $4 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}$.

Further, the size of fractions and numbers appears to be complexity factor. We expect that students are more familiar with rational numbers between 1 and 2, than with larger mixed numbers. We also expect that larger numerators are more difficult.

**Number line**

Research has shown that the length of the number line can influence the performance of students (Larson, 1980; Behr et al., 1983). A number line –one unit long– can be interpreted as a part-whole model more easily, the unit being the whole. Recognition of the unit is found to be more difficult if the number line consists of more units. Students’ assignment of the total length to the “whole” of the part-whole model is a common mistake. The
studies of Behr et al. (1983) and Larson (1980) also showed that the subdivision of the unit influenced difficulty of tasks. If the subdivision of the unit is equal to the denominator of the fraction, this is much easier than if they do not match. There are some ways in which the denominator and subdivision can differ. We distinguish between a subdivision that is finer than the denominator (e.g. place $\frac{1}{4}$ on a number line with a subdivision in 6ths) and the reverse where the denominator is a multiple of the subdivision. A last category is denominator and subdivision that do not have a common factor. Furthermore, we expect that subdivisions of halves, quarters, fifths, and tenths are more easy than other subdivisions.

Reduce & complicate

In this subdomain, various types of problems are known, such as missing value problems, reduction to smallest terms and asking for an equivalent fraction. These type of problems differ in the concepts that must be used. As discussed with ordering previously, we expect that the factor between the numerators influence complexity for the missing value problems. Furthermore we expect a difference between assignments to either reduce or complicate.

2.3.3 Basic operations

We consider the basic operations addition, subtraction, multiplication and division.

Addition and subtraction

Key in the addition and subtraction of fractions is that the numerators can be added or subtracted if denominators are equal. Not surprisingly, addition is found to be easier if the two operands have common denominators. Brueckner (1928) found that 8.3% of the errors in addition are caused by difficulty in changing fractions to a common denominator. We further expect that not all types of unlike denominators are equally difficult. That is, denominators that differ a factor of 2 were expected to be the most simple (e.g. $\frac{1}{2} + \frac{4}{5}$). Of numbers that are relative prime the least common multiple (lcm) is the product of these numbers. So, only if denominators have a common factor, there is an advantage in reducing denominators to the lcm instead of to the product, e.g. $\frac{5}{14} + \frac{8}{21} = \frac{15}{42} + \frac{16}{42}$ instead of $\frac{105}{294} + \frac{112}{294}$. The smaller numbers in the ‘lcm strategy’ may reduce the chance on calculation errors. Thus, the relation between denominators is considered to be a complexity factor. For subtraction difficulty in borrowing is a common error. Brueckner (1928) found this error in 24.3% of the errors on subtraction tasks. Similarly, we expect a difference in complexity of tasks for addition between a sum less than 1 or larger, making carry a complexity factor.

Multiplication

For multiplication tasks the order of the operands is of importance. Our analysis of Dutch textbooks showed that multiplication of fractions is connected with four types of (in)formal strategies based on number specific characteristics (Chapter 6). If students attribute mean-
ing to numbers on the basis of their position (e.g. multiplier and multiplicand\(^1\)), the character of the numbers at each position determines the interpretation of the multiplication sign. The strategy “repeated addition” is used if the multiplier is a whole number. Similarly, multiplication as “part-of” operation is only meaningful for proper fraction multipliers. Additionally we expect that various types of problems such as “part-of”, “×” or “handig rekenen” (“flexible arithmetic”), to have different difficulties. Multiplication can be simplified by intermediate cancelation of common factors, e.g. \( \frac{2}{15} \times \frac{5}{8} = \frac{1}{3} \times \frac{1}{4} \). Although this is often referred to as a trick, it can also be an indication of a flexible interpretation of the fraction as a combination of multiplication and division.

### Division

Depending on the strategy used for division, the difficulty of a task is influenced by the form of the outcome. A division strategy connected with ‘repeated subtraction’ or ‘pacing out’ is more difficult if the outcome is not a whole number (e.g. Rittle-Johnson and Koedinger, 2001). Again, “cancelation” can simplify calculations.

### 2.3.4 Application of fraction knowledge

Whereas, applications of fractions are, in principle, infinitely diverse, we cannot address it in its entirety here. Instead, we describe some common type of applications and common structures that play a role in primary and secondary education.

#### Proportion

There are several ways to represent a proportion. Examples of proportions in daily live are dilution ratios (e.g., 1 part syrup on 8 parts water or 1 to 8), and scale (e.g., 1:100.000). These two examples show that proportions can concern much more than part-whole relations. A task can be more complex once the numbers in the ratio have to be derived from other numbers in the context of the task. An example is to go from a dilution ratio (solute to solvent) to a dilution factor (solute to final volume). Thus, again we expect difference between type of problems.

#### Percentages

Regarding the relation between percentages and fractions we distinguish two types of problems. Among others, percentages can be used to express the ratio between two quantities, e.g. 54 % of the group were boys. Percentages can also be used in multiplication contexts, e.g. you get a 25% discount. Percentages are hardly used for addition. Furthermore, a task can become more complex if the standard of 100 % is not directly expressed in the tasks, e.g. to calculate the VAT if the price including VAT is given.

\(^1\)Addressing the multiplier and multiplicand appears not to be universal. In the Netherlands \(5 \times 3\) is interpreted as \(3 + 3 + 3 + 3 + 3\), 5 being the multiplier. In contrast \(5 \times 3\) is interpreted as \(5 + 5 + 5\) in some other countries. In this case 3 is the multiplier.
Algebraic expressions

The role of fractions in algebraic expressions can vary. Again, the fraction can play more than one role in the same task. Roles that can be distinguished are that of for example rational number, factor, division, algebraic fraction. Examples of tasks with different roles for the fraction are: 

$$\frac{2}{3}x - 1 = x - \frac{2}{3}$$, 

$$\frac{a}{b} + \frac{c}{d}$$ and 

$$\frac{3}{x-1} + 4 = 0$$.

Domains other than mathematics

In the domains of physics and chemistry the structure of \( a = \frac{b}{c} \) is very common. A well known example is “density = \( \frac{\text{mass}}{\text{volume}} \)”. If both the density and the volume of a substance are known, then its mass can be calculated. We expect that not all such problems are of equal difficulty. In case of density, calculating the density is easiest, then mass and finally volume, since the latter involves division. Proportion and percentages represent other common usages of fractions outside the mathematical domain.

2.4 Conclusion

The aim of this study was to develop a test that can be used to evaluate students’ proficiency in the domain of fractions and that can provide empirical detail required for improvement of domain-specific instruction. For this purpose we developed a framework to systematically vary tasks and cover the domain of fractions. Pivotal in our framework are so-called complexity factors, which are characteristics determining the difficulty of tasks. We formulated big ideas that were to guide the selection of tasks and the analysis of the students’ deeper understanding based on their results. In so doing, we established tight links to existing research on fractions. These big ideas and complexity factors are summarized in Tables 2.1 and 2.2.

Thus, the systematic construction of a framework of complexity factors is to allow the analysis of the test results at two different levels. The first level of analysis is an item per item analysis of the types of tasks students have (not) mastered. A more complex, second level of analysis involves the combination of tasks, aiming at insight of students’ understanding of certain concepts underlying fraction proficiency. This will be discussed in the next chapter.

In that chapter, the question whether this approach results in an assessment instrument that can fulfill its requirements. Thus if it can be used to evaluate the development in students’ proficiency. We will evaluate our test according to three criteria: 1. Can we construct a single linear scale for fraction proficiency on which all our items can be ordered according to their difficulty? 2. Can the development of proficiency from grade 4 to 9 be described with the results of this test? and 3. Does this provide diagnostic data which can be used for improving instruction?
### Table 2.1: Overview of big ideas.

<table>
<thead>
<tr>
<th>subdomain</th>
<th>Complexity factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>general</td>
<td>familiarity of numbers support context or model reduction of the answer form of fraction representation direction of the tasks</td>
</tr>
<tr>
<td>part-whole</td>
<td>discrete / continuous partitioning of the unit type of objects</td>
</tr>
<tr>
<td>order</td>
<td>type of problem whole number dominance relation between fractions equivalence-factor size of fractions/numbers subdivision of the unit reduce or complicate</td>
</tr>
<tr>
<td>mixed &amp; improper</td>
<td>direction length number line direction type of problem factor</td>
</tr>
<tr>
<td>number line</td>
<td>reduce &amp; complicate type of problem factor</td>
</tr>
<tr>
<td>add &amp; sub</td>
<td>relation denominators carry type of problem form of the outcome cancellation</td>
</tr>
<tr>
<td>multiplication</td>
<td>order of operands cancelation form of the outcome cancelation</td>
</tr>
<tr>
<td>division</td>
<td>form of the outcome representation type of problem role derived</td>
</tr>
<tr>
<td>proportion</td>
<td>representation type of problem</td>
</tr>
<tr>
<td>percentages</td>
<td>type of problem role 100 % directly expressed</td>
</tr>
<tr>
<td>algebra</td>
<td>unknown in $a = \frac{b}{c}$</td>
</tr>
<tr>
<td>other sciences</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2: Overview of complexity factors.
“Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigor should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere.”

(W.S. Anglin)
Construction and evaluation of the proficiency test

In the previous chapter we discussed a framework of big ideas and complexity factors that is to serve as basis of our test construction.

This test would have to provide insight in the nature of problems with basic mathematical skills and offer footholds for improving instruction in a specific domain. In this chapter we investigate whether the test based on this framework meets its goals. In this chapter we evaluate this test on three criteria: 1. Is it possible to construct a single linear scale for fraction proficiency on which all our items can be ordered according to their difficulty? 2. Can the development of proficiency from grade 4 to 9 be described with the results of this test? and 3. Does this provide diagnostic data which can be used for improving instruction?

The outline of this chapter is as follows. In Section 3.1 we start with a description of how we constructed a test on the basis of the framework of complexity factors that we derived in Chapter 2. Given the intended use of our assessment instrument we chose for a Rasch model to analyse our data. The objective of this Rasch analysis is to produce a proficiency scale for the difficulty of tasks and student ability simultaneously (Section 3.2). Such a scale allows for a quantitative description of the development in fraction proficiency over a number of grades. Furthermore it offers the possibility to qualify that proficiency. In Section 3.3 we explore the usefulness of the test as a diagnostic instrument that may provide footholds for improvement. Finally we discuss our findings in light of the functions we want this test to fulfill (Section 3.4).

3.1 Test construction

The basis of our test construction is a framework of so-called complexity factors. Complexity factors are characteristics of tasks that theoretically determine their difficulty. In Chapter 2 we described these factors which we developed on the basis of existing research on fractions, the formal curriculum and the mathematical structure of rational numbers and fractions. The systematic construction of our test based on this framework serves two goals which are related to two levels of analysis. At the first level, the item level, we want to be able to characterize students proficiency in terms of the type of tasks that students mastered. The more fine grained and systematic the coverage of the domain, the more precise are the statements we can make on the type of tasks students mastered. At the second level, the concept level, we want to be able to characterize students development in proficiency in terms
of the underlying fraction concepts they got to grips with. By ensuring a well-considered
variety in tasks, and by analyzing how students’ performance varies over these tasks, we
try to get a handle on students’ understanding of underlying concepts. For this purpose
we choose the following five big ideas as a framework of reference, relative comparison,
equivalence, rational number as an object, the transition from $\mathbb{N}$ to $\mathbb{Q}$ and the relations with
division and multiplication (Section 2.2).

3.1.1 Design

To construct the test we formulated some design criteria in addition to our framework of
big ideas and complexity factors. Since the goal of this study is to analyse the development
of fraction proficiency from grade 4 to grade 9, the test has to represent the whole domain.
At the same time, however, the test has to match with the students’ level. We therefore
decided to use an anchor design of subtests per grade. In this manner, the students will only
have to answer those items that correspond to the level of their grade. At the same time
sufficient overlap between the subtests will allow for comparison between grades. Taking
the grade level into account does not imply however that only items that directly correspond
with tasks in the curriculum were incorporated in the subtest. To enable an analysis of the
skill level with more precision, unfamiliar items were incorporated to subtests for further
understanding. Moreover, we wanted the difficulty of tasks to vary enough to ensure for
discriminatory power.

To avoid guessing and to get more insight in the strategy use of students, the items in
the test are open ended tasks. Students are not allowed to use calculators and they are en-
couraged to write down the intermediate steps and to use the extra space as scrap paper.
Although, our analysis primarily focussed on correct/incorrect scores, the written answers
especially the intermediate steps - were used in the scoring. That is, if an incorrect proce-
dure by accident led to a correct answer this item was still coded as incorrect. Finally the
subtests were designed to be completed within one school hour (50 minutes). The length of
each subtest would have to correspond to this time frame.

3.1.2 Construction and validation

In 2007, we started with the subtests for grade 5 to 7. These grade versions of the test were
based on our framework of complexity factors and the design criteria described before. The
subtests were presented to a panel of experts on primary education, secondary education
and mathematics educational research. They were asked to advise about the length of the
subtests, the suitability of the items for the intended group of students (face validity) and
representation of the domain (content validity). Based on their advise, a final version of
each subtest was made and administered. The students in grade 5 and 6 had enough time to
finish all items. This was not the case in grade 7. We decided to shorten this sub-test since
we felt that the time restraint might frustrate students. Given a Cronbach’s alpha of 0.942
for the original grade 7 test of 78 items, and using the Spearman-Brown formula (Lord and
Novick, 1968) we calculated that, with a minimum length of a subtest of 44 items we could
expect a reliability of 0.9. Relating this to the number of items the students completed in
the first round, we concluded that more than 90% of these students would have been able to finish a test of 44 items within 50 minutes.

3.1.3 Data collection

In the next round of data gathering we used the revised subtest for grade 7. An overview of the data collection is given in Figure 3.1.

<table>
<thead>
<tr>
<th>Grade</th>
<th>06/07</th>
<th>07/08</th>
<th>08/09</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>October</td>
<td>October</td>
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<tr>
<td></td>
<td>January</td>
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</tr>
<tr>
<td>grade 4</td>
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<td>24</td>
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<td>grade 6</td>
<td></td>
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<tr>
<td>grade 7</td>
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<td>99</td>
<td>151</td>
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<tr>
<td>grade 8</td>
<td></td>
<td>101</td>
<td>195</td>
</tr>
<tr>
<td>grade 9</td>
<td></td>
<td>97</td>
<td>162</td>
</tr>
</tbody>
</table>

Figure 3.1: Cross-sectional and longitudinal data collection - number of students per grade.

In total seven primary schools (grade 4 to 6) and one secondary school (grade 7 to 9) participated, all in the South of the Netherlands. Of the participating primary schools, two were high socioeconomic status schools, three were average in this respect and two were low socioeconomic status schools. In primary education all students participated. In contrast, of the secondary-school students, only the students in the streams of pre-university education (VWO) and pre-higher vocational education (Havo) participated. In the school year 2008/2009 all HAVO/VWO classes participated at the start and the end of that year. The gymnasium students, who follow a VWO programme with Latin and ancient Greek, participated at the end of that school year only. The majority of the students who participated in 2007 in grade 7 were also assessed in grade 9 in 2008/2009. The data collection allows for cross sectional comparison between grades (grade 4 to 9) and a longitudinal analysis of the development of individual students within each of the grades and from grade 7 to 9.

3.2 Creating a scale for proficiency in fractions

To evaluate our test we used the Rasch model (Rasch, 1980), a one parameter Item Response Model (IRT). Rasch analysis and other IRT models have proven to provide a good scale for student achievement in mathematics in large scale assessment projects such as TIMMS, PISA (e.g. Boone and Rogan, 2005) and PPON (Janssen et al., 2005). This type of analysis
nicely fits the goals of our test. In a Rasch analysis one linear scale is created on which both students and items can be arranged according to their “ability” respectively “difficulty” (e.g. Wright and Stone, 1999). Thus, not just the order of the items, but also the distance between items on the scale has a meaning. Furthermore, the Rasch scale—with as its unit the so-called logit—can be used to relate item difficulty directly to student ability. Consequently, ability is not just a number, it can also be qualified in terms of tasks. Another advantage is that the model can be used for anchor-designs, since the difficulty of items is independent of the selection of students that took the test and vice versa (e.g. Bond and Fox, 2007). This property facilitated our semi longitudinal anchor design in that it can handle missing values. Finally, the fit of a test to the uni-dimensionality of an IRT model provides empirical evidence for the construct validity of that test (e.g. Glas, 1998). The fit of the model to the data implies that observed responses can be attributed to person and item parameters that are related to some uni-dimensional latent dimension. In this sense Rasch analysis provides a means for analyzing and improving test instruments (e.g. Boone and Rogan, 2005).

Underlying the Rasch dichotomous model is a logistic function that relates the probability of success \( P_{ni} \) of a student \( n \) answering an item \( i \) to the difference between the students ability \( B_n \) and the item difficulty \( D_i \):

\[
P_{ni} = \frac{e^{B_n - D_i}}{1 + e^{B_n - D_i}}.
\]

Thus instead of assuming the probability on success to follow a threshold function or step function, the probability of success is assumed to gradually grow with student ability (Figure 3.2).

![Logistic ogive: relation between relative ability and probability of success.](image)

From this function it follows that when \( B_n = D_i \) or \( B_n - D_i = 0 \), the probability of success is 0.5. In the context of basic mathematical skills we arguably consider mastery starting from a 0.8 probability of success. From \( \log_e \left[ \frac{P_{ni}}{1 - P_{ni}} \right] = B_n - D_i \), it can be derived that \( P_{ni} = 0.8 \) corresponds with \( B_n - D_i = 1.39 \). To summarize, fit of the model to our data will provide a scale for ability of students and difficulty of items. The model specifies
a unit for the scale, that determines the relative distance between items and students, but not its origin. In practice the origin is usually set at the mean of item difficulty.

3.2.1 Fit

By construction, the model does not allow for items and students with a maximum (all correct) or minimum (none correct) scores. Therefore, we removed three students and 4 items with minimum scores from the data set. Based on the 169 remaining items and 1485 students a Rasch analysis was performed. We solved the issue of the length of the grade 7 test in 2007, by regarding the tasks that the students left open at the end of the test as missing value. In contrast, other blanks in this test as well as all blanks in the other subtests were regarded as incorrect answers.

The fit of the model to our data was evaluated in a number of steps: item polarity, infit and outfit, reliability and multi dimensionality.

The first step –item polarity– comprised of a check of point-measure correlations. These values reflect the extent to which the responses of the items align with the ability of the persons. Negative correlations usually indicate that the responses to the item contradict the direction of the latent variable. That is, students with a higher ability perform worse on the task than students with a lower ability. We removed two items with a small negative point-measure correlation from the data set.

In the next step we looked at the outfit and infit measures (e.g. MNSQ). We found 28 students with high outfit scores. Outfit implies that these students (in majority low-achievers and high-achievers) responded idiosyncratically to items far from their own ability. Since we did not see a clear pattern in their answers, we kept them in the analysis.

With regard to summary statistics we found a person reliability of 0.93. This Rasch reliability value can be compared with more traditional reliability values such as KR-20 or Cronbach’s alpha. In this case they indicate consistent behavior of persons.

Finally, we investigated multi dimensionality. We found that the variance in the data explained by the Rasch measure based on the empirical data is 85.2 %, and based on the perfect fit of the model 85.4 %. Since these values are high and very close, and since we could not find a factor in the decomposition of the ‘unexplained variance’ we found no reason to assume that there is another latent trait. In summary, we may conclude that all measures indicated a very good fit of the data to the Rasch model.

3.2.2 Rasch scale

Figure 3.3 is a plot of the data on the Rasch scale. Below the scale we plotted the distribution of student ability in each grade according to percentiles. We recall that in primary education all students participated (grade 4 to 6), and in secondary education only the streams preparing for higher education, representing about 40% of the best performing students. Above the scale each items is marked, sorted per subdomain. Each mark represents the point where a student with that ability has a 0.8 chance of answering that task correctly. On the Rasch scale a lower measure indicates less ability or lower difficulty. The evenly scat-
Figure 3.3: All items and students on the Rasch scale. Ordered by grade and subdomain. Items are marked based on a 0.8 probability of success.
tering of items along the Rasch scale within subdomains indicates that the test has sufficient discriminatory power.

There are several ways to use this scale for analysis. The scale provides the opportunity to compare students ability by grade and per students since some students participated two or three times. Figure 3.3 illustrates that there is a considerable progress in proficiency from grade 4 to 6, at which point it comes to a halt. By connecting the proficiency scores to the corresponding test items we may find out what characterizes the point at which the progress in proficiency stagnates. Finally the interrelation of items on the scale can provide insight at the concept level of analysis. That is, by comparing the difficulty of closely related tasks the development of conceptual understanding might be described.

3.3 Analysis of the development of fraction proficiency

In the above we have evaluated the validity and reliability of our test. It showed that the test fits the uni-dimensionality of the Rasch model. That means that the test meets the first of our criteria; we measure one latent trait and we can construct a single scale for proficiency. In this section we will address the questions if we can describe the development of proficiency over time and if the data allow for an analysis that aims at improving domain specific instruction. The first question is answered by Figure 3.3, which shows the development of student proficiency over a number of years.

We will explore the potential of the test for curriculum improvement further by analyzing the data on item and concept level. We will start the analysis at the item level in the subdomain addition and subtraction. The second level of analysis concentrates on three of the big ideas related to initial fraction concepts: unit, equivalence and rational number. We stress that these are only examples of possible analyses.

3.3.1 Item level analysis: addition of fractions

Figure 3.3 shows that in grade 4, the average students have just mastered part-whole models. In grade 5, ordering of simple fractions and the simplest tasks on multiplication and division are mastered. In grade 6, students made progress in tasks on reduce and complicate, and the basic operations on fractions. In secondary education proficiency hardly grows.

In Figure 3.4 we listed the tasks on addition and subtraction. For reference, the ability distribution of the grade 6 students is added to the figure. The rectangles show the probability of success to answer a task correctly ranging from 50% to 80%. The figure illustrates that in grade 6 most of the students mastered simple addition with proper fractions and no carry. The better half of the students also mastered addition with common denominators for all sorts of fractions. Only the best achieving students mastered simple addition of fractions with unlike denominators (primarily with factor 2).

We may conclude from this figure that the students did not reach full mastery of adding fractions. We found that we could draw similar conclusions for the other subdomains. However, for more insight in its background, we need a more conceptual analysis, which is the topic of the next paragraph.
3.3.2 Concept level analysis: development of fraction understanding

In the concept level analysis we focus on three of the big ideas, namely understanding of the unit (related to relative comparison), understanding of equivalence and developing a full understanding of rational number.

Understanding the unit

In the Netherlands, learning fractions starts with part-whole models. This is reflected in Figure 3.3. From this figure we learn that tasks on part-whole models are among the easiest items. Items relating to the big idea of “understanding of the unit” are in the subdomains part-whole, number line and application. In grade 7, not all the students mastered such items. In Figure 3.5, a selection of representative tasks is placed on the Rasch scale.

The easiest items concern naming a part of a part-whole model that is equi-partitioned. The next step is to represent such fractions. In both cases the whole coincides with the natural unit of one object\(^1\). Then follow part-group tasks, which in turn are followed by naming parts of objects that are unequally partitioned and tasks that concern the conceptual mapping of object and persons in the context to the fraction notation (e.g. Charles and

\[^1\]There appears to be a striking difference between circular and rectangular models. To name a fraction is significantly more difficult for rectangular than circular models. The same holds for the use of these models in the context of sharing multiple objects. In contrast, the rectangular model is easier in case the unit is not equi-partitioned.
Tasks that concern naming a point on a number line with a unit that is subdivided in six parts appear to be too difficult. This represents the conceptual mapping from a more abstract part-whole model to fraction notation. Not so much the recognition of the unit of the number line seems to be a problem, but rather its less usual subdivision. That is, tasks on number lines with length 1 and 2 coincide. In a way this contradicts with the work of Larson (1980), who found that the length of the number line did matter. However for the students to work with a subdivision of the unit in 6 parts rather than the standard 2, 4, 5 or 10 may be more difficult than the recognition of the unit.

Finally we zoom in on the tasks that are in the region of mastery of grade 6 students. We have listed these tasks in Table 3.1. There is a significant gap between P14 and P16. Although the majority of the students is able to work with unit fractions and familiar denominators, they appear to have problems with more unfamiliar tasks.

<table>
<thead>
<tr>
<th>tasks</th>
<th>difficulty</th>
</tr>
</thead>
<tbody>
<tr>
<td>P18: 3 children share 5 bars of chocolate. How much does each of them get? + illustration of 5 chocolate bars of 5 times 3 pieced.</td>
<td>0.01</td>
</tr>
<tr>
<td>P16: 6 children share 4 pancakes. How much does each of them get?</td>
<td>-0.42</td>
</tr>
<tr>
<td>P14: 4 children share 7 pizza’s. How much does each of them get? + illustration of 7 small pizza’s.</td>
<td>-1.12</td>
</tr>
<tr>
<td>P13: 6 children share 2 bars of chocolate. How much does each of them get?</td>
<td>-1.55</td>
</tr>
</tbody>
</table>

Table 3.1: Tasks on conceptual mapping.
Understanding of equivalence

We may argue that the concept of equivalence is more than to reduce a fraction to smallest terms or reduce fractions to the same denominator. This can be mastered in a rather procedural manner. At the conceptual level, however, it involves the notion that there is a class of fractions representing the same rational number. In Figure 3.6 we plotted a selection of representative items on the Rasch scale. We can characterize the development of the notion of equivalence as follows. The first stage constitutes of equivalence with a whole factor between numerators, starting with unit fractions, later with proper fractions. Then follow open ended tasks with difficult numbers. Tasks with equivalent fractions that do not differ a whole factor are highest on the scale. Such tasks are mastered by less than 50% of the students in grade 6. Again it appears that only procedural tasks are mastered and that there is no real understanding of the concept of equivalence.

Developing a full understanding of rational numbers

With respect to the development of fractions towards rational numbers we focus on two aspects. These aspects are the transition from proper fractions to improper fractions (or mixed numbers) and the transition of a fraction as a process of sharing to a number as an object-like entity. These transitions are visible in several subdomains. In Table 3.2 we listed the easiest tasks on improper fractions and mixed numbers from each of these subdomains.

Improper fractions with small numbers are around -1 logit on the Rasch scale. This concerns the interpretation of an improper fraction as a mixed number. Larger numerators are more difficult (-0.41 logit). Around 0.35 logit, which is already above the level of mastery of almost all students in grade 6 and 7, fractions become measures that can be added and subtracted (repeatedly). Around the same logit value also the improper fraction on the number line is recognized. This value has be taken with great care however, since we have reason to believe that the major difficulty of these tasks was the subdivision in 6 parts. Still the number line tasks with proper fractions appeared to be easier than those with improper fractions. Finally around a logit of 2, tasks appear that represent multiplication and division on mixed numbers as rational numbers. We may conclude that the students...
test validation

<table>
<thead>
<tr>
<th>task</th>
<th>difficulty</th>
</tr>
</thead>
<tbody>
<tr>
<td>D21</td>
<td>( \frac{4}{5} \div 8 = )</td>
</tr>
<tr>
<td>M45</td>
<td>( \frac{1}{2} \times \frac{2}{3} = + \text{ area model} )</td>
</tr>
<tr>
<td>D15</td>
<td>( \frac{6}{5} \div \frac{2}{3} = )</td>
</tr>
<tr>
<td>N5</td>
<td>name ( \frac{1}{6} ) on number line length 2 units</td>
</tr>
<tr>
<td>M22</td>
<td>( 4 \times \frac{2}{7} = )</td>
</tr>
<tr>
<td>D6</td>
<td>( 6\frac{3}{4} \text{ m rope, how many pieces of } \frac{3}{4} \text{ m} )</td>
</tr>
<tr>
<td>A9</td>
<td>( \frac{7}{5} - \frac{3}{2} = )</td>
</tr>
<tr>
<td>I3</td>
<td>( \frac{58}{9} = )</td>
</tr>
<tr>
<td>I1</td>
<td>( \frac{10}{7} = )</td>
</tr>
<tr>
<td>P14</td>
<td>4 children share 7 pizza’s. How much does each get? + model</td>
</tr>
</tbody>
</table>

Table 3.2: Tasks that mark the transition to rational number.

did not reach full understanding of a fraction as a object or rational number at the end of primary education, nor in lower in secondary education (Figure 3.3).

3.4 Discussion

In this chapter we evaluated the quality and usefulness of our test on fraction proficiency. We used three criteria to evaluate the test: 1. Is it possible to construct a single linear scale for fraction proficiency on which all our items can be ordered according to their difficulty? 2. Can the development of proficiency from grade 4 to 9 be described with the results of this test? and 3. Does this provide diagnostic data which can be used for improving instruction? In this section we summarize our findings on these three criteria.

3.4.1 Test construction: uni-dimensionality, validity and reliability

The content validity was verified by a team of experts. A Rasch analysis showed that the test items can be ordered on a linear scale, which means that our test measures one latent trait which we denote fraction proficiency. We found a person reliability of 0.93 which indicates consistent answers of the students. Furthermore, the test proved to have good discriminatory power, judging by the spread of the tasks and the students on the Rasch scale (Figure 3.3). The test proved to be adequate to collect data on a large number of students and still be able to analyse this data both quantitatively and qualitatively.

3.4.2 Describing the development of proficiency

We illustrated how the test can be used to analyse the development of proficiency at two levels, that of items and that of concepts. For the first type of analysis we took the addition of fractions as an example. We showed how the development of proficiency over the grades could be related to the mastery of certain types of tasks. We were able to describe this
development in terms of item characteristics. The development of proficiency stagnates in grade 7. We illustrated that at this point the students are far from full mastery of the addition of fractions. The same holds for the other subdomains. In the second type of analysis we described the development of proficiency in terms of the underlying concepts, using big ideas that we expressed in Chapter 2. We found that the students in grade 6 and 7 only seem to have reached proficiency of tasks on a superficial level. Regarding the big idea of unit, most students did not master conceptual mapping in the context of the number line. They were also not able to apply this knowledge. Regarding the idea of equivalence, the students were able to reduce fractions to its lowest term, but had difficulty in recognizing fractions as equivalent when there was no whole number factor between numerators. Finally, regarding the development of rational number, the students appear to have mastered improper fractions at the level of process, but are not able to use improper fractions/mixed numbers to operate with. For the students, fractions appeared not to have become rational numbers with the character of objectlike entities that can be used as a number. It appears that the students are capable of solving tasks that ask for reproduction and procedural use of symbols and operations. However, tasks that differ from standard and ask for more conceptual understanding lie above the ability of most of the students.

3.4.3 Diagnostic value of the test

The analysis onto three big ideas illustrated the practical value of the test. The test proved to be constructed in such a manner that it allowed for analysis both at the level of individual tasks, and at the conceptual level. The latter could be done by comparing the responses of the students on items that varied in small but significant aspects. Insights in how the proficiency of the students develop, both in the fraction domain as a whole and in a certain area, can provide footholds for curriculum improvement. Arguably there is a need for deepening the students understanding of fractions. In relation to this, we would argue for more emphasis on underlying concepts of the fraction domain, rather than only on the rules for arithmetic. The test did also provide insight in the relation between underlying fraction concepts and certain types of tasks. Although these results will be in part curriculum and thus nation specific, we assume that they represent general trends in the learning of fractions caused by the nature of this topic.

3.4.4 Further research

In this chapter we reported on the evaluation of a test that was constructed on the basis of a framework of complexity factors. As part of the evaluation we have shown that the data gathered with this test made diagnostic analyses on student proficiency in the fraction domain possible. The motivation for our test construction was the need for tools to examine the causes of a lack of proficiency of Dutch secondary school students. We chose the domain of fractions for developing a test that would help us in analyzing how student proficiency develops over time. We already found some interesting results in our first round of analysis. These call for more qualitative follow-up research. Such an analysis is discussed in the next chapter.
“Mathematics is the art of giving the same name to different things.”

(J. H. Poincaré)
4

Progress of proficiency in secondary education

We developed a test for proficiency in the domain of fractions in Chapter 2 and 3. In this chapter, this test is used to analyze the development of proficiency in lower secondary education and the transition from arithmetic to algebra. These grades (7 through 9) represent the period between initial fraction learning and algebraic use and application of fraction knowledge. Fractions are arguably a critical concept in the transition from early mathematics to more advanced mathematics and the use of mathematics in other fields. They are a common link between topics like ratio, proportion, quotient and rational number. Proportional thinking underlies notions such as rate of change, scale and algebraic fractions (e.g. Confrey and Scarano, 1995). Fraction knowledge finds its application in many fields such as economics, physics, biology and chemistry in topics involving ratio, percentages, measure, standardization and division.

The outline of this chapter is as follows. The performance of students on the test constitutes the empirical core. This test is designed to cover a wide range of fraction concepts on the basis of known difficulties with, and confounding concepts of, fractions (Chapter 2 and 3). We will sketch the design of this test and the data collection and describe the Rasch model that has been applied to fit our data to a common scale for both item difficulty and student ability (Section 4.1). Next we describe how this scale can be used to quantify the growth of individual students (Section 4.2) as well as to qualify this growth in terms of the type of skills and understanding students develop (Section 4.3). We conclude that there is little progress in the proficiency levels of the students in general and that the 9th grade students are not well prepared for the transition to algebra. We discuss the implications of these findings (Section 4.4).

4.1 Methodology

4.1.1 Test construction and data collection

To assess the proficiency of the students in the domain of fractions we developed a test based on a framework of big ideas and so-called complexity factors (Chapter 2). This framework created a basis for covering the domain of fractions in a systematic manner. The systematic test construction allowed for two levels of analysis. The first level of analysis constituted of the plain recognition of tasks that students either can or cannot perform.
progress in secondary education

We refer to this level as the item level of analysis. On the second level of analysis, the concept level, we focussed on the understanding of concepts and structure of the fraction domain and on interrelations between those concepts. This concept level comprised of the interplay between tasks, that is, students’ understanding is analyzed by considering students’ responses to related tasks.

The test was a paper and pencil test, providing the means to assess for a wide range of concepts and topics and to assess the proficiency of a large number of students. The test followed a so-called anchor design which made it possible to give the students in each grade tasks that matched with their grade level, while the anchoring items provided sufficient overlap between the grades to make comparison between grades possible. One school for secondary education in a rural part of the Netherlands participated in our study. This school offers education in all streams to in total 2200 students with a wide variety of backgrounds from the whole region. In our study all students in the streams preparing for higher professional education (HAVO) and for university (VWO) participated. Figure 4.1 summarizes the selection of data that will be used in this chapter. In 2007 only students in grade 7 participated. In addition, tests were administered in October 2008 at the start of the school year and in May 2009 at the end of that same school year. At the beginning of the school year only students in the HAVO/VWO stream participated. At the end of that year the Gymnasium stream participated as well.

<table>
<thead>
<tr>
<th>School year</th>
<th>grade 6</th>
<th>grade 7</th>
<th>grade 8</th>
<th>grade 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>06/07</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>October</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>May</td>
<td></td>
<td></td>
<td></td>
<td>218</td>
</tr>
<tr>
<td>07/08</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>October</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>May</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>08/09</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>October</td>
<td>99</td>
<td>195</td>
<td>162</td>
<td>250</td>
</tr>
<tr>
<td>May</td>
<td>151</td>
<td>512</td>
<td>273</td>
<td>347</td>
</tr>
</tbody>
</table>

Figure 4.1: Data collection: number of students per grade.

This setup of the data collection has both cross-sectional and longitudinal characteristics. Longitudinal data were collected ‘within the grade’, i.e. beginning and end of one schoolyear, and from grade 7 to 9, i.e. grade 7 in 2007 and grade 9 in 2009. Cross-sectional data were collected ‘between the grades’, i.e. groups of students can be compared based on their grade. In 2007 the test appeared to be too long for a part of the students, therefore blanks at the end of this test were regarded as missing values. All other blanks were regarded as incorrect answers (Section 3.2.1). Later tests were adjusted with respect to their length. In 2008 and 2009, the test for grade 6 and 7 were identical and contained 39 items.
Tests for grade 8 and 9 consisted of 48 and 35 items respectively. All were to be finished within 50 minutes. In 2008 and 2009 this was sufficient time for the students to consider all items.

In addition to the school for secondary education, several schools for primary education participated in our study. However in this chapter the focus is on the results of the students in secondary education. Results from grade 6 are only added for reference.

4.1.2 Scale for proficiency in fractions

To evaluate the students’ results on the test we used Rasch analysis (Rasch, 1980). The Rasch model is a one parameter item response theory (IRT) model providing a scale that is common for both students and items. Furthermore, the Rasch model facilitates anchor designs. This implied that the difficulty of the items could be matched with each grade, while the results from the different grades could still be compared. Finally, the ability of students can be related to the difficulty of items since the Rasch scale is common for both items and students.

The Rasch dichotomous model specifies the probability \( P_{ni} \) that a student \( n \) answers an item \( i \) correctly depending on the difference between students ability \( B_n \) and item difficulty \( D_i \):

\[
P_{ni} = \frac{e^{B_n-D_i}}{1+e^{B_n-D_i}} \quad \text{(Section 3.2)}.
\]

We applied the model to all our data. That is, all 1485 students from grade 4 to 9 and all 169 items were included in the analysis. In this chapter we concentrate on the development of the students’ proficiency in lower secondary education and specifically on the students’ proficiency level at the end of grade 9.

This resulted in a scale on which items and students are distributed by their difficulty and ability respectively (Figure 3.3). With this figure we illustrated how the ability of the students is distributed in each grade and how these scores relate to probability of success on the items in each of the subdomains.

4.2 Development of proficiency

4.2.1 Cross-sectional analysis

To analyze the progress of the students’ proficiency we compared the distribution of the students’ ability in each of the grades. Figure 4.2 illustrates these distributions for the HAVO/VWO and Gymnasium streams. As noted in the previous section the gymnasium classes only participated at the end of the school year (Figure 3.1), while HAVO/VWO students participated at the beginning and the end of the school year. Thus in Figure 4.2, the distributions of HAVO/VWO students at the beginning and the end of a grade represent the same group of students, and are thus not independent. For the cross sectional analysis we consider the differences between the grades. Data from the 6th grade students is added to the figure for reference, based on their score on the nationwide test at the end of primary education. From the distributions in Figure 4.2 we may conclude that there was almost none or only marginal overall progress in the first three years of secondary education. After a drop in grades 7 and 8, there seems to be a slight improvement in grade 9.
progress in secondary education

In grade 7 and 8, the school combines the HAVO and VWO streams. These streams are separated in grade 9. In our figure we combined the data of these streams to make comparison with grade 7 and 8 possible.

Figure 4.2: Distribution of students’ ability per grade and stream in 2008 and 2009.

4.2.2 Longitudinal analysis

To further describe the progress of the students we analyzed the longitudinal data of individual students. This comprised of the ‘within grade’ as well as the ‘grade 7 - 9’ data. We started with the analysis of the within grade progress of individual students. Each of the marks in Figure 4.3 represents one of the students participating both at the beginning (x-value) and at the end (y-value) of the school year. We represented no change with a dashed line in the figure. We considered the 95% confidence intervals around this line, based on the SE-values given by the Rasch model, to determine whether the difference in ability of the first and second test is significant. These longitudinal comparisons “within grade” 7, 8 and 9 revealed that there is not a uniform development within the group of students. The great majority of students did not significantly progress in fraction proficiency (e.g. 60 out of 91 students in grade 7, 56 students out of 89 students in grade 8 and 42 out of 68 students in grade 9) That is, they did not obtain a significantly higher or lower ability at the end of the school year.

We performed a similar analysis on the results of the students who participated both at the end of grade 7 (in 2007) and the end of grade 9 (in 2009). The progress of each of these 160 students in plotted in Figure 4.4. It follows from the data that 49 out of the 160 students improved significantly in skill level. The great majority of students (105 students or $\frac{5}{8}$th of the total) did not show significant progress. The remaining 6 students fell back considerably in their skill level.

Thus, from both the cross sectional and longitudinal analysis we may conclude that the majority of the students in lower secondary education did not improve their proficiency.
The progress made during the school year appears to be undone during summer vacations (Figure 4.2).

4.3 Level of understanding

In this section we describe how the results of the students in grade 9 relate to the tasks on the scale, both at item and concept level.

4.3.1 Item level analysis

To get an impression of the proficiency of the students in grade 9 we started with an item level analysis. From Figure 3.3 we may conclude that in general students do not fully master the basic operations on fractions and that the application tasks are out of their reach. In what follows we describe our first focus, that is on the subdomains of addition and subtraction, multiplication and division. For each of these subdomains we describe the succession of tasks in terms of characteristic difficulties and concept forming. In doing so we create a qualitative description of progression on our Rasch scale. For this analysis we consider both the order of items on the Rasch scale and distance between items on this scale.

Addition and subtraction

In Figure 4.5 we listed a representative selection of tasks on the addition and subtraction of fractions. The rectangles behind each of these tasks represent the chances on answering this task correctly ranging from 50% to 80% on the Rasch scale. For reference the distribution of the ability scores of the grade 9 students is added to this figure.
Our analysis shows that in grade 9, most students mastered simple addition with proper fractions and no carry. The better performing half of the group also mastered addition with common denominators for all fractions. Only the best performing students in grade 9 mastered simple addition of fractions with unlike denominators (primarily with factor 2). We conclude that for these students the “commonness” of the denominators (common denominators versus unlike denominators) is the salient complexity factor in the addition and subtraction of fractions. To conclude, in grade 9, the students are far from general mastery of the addition of fractions.

To get a better understanding of the problems these students experienced with addition tasks, we analyzed the answers of one of these tasks. We listed the type of answers for task A10: \(5\frac{1}{4} + 3\frac{1}{2}\) in Table 4.1. From the table we can derive that at least 18% (7% + 4% + 5% + 2%) of the students at the end of grade 9 started by transforming the mixed numbers into an improper fraction. Furthermore, we can conclude that a considerable proportion of the errors can be attributed to blanks.
progress in secondary education

A16: $4 \frac{2}{3} - 3 \frac{2}{3} = \quad$ unlike denominators, subtraction, carry
A15: $5 \frac{1}{3} - 2 \frac{1}{3} = \quad$ common denominator
A14: $3 \frac{1}{2} + 4 \frac{1}{2} = \quad$ addition, no carry
A13: $5 \frac{1}{2} + 2 \frac{1}{2} = \quad$ common denominator
A12: $8 \frac{3}{4} + 5 \frac{3}{4} = \quad$ unlike denominators, addition
A11: $6 \frac{1}{3} + 3 \frac{1}{3} = \quad$ no carry
A10: $5 \frac{1}{2} + 3 \frac{1}{2} = \quad$ common denominator
A9: $1 \frac{2}{3} - 3 \frac{2}{3} = \quad$ proper, common denominator
A8: $\frac{3}{4} - \frac{1}{4} = \quad$ subtraction, carry
A7: $\frac{3}{4} + \frac{1}{4} = \quad$ improper and stuck
A6: $5 \frac{1}{4} - 2 \frac{1}{4} = \quad$ improper and stuck
A5: $\frac{4}{4} + 2 \frac{1}{4} = \quad$ different error
A4: $\frac{4}{4} + 2 \frac{1}{4} = \quad$ different error
A3: $2 \frac{1}{11} + 3 \frac{1}{11} = \quad$ different error
A2: $\frac{5}{9} - \frac{4}{9} = \quad$ different error
A1: $\frac{2}{7} + \frac{4}{7} = \quad$ different error

Figure 4.5: Item analysis - addition and subtraction.

<table>
<thead>
<tr>
<th>number of students</th>
<th>percentage of students</th>
<th>strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>129</td>
<td>52</td>
<td>$8 \frac{17}{20}$ directly</td>
</tr>
<tr>
<td>17</td>
<td>7</td>
<td>$8 \frac{17}{20}$ via $177 \frac{20}{20}$</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>$177 \frac{20}{20}$</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
<td>error in reducing or complicating with $177 \frac{20}{20}$</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>improper and stuck</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>addition numerators and denominators</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>different error</td>
</tr>
<tr>
<td>64</td>
<td>26</td>
<td>blank</td>
</tr>
</tbody>
</table>

Table 4.1: Strategies to solve $5 \frac{1}{2} + 3 \frac{1}{3}$ (end of grade 9).

Multiplication

For the multiplication items the interpretation of the multiplication sign appears to be a salient complexity factor (Figure 4.6).
progress in secondary education

Figure 4.6: Item analysis - multiplication.

<table>
<thead>
<tr>
<th>number of students</th>
<th>percentage of students</th>
<th>strategy for $4 \times \frac{2}{7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>49</td>
<td>$1\frac{1}{7}$</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>$\frac{8}{7}$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>correct via common denominator</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>incorrect via common denominator</td>
</tr>
<tr>
<td>16</td>
<td>6</td>
<td>$\frac{8}{21}$</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>reverse one of the fractions</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>addition</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>other</td>
</tr>
<tr>
<td>69</td>
<td>28</td>
<td>blank</td>
</tr>
</tbody>
</table>

Table 4.2: Strategies to solve $4 \times \frac{2}{7}$ (M22) (end of grade 9).

The students in grade 9 have mastered multiplication as “part of a large amount”. On the Rasch scale these tasks are followed by multiplication as “part of part-whole” and a “whole number $\times$ fraction”. More difficult are tasks that can be characterized as “proper $\times$ proper”
and “proper × whole number”. Most difficult are the tasks that involve the multiplication of mixed numbers. It looks as if the students are not capable of applying their strategies to more complex numbers. To get more insight in the type of strategies the students used and the type of errors they made, we focussed on two particular tasks, M22: \(4 \times \frac{2}{7}\) and M35: \(31 \times \frac{17}{31}\) (Table 4.2 and 4.3).

<table>
<thead>
<tr>
<th>number of students</th>
<th>percentage of students</th>
<th>strategy for (31 \times \frac{17}{31})</th>
</tr>
</thead>
<tbody>
<tr>
<td>46</td>
<td>18</td>
<td>17</td>
</tr>
<tr>
<td>64</td>
<td>26</td>
<td>(\frac{327}{31} = 17)</td>
</tr>
<tr>
<td>29</td>
<td>12</td>
<td>(\frac{327}{31})</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>error in reducing (\frac{327}{31})</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>(\frac{16337}{961})</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
<td>(\frac{351}{961})</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>incorrect via common denominator</td>
</tr>
<tr>
<td>35</td>
<td>14</td>
<td>other</td>
</tr>
<tr>
<td>41</td>
<td>16</td>
<td>blank</td>
</tr>
</tbody>
</table>

Table 4.3: Strategies to solve \(31 \times \frac{17}{31}\) (M35) of the students at the end of grade 9.

For task M22, a considerable number of students (28%) left the item blank. About 7% of the students converted the fractions to common denominators. About 6% of the students multiplied both the numerator and the denominator with 4. It appeared that a lot of the incorrect answers were caused by variants of other procedures for fraction arithmetic. An example is to reduce both operands to fractions with a common denominator. The same goes for M35. A considerable number of the students (43%) just followed a straight forward procedure of multiplying the whole number with the numerator (\(\frac{327}{31}\)). Of these 105 students (64 + 29 + 12), 64 students were able to derive the correct answer from this (61%). However, it also led to many errors. Similar to task M22, a lot of students converted the fractions to fractions with the same denominator.

Division

A selection of the division items is listed in Figure 4.7. The complete list of tasks in this subdomain can be found in Appendix A.8. The easiest items appeared to be the items where division by a unit fraction can be interpreted as a multiplication (e.g. “A tank contains 21 liter of oil. How many cans of \(\frac{1}{4}\) liter can you fill with this amount?” (D3)). Next in difficulty were items without a context, where the answer is a whole number. These tasks were for most of the students already too difficult. The most difficult items were those tasks where the answer is not a whole number. We concluded that formal division was not mastered by most of the students.
4.3.2 Concept level analysis

In the concept level analysis we focus on the big ideas of the relation between division and multiplication and ratio-rate. Corresponding tasks are mainly in the subdomain of the application of fraction knowledge. From Figure 3.3 we learned that most of the tasks in this subdomain have high difficulties. In the following we look more closely at the big ideas.

Relation division-multiplication

One of the big ideas underlying fraction understanding is the idea of a fraction as a division and its relation with multiplication. These are tasks that relate for example to the inverse of a fraction, or to a flexible use of fractions. In Figure 4.8 we list some characteristic tasks related to this big idea. The original tasks can be found in Appendix A.9.

From the figure we learn that all such tasks lie above the ability of the majority of students and we may conclude that students appear not to see fractions as a way to denote a division. They appear not to be familiar with the concept of inverse and cannot operate on fractions flexibly, regarding it a division.

Ratio-rate

Another big idea is that of “relative comparison”. We focussed our analysis on what is sometimes called the ratio-rate duality, which refers to the duality of a fraction as both the interplay of two numbers and the single number that relates them. In Figure 4.9 we listed some characteristic tasks relating to this concept.
Figure 4.8: Tasks related to a fraction as a division.

This figure shows that tasks where the ratio is used as a factor (or rate) were above the proficiency level of most students. Since we found problems with the basic operations (Figure 4.6) there might be two reasons for problems with such tasks. Students may have had trouble in translating the context into a mathematical operation, or with calculating the result of this mathematical operation. In Table 4.6 we listed the strategies that students used to solve task T9: “The density of alcohol is 0.80 g/ml. How much does 55 ml weigh?”.

From the list of answers, we may conclude that the majority of the students did not know...
how to translate the contextual problem into “0.8 × 55”. Only a small part of the students translated the problem in the proper mathematical operation, e.g. 43% at the end of grade 9. A large part of the students who translated the problem into the proper mathematical operation appeared to have trouble in performing the mathematical operation. At the end of grade 9, 24 students answered correctly and 22 students made a calculational error or did not know how to calculate “0.8 × 55”. In fact, also a task such as “$\frac{3}{7} \times 49$” (M19) was just above the ability of the average grade 9 students.

### 4.4 Discussion

#### 4.4.1 Development of proficiency

In the previous sections we report on the results of a group of Dutch students in lower secondary education on a fraction proficiency test. Both in the cross-sectional and in the longitudinal analysis we found hardly any progress in fraction proficiency. Students in grade 9 were limited in the ability to perform the basic operations on fractions. That is, for the addition of fractions, mixed numbers with unlike denominators were too difficult to handle for most students. Furthermore, most students could solve only very simple multiplication tasks, and contextual division problems. The analysis focussing on big ideas, revealed that the students had little understanding of the relations between fractions, division, and multiplication, and were not able to handle ratio-rate duality.

These results are very disappointing even if we take into account that only part of these students would take more formal mathematics courses in the subsequent two or three years of secondary education. Still, for all these follow-up courses, whether there is an orientation to statistics or a more formal focus on analysis, a proper understanding of fraction arithmetic is required. For the more formal oriented mathematics courses, an even deeper understanding of how fractions, division and multiplication relate as well as understanding of ratio-rate duality is required.

Of course we have to be careful in generalizing these results since all students in our sample attended the same secondary school. However, exploratory research under first year
university science students corroborated these findings (Bruin-Muurling and Van Stiphout, 2009). Furthermore, in the textbooks, we found that there is a general lack of explicit attention for this topic, both in the textbook series used at the participating school (Moderne Wiskunde) and in the other textbook series (Getal en Ruimte). Together these two textbook series cover almost the complete market. In addition it may be noted that Dutch teachers tend to follow their textbooks very closely (Hiebert et al., 2003). The lack of attention to fractions as a topic in and of itself may be one of the main reasons for the lack of progress of the students. This, in turn, may have its origin in a lack of awareness in secondary education of how the primary school curriculum has changed in recent years. The current mandated attainment targets for primary school (Greven and Letschert, 2006) do not include operations with fractions, other than in contextual problems. Both the results of our test and an analysis of primary and secondary school textbooks (Chapter 5) suggest that secondary school teachers, and textbook authors do not realize that in primary school a formal level of operating on fractions is not strived for.

4.4.2 Preparation for higher secondary education

Apart from the need to support students in making the transition from working with contextual problems towards operating with fractions on a more formal level, we would argue that an additional step is required to prepare students for dealing with fractions, or rational numbers, in the context of algebra. Especially this applies for those students who attend the more formal mathematics courses in upper secondary education and continue in the direction of the sciences. This requires a deeper understanding of fractions as object like entities, rational numbers that can be reasoned with. Such deeper understanding entails the understanding of how this all relates. This is what Van Hiele (1986) named the level of “relations between relations.”

We already touched upon these issues in the concept level analysis. The relation between fractions, division and multiplication and the ratio-rate duality are examples of what is needed for deeper understanding. A central notion of fractions is that it denotes a division. This idea supports for example the connection between many contextual problems and its mathematical translation, such as for example linking “per” – as in “share per person” – to fractions. This notion is also required for understanding shortcuts in fraction multiplication, for example in canceling 4 in \( \frac{4}{7} \times \frac{5}{17} = \frac{4}{7} \times \frac{5}{7} \) or directly knowing that \( 31 \times \frac{17}{31} = 17 \), since \( 31 \times \frac{17}{31} = 31 \times 17 \div 31 \). It is also required when fractions are used as factors. When transforming the price per 320 grammes into the price per kilo for example, several strategies can be followed. A ratio table can be used, in which one can work towards the “per 100 gramme” (or “per 1 gramme”) price and then calculate the price per kilo. One can realize that to get from 320 grammes to 1000 grammes, one has to multiply with \( \frac{1000}{320} \) because \( a \times \frac{7}{5} = b \). Or one could see it as a problem of standardization, and solve the task by dividing by 0.32 (or \( \frac{320}{1000} \)) since one has to standardize 320 grammes to 1000 grammes. Understanding fractions as a division helps in recognizing that these solutions all boil down to thinking of fractions as a factor, and realizing that these solutions are all essentially the same.
Another example is task T10: \( \frac{3}{x-1} + 4 = 0 \). In this task the fraction symbol can be interpreted as a number, a division and maybe even representing a factor between denominator and numerator. Steps in solving this task involve \( \frac{3}{x-1} = -4 \) and \( x - 1 = -\frac{3}{4} \), regardless if solving such an equation with “global substitution” or “balanced scale method”. To get from \( \frac{3}{x-1} = -4 \) to \( x - 1 = -\frac{3}{4} \), in global substitution, a student must find the \( y \) that solves \( \frac{3}{y} = 4 \). Since this \( y \) is not a whole number, the fraction must be interpreted as a division, i.e. “3 divided by what gives 4?” In the balanced scale method, both sides are multiplied with \( x - 1 \) and divided by \( -4 \). This implies that the student understands how to “get rid of” the denominator. Thus, the student must have some understanding of the relation between division and multiplication and fraction as a division.

The last example relates closely to the ratio-rate duality. In the use of fractions in other sciences, often the relative comparison of two measures determines another composite measure, e.g. speed or density. For example, to find the mass of 55 ml of alcohol, knowing the density of alcohol to be 0.80 g/ml (item T6) represents a very common structure of fraction problems that are encountered in the natural science. The ratio of mass and volume of a sample of alcohol refers to the density, the “rate” that applies for every amount of alcohol. Density is an entity on its own. It represents also the factor between mass and volume. To solve problems, such as the example on density, with understanding students need to have understood rate. The same structure is also found within mathematics itself, such as in tasks on goniometry, when for example the length of one side in a right triangle is to be calculated from the length of another side and one of the angles.

Both aspects, ratio-rate and division, relate to the ambiguity of fractions. Ambiguity has been described in the work of for example Byers (2007). The author describes the idea that in mathematics, ideas, concepts and symbols can be simultaneously interpreted from different perspectives. The strength and beauty of mathematics lies in transcending ‘the conflict’ between these perspectives and to flexibly go back and forth between these interpretations (Meester, 2009). Or in the words of Byers (2007): “..., I maintain that what characterizes important ideas is precisely that they can be understood in multiple ways; this is the way to measure the richness of the idea”. An example of one of the ambiguous aspects of fractions is that it has at least two sides, namely as a process and a number. This is an example of what Sfard (1991) calls the ‘dual nature of mathematics’ and Gray and Tall (1994) name ‘procept’. Both sides (process and object) are considered to be necessary for a deep understanding of mathematics. Sfard (1991) describes how processes become objects in the historical development of mathematics and argues that this process must be mirrored in the learning process of students. Tall (1991) capture the relation between the proces-object interpretation and the fraction symbol in the notion which they call ‘procept’. We feel that in general the ambiguity of mathematical symbols, which is broader than only process-object duality, has been insufficiently addressed in education.

To conclude, our results point to important aspects of the transition from arithmetic to algebra in the domain of fractions. This transition involves at least three areas of special interest. The first is a transition from informal stated tasks via “bare tasks” (e.g. \( 4 \frac{1}{2} + 3 \frac{1}{2} = \)) to algebraic expressions. The second is the transition from informal ways of solving tasks to more general procedures. Thus informal strategies must evolve in general rules for arithmetic. These two first areas involve transition from informal to formal, both for
the way tasks are formulated and the strategies to solve them. We have shown that this transition does not evolve naturally from an informal introduction. The final area of interest concerns the ambiguous interpretation of the fraction symbol and mathematical operators. We argue that this aspect of ambiguity or versatility represents the most prominent hurdle in the transition from simple fraction arithmetic to the application of these skills and the use of them in an algebraic setting. We think that students at the end of grade 9 in the streams preparing for higher education should at least be able to perform the basic operations on fractions. For the students who proceed in the direction of the natural sciences, a deeper level of understanding is required that also includes the ambiguity of fractions.
Part II

Textbook analysis
“The real voyage of discovery consists not in seeking new landscapes but in having new eyes.”

(Marcel Proust)
Incoherence as reflected in primary and secondary education textbooks

There has been much research on initial fraction concepts and the basic operations on fractions, which has provided us with many insights in the learning of fractions and in the design of a fraction curriculum (e.g. Olive and Vomvoridi, 2006; Streefland, 1991). However, problems may also be caused by the fact that the fraction curriculum stretches over a large number of years. In the Dutch curriculum, for example, it even stretches over two educational systems, namely primary and secondary education. There is reason to believe that some of the problems in the domain of fractions stem from differences in the educational practices in the two educational systems. The two systems each have their cultural and historically determined traditions, which express themselves in educational practices and the use of artifacts. As a consequence, these differences in traditions may lead to confusing differences in the epistemological messages expressed (e.g. Raman, 2004). Given the potential existence of such incoherencies, the purpose of this study is to research the coherence in the fraction curriculum in the transition from primary to secondary education. In doing so, we draw on cultural-historical activity theory (CHAT) because this perspective takes the cultural-historical nature of human practices as its unit of analysis and hence provides a tool for understanding how traditions express themselves in educational practices (e.g., Roth and Lee, 2007).

Although we are analyzing the Dutch curriculum, the results have a broader significance. In the Netherlands, Realistic Mathematics Education (RME) (Gravemeijer, 1999) has been the primary source of inspiration for all primary school textbooks, and also has had its influence on secondary school textbooks. In this sense, the Dutch curriculum is in the international vanguard of mathematics education reform. Secondly, issues that concern the transition from one educational system to the other are not unique to the Netherlands (e.g. in Germany, Haggarty and Pepin, 2002) nor to primary and secondary education (e.g., Raman, 2004). It is generally known that there are discrepancies in the way in which content is presented in consecutive educational systems. In the research presented here we aim at a better understanding of the causes of such inconsistencies and its generally unnoticeable consequences. Deeper insight in the mechanisms that lie behind this in one specific case, here fractions in the Netherlands, may also add to the general understanding of such transition issues.

This chapter is organized as follows. We will start with an outline of how we used cultural-historical activity theory (CHAT) as our perspective to identify and explain dis-
continuities in the transition from one educational system to the other. Drawing on this theoretical perspective, we will outline the main characteristics of mathematics education communities in primary and secondary education in The Netherlands. Subsequently, we illustrate the construction of a data set consisting of selected fragments of textbooks as representative accounts of either primary or secondary mathematics curricula. We then report on the analysis of these textbooks, which consisted of a coarse-grained analysis, where we found a difference in the use of inscriptions and texts, and a fine-grained analysis where we looked into more detail at the multiplication of fractions. The latter analysis particularly illustrates a tension in meaning of artifacts between the two communities of primary and secondary education that can be explained by the traditions and rules of these two communities. Finally, we discuss the educational implications of our findings.

5.1 Background

5.1.1 Theoretical lens

Educational practices have a strong cultural-historical component, since education reproduces itself—including its typical traditions and artifacts—continuously through the people that participate in it. That is, no participant starts from scratch in the practice of education (Lave, 1988). Teachers, for example, rely in their teaching on the experiences they had as student in primary, secondary, and undergraduate education, as well as in teacher training. Cultural-historical activity theory (CHAT) provides a lens to analyze these typical ways in which traditions and artifacts are (re)produced in practices such as education. In this chapter we will especially use Roth’s (2004) elaboration of CHAT for analyzing the role of artifacts.

CHAT is rooted in the work of Soviet psychologists who maintained that human action is always mediated by cultural-historical artifacts (tools, signs) (Vygotsky, 1978). As such, artifact-mediated human action occurs in typical cultural-historical determined ways that collectively make up comprehensive practices such as mathematics education. In turn, human actions cannot be understood independent of practices, for these practices enable and constrain the possible meanings of these artifacts and hence the ways in which they mediate human action. Thus there is a dialectical relationship between individual actions and collective activity. Such a relationship also exists between unconscious operations that constitute conscious human actions, and the human actions themselves. An action like solving a complicated fraction problem, for instance, requires typical unconscious operations such as counting and adding. In turn, these operations collectively determine which actions can be performed and hence whether the problem can be solved. Such invisible and conditioned operations have a cultural-historical origin as former goal-directed actions that are internalized by means of language and other forms of communication (Vygotsky, 1978). Thus, in CHAT, human activity (practice) is conceived dialectically as a relational system in which different units such as artifacts, actions, and operations mutually presuppose each other.

There is a distinction between operation in CHAT terminology and mathematical operations like addition and multiplication.
Taking human activity as the unit of analysis, artifacts such as textbooks cannot be considered separately from other aspects of the activity system such as the traditions and rules that hold for the community by which they are used. For our research this implies that all these aspects are reflected in the textbooks. At the same time, we assume that textbooks determine to a large extent instructional practices. Analysis of Dutch textbook use in mathematics education showed a strong connection between textbook and the practice of teaching (De Vos, 1998; Gravemeijer et al., 1993). Also in science education it has been found that teachers’ ideas of the practices of science are based on the views expressed in science textbooks on which they commonly over-rely when planning lessons (Weiss, 1993; McComas et al., 1998). This conforms to Thomas Kuhn’s saying that “more than any other single aspect of science, that pedagogic form (that is, the textbook) has determined our image of the nature of science” (Kuhn, 1970, p. 143). In addition we may note that teachers not only rely heavily on the textbook regarding content and pedagogy, textbooks can also be regarded as the product of the culture in each specific activity system. That is, mathematics textbooks in secondary education are written by teachers. In primary education most textbook authors are affiliated to the teacher training institutes or school advisory services. Given this strong relationship between instructional practices and textbooks, we may assume that actual incoherencies between practices in primary and secondary mathematics education can be observed in the textbooks. Therefore, we take textbooks as valid accounts of the practices in primary and secondary mathematics education in order to study the incoherencies that may exist between those practices.

A CHAT-perspective suggests not to limit oneself to single instances of human activity but rather to consider a network of activity systems and the way in which different practices interact. In this respect, we may suspect that the differences between traditions in primary education and secondary education both in the classroom itself and in curriculum development, may express themselves as incoherencies in the transition between the two educational systems. This broader perspective corresponds with the so-called third generation CHAT, in which activity systems are collectively considered in a network that constitutes human society (Engeström, 1987). In this study we consider the activity systems of primary education and secondary education. The students are considered as boundary subjects that cross the boundary between these two educational systems (Engeström et al., 1995). Any inconsistency between the two systems may affect the learning in domains that stretch over these two systems. In terms of networks, other activity systems may play a role in the learning process as well. For instance, several researchers have pointed to the difference between mathematics and mathematics education (e.g., Raman, 2002). In mathematics the rules of logic have to be satisfied, whereas in education comprehensibility for students is determinative. This asks for a balance between the more formal character of mathematics and the more informal character of appropriating mathematics, which will be particular for each type of education.

As we will discuss in Section 5.1.4 in the secondary education system, university mathematicians are peripheral participants, whereas in primary education educational researchers are more involved. Cultural differences in beliefs and values on what mathematics is and
what engaging in mathematical activity means in these two communities are likely to permeate the practices in the mathematics classroom (e.g., Arcavi, 2003).

5.1.2 Meaning of artifacts

In this study we particularly focus on the meaning of artifacts when analyzing the coherence of the fraction curriculum in the transition from primary to secondary education. In analyzing the meaning of artifacts we follow Roth’s interpretation of CHAT (e.g., 2004). The artifacts in mathematics textbooks that we are particularly interested in are inscriptions\textsuperscript{2}, such as drawings, models, schemes, tables, and the likes (Latour, 1987). The way in which the concept of meaning of things such as tools and inscriptions is theorized in CHAT, draws on the notion that meaning is always (re)produced in activity (Roth, 2004).

Given the dialectical approach of artifact-mediated activity, the meaning of artifacts is in this view understood as resulting from the way in which they simultaneously mediate operations, actions, and activity. The meaning of artifacts can thus be theorized on two different planes. One plane involves the interplay between action and activity and is denoted with sense. That is, artifacts may have a different sense in regard to the wider societal aims of the activity to which the mediated action contributes. This aspect of meaning explains that particular symbols have a slightly different meaning in either mathematics education or mathematics. The other plane at which meaning is (re)produced in human activity, pertains to the interplay between actions and operations, which is denoted reference. Accordingly, the meaning of an artifact is determined by the patterned unconscious operations that collectively constitute the human action mediated by the artifact. For instance, the meaning of common signs used for multiplying (e.g., ‘×’, · or ‘∗’) has its reference in actually carrying out the action of multiplying in practices such as mathematics education, mathematics, or computer science. The meaning of an artifact, in turn, is theorized as the interplay between sense and reference, and is thus determined by both the wider societal activities to which the artifact-mediated actions contribute and the operations that make such action possible. In the remainder of this chapter, we use the terms meaning, sense and reference in accordance with the CHAT perspective.

5.1.3 Organization of the Dutch educational system

In the Dutch educational system there is an early differentiation at the end of primary education (age 12) into three main streams. The selection for these streams is based on the advice of the primary school and the results of a nation-wide test (CITO-test). In this paper we address the streams of pre-university education (in Dutch: VWO) and pre-higher professional education (in Dutch: HAVO). These encompass about 40% of the population that leave primary education. In our research the fraction curriculum is considered as a link between the activity systems of primary and secondary education since this topic is taught in both educational systems in the Netherlands. The initial fraction concepts are mainly addressed in primary education, whereas the formal level of mathematical operations on

\textsuperscript{2}Although technically text also consists of inscriptions, we will reserve the term ‘inscriptions’ here for inscriptions other than text.
rational numbers and the transfer to algebra are typically part of the secondary curriculum. Hence learning about fractions is an object of the study in both primary and secondary education design and research communities. A salient difference between primary and secondary education is the specialization of teachers in disciplines in secondary education. In primary education the teachers teach all disciplines, with an exception for sports at some schools. In secondary education there is a specialized teacher for each of the disciplines. The background of teachers in primary and secondary education thus differs with respect to the subject-specificity of their initial teacher training.

5.1.4 Activity systems in Dutch education

In the Netherlands, traditions in instructional design and education differ in primary and secondary education. This is the result of a difference in the culture and history of the professional communities involved. In primary education, the work of the Freudenthal Institute on the theory of Realistic Mathematics Education (RME) has been widely adopted by textbook authors, which are in general affiliated to teacher training institutes. All common primary school textbooks indicate that they are written in this tradition. In secondary education, it is difficult to precisely distinguish the role of different pedagogical approaches in the traditions of instructional design. For example, the two major textbook series for pre-university education, which we analyzed in this study, differ considerably in regard to the different instructional approaches applied. In general we can distinguish the influence of the most recent major reform in the first years of Dutch secondary education; the so-called “basic secondary education” in 1993, in which a curriculum for the first three years from all levels of secondary education was designed. This general reform of the foundation phase in secondary education was combined with a reform of the mathematics curriculum by the project group “W12-16” (translated Mathematics age 12 - 16), which was in the tradition of RME. Yet, in secondary education, the relevance for higher education has traditionally been the major consideration for choosing the contents of the curriculum. The influence of RME can be observed in secondary education textbooks, but this instructional approach is not as consistently implemented as in primary education. The influence of higher education still seems to dominate and textbooks come across more traditional. So, primary education is mainly influenced by the pedagogical/didactical RME community. RME is grounded on understanding from a broad phenomenological exploration and is in service of broad application of mathematics for all students. Secondary education is more directed to preparation for higher education. This implies an orientation on mathematics with more emphasis on understanding mathematical structures and recognizing situations as structurally the same.

RME originated in the early 1970s. Keywords are “guided reinvention” and “progressive mathematization” (Freudenthal, 1991). Two concepts are important in this respect: context and model. This is reflected in the theory of emergent modeling that classifies mathematical activity into four levels, namely task setting, referential level, general level and formal level (Gravemeijer, 1999). The main idea is that students’ thinking shifts from reasoning about the context of a problem to reasoning about the mathematical relations involved. This process is supported by the introduction of proper models. Ideally, these models initially come to the fore as models of informal situated activity, and later gradu-
ally develop into models for more formal mathematical reasoning. In older, “traditional” approaches to mathematics education, contexts and models play another role which is still reflected in secondary school textbooks. Contexts are used as an application of the learning process (Gravemeijer and Doorman, 1999), while models are rather used as mathematical models in problem solving and proving. This approach is more akin to the approach of academic mathematics, which may be a consequence of the influence of the final central examinations of secondary education. Thus, in secondary education mathematics has a more academic character. In primary education and in the design of primary education curricula, the focus is on understanding. This understanding is to be grounded in the experiential reality of the students, and on the application of mathematics in everyday-life situations. The large group of students who will continue with (pre) vocational education is of legitimate concern. An important goal of the primary school curriculum is to enable all students to learn mathematics in a way that allows them to use it in daily live. This may explain why the formal mathematical operations on fractions have been removed from the core curriculum in 1998 (Greven and Letschert, 2006). Even though advancing towards this formal level is advocated for the better achieving students, by mathematics educators (Van Galen et al., 2008). Finally we found a difference in the role of the teacher in each of the educational systems. In primary education the teacher plays a central role in orchestrating the learning process. In secondary education more responsibility is given to the student for his own learning. In short, a survey of traditions in Dutch mathematics education reveals pedagogical differences between primary and secondary mathematics curricula.

5.1.5 Focus of this study

The purpose of this study is to analyze the coherence in the fraction curriculum with a focus on the transition from primary to secondary education. Given the differences in traditions in the activity systems of primary and secondary education as we sketched, we take a cultural-historical perspective to better understand the way in which these incoherencies manifest in mathematics curricula. Drawing on this perspective, we consider textbooks as representative accounts of these curricula. Using these textbooks as our primary data source and by focusing on the use and meaning (sense and reference) of signs such as texts and inscriptions in sections on fractions, we analyzed the coherence between primary to secondary education.

5.2 Dataset construction

5.2.1 Selecting textbooks

We constructed a dataset from the major mathematics textbooks in the Netherlands. We selected the textbooks that together represent almost the total market share of the last year of primary education and the first year of secondary education that prepare for higher education (in Dutch: HAVO/VWO). Table 5.1 lists these textbooks for both primary (PE) and secondary education (SE).
5.2.2 Units - text and inscription

Drawing on methods employed in other textbooks analyses that focused on their use as artifacts in the curriculum (Van Eijck and Roth, 2008), we took as a unit the combination of text and if applicable inscription that forms a natural unit. In this respect we considered the tasks that are denoted with a separate number (not the sub questions as separate), pieces of text in the summary of a chapter that have a separate heading/title and pieces of theory between problems, as individual units. Next, we selected all units where calculation with fractions is required or that handle fraction concepts that constitute these calculations. In RME fractions, percentages, ratios and scale are seen as strongly related. In this research, however, given our focus on the streams preparing for higher education (HA VO and VWO) we only selected units on percentages, ratios and scale if they involved the use of fraction properties to ease manipulations. An example would be a calculation with percentages that represent known fractions, and where a multiplication with this fraction is asked for (e.g. 25% of 200). Table 5.2 shows the number of units that were included in the data set.

5.3 Coarse-grained analysis: common structures in the textbooks

As first phase, we performed a coarse-grained analysis of our data set to get an understanding of the common structures of texts and inscriptions therein and the resulting differences between primary and secondary education textbooks. Drawing on CHAT perspective, we distinguished various types of units consisting of texts and inscriptions on the basis of genres that are common in mathematics textbooks (Russel, 1997). We first made a coarse-grained classification in exercises and main text on the one hand and units with and without inscriptions on the other hand. With main texts we denote the texts between exercises and texts in the summary, that should be read by students but that do not actually pose a question to be solved by the student. Table 5.2 summarizes the percentages of each of these types compared to the total number of units per textbook.
One difference that emerged from this coarse-grained classification was that primary education textbooks do not include texts between tasks and also do not use summaries, so there is no “main text”. In secondary education, the textbooks are rather organized into topics and have a summary for each chapter that contains a single topic. In primary education, the structure of the book follows the lessons of the week. Chapters encompass chunks of weeks and several topics are addressed in a mixed manner. In addition to the difference in organization, we found a difference in the number of inscriptions in primary and secondary textbooks. Because inscriptions play a major pedagogical role in mathematics education, we decided to further analyze the type of inscriptions used in each of the textbooks. Here, we distinguished several genres of inscriptions, that is, (1) decorative inscriptions, (2) inscriptions that are schemes to complete, (3) inscriptions that contain numerical information that must be used to solve the exercise, (4) inscriptions that support reasoning as proposed in emergent modeling in varying abstractions (5) worked-out-examples and clues, and (6) combinations of these genres.

Table 5.3 shows that in some primary-school textbooks a large number of inscriptions is merely decorative. Especially in Pluspunt (PP) we see an extraordinary amount of inscriptions that do not have a real function in the learning process. In secondary education inscriptions seem to be used rather with a pedagogical purpose. They function mainly as worked-out-example (genre 5) or as models for reasoning (genre 4). These results reflect a difference in the function or role of inscriptions in primary and secondary education. The difference in Table 5.2 can partly be attributed to the central role of the teacher in primary education school, guiding the reinvention process of students. In secondary education it is more common that the theory is explicated in the textbooks. As already described in Section 5.1.4, students are given more responsibility for their own learning in secondary education. ‘Getal en Ruimte’ (GR) chooses for an approach of presenting examples where ‘Moderne Wiskunde’ (MW) leaves students more room for their own explorations, although in small steps. This corresponds with the difference in genres in Table 5.3.

We felt that a fine-grained analysis focusing on the use of these inscriptions would give
5.4 Fine-grained analysis: meaning of multiplication

For our fine-grained analysis we selected those units of our data set that involved the multiplication of fractions. This left us with the following amount of units for the primary-school textbooks: AT 12 units, PP 12 units, WG 32 units, RR 26 units. For the secondary-school textbooks we found 10 units for GR and 5 units for MW. These figures reflect the attention that fraction multiplication receives in each of the textbook series. We started the fine-grained analysis with a selection of units from one of the primary-school textbooks (Alles Telt, AT) and from one of the secondary-school textbooks (Getal en Ruimte, GR). In what follows, we focus on the meaning of inscriptions from a CHAT perspective in these two textbooks. We then show that there is an overlap in the physical features of the inscriptions used, but that their meaning differs. Next, we illustrate how we expanded our analysis, assuring that the selected units are representative also for the other textbooks, and we finally present our conclusions on the coherence of fraction multiplication in the transition from primary to secondary education.

5.4.1 Description of analysis

Our fine-grained analysis focused on the meaning of artifacts present in units on fraction multiplication. Figure 5.1 presents a representative example of the units we analyzed in the primary education textbook (Alles Telt, AT).

We took our point of departure in the observation that in each unit a particular action (strategy/procedure) was expected from the students in order to solve a problem or to meet other goals of the unit. For instance, the upper left unit of Figure 5.1 supports the strategy of multiplication of fractions as repeated addition when multiplying a small whole number and a proper fraction (e.g., $5 \times \frac{3}{4} = \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4}$). We denote this type of task as ‘small $n \times p$’ ($n$ stands for natural number and $p$ for proper fractions). The whole number

<table>
<thead>
<tr>
<th>Type of inscription</th>
<th>primary education</th>
<th>sec. educ.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PP</td>
<td>WG</td>
</tr>
<tr>
<td>1 - decorative illustration</td>
<td>45</td>
<td>29</td>
</tr>
<tr>
<td>2 - schemes to complete</td>
<td>10</td>
<td>28</td>
</tr>
<tr>
<td>3 - numerical information</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>4 - inscription supporting reasoning</td>
<td>14</td>
<td>13</td>
</tr>
<tr>
<td>5 - clue or worked out example</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>6 - combination of genres</td>
<td>18</td>
<td>24</td>
</tr>
</tbody>
</table>

Table 5.3: Percentages of use of inscriptions in textbooks per year of each type of inscription as compared to the total number of inscriptions.
is relatively small, otherwise repeated addition would not be efficient. Besides this strategy, we found three more strategies. The strategy of multiplying fractions by repeated addition becomes problematic when a proper fraction precedes a whole number. For such cases, *fair sharing* is introduced (Figure 5.1, bottom left). For instance, $\frac{1}{4} \times 28$ is similar to taking $\frac{1}{4}$ part of 28. In these types of tasks the whole number following the multiplier is generally larger than the whole number in the previous type of tasks. The outcome is usually a whole number as well. The expected action here is to first take a *unit part* (e.g. $\frac{1}{4}$ parts of 28) and then evaluate the proper fraction (e.g. $\frac{3}{4}$ of 28 is 3 times as much as $\frac{1}{4}$ of 28). In the remainder of this paper we denote this type of task with ‘$p \times \text{large } n$’. The third strategy comprises of the product of two proper fractions (see Figure 5.1, top right). This type of task is supported with an area model of squares. These models have a rectangular shape and multiplication is presented as *part-whole* regions. In Figure 5.1 the product $\frac{1}{3} \times \frac{1}{3}$ corresponds with the second set of squares. First $\frac{1}{3}$ of the region is shaded, that is, the top row (4 tiles) of the region of 3 by 4 tiles. In the second step $\frac{1}{3}$ of these tiles is shaded in a darker color. The rectangular shape represents the unit. In primary education this model is used only with one rectangle representing one whole and thus only applicable for proper fractions. In the remainder of this chapter we denote this type as $p \times p$. The fourth strategy deals with the multiplication of mixed numbers. Here, *splitting* the mixed number in an integer and a proper fraction is promoted. In a way this boils down to using the distributive property for multiplication, although not presented in a formal manner. In the remainder of this chapter we denote this with ‘$m \times a$’ and ‘$a \times m$’ ($m$ for mixed number and $a$ for any number).
A close analysis of these units of the textbooks revealed that each of these four types of strategies corresponds to a certain type of tasks and is always accompanied by a specific inscription. For instance, the number line always accompanies the strategy repeated addition and the rectangles always support the multiplication of two proper fractions. Thus, while using these textbooks, each of these specific artifacts may obtain a very specific reference because it is used with particular actions that are expected to crystallize into unconscious operations during the learning process. Furthermore, the inscriptions used in these units often represent artifacts that are commonly used in activity systems in which students participate in daily life. For instance, in case of repeated addition, this is reflected in the carton with cream that can be encountered at home or in a grocery shop. Thus, in terms of a CHAT perspective, the inscriptions used in these textbooks also get a particular sense. The meaning of artifacts present in units of fraction multiplication thus results from both the sense and reference of the inscriptions used.

5.4.2 Comparison of primary and secondary education textbooks

In the next step of our fine-grained analysis, we performed a similar analysis in the secondary education textbook (Getal en Ruimte, GR). This analysis revealed two major findings, which we illustrate in Figure 5.2.

This figure represents two units that are part of the main text in which fraction multiplication is introduced. At the left hand side the multiplication $\frac{1}{2} \times \frac{3}{4}$ is connected to the area model. At the right hand side other examples of fraction multiplication and their outcomes are given. These examples are used as “evidence” for the rule that multiplying fractions is tantamount to multiplying the numerators and the denominators. This is made explicit as a formal rule, “breuk \times breuk = \frac{teller \times teller}{noemer \times noemer}” (fraction \times fraction = \frac{numerator \times numerator}{denominator \times denominator}.)
Incoherence in Textbooks

In the remainder of this piece of main text examples are given how to apply this rule to mixed numbers (e.g., \( \frac{1}{4} = \frac{5}{2} \)) and to whole numbers (e.g., \( 3 = \frac{3}{1} \)). Apparently, the aim of these two fragments is to introduce a formal rule for the multiplication of fractions. The area model is used to make this rule plausible. Although the examples for the area model involve proper fractions, the formal rule is presented as applicable for all rational numbers. The word “breuk” (fraction), refers to all rational numbers. This is explicated by examples of whole numbers and mixed numbers.

As a first observation we may note that similar inscriptions are used in the primary and the secondary education textbook. The inscriptions by which the multiplication \( \frac{1}{2} \times \frac{3}{4} \) is represented, are akin to the inscriptions used in the primary education textbook (Figure 5.1, right hand side). Thus, there is an overlap in artifacts used in textbooks of primary and secondary education. A second observation is that although similar inscriptions are used in primary and secondary education textbooks, their meaning differs considerably. This difference in meaning is the result of differences in both the sense and reference of these inscriptions.

Regarding reference, the example suggests that the use of inscriptions in secondary education textbooks is connected to a formal rule. Thus, the sense of inscriptions in secondary education textbooks derives from the activity of formal mathematics. This is in contrast with activity systems that students encounter in daily life, which are connected with inscriptions in primary education textbooks. Furthermore, whereas the inscriptions in primary education textbooks are connected with particular, number-dependent strategies, this is not the case for secondary education textbooks. The area model in the given example, for instance, is used to make a formal rule plausible that holds for any rational number, rather than to support particular strategies or reasoning.

In the given textbook examples from primary and secondary education these differences in meaning boil down to the following. In primary education, the word “breuk” (fraction) is used for a proper fraction, whereas in secondary education it denotes a rational number. The multiplication sign has a reference to fair shares, repeated addition, part-whole and splitting in primary education. In secondary education fraction multiplication refers to the multiplication of natural numbers as an object-like entity. Regarding mixed numbers, different strategies for multiplication are promoted. In primary education, the mixed number is split in a natural number and a proper fraction. In secondary education the mixed number is transformed into an improper fraction and the formal rule is used for multiplication. These choices for strategies to multiply mixed numbers each correspond to the particular meaning that is attributed to the word “fraction” in each of these educational systems. Like mixed numbers, natural numbers are treated differently in the examples. In primary education, the natural number refers to an amount or factor. In secondary education, natural numbers are treated as rational numbers. Finally, differences in the activity results in subtle differences in the sense of the area model. We take as an example the task \( \frac{1}{2} \times \frac{3}{4} \). In primary education, \( \frac{3}{4} \) would correspond with 6 tiles of in total 8 tiles. The next step would be, to take half of these 6 tiles. In a way this corresponds to \( \frac{1}{2} \times \frac{6}{8} = \frac{1 \times 6}{2} \). In secondary education however, “\( \frac{1}{2} \times \)” is connected with making a step from a rectangle with quarters to one with eights. By using two rectangles to illustrate multiplication, the link with the formal rule is made.

76
5.4.3 Findings

We expanded this analysis to all other textbooks in our dataset. As explained previously, we analyzed the kinds of strategies in the text and the accompanying inscriptions. The outcomes of this expanded analysis are given in Table 5.4.

<table>
<thead>
<tr>
<th>book</th>
<th>operands</th>
<th>expected action</th>
<th>artifact</th>
<th>bare*</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>primary education</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AT</td>
<td>small $n \times p$</td>
<td>repeated addition</td>
<td>number line</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>$p \times p$</td>
<td>'part-whole'</td>
<td>area model</td>
<td>no</td>
</tr>
<tr>
<td></td>
<td>$p \times$ large $n = n$</td>
<td>fair shares</td>
<td>rectangle</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>$m \times a$ &amp; $a \times m$</td>
<td>splitting</td>
<td></td>
<td>yes</td>
</tr>
<tr>
<td>PP</td>
<td>small $n \times p$</td>
<td>repeated addition</td>
<td>rectangular object</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{p} \times m$</td>
<td>find the middle</td>
<td>number line</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$p \times$ large $n = n$</td>
<td>fair shares</td>
<td>-</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>$m \times a$ &amp; $a \times m$</td>
<td>splitting</td>
<td>-</td>
<td>yes</td>
</tr>
<tr>
<td>RR</td>
<td>$p \times p$</td>
<td>times numerator**</td>
<td>number line</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$p \times$ large $n = n$</td>
<td>fair shares</td>
<td>-</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>$m \times a$ &amp; $a \times m$</td>
<td>splitting</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>WG</td>
<td>small $n \times p$ or $m$</td>
<td>repeated addition</td>
<td>measuring mug</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>$p \times$ large $n = n$</td>
<td>fair shares</td>
<td>-</td>
<td>yes</td>
</tr>
</tbody>
</table>

| **secondary education** |          |                         |                 |       |
| GR     | $a \times a$          | formal rule | area model | yes |
| MW     | $p \times p$          | formal rule | area model | yes |
|        | $p \times$ large $n = n$ | fair shares | - | yes |

Note: abbreviation for type of numbers:

$n =$ whole number - $m =$ mixed number - $u =$ unit fraction - $p =$ proper fraction - $a =$ any number

*tasks without supporting artifact, e.g. $4 \times \frac{2}{3} =$

**woe: worked out example

***times numerator: Eg. $\frac{1}{2} \times \frac{2}{3} = \frac{1}{2} \times \frac{6}{9} = \frac{2}{9}$

Table 5.4: Description textbooks in terms of operands, expected action and artifacts.

In Table 5.4, one can observe that the outcomes of the analysis as described for Alles Telt (AT) and Getal en Ruimte (GR), are akin for the other textbooks. Each of the primary education textbooks presents fraction multiplication such that specific inscriptions are connected with particular expected actions. Each inscription thus gets a specific meaning that is mediated through the concrete realizations of individual actions. It must be noted, however, that for secondary education, Getal en Ruimte (GR) and Moderne Wiskunde (MW) differ somewhat in their treatment of fractions. In Moderne Wiskunde (MW) the formal rule is also presented, however this rule is not applied to all rational numbers. In a way it is a mix between what we saw in the primary education textbooks and the examples above from Getal en Ruimte (GR). These observations lead to the conclusion that although there is an overlap in the inscriptions that are used, the meaning of these inscriptions differs between the two educational systems. This conclusion is corroborated by a brief inspection of the teacher guides of the textbooks involved. Thus there is a discontinuity in the transition from primary to secondary education in this respect. In primary education, the meaning of
inscriptions used in the teaching of fraction multiplication support expected actions salient in fraction multiplication as it is connected to activity of daily life. The operations by which the actions are realized are those of repeated addition, part-whole, division and splitting. There is no indication of generalization towards the formal rule of multiplication of fractions. This does not match with the meaning the inscriptions are expected to have in secondary education. There they are connected to the activity of formal mathematics and do not support particular number-specific strategies. The action of applying the formal rule is realized by the operation of whole number multiplication. In this respect the multiplication sign itself has also different meanings in primary and secondary education. An important consequence of this is that in primary education, because of the number specific meaning of the multiplication sign the commutative property of multiplication does not have meaning. This too, contrasts with the use of the multiplication sign in secondary education.

5.5 Discussion

In this chapter we discussed the meaning of inscriptions (artifacts) on fraction multiplication in Dutch textbooks. We found differences in sense and reference which pointed to an incoherence in the transition from primary to secondary education. This incoherence in meaning is congruent with the different traditions in these two activity systems. In primary education, the aims of mathematics education are more oriented towards the whole student population and thus also to the use of mathematics in daily life, in addition, foundations for understanding build on experiences in daily live. The aims of pre-university (VWO) and higher pre-vocational streams (Havo) secondary education, are more oriented towards higher education and hence influenced by academic mathematics. The resulting differences in the traditions between both educational systems may hamper the transition from primary to secondary mathematics education. That is, there is no continuous learning trajectory along which the informal contextual approaches that are the starting point in primary education lead to the more formal mathematical reasoning with numbers in secondary education. This study has shown that although there appears to be coherence in the artifacts that are used in two educational systems, there can be a great incoherence in the meaning that is attributed to these artifacts. We found a difference in both the meaning and the use of artifacts. As a consequence, such differences may not even be noticed by primary and secondary school teachers, teacher trainers or researchers. These hidden differences however, may pose considerable problems for students’ transition from one educational system to the other.

This study adds to work of other researchers who have described discontinuities between academic mathematics and the mathematics in school teaching (e.g. Moreira and David, 2008). We have to acknowledge, however, that the results of this analysis are based on the textbooks. We did not observe what actually happens in the classroom. However we may assume that textbooks are leading in terms of both the topics that are addressed in the textbooks and the solution strategies that are taught in the classroom. Although we cannot recognize the exact direct and indirect influence of the textbook on the actual classroom practice, coherency of textbooks can be considered as an aspect of quality in and of itself.
5.6 Implications

To conclude, we think that the transition from primary to secondary education in the Netherlands needs to be improved. The differences between the traditions in primary and secondary education can be characterized as informal contextual solution procedures versus more formal mathematical reasoning with numbers as mathematical objects. This transition, a generalization process sometimes called vertical mathematization is one of the most difficult steps in learning fractions. The gap between textbooks can be considerably narrowed if generalization is addressed explicitly in either one. Interestingly enough, the RME theory that appears to have informed textbook design, anticipates such a transition, from informal contextual solution procedures, to more formal mathematical reasoning with numbers as mathematical objects (Gravemeijer, 1999). This aspect of RME theory, however, is not systematically worked out, neither in primary or secondary school textbooks, nor in the combination thereof. It is not something that is strived for in the national primary school curriculum goals (Greven and Letschert, 2006), and this transition is not addressed in secondary school either. We would argue that remedying this could largely improve the instruction on the multiplication of fractions. Context-specific solution procedures that are central in the primary school system with its focus on contextual understanding and applications, have become a goal in and of itself. Instead we would argue, primary school teachers will have to focus on support of students in coming to see the relations between the number-specific procedures. Secondary school teachers in turn have to realize that the basic operations (“+”, “-”, “×” and “÷”) with fractions are not just simple procedures necessary for more interesting mathematics. They will have to try to build on the meaning that students have developed of operating with fractions. In addition, the meaning of artifacts—both reference and sense—and relating underlying concepts should be a topic in classroom discourse.

We want to close by stressing the importance of taking the transition from one educational system to the other into account in curriculum design and research, particularly in regard to the meaning of artifacts. With respect to topics that stretch over the two educational systems a coherent learning trajectory for the entire learning strand needs to be designed.
“As a kid, I loved playing with numbers. I like taking problems and pulling them apart. And not just doing the problem one way, but trying several different ways to do the same problem. And amazingly you always get the same answer. I found that consistency of mathematics to be absolutely beautiful and I still do today.”

(Arthur Benjamin)
The use of contexts in primary education

The analysis of the textbooks in primary and secondary education, which we reported on in Chapter 5, revealed a breach between primary and secondary education curriculum. In primary education the focus was on context related solution procedures for specific cases, whereas in secondary education the focus was on the formal rule for multiplying arbitrary rational numbers—which boils down to multiplying the nominators and multiplying the denominators after turning both multiplier and multiplicand into (im)proper fractions. In reflection, we believe we hit on a fundamental problem in the transition from informal, situated solution procedures to more formal, abstract mathematics. A problem that might be inherent to current elaborations of reform mathematics.

Many current reform recommendations suggest that mathematics instruction should build on the informal, often contextual, knowledge of students. This contrasts with more “traditional” views on mathematics education in which only prior (formal) knowledge from previous mathematics courses is taken into consideration. Building on a foundation that is meaningful in terms of previous experiences of the learner is seen as a prerequisite for learning with understanding. Consequently, a central question for research is how to support students in building upon their prior knowledge in such a manner that they develop more abstract, conventional mathematics. Over time various theories have been developed to tackle this problem. These theories have to be translated into practice, and problems may occur in the translation. Conversely, problems that come to the fore in instructional practice may point to problems in the corresponding theory. In this chapter we reason from the premises that the breach we found in the fraction curriculum is a problem that may point to a weakness in the underlying theory and/or in its elaboration. In order to shed light on the roots of this problem, we will investigate in the following how the underlying theory—in this case, RME theory—is translated into the Dutch fraction curriculum.

The Dutch curriculum provides a good opportunity to survey the whole trajectory from theory, through implementation, to learning outcomes, because of its long tradition in reform mathematics. RME theory offers a concrete elaboration of the idea of supporting students in developing more formal mathematics. It is rooted in Freudenthal’s notions of mathematics as an activity, guided reinvention, and didactical phenomenology. In the tradition of RME, design research is employed to develop local instruction theories and prototypical instructional sequences as part of a research program for developing and refining the RME instruction theory (e.g. Gravemeijer and Cobb, 2006). Moreover, all Dutch primary education textbooks claim to be based on RME. The outline of this chapter follows cur-
riculum levels from research outcomes (ideal curriculum) to the achievements of students (attained curriculum). We start with a general overview of theoretical points of departure and how these ideas are elaborated in RME theory (Sections 2 and 3). Then, we focus on fractions and describe Dutch design research on this topic (Section 4). Next, we present the outcomes of an analysis of textbooks (Section 5) which we collate with students’ answers to a test on fraction proficiency (Section 6). This leads us to conclude that there are risks of compartmentalization in contexts that bear clear references to only one informal strategy. Finally, we reflect on these findings in Section 7.

6.1 Theoretical considerations on learning mathematics

As early as the 1960s, Freudenthal (1968) advocated for mathematics education in which students would be actively involved in mathematizing mathematics. Today, the idea of students constructing mathematical knowledge is broadly accepted in the community of mathematics educators and is often referred to as ‘reform mathematics’. This view on the learning of mathematics implies that students have to build upon their informal knowledge to construct more formal mathematics. Learning mathematics as a process of knowledge construction has been studied extensively.

Many studies have proposed level transitions in this process (e.g. Van Hiele, 1973, 1986). Freudenthal (e.g. 1971) does not point to fixed levels but emphasizes on the role of levels in the process of mathematical development. He saw the development from one level to another as occurring when students become conscious of the rules applied unconsciously or, at a later stage, when such awareness evolves in the analysis of their own activity. This means that the activity at one level becomes the subject of analysis at the next. In other words, the operational matter on one level becomes the subject matter at the next level. In the same vein, (Sfard, 1991) discusses a transition from computational operations to abstract objects. She points to parallels between the progression of students’ mathematical understanding and the historical progression of academic mathematics. She argues that mathematics has a dual nature in terms of process and object. In the transition from process to object she distinguishes the phases of interiorization, condensation, and reification. She considers this last step of reification as an unpredictable jump that is most important in the learning of mathematics. Many mathematicians have described such a moment in terms like ‘the penny has dropped’ or ‘the light is turned on’ and they consider this step as reaching understanding (e.g. Byers, 2007) From that moment onwards one can flexibly attribute both a process and an object interpretation to the mathematical concept. Ambiguity in process and object is what (Tall, 1991) calls procept. Pirie and Kieren (1994) characterize growth in understanding in eight steps from primitive knowing, that represents the knowing at the beginning of the process, to the level of inventising, the outermost level of full structured understanding. The authors consider this last level as the starting-point for new questions that might lead in turn to growth into new concepts. They also take this as the point where the student can break away from the preconceptions that lead to this level of understanding, giving way to these new questions. Similarly to Van Hiele (1986) and Sfard (1991), Pirie and Kieren (1994) consider the last stage of the process as the starting-point of a new process. In this sense, there is a hierarchy in learning mathematics, i.e.
the 'stacking' of mathematical ideas. Although these (and other) authors discern different steps, they all describe a process in which students build upon and extend their ways of knowing that initially may relate to their experiences both inside and outside school. Long before attending school, children experience forms of mathematical activity in their daily lives. They develop an understanding of concrete instances of these mathematical concepts and find informal strategies to solve problems. Later, experiences within school contribute to their informal knowing. What is experientially real to students can thus be part of both daily life and mathematics itself.

This view on learning mathematics has implications for teaching. Learning should start with situations that are experientially real to students in the sense that it are situations in which they can act and reason sensibly. Yet the question of how to overcome the 'contradictions inherent in weaving together respect for mathematics with respect for students in the context of the multiple purposes of schooling and the teacher's role' (Ball, 1993, p. 7) poses a dilemma. Likewise, the question remains how to organize education in such a way that students truly make the transition from informal to formal mathematical knowledge. According to Freudenthal (1971) the source of the principal dilemma of teaching mathematics is in the process of mathematizing mathematics. As a consequence of this process, modern mathematics has reached such generality that it applies to a rich variety of situations and enhances flexibility. He argues, however, that whereas the most abstract mathematics is undoubtedly the most flexible, it is useless to those who cannot take advantage of that flexibility (Freudenthal, 1968).

Starting with carefully selected contexts that are experientially real for students and offering them the opportunity to build upon their own insights and experiences as a means of constructing more formal mathematics is considered a solution to this dilemma. Even though contexts have this potential in theory, empirical findings suggest that there might be a problem in selecting the right contexts. One such finding is that students tend to focus on aspects of contexts that are superficial or irrelevant from a mathematical perspective. For instance, it has been observed that students choose a mathematical operation on the basis of an apparent structure of tasks instead of the actual mathematical structure (Van Dooren et al., 2009). Alternatively, students unfamiliar with the underlying mathematical concepts appear to focus on aspects that remain unnoticed by experts (Arcavi, 2003). In sum, there is a tension between the way students think and act and the way theorist or instructional designers expect them to. In order to investigate this tension, we look at an example of how reform mathematics theory works out in practice. The example we discuss in the following concerns the learning strand in the multiplication of fractions in RME-based textbooks in the Netherlands.

### 6.2 Realistic mathematics education

The domain-specific instruction theory for realistic mathematics education (RME) originated in the early 1970s in the Netherlands. We take RME as an example of curricular reform that aims at supporting students in constructing mathematical knowledge. RME research has been influential in both theoretical work and the design of education, especially in the Netherlands but also in other countries (e.g. Sembiring et al., 2008; Tzur, 2004).
RME, mathematics is seen as ‘a human activity’ of mathematizing. This is eventually to lead to generalized and abstract end products. Freudenthal (1968) stressed that students have to mathematize mathematics. The end results of mathematical activities of others should not be taken as a starting-point for learning mathematics. For that would create an ‘anti-didactical inversion’. Mathematics should not be presented as a ready-made product, but as something that students construct from their experiences. Students are to be guided in such a manner that they can reinvent mathematical concepts while building upon the knowledge they already have (Freudenthal, 1991). This guided reinvention principle - expressing both students’ activity in reinventing mathematics and teachers’ guiding role - is at the core of RME. In RME, progressive mathematization is seen as the counterpart of guided reinvention (e.g. Treffers, 1987).

The objective of supporting students’ mathematical reasoning is that students will eventually be able to participate in established mathematical practices that have grown out of centuries of exploration and invention (cf. Cobb, 2008). Accordingly, a challenge is to design or to identify problem situations in which phenomena, in the words of Freudenthal, ‘beg to be organized’ by the ‘mathematical thought objects’ that students need to develop (Freudenthal, 1983, p. 32). In relation to this, Freudenthal puts forward the heuristic of a didactical phenomenology, according to which one has to study how the aforementioned thought things organize the corresponding phenomena. A phenomenological analysis reveals how certain phenomena are organized by a given ‘thought-thing’ and what contextual problems might create a need for student for organizing these phenomena (Freudenthal, 1983).

The intertwinement of learning strands (Streefland, 1988), which is an RME principle that is relevant to our study, evolves in a natural manner from didactical phenomenology. That is, the relations between mathematical domains, as they manifest themselves in applied problems will be reflected in the results of a didactical phenomenological analysis. Consequently instructional designers are to exploit the relation between mathematical domains by intertwining learning strands to avoid a curriculum of separate strands (e.g. Streefland, 1988). For the fraction domain for instance this means that fractions, ratio, percentage and decimal numbers are linked to each other from the very start (e.g. Van Galen et al., 2008).

The instructional design heuristic of emergent modeling describes how models may support a shift from reasoning about the experientially real contexts of problems to reasoning about the mathematical relations that are involved (Gravemeijer, 1999). Ideally, models that come to the fore as models of informal situated activity will later gradually develop into models for more formal mathematical reasoning. The first level of this modeling process concerns activity in the task setting, which is based on knowing how to act and reason in that setting. At the next level, well-chosen contextual problems offer students opportunities to use informal, context-specific strategies in conjunction with ‘models of’ activity in the contextual setting. At this level, which is denoted (referential level), the model will derive its meaning from the situation it refers to. Next the teacher tries to foster a shift in attention from the contexts towards the mathematical relations involved. Consequently, for the students the model will start to derive its meaning from those mathematical relations. In this manner, the ‘model of’ may gradually develop into a ‘model for’ more formal mathematical reasoning which characterizes the general level of activity. Finally this is to be followed by
the formal level, at which mathematical reasoning is no longer dependent on the support of these models.

In the following section we will proceed to the curriculum levels and describe how these theoretical considerations have been worked out in the development of teaching materials.

6.3 Design research on fractions

In the tradition of RME, theoretical work has been combined with the development of teaching materials. The many prototypical teaching materials developed by RME researchers have found their way into Dutch textbooks. The fraction domain on which we focus here has been central to the work of Streefland (1983, 1991), Treffers et al. (1994), Keijzer (2003) and Van Galen et al. (2008). In this chapter we focus on the foundational work of Streefland and Treffers.

The design by Streefland (1991) of a fraction curriculum based on the basic principles of RME was part of Wiskobas, a project to reform primary mathematics education which started in 1968. Streefland considered as his main design principles (1) a broad phenomenological exploration of the fraction concept (veelsporigheid), (2) the use of contexts as source for and field of applications of fractions, (3) the use of models and schemas to support algorithmizing or mathematization, (4) the retention of insight by ‘keeping the sources open’, and finally (5) progressive schematizing and mathematizing (Streefland, 1983). In his work the many manifestations of fractions are used in the first concept-forming phase. There is a close relation between these many manifestations and the diversified use of contexts, albeit that the latter does not necessarily implicate meeting the requirement of the first. Comparing a variety of ways of equal sharing, for example, may lead to a network of number relations such as $\frac{3}{4} = \frac{1}{2} + \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 3 \times \frac{1}{4} = 1 - \frac{1}{4}$, which may come to the fore when describing what each individual gets when four children share three pizzas (Streefland, 2003). For the algorithmization of operations one manifestation is chosen, in this project usually the measurement aspect. The equivalence of fractions, which follows naturally from ordering fractions on a number line, is chosen for the reinvention of algorithms for addition, subtraction and division of fractions. To support this process Streefland proposes a number line with two or more scales, a ratio table and isomorphism between the circular clock model and the linear model of the timeline. The process for developing formal multiplication is supported by a grid or area model. Using this model, Streefland discusses the commutative property of multiplication.

In conjunction with the research by Streefland, a series of Proeven (translation: ‘exemplary practices’) was developed as a guideline for a national curriculum, and one of these publications was dedicated to fractions (Treffers et al., 1994). According to Proeve 3a the first step has to be to develop a fraction language and the meaning of the fraction notation. From the start of the learning strand, five (phenomenological) conceptions of fractions are developed which are similar to the subconstructs of Behr et al. (1984) and Kieren (1980). These conceptions involve part-whole relation, operator, measure, ratio and

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1 Wiskobas is an acronym for the Dutch translation of ‘mathematics in primary school’
role of contexts

The number line is introduced to prepare for global estimating. Equivalence is introduced through contextual models. The third goal is to teach students to operate in fractions. The authors suggest to start with fractions embedded in model contexts like measuring with paper strips and partitioning chocolate bars. The latter context especially is used to introduce a general solution strategy for addition and subtraction, denoted as ‘finding an appropriate sub-unit’ (passende ondermaat). Instead of taking the chocolate bar as a unit and working with fractions, a smaller unit is chosen, corresponding to subdivision of the chocolate bar. This enables one to make a detour via working with natural numbers, which anticipates the method of finding a common denominator. Students will be working with concrete numbers (e.g. 9 pieces of a chocolate bar of 12 pieces) rather than abstract numbers (e.g. $\frac{3}{4}$). Finally, the Proeve suggests using ‘fraction bars’ or double-sided number lines to connect fractions, decimal numbers and percentages.

To summarize, in line with the RME principle of a broad phenomenological exploration, Streefland (1988) and Treffers et al. (1994) have exploited many phenomenological appearances of fractions to give meaning to basic operations for fractions, and to intertwine learning strands (fractions, ratio, decimal numbers and percentages). This contextualized approach is sufficient to fulfill the requirements of the formal curriculum for primary school, in which formal arithmetic with bare fractions is not one of the attainment targets. These learning strands are to create the basis for a process of generalizing and formalizing, in which general algorithms are developed. This last step is left for secondary education, although it appears that textbook author’s and teachers in secondary education expect a more formal level of understanding (Chapter 5). In what follows, we look at how the rather general guidelines based on the principles of RME are elaborated more or less idiosyncratically in the design of textbooks.

6.4 Textbooks

All major textbook series in Dutch primary education claim to be based on the ideas of RME. Indeed, we can recognize much of the prototypical work and recommendations in the textbooks. Textbooks in turn can be seen as a good indicator of what happens in Dutch classrooms, since Dutch teachers rely heavily on the content of textbooks when it concerns both the topics and the teaching thereof (De Vos, 1998; Gravemeijer et al., 1993; Hiebert et al., 2003). In Chapter 5, we described how we collected and analyzed textbook segments on fractions in the sixth-grade textbooks of the four leading textbook series and from the two leading textbook series in seventh-grade. These textbook series together cover about 80% of the market (Janssen et al., 2005; Ctwo, 2009) in both primary and secondary education. The first analysis showed that these fragments contained both contexts that can function as starting-points for mathematization and models that are intended to support this process. In this analysis we focus on multiplication in primary education. We use fragments from the following textbooks: Alles Telt (Boerema et al., 2001), Pluspunt (Van Beusekom et al., 2000), Rekenrijk (Bokhove et al., 1999) and De wereld in getallen (Huitema et al., 2000).

In analyzing these fragments we found that in each of these textbook series, fraction multiplication is strongly connected with informal strategies, contexts and models. Although textbooks differ, we distinguish four specific forms of fraction multiplication that
characterize all of them (Figure 6.1). Typical of those specific forms is that they are number-specific, that is to say, they depend on the type of fractions involved.

Similar to the multiplication of whole numbers, the multiplication of a ‘whole number times a proper fraction’ is presented as ‘repeated addition’. In these types of tasks the whole number is typically small. In the various textbooks different models are chosen to represent this. Examples are the number line or natural predecessors thereof and contexts such as measuring jugs.

For the form ‘proper fraction times a whole number’, multiplication is translated into ‘part of’. In these kinds of tasks the whole number is usually large and refers to an amount of something, e.g. 200 grams of flour or 500 tins. In practice the answer is always a whole number in these textbooks. In many cases first a multiplication with a unit fraction is presented and then the numerator is changed in the next exercise, e.g. \( \frac{1}{4} \times 200 = \) is followed by \( \frac{3}{4} \times 200 = \). A (paper) strip is often used for visualization.

The multiplication of two proper fractions is illustrated in two different ways. Most textbooks illustrate these tasks with so-called ‘rectangle multiplication’ or an ‘area model’. Both fractions are represented in separate directions in a rectangle representing the unit. The intersection of these two regions is modeled as the product of the two fractions. Fractions in these tasks have small denominators and the model is given. Another strategy is presented in one textbook. Here, an equivalent fraction of the second operand is proposed, with a numerator which, divided by the denominator of the first operand, produces a whole number (e.g. \( \frac{1}{2} \times \frac{1}{4} = \frac{1}{2} \times \frac{2}{6} \)). The equivalence in this case is represented as a position

\[ \frac{1}{5} \times \frac{3}{4} \]

\[ \frac{4}{5} \times \frac{1}{2} \]

\[ 4 \times \frac{1}{3} \]

\[ 4 \times \frac{1}{2} = \]

\[ \frac{1}{3} \times \frac{1}{4} = \]

Interestingly, the interpretation of the multiplier seems not to be universal. In the Netherlands it is interpreted as \( 3 + 3 + 3 + 3 + 3 \), thus being the multiplier. In other countries this is the other way around and is interpreted as \( 5 + 5 + 5 \) with 3 being the multiplier.
on the number line. In neither case did we find ‘cancelling’ of common factors to reduce calculation effort, e.g. $\frac{10}{27} \times \frac{9}{25} = \frac{2}{5} \times \frac{1}{5}$.

The multiplication of mixed numbers can involve splitting the mixed number in a whole number and a proper fraction (e.g. $4\frac{1}{2} \times \frac{1}{2} = 4 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}$) or alternatively in a suitable whole number and a mixed remainder (e.g. $\frac{1}{2} \times 7\frac{1}{2} = 7 \times 6 + \frac{1}{2} \times 1\frac{1}{2}$). We did not find a strategy to turn the mixed number into an improper fraction (e.g. $\frac{1}{2} \times 7\frac{1}{2} = \frac{1}{2} \times \frac{15}{2}$) in these textbooks. The use of the distributive rule is proposed, albeit not formally.

To summarize, we found a range of contexts and models in the textbooks that originate from design research. They represent multiple perspectives on fractions and the multiplication of fractions, but they are not linked to each other. What multiplication entails, varies with the type of numbers that constitute the multiplier and the multiplicand. In this manner students learn four procedures for multiplication instead of one that applies to all rational numbers. This is not in line with the original theoretical intentions of the use of such contexts. We argue that this compartmentalization may hinder the process towards more formal understanding of multiplication of fractions. That is, we have established that the way fractions are presented, and the solution procedures that are practiced in the primary-school textbooks, can be characterized as compartmentalized into a set of number-specific solution procedures. As the common Dutch practice is that teachers follow the textbooks closely, we may expect this compartmentalization to affect the students’ way of reasoning with fractions. To check this assumption, we looked for items in a test on fraction proficiency that might reveal this.

6.5 Student performance

As part of a larger study, in which we analyzed the development of fraction skills over several years, tests were administered in primary and secondary schools. For a description of this test we refer to Chapter 3, in which we discuss the reliability and validity is discussed.

To reflect the proficiency level at the end of primary education we used the results of the students in grade 6 (151 students) and in the start of grade 7 (99 students). That is, at the start of secondary education the subject of fraction multiplication had not yet been taught. We used the test results of 413 students at the end of grade 7, to get some clues on how the introduction of more formal multiplication had influenced the students’ proficiency level. Our tests were designed such that they contained series of items that were similar with respect to the underlying formal structure and mathematical concepts, but differed with respect to details of formulation of the task. In this chapter we use such tasks that are closely related. We will discuss two types of differences.

The first of the differences concerns the abstraction of the problem statement (e.g. ‘half of $\frac{5}{8}$ of a piece of cake’ versus ‘$\frac{1}{2} \times \frac{5}{8}$’). The second difference concerns the ordering of the operands (e.g. ‘$\frac{3}{4} \times 49$’ versus ‘$4 \times \frac{3}{4}$’). We assume that order and abstraction should not matter when a student has reached a certain level of understanding. One example of a series of items that differed in abstractness but were based on the same mathematical expression
The role of contexts

Table 6.1: Comparison of strategy use in concrete and formal formulation of the same task.

<table>
<thead>
<tr>
<th>year</th>
<th>task</th>
<th>correct in %</th>
<th>similar in %</th>
<th>different in %</th>
<th>similar in %</th>
</tr>
</thead>
<tbody>
<tr>
<td>end grade 6</td>
<td>de helft van $\frac{3}{8}$ reepkoek is</td>
<td>39</td>
<td>34</td>
<td>113</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2} \times \frac{5}{8}$ reepkoek is</td>
<td>10</td>
<td>8</td>
<td>82</td>
<td>9</td>
</tr>
<tr>
<td>start grade 7</td>
<td>$\frac{1}{2} \times \frac{3}{8}$ reepkoek is</td>
<td>34</td>
<td>73</td>
<td>131</td>
<td>31</td>
</tr>
<tr>
<td>end grade 7</td>
<td>de helft van $\frac{5}{8}$ reepkoek is</td>
<td>38</td>
<td>65</td>
<td>153</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2} \times \frac{5}{8}$ reepkoek is</td>
<td>38</td>
<td>65</td>
<td>153</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2} \times \frac{3}{8}$ cake is</td>
<td>49</td>
<td>59</td>
<td>159</td>
<td>27</td>
</tr>
</tbody>
</table>

The second type of variation in items is in the order of the operands; e.g. $8 \times \frac{3}{4}$ and $\frac{3}{4} \times 49$. We found that the most typical error in the multiplication of a whole number with a proper fraction is that students multiply the whole number both with the numerator and denominator (e.g. $8 \times \frac{3}{4} = \frac{24}{4}$). We compared the number of errors and the proportion of this typical error in the total of incorrect answers for each of the prototypical tasks (Table 2). The order of the operands appears to be a factor in the performance of these students.

In Dutch there are two different words for $\frac{1}{2}$. Helft implies taking a part of an amount. Half or halve is used for the number $\frac{1}{2}$ and part of an object like half an apple.
Table 6.2: Use of “whole number times both numerator and denominator”, e.g. $4 \times \frac{2}{7} = \frac{8}{28}$.

We notice a difference in the number of errors and in the nature of these errors when we compare $4 \times \frac{2}{7}$ and $8 \times \frac{3}{4}$ on the one hand and $\frac{3}{7} \times 49$ on the other hand. That is, there is a difference in the percentage of correct answers (e.g. 56% versus 21% at the start of grade 7) and in the percentage of students multiplying both numerator and denominator (e.g. 23% versus 3% at the start of grade 7). It appears that, students experiences these differently ordered but mathematically similar tasks as very different mathematical problems. Again, this can be explained as a compartmentalized notion of fraction multiplication.

To summarize, in our fraction test we found differences in student performance on items that differ with respect to a small variation in formulation (i.e. level of abstraction and order of operands) but which have the same mathematical structure. The data suggest that the students could not flexibly switch the multiplier and the multiplicand. Moreover, the results at the end of the 7th grade suggest that the students could not apply the formal rule for multiplying fractions, which is taught halfway the 7th grade.

6.6 Reflection

In the current views on mathematics education students are expected to build on their own informal situated knowledge to construct more formal mathematics. The central question of this chapter is how this constructivist stance of students constructing their own knowledge is brought into practice and what problems arose in this process. In the previous sections we showed that exemplary tasks that were developed in design research in the tradition of RME were incorporated in Dutch textbooks. We found that the way this was done led to a number of very distinct procedures for multiplying fractions. These findings were compared with empirical data on the performance of students in this domain.

The results suggest that the compartmentalization of the textbooks in primary school affects the way the students try to solve multiplication tasks, and that the students do not apply the formal rule they have learned in secondary school. As we showed in Chapter 5 these problems are in part the result of a lack of alignment of primary and secondary school textbooks, and the difference between the primary and secondary school traditions. We
would argue, however, that these results also point to a barrier in the transition from informal situated solution procedures to more formal mathematics. Alternatively, we may take the manner in which fraction multiplication is treated in primary and secondary school, as exemplary for the lack of awareness of such a barrier. The barrier we have in mind here, concerns the fact that the students have to mathematize their informal mathematical activity to arrive at a more formal level. In this case they have to generalize and formalize the solution procedures for multiplying fractions and whole numbers, they appropriated in primary school, in order to arrive at the formal rule for multiplying arbitrary rational numbers. The naturalness of the context and number specific solution procedures, however, may become a barrier for such a process of generalization and formalization.

6.6.1 Contexts as starting-points

The rationale for starting with contextual problems is rooted in the notion that formal mathematics is not directly accessible but has to be constructed. In relation to this, it may be helpful to distinguish between the student’s view as actor and the mathematics educator’s view as observer. The mathematics educator does not realize that he uses mathematical insights that students do not already have. Students on the other hand can only reason from their own knowledge and experience. From the designer’s point of view, one has to look for contexts and models that are experientially real to the students, and are justifiable in terms of the potential endpoints of a learning sequence (e.g. Gravemeijer, 1999; Pirie and Kieren, 1994). Thus, researchers/designers will be thinking of the relation between one solution procedure and the potential endpoint when they design instruction. Students, however, will primarily see the relation between (the characteristics of) the problems and the corresponding solution procedures. Consequently they may not construct the relationships the researchers had in mind. Instead they may focus on aspects that are mathematically irrelevant (e.g. Magidson, 2005) - aspects that are even unnoticed by those who understand the underlying concepts of the context (e.g. Arcavi, 2003). As a result, inherent constraints of implicit models contained in contexts can in turn constrain concept construction (e.g. Fischbein et al., 1985; Prediger, 2008). In addition, since students do not know the intended endpoint they may very well focus on superficial features of tasks when trying to solve a problem. This may involve finding ‘keywords’ for certain operations or the structure of the task (e.g. Prediger, 2009; Van Dooren et al., 2009). A similar problem may occur when examples are too limited, as students may generalise an accidental but mathematically irrelevant commonality in the examples to a general structure (e.g. Guy, 1988).

This tendency may be strengthened by the common approach to practice that we found in textbooks. Typically, tasks in textbooks were clustered in sets of similar types of tasks. This clustering, of course, makes sense in terms of the need to develop proficiency. For students, however, it may be effective in such cases to be guided by superficial task features.

Actor-observer conflicts may also appear in the use of language and mental models. A clear example is the connection between ‘taking a part of’ and fraction multiplication. For students this may not be as natural as it is for experts (e.g. Streefland, 1991). Based on their experiences with natural numbers students associate multiplication with repeated addition, a conception of multiplication they have to adapt to include multiplication with fractions.
We may also observe a paradox in the use of exemplary context problems. Such tasks are selected on the basis of the naturalness of corresponding solution methods. When we are calculating the total content of 20 milk cartons, holding \( \frac{3}{4} \) liter each, for instance, repeated addition presents itself as a natural solution method. Whereas the task of calculating \( \frac{3}{4} \) of 20 kilograms lends itself very naturally to dividing 20 kilograms into four parts of 5 kg and taking three parts. This self-evident character of the solution methods makes such exemplary context problems very powerful - initially. Later, however, when the students have to generalize those solution methods, the strong links between contexts and methods may prove to be a severe hindrance.

Finally there is a problem in that textbooks offer paved roads. That is, contexts may bear such strong references to only one strategy that students are prevented from coming up with a solution of their own that is real for them and forms an individual basis for knowledge construction. This issue closely relates to the challenge to create a learning environment in which there is a proper balance between guidance (with knowledge of the goal) and students’ own inventions. In conclusion, we may say that the transition from informal contextualized solution methods to formal mathematics is problematic; students will not spontaneously generalize in the right direction.

### 6.6.2 Formal mathematics as a goal

Fostering a broad phenomenological exploration, in which students develop a variety of solution methods, will not be sufficient to ensure a process in which students construct formal mathematical knowledge. Teachers will have to play an active role in fostering and guiding the intended process of generalizing and formalizing. Students will have to be stimulated to compare and link solution methods, and teachers will have to orchestrate classroom discussions in which mathematical issues and mathematical relations become topics of discussion (Cobb, 1997). One could argue that the problems Dutch students have, are the result of a lack of generalisation in primary education and the lack of coherence between the primary school and the secondary school curriculum. In primary school textbooks, contexts and models which were developed in design research as a means of support for a long-term learning process have become goals in and of itself instead of starting points for developing more general strategies. Secondary education textbooks also do not support this process towards more formal mathematical knowledge. These circumstances certainly play a role, but we believe that it only magnifies a general problem. Even though we agree that a well-considered learning route would certainly diminish the problem. In fact we have shown that the theory of RME offers various design heuristics, such as ‘guided reinvention’, ‘didactical phenomenology’ and ‘emergent modeling’ (Gravemeijer, 1999) that aim to guide students towards more formal and general solution strategies. In addition an actual reinvention route for the multiplication of fractions based on the idea of “finding and appropriate sub-unit” can be found in (Streefland, 1991).

A limitation of the latter, however, is that helping students to reinvent the conventional method for multiplying fractions on the basis of a specific model offers only a partial solution. First, this conventional formal method may still have to compete with the number-specific solutions the students developed earlier in concrete instances. Second, the fraction
role of contexts

curriculum also has to prepare for algebra. This asks for a much higher level of abstraction
and flexibility than the conventional method offers. We may elucidate this second argu-
ment with the strategies for multiplication and division that are described in the work of

The proposed method for multiplication is to multiply numerators and denominators.
For division students are directed towards a method of first reducing the fractions to the
same denominator and then interpreting the division in the same way as the division of the
numerators (e.g. \( \frac{2}{3} \div \frac{3}{4} = \frac{8}{12} \div \frac{9}{12} = 8 \div 9 = \frac{8}{9} \)). Both methods offer good general
strategies for solving multiplication and division tasks, but they do not directly support rea-
soning about the relation between division and multiplication. For a deeper understanding
of rational numbers students will have to come to understand that every division is related
to a multiplication with its inverse. Gifted students could even come to understand that this
is a consequence of the extension of the natural numbers or integers to (positive) rational
numbers. Eventually, fractions should evolve into rational numbers which find their appli-
cability in a variety of situations. This also involves a process in which the product of two
rational numbers becomes an object that students can act and reason with. This would for
instance imply that students would no longer consider \( \frac{1}{2} \times \frac{3}{4} \) as the assignment to multiply
\( \frac{1}{2} \) with \( \frac{3}{4} \). Rather \( \frac{1}{2} \times \frac{3}{4} \) is again a number, namely the product of \( \frac{1}{2} \) and \( \frac{3}{4} \). Consequently
\( \frac{1}{2} \times \frac{3}{4} \) and \( \frac{3}{4} \times \frac{1}{2} \) are just different names for the same thing. These examples that focus
more on the mathematical structure of the topic and require a higher level of abstraction
and flexibility refer to concepts that are needed in algebra. We found, however, that these
aspects receive little attention.

On reflection, we may offer two recommendations in answer to the challenge of help-
ing students to construct formal mathematics. First, we have to be aware of the fact that
there are strong competing forces that lead students away from the path of generalizing and
formalizing that we as educators envision for them. This implies that teachers will have to
work hard to keep students on the right track. Teachers will have to invest systematically in
making mathematical issues and mathematical relations a topic of discussion. Second, we
have to acknowledge that we have to aim for a much higher level of mathematical think-
ing and reasoning than a conventional view of the fraction curriculum suggests. We may
take the Dutch curriculum as a point of reference again. Here we may observe that the pri-
mary school goals are set in terms of developing a framework of number relations in which
fractions become mathematical objects that derive their meaning from this framework of
number relations (Van Galen et al., 2008). This corresponds with Van Hiele (1986) level
of ‘mathematical objects as junctions in a framework of relations’. In secondary school the
common opinion seems to be that it is sufficient to teach the conventional rules for fraction
arithmetic. What is needed in upper secondary school however concerns the relations be-
tween multiplication and division, and the relation between those operations and fractions.
These are goals on the next Van Hiele level that deals with ‘relations between relations’.
Thus, although much can be improved in the generalization of algorithms, we argue that for
a good preparation for algebra a much higher level of abstraction is required.
“Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.”

(Jean Baptiste Joseph Fourier)
7 General conclusion and discussion

The general aim of the studies in this dissertation was to explore the nature and causes of problems with mathematical skills in the Dutch curriculum regarding fractions. More concretely, our aim was to investigate the development of the students’ proficiency from grade 4 through 9 and to analyze how textbooks support students in the transition from primary to secondary education and in reaching the formal understanding that is needed for the transition from arithmetic to algebra. In doing so we searched for possible footholds for improvement of instruction and contributions to theory building. We based our findings on two main areas of analysis, the students’ results to a test for proficiency and the textbooks. Thus, we looked at the outcomes of the fraction curriculum (attained curriculum) and the input of that curriculum especially the content of textbooks (formal curriculum) and its relations with the theoretical foundation of reform mathematics (ideal curriculum).

In this final chapter we give an overview of the main findings of each of the previous chapters, discuss limitations of the research as it was performed, and discuss our study in a broader perspective. We conclude this chapter by returning to the Dutch context and make recommendations for Dutch education.

7.1 Main findings

In this section we discuss the main findings of our studies that were reported in this dissertation, along two lines of analysis: development of proficiency and textbook analysis.

7.1.1 Development of proficiency

Test for proficiency in the domain of fractions

To better understand problems with basic mathematical proficiency, there is a need for detailed empirical data on the proficiency of the students. For this purpose we developed a test to assess student proficiency in the domain of fractions that holds the middle between large scale assessment and domain-specific research (Chapter 2). This test has been the basis of our analysis of student proficiency. The test was designed as a paper and pencil tests in an anchor design to allow for efficient assessment of a large number of students in grades 4 through 9. The theoretical framework of this test was based on literature on the learning of fractions and an analysis of the fraction domain (Chapter 2). We identified five big ideas that
describe the domain at the level of underlying concepts: relative comparison, equivalence, reification, from natural to rational numbers and relation division-multiplication. This description according to the big ideas had two functions. During the test construction it was used to select items such that the test covers the domain evenly. Later, in the analysis phase, the big ideas were guidelines for the analysis at the concept level. The appearance of the items is described in a list of so-called ‘complexity factors’. These complexity factors are the external characteristics of tasks that influence their difficulty as described in literature. The systematic construction allowed analysis of test results at two different planes. The first level of analysis involved an item per item analysis of the types of tasks the students can or cannot answer correctly. The second, more complex level of analysis involved the combination of tasks, aiming at insight on the students’ understanding of the identified concepts underlying fraction proficiency. These two levels of analysis relate to the complexity factors respectively the big ideas.

We used three criteria to evaluate the quality and usefulness of our test: 1. Is it possible to construct a single linear scale for fraction proficiency on which all our items can be ordered according to their difficulty? 2. Can the development of proficiency from grade 4 to 9 be described with the results of this test? and 3. Does this provide diagnostic data which can be used for improving instruction?

In Chapter 3 we have exemplified, how our test fulfills these requirements. We employed a Rasch model to create a scale for fraction proficiency. Analysis and evaluation of the test results of 1485 students, resulted in positive conclusions about the validity and the reliability of the test. The data fitted to this model and based on the polarity, infit and outfit, reliability values and uni-dimensionality we concluded that our test measures one latent trait. The Rasch scale based on our test served as scale for proficiency in fractions in the remainder of our studies.

In Chapter 3 and 4 we illustrated how test results can be used to diagnose curricular problems and offer footholds for improvement. We showed how the development of proficiency over the grades could be related to the mastery of certain types of tasks and be described in terms of item characteristics. We were also able to describe the development of proficiency in terms of the underlying concepts, using the big ideas as expressed in Chapter 2. Thus, the test allowed for analysis both at the level of individual tasks, and at the conceptual level. The practical value of the test was illustrated by the analysis of three of these big ideas (Chapter 3).

Student proficiency

We evaluated the proficiency of students in grade 6 regarding the addition of fractions and three of the big ideas, namely unit, equivalence and the development of rational number (Chapter 3). The analysis showed that the students were able to add and subtract fractions with a common denominator, but had not yet mastered the addition and subtraction with unlike denominators. Regarding the big idea of unit, most students were able to answer items on naming parts in and representing fractions with part-whole models. Most of the students did not master conceptual mapping in a context or in relation to the number line. Regarding the idea of equivalence, the students were able to reduce fractions to its lowest
term, but had difficulty in recognizing fractions as equivalent when there was no whole number factor between numerators. Finally, regarding the development of rational number, the students mastered improper fractions on the number line, but were not able to use improper fractions/mixed numbers in multiplication and division tasks. For the students, improper fractions appeared to have not become rational numbers that have the character of object-like entities that can be used as a number. It showed that the students were capable to solve tasks that ask for reproduction and procedural use of symbols and operations. However, tasks that differ from standard and ask for more conceptual understanding surpassed the ability of most of the students.

In Chapter 4 we reported the results of Dutch students in the first three years of secondary education. We expected to find that students would have deepened their understanding thanks to their experience in using fractions in a variety of tasks. However, it appeared that there was no significant progress in fraction proficiency in lower secondary education. This result mirrors the lack of explicit attention to fractions in textbooks.

We found problems in two different lines. On the item level we found that the grade 9 students did not have developed generalized strategies for the basic operations. Instead we found that the students developed number specific strategies, which they could not generalize over various situations. This finding corresponds with our findings on the textbooks (Chapter 5 and Chapter 6). The analysis at concept level indicated that the students lacked the notion of a fraction as a division and they found it difficult to interpret a fraction as both two numbers and one, an interpretation that relates to the so-called ratio-rate duality. We concluded that most of the students did not develop such an dual notion of fractions.

Since the students in grade 9 are still in the process towards mastering general ways of solving fraction tasks that involve addition, subtraction, multiplication and division, we concluded that the proficiency level of most students was insufficient for the transition to algebra. That is, for algebra, plain knowledge of these general rules for arithmetic as well as conceptual understanding of these operations and their relations with other concepts underlying the domain of fraction, are a pre-requisite.

Although our results were based on the students of one school and this one textbook series, we argued that the results are representative. Our analysis of textbooks shows that the two major textbook series (with a current estimated total market share of 90% (Ctwo, 2009)) do not differ much in respect to the explicit attention to fraction arithmetic and understanding (Chapter 5). Furthermore, the results were corroborated by data that we collected in tertiary education, where we tested 97 first year science students using a selection of our items (Bruin-Muurling and Van Stiphout, 2009).

7.1.2 Textbook analysis

We analyzed not only the students’ proficiency but also the formal curriculum. Our aim was to find causes of problems in the curriculum and provide footholds for the improvement of those curricula. In the analysis we considered textbooks as representative for the curriculum. Hence, we analyzed the texts and inscriptions on fractions in mathematics textbooks for grade 6 and 7. Additionally, we studied how these textbooks relate to prototypical sequences and tasks produced in design research, and to the principles of RME.
Following CHAT theory, we assumed that aspects of an educational system, such as its traditions and rules, are reflected in the textbooks. At the same time, we assumed that textbooks determine instructional practices to a large extent. In the Netherlands this connection between educational practice and the content of the textbooks may even be tighter than in other countries. Analysis of Dutch textbook use in mathematics education showed a strong connection between textbook and the practice of teaching, both in primary and secondary education (De Vos, 1998; Gravemeijer et al., 1993; Hiebert et al., 2003). In secondary education, students' first experiences with new concepts and procedures might come directly from the textbook. Students' are expected to take responsibility for their own learning and work individually on sets of problems closely. Research has shown that teachers are afraid to deviate from the pace and content of the textbooks, even if there is a need to "compact" the curriculum to accommodate the better performing students (Noteboom and Klep, 2004). This link between textbook and educational practice is dialectic, in that teachers not only rely heavily on the textbook regarding content and pedagogy, textbooks can also be regarded as the product of the culture among teachers. That is, mathematics textbooks in secondary education are written by teachers. In primary education most textbook authors are affiliated to the teacher training institutes or school advisory services. Given this strong relationship between instructional practices and textbooks, we assumed that actual incoherencies between practices in primary and secondary mathematics education can be observed in the textbooks.

**Transition to secondary education**

In Chapter 5, we focus on the transition from primary to secondary education. We hypothesize that part of the problems with fraction proficiency may be explained by the fact that the fraction curriculum stretches over a large number of years. In the Dutch curriculum, for example, it even stretches over two educational systems, namely primary and secondary education. There was reason to believe that some of the problems stem from differences in the educational practices in the two educational systems. That is, we hypothesized that the cultural and historically determined traditions of both primary and secondary education, would express themselves in the educational practices and the use of artifacts, leading to confusing differences in the epistemological messages expressed.

We found that differences between these educational systems resulted in a gap between primary and secondary education textbooks with regard to fraction multiplication. We argued that this gap is difficult for students to overcome. Given the different traditions in both educational systems, we showed that similar artifacts get a different meaning in the context of each of these traditions. In secondary education textbooks, students are expected to use different strategies for solving tasks which require a more formal understanding of fractions, than can be expected on the basis of the primary school textbooks. Our analysis also pointed to the fact that the differences between primary and secondary education imply that students develop a “operational conception” (to use Sfard (1991) terms) of fractions in primary school, whereas they are expected to reason with a “structural conception” of fractions in secondary school. In neither systems students are supported in making the transition to a structural conception. Moreover, primary school textbooks aim at training
number specific solutions which are likely to become a barrier for the generalization and formalization that is needed to come to understand fractions at a more formal level. We indicated that these differences stay hidden because textbooks in primary and secondary school use similar artifacts, with different meanings. We also argued that these differences are usually unnoticeable for participants in each of the systems, because they bring with them different frameworks of reference.

The use of context in primary education

In Chapter 6 we addressed the content of the textbooks in primary school and its relation to prototypical work in design research on fractions and to the basic principles of RME. We found, that the way this prototypical work was incorporated in the textbooks led to a number of very distinct procedures for multiplying fractions. That is, multiplication of fractions became a set of number specific procedures. We distinguished between four compartmentalized interpretations of the multiplication sign. Tasks on a “whole number times proper fractions” are related to repeated addition. Tasks on “proper fraction times whole number = whole number” are to be interpreted as part of. Tasks on “proper fraction times proper fraction” are connected with multiplication as an area and finally “multiplication with mixed numbers” is related to splitting. We found that this compartmentalization was still visible in the empirical data on the performance of students in grade 6 and 7. The outcomes showed that the success rate and type of errors were related to the order of operands and the abstraction of the problem formulation. We conclude that this compartmentalization may hinder the process towards more formal understanding of multiplication of fractions. Thus, the versatile approach of mathematical concepts is not yet elaborated such that it allows students to come to grips with the conventional mathematical method for multiplying fractions. In reflection, we showed that the use of contexts has two sides. On the one side it affords/supports students’ building upon informal knowledge. On the other side however, contexts can hinder mathematization when their inherent characteristics become constraints in the generalization process. We will elaborate on this in Section 7.4.

7.1.3 Overall findings

We found that problems in the fraction domain start already early in education and continue in later grades. Although the proficiency of the students developed considerably from grade 4, the students’ proficiency level at the end of grade 6 showed lack of deeper understanding of fractions. Corresponding to the textbooks, the teaching of fractions and fraction arithmetic in primary education, is directed towards procedural problem solving of standard tasks. The students did not develop deeper understanding of underlying concepts like unit, rational number and equivalence. Moreover, strategies that students did learn appeared to be number specific. The consequence of this unconnected knowledge might be that their proficiency level is unstable. In the transition from primary to secondary education, problems arise because of an apparent misunderstanding about the intended level of proficiency at the end of primary school. This may have its origin in a lack of awareness in secondary education of how the primary school curriculum has changed in recent years. The current mandated attainment targets for primary school (Greven and Letschert, 2006) do not include
operations with fractions, other than in contextual problems\(^1\). Both the results on our test (Chapter 4) and an analysis of primary and secondary school textbooks (Chapter 5) suggest that secondary school teachers, and textbook authors, do not realize that the fraction curriculum is not completed in primary school. The resulting lack of attention to fractions as a topic in and of itself, may be one of the main reasons for the lack of progress of the students proficiency in the first three years of secondary education. In addition, secondary school teachers and textbook authors do not seem to realize that once the students have developed general strategies for fraction arithmetic, a second step in formalization and generalization is required, in order to create a basis for algebra.

7.2 Limitations

This research has some limitations following from the choices we have made in our focus and delineation of the domain. In this section we will discuss these limitations.

7.2.1 Preparing for higher education

In secondary education we focussed on students preparing for higher education, i.e. pre-higher professional education (HAVO) and pre-university education (VWO). For this group of students a more formal level of understanding fractions is important. These streams together represent 40\% of the students.

The remainder of the Dutch students attends the third stream: VMBO. For this heterogeneous group of students different goals for the mathematics curriculum are formulated. Such goals are more related to the use of mathematics in daily and professional life. In addition, in these streams educators are confronted with students who finished the mathematics course in primary education only at the level of grade 4 or 5. The textbooks for two of the more theoretically oriented substreams in VMBO (TL and GL), are similar to those of the HAVO, and in this sense some of our conclusions, especially regarding the coherence in textbooks from primary and secondary education, also apply for these substreams.

In primary education, we did not make this distinction between streams, neither in the analysis of the textbooks nor in the assessment of students. Our sample contained students from all streams although there was a slight over representation of higher achieving students.

Thus additional research is needed for VMBO, however with a focus on the application of fractions in daily life and taking future vocation into account.

7.2.2 The Dutch curriculum

Our study centered on the Dutch curriculum. We tested the proficiency of Dutch students and analyzed the content and use of Dutch textbooks. This implies that direct conclusions

\(^1\)“De leerlingen leren structuur en samenhang van aantallen, gehele getallen, kommagetallen, breuken, procenten en verhoudingen op hoofdlijnen te doorzien en er in praktische situaties mee te rekenen”; translated: Students learn the main features of the structure and coherence of amounts, whole numbers, decimal numbers, fractions, percentages and ratio and are able to calculate them in practical situations
can only be made for the Netherlands. We realize that the order of items on the Rasch scale is influenced by two main factors, the curriculum and the natural ordering of concepts that lies within the topic itself. Testing only Dutch students implies that the ordering on our Rasch scale is to some extent the outcome of the national curriculum.

Nevertheless, we believe that the Dutch curriculum can serve as a paradigmatic case for other countries. The Dutch discours on basic mathematical skills mirrors the discours in many western countries. Our results point to specific problems and give direction for improvement in the Netherlands. They can serve as a starting point for similar analysis in other countries.

In the Netherlands, Realistic Mathematics Education (RME) (e.g. Gravemeijer, 1999) has influenced all primary school textbooks to a great extent, and it also has had its influence on secondary school textbooks. In this sense, the Dutch curriculum is in the international vanguard of reform mathematics. In finding the source of some problems in the Dutch school system we may inform others.

Finally, our study can also be seen a paradigmatic case for issues that come up in the transition from one educational system to another. Issues concerning such a transition are not unique to the Netherlands (e.g. in Germany, Haggarty and Pepin, 2002) nor for primary and secondary education (e.g., Raman, 2004). It is generally known that there are discrepancies in the way in which content is presented in consecutive educational systems. In the research presented here we shed new light on the causes of these inconsistencies and its generally unnoticeable consequences. We found that these problems originate in the different orientations in primary and secondary education of a more general pedagogical approach versus respectively a more content-oriented approach. Deeper insight in the mechanisms that lie behind one specific case, here fractions in the Netherlands, may also add to the general understanding of such transition issues.

7.2.3 Limited number of participating schools

Due to organizational reasons, we were only able to assess students at a limited number of schools, i.e. seven schools for primary education and one school for secondary education. This urges for some precaution with our results. Of main concern are the variation between textbook series and differences between teachers.

The participating primary schools did use different textbooks. We could validate the data with the results from the start of the first year of secondary education. Given the regional character of this secondary school together with its size, at least 26 different primary schools are represented in the data from secondary education. Comparison of student ability per school did not reveal significant differences between schools. Instead, we found great differences between students within the same primary school (Bruin-Muurling et al., 2009).

We do of course have to be careful with generalizing the results for secondary education since all students in our sample attended the same secondary school. However, exploratory research under first year university science students corroborated our findings (Bruin-Muurling and Van Stiphout, 2009). Furthermore, if we look at the textbooks, it shows that there is a general lack of explicit attention for fractions. This is not just the case
conclusion

for the textbook series used in this school (Moderne Wiskunde) but also for the other textbook series (Getal en Ruimte). Together these textbook series represent the vast majority of the market. In addition it may be noted that Dutch teachers tend to follow their textbooks very closely (Hiebert et al., 2003). Given the size of the school, multiple classes (up to nine parallel classes per grade) and thus multiple teachers participated in our study.

7.2.4 Textbooks as representatives of the curriculum

In our studies we considered textbooks as representative for the curriculum, since there is a close relation between textbooks and educational practice in the Netherlands. Analysis of Dutch textbook use in mathematics education showed a strong connection between textbook and the practice of teaching (Gravemeijer et al., 1993; De Vos, 1998; Hiebert et al., 2003). In addition, teachers not only rely heavily on the textbook regarding content and pedagogy, textbooks can also be regarded as the product of the culture in each specific activity system. That is, mathematics textbooks in secondary education are written by teachers. In primary education most textbook authors are affiliated to the teacher training institutes or school advisory services. In other words, the tasks, content, pace, and strategies for solving problems are part of a tradition in which they are produced and used. Given the strong relationship between instructional practices and textbooks, we may assume that textbook analysis may reveal actual incoherencies between practices in primary and secondary mathematics education. We found support for this assumption in the results of the students that fitted with the results we found in the textbook analysis.

Although we have strong pointers that justify our approach, we have no precise knowledge of how teachers act. The role of the teacher in supporting students in constructing mathematical knowledge in relation to the Dutch textbooks might be a topic for further research, as is the influence of the teacher’s own understanding of the topic and his own beliefs on effective teaching of the topic. Such new research could for example be related to ‘mathematical knowledge for teaching’ (Ball et al., 2008).

7.3 Future research

Given the main findings and the limitations of the studies that are described in this dissertation, we see several directions for future research.

Extension of data collection

In the limitations we addressed the data collection in one school for secondary education and the specificity of our scale on proficiency in fractions for the Dutch curriculum. Therefore, we propose for two extensions of the data collection in future research. The first one is to assess students from a large number of schools. Secondly, since we expect that the order of the items on the scale will be influenced by the curriculum, we assume that international comparison of test results will make it possible to distinguish between problems that are specific to a national curriculum and issues that are inherent to the topic of fractions. Comparing the Rasch scales for different countries would also make it possible to distinguish
between instruction that does or does not have a positive effect on the development of proficiency. This could give us more insight in the influence of curriculum components and can point to best practices.

**Classroom observations**

In the previous we comprehensively discussed our arguments for taking the textbooks as reference points of the curriculum. However, although the students’ results correspond with the outcomes of our textbook analysis, we have no knowledge of what the teacher contributed exactly. Therefore, in future research, this aspect of the current role of teachers can be addressed. Such research could involve an analysis of the extent to which teachers follow the compartmentalized manner of fraction multiplication in the textbooks. This could be related to the teachers’ experience, knowledge of the fraction domain, and beliefs on mathematics pedagogy. However, research could also be directed to how the teacher can contribute to students’ deeper understanding.

**VMBO**

We focussed in this dissertation on students preparing for higher education, and thus on HAVO and VWO. This led to a focus on a more formal level of understanding fractions. Further research is needed for the other stream (VMBO), where the focus is more on the use of mathematics in daily and professional life. Some of our results hold also for this group of students, especially regarding compartmentalization in primary education and the coherence between primary and secondary education. Regarding the developments in lower secondary education, in VMBO other considerations are made than the formal focus of pre-higher education.

**Follow-up of the current research**

We chose the domain of fractions for developing a test that would help us in analyzing how student proficiency develops over time. Our studies showed that the test of fraction proficiency proved useful in diagnosing development in proficiency. For future research the same procedure could be followed to develop tests for other topics.

In our studies we found some interesting leads for future research, usually in the direction of more qualitative research. A starting point could be an analysis of the written answers. This could provide us with insight in the strategies that students followed and typical errors they made. To identify a more detailed developmental pathway, in which common errors are recognized a so-called partial credit Rasch model could be used (Masters, 1982).

A related line of research might aim at improving the test. Transitions that proved more difficult and crucial then we anticipated beforehand, for instance, might require a higher task density. In addition, some items were located at a different location on the scale than anticipated. Further research could investigate for the causes of these issues.

Finally our analysis illustrated the difficulties in reaching a conceptual higher level of understanding. Future research could aim at new insights on how this could be reached.
7.4 Reflection on our findings

In this section we will elaborate on our findings and reflect on them in a broader perspective. We argue that the results presented in this dissertation indicate that some aspects of reform mathematics are more difficult than anticipated. The transition from primary education to secondary education can be characterized as a transition from contextual problem solving to the use of more formal mathematics and general rules. Our results show that formalization and generalization received relatively less attention than the initial stages of concept forming. The process, referred to as vertical mathematization does not spontaneously emerge from horizontal mathematization but needs separate attention. In Chapter 6 we illustrated that the balance between the two sides of the use of contexts – its affordances and its constraints – need special attention. We will elaborate on these findings in two areas: the educational setting and the nature of mathematics.

7.4.1 Educational setting

In Chapter 6 we concluded that the versatile approach of mathematical concepts is not yet elaborated in such a manner that it allows students to come to grips with the conventional mathematical method for multiplying fractions, let alone that students develop a deeper understanding of the structure and the interrelations of the fraction domain. In Chapter 3 we have shown that this conclusion was in line with the overall proficiency level of the students in our study. In this section, we argue that there are some adverse factors in the educational setting that impair the initial intentions and ideas of reform mathematics. We discuss here the influence of proceduralization, local analysis, contexts, and the role of teachers.

Proceduralization

“Rather than being taught the rules, they should have been taught to argue their intuitions, to reflect on what appears to be obvious. ..” (Freudenthal, 1991, p. 113)

This is one of the basic ideas of reform mathematics. We can regard reform mathematics as a reaction to a drill and practice method for learning nothing more but rules. In general it means an orientation towards understanding. In the Netherlands this focus is reflected in the formal attainment targets in primary education (Greven and Letschert, 2006). It is stated that ‘Students learn to solve practical and more formal mathematical problems and to articulate their reasoning clearly’<sup>2</sup> and ‘Students learn to substantiate their strategies in solving mathematical problems and to judge solutions’<sup>3</sup>. Thus, in the attainment targets there is a strong emphasis on mathematics as an activity and on mathematical discourse. In the same attainment targets it is articulated what students should learn about numbers and

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<sup>2</sup>translation of “De leerlingen leren praktische en formele rekenwiskundige problemen op te lossen en redeneringen helder weer te geven”

<sup>3</sup>translation of “De leerlingen leren aanpakken bij het oplossen van reken-wiskunde problemen te onderbouwen en leren oplossingen te beoordelen.”
Students learn the main features of the structure and coherence of amounts, whole numbers, decimal numbers, fractions, percentages and ratio and are able to calculate them in practical situations. The words “practical situations” may be seen as a characterization of the direction that has been taken: away from original intentions of reform mathematics. In Chapter 6 we illustrated that textbooks stay at the level of informal strategies, meeting the standards in “arithmetic in daily life”. General rules or procedures appear to be avoided and understanding appears to be interpreted as linking contextual meaning to formal mathematical notation.

Taking the quote of Freudenthal as a guideline, we see two problems in this approach. The first is that by removing general procedures or rules from a curriculum, proceduralization is not automatically abandoned. In the educational setting there is a tendency to teach procedures to produce the right answers. There is an orientation on the right answer rather than on conceptual aspects that lie behind it (e.g. Nijland, 2007). This is not surprising. The outcome of schooling and its effectiveness is primarily measured by success rate on tests consisting of standard tasks. Students are judged on the correct answer or on correctly following the standard procedure.

Freudenthal continued the statement that we quoted at the beginning of this section with: “...But this requires more patience than teachers can afford.” Indeed to prepare students for the final nationwide test at the end of primary education, and to the, in most schools obliged, standardized tests that are held twice a years in all grades of primary education, teachers easily fall back to teaching students recipes to solve certain types of tasks. This is an effective and successful way to prepare students for such tests. The tests do not assess the other aspects of the attainment targets such as those that refer to mathematical discourse and the structure and coherence. Moreover, this routine of portraying mathematics as a set of “recipes”, can be found in books for primary education (Chapter 6), books for teacher training (e.g. De Moor et al., 2009) and in the development of new prototypical instructional material for primary and secondary education (e.g. Dekker and Wijers, 2009). The combination of preparing students for standard tasks and the disappearance of general traditional rules for arithmetic, number specific strategies may be proceduralized instead. Even more so, a strong focus on correct answers or standard procedures may stand in the way of arguing on intuitions and reflecting on findings.

The second issue is that Freudenthal criticizes the way students learn rules rather than learning rules per se. In his work, Freudenthal (e.g. 1973) stresses on the importance of abstraction and generalization. It is as if the baby was thrown out with the bath-water. With the break with the tradition of the method of drill and practice to learn arithmetic rules, these rules were no longer a goal. What is eventually needed in upper secondary school for a good preparation for algebra concerns the relations between mathematical objects, i.e. a higher Van Hiele level that deals with ‘relations between relations’ (Van Hiele, 1986). To achieve such a level at least the knowledge of these “relations” is required. Thus, this requires both knowledge of the general rules of arithmetic and knowing why these rules apply. Thus, there is a risk in the current call for more practice and the focus on general rules of arithmetic. Practice of compartmentalized mathematics may make generalization...
even more difficult. A focus mainly on general rules of arithmetic may leave less room for achieving higher levels of understanding.

Local analysis

Research on fractions is virtually infinite. This research has made a considerable contribution to the improvement of instructional design. This research was usually carried out around certain topics or problems (Pitkethly and Hunting, 1996). This was needed to reduce the complexity of the studies. Such studies have been successful in finding didactical/pedagogical approaches to help students overcome certain difficulties. A problem arises when the goals that are to be reached by such local pedagogical solutions are not in line with the goals for the entire curriculum. Since mathematical topics are so much interrelated, small differences in the goals for one subject may have a large influence on the whole. Our study indicates that next to the local analysis of topics, an analysis on the global impact is needed.

One example is the explicit meaning that is attributed to the two operands in a multiplication, that is, as multiplicand and as multiplier. In the short term, or local focus, this distinction between multiplicand and multiplier may be an appropriate way to build upon students' intuitive knowing of multiplication. In the longer run however, such a rigorous distinction between operands, may hinder a deeper understanding of multiplication, if these initial aids are not abandoned in time. That is, an asymmetry is created that contradicts with the mathematical properties of multiplication. The same goes for other topics such as a too strict distinction between "partitive division" and "measurement division".

Thus problems can arise when phenomenological distinctions are made that are not transcended to arrive at the mathematical concept at hand. The same goes for approaches that in some way take a d-tour around the difficulties at hand. Examples are the several methods that have been used to avoid division with fractions. One such method is to first reduce to common denominators in divisions tasks (e.g. \( \frac{2}{3} \div \frac{3}{4} = \frac{8}{12} + \frac{9}{12} \)). Disadvantage of such solutions can be that it may prevent one from thinking about the underlying structure, such as the relation between multiplication and division. It can however also be used to start a reflection on division if the teacher consciously makes it a topic of discussion.

Contexts

“Guiding means striking a delicate balance between the force of teaching and the freedom of learning.” (Freudenthal, 1978)

In Chapter 6 we have discussed the role of contexts. We argued that the use of contexts is two-sided, it creates both affordances and constraints. On the one hand it affords/supports students’ building upon informal knowledge. On the other hand however, contexts can hinder mathematization when inherent characteristics become constraints in the generalization process.

To make optimal use of the affordances and to limit the constraints, contexts must be selected very carefully. For a teacher to have a chance on success in orchestrating a classroom discussion that allows for guided reinvention, there have to be differences in the solutions of
various students and in the level of these solutions (Gravemeijer, 1994). That is, productive classroom discussion can only take place if there is anything to discuss about c.q. students have come up with different solutions. Several levels of solutions are needed to provide a good reflection of the learning path. If the solutions reflect different levels of understanding the underlying mathematical issues can become a topic of discussion (Cobb, 1997).

The contexts in the primary school textbooks that we have analyzed do not meet these criteria. Instead, the problems offer paved roads and the contexts are too strongly related to just one type of solution. There is hardly room for diversity in the approaches and neither in the level of these solutions. Initially, this self-evident character of the solution methods makes such exemplary context problems very powerful. Especially since this is the level that is aimed for and tested for in primary education. Later, when the students are to generalize over those solution methods, the strong links between contexts and methods may prove to be a severe hindrance. Isolating specific solution methods adds to this problem.

Apart from these basic criteria for contexts there is another aspect of contexts that may counteract with the aim of its use. This can be explained by distinguishing between the actor’s point of view of students and the observer’s point of view of the mathematics educator. The mathematics educator is guided by the mathematical insights that students do not have. Students on the other hand can only reason from their own knowledge and experience. In looking for contexts and models that are experientially real to the students and that are justifiable in terms of the potential endpoints of a learning sequence (e.g. Gravemeijer, 1999; Pirie & Kieren, 1994), researchers/designers will be thinking of the relation between one solution procedure and the potential endpoint. Students, however, will primarily see the relation between (characteristics of) the problems and the corresponding solution procedures. It is known that students may focus on superficial features of tasks when trying to solve a problem. This may involve for example finding “keywords” for certain operations or the structure of the task (e.g. Prediger, 2009; Van Dooren et al., 2009), but may also consist of connecting accidental but mathematically irrelevant commonalities in examples. In any case, students do not know the intended endpoint. As a result, students may attribute inherent constraints of the context in which a concept is presented to his understanding of the concept itself, which in turn may hinder the generalization process. Actor-observer conflicts may also appear in the use of language and mental models. A clear example is the connection between “taking a part of” and multiplication. For students this may not be as natural as it is for experts (e.g. Streefland, 1991). In Chapter 6 for example we found that students had not connected “taking half of” an amount with \( \frac{1}{2} \times \) that amount. To conclude, we can not just assume that students will spontaneously generalize in the right direction, which brings us to the role of the teacher.

Role of teachers

“Teachers (or peers in learning groups) need to insist on justification of what appears to be new knowledge or new procedures, thereby requiring the inventor to reflect on what he - consciously or unconsciously - performed.”

(Freudenthal, 1991)
As we discussed, in reform mathematics, contexts are to support the teacher in successfully orchestrating a classroom discussion that allows for guided reinvention. A productive classroom discussion requires a variety of solutions. The role of the teacher is to choose appropriate problems, and to make sure that differences between the solutions of students give rise to a discussion on the underlying mathematics. However, although developing suitable contexts or problems is essential for success, it is not sufficient. It is also needed to create a classroom climate that supports students in bringing about new ways of solving problems. This asks for a change in the “didactical contract” (Brousseau, 1990) between students and teacher. That is, in order for realistic mathematics to work the traditional expectations on the role of teacher and students need to be changed. Yackel and Cobb (1996) call this changing the “social norms”. In the light of our earlier discussion on proceduralization (Section 7.4.1), it asks for a change of perspective. That is, the mathematics behind the tasks at hand should be subject of discussion rather than the correct answer to the problem. In addition, “socio-math norms” need to be developed that concern mathematics and beliefs on what mathematics is (e.g. Gravemeijer, 1995; Yackel and Cobb, 1996). Such socio-math norms may concern mathematical valuable or less valuable strategies to solve problems. In prototypical work, the different levels of solutions that students come up with have been used to make the learning path visible. This may carry with it the risk of an orientation that is too much on algorithmization, in the sense of shortening those strategies and make them more efficient. However, the goal of mathematics education is more than only learning algorithms and strategies for solving problems. For deeper understanding, also questions like “Why do these strategies work?”, “Why do these strategies boil down to the same”, “Does it always work?” or “Why do these different views still describe the same concept or structure”, deserve attention. This implies a discussion on different ways of solving problems, rather than a discussion only directed towards the most efficient or otherwise more appreciated strategy (Cobb, 1987).

This asks a lot of the teacher. We argue that such an approach is almost impossible if the teacher leans too much on the textbook. The classroom discourse that is to allow for guided reinvention and to reach a higher level of understanding, asks for an active guiding role of the teacher. It also asks for a new orientation on the goals for mathematics education that is less procedural. To accomplish these goals, the teacher must have a deep understanding of the mathematics involved. The question is if this level of understanding is reached in current teacher training. It appears that the understanding that institutes for teacher training aim at, is related to “gecijferdheid” (numeracy) (e.g. Van Zanten, 2006; Van Zanten et al., 2010) and not to deeper understanding of the mathematics behind the arithmetic.

Another, yet more prominent problem is that the compartmentalization that we found in primary education textbooks, is likely to represent also the traditions in teacher training, since the textbook authors are mainly affiliated with institutes for teacher training. Indeed, we found compartmentalization in educational resources for teacher training (e.g. De Moor et al., 2009).
7.4.2 Nature of mathematics

In our study we have found many examples, where school mathematics and mathematics as a discipline disperse. In this section we argue that this gap is too large in some respects. We plead for more focus on three basic elements of mathematics: structure, coherence and ambiguity.

Structure

"Les mathématiciens n'étudient pas des objets, mais des relations entre les objets; il leur est donc indifférent de remplacer ces objets par d'autres, pourvu que les relations ne changent pas. La matière ne leur importe pas, la forme seule les intéresse.", (Poincaré, 1902)

As Poincaré formulated in the above, mathematics is often called a science of structures. The importance of structure was beautifully exemplified by Sfard (1991).

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**Definition:** *Promenade* is the set $P$ of all natural numbers from 1 to 25 together with the following four functions:

- $S(x) = x + 5$, allowed only for $x \in P$, $x \leq 20$
- $N(x) = x - 5$, allowed only for $x \in P$, $x > 5$
- $E(x) = x + 1$, allowed only for $x \in P$, $x$ mod 5 = 0
- $W(x) = x - 1$, allowed only for $x \in P$, $x$ mod 5 = 1

Any composition of the above functions is called a stroll. We say that stroll $s$ leads from $a$ to $b$ if $s(a) = b$.

**Example:** the stroll $S \circ W^3 \circ S^3$ leads from 5 to 17:

$$(S \circ W^3 \circ S^3)(5) = (W^3 \circ S^3)(10) = (S^3)(9) = \ldots S^3(7) = S(12) = 17$$

**Tasks:**

1. Give an example of a stroll which would lead from 11 to 3.
2. Find all the numbers which can be reached by strolls from 9 without using the steps $N$ and $W$.
3. Without looking into the answer you gave to the question 1 above, give an example of a stroll from 11 to 3 once again.

---

Figure 7.1: First experiments of Sfard (1991) - unstructured.
She shows that questions that are easily answered if the structure is known (Figure 7.2) are much harder if the same structure is presented as a set of four seemingly unrelated rules (Figure 7.1). This example shows also that it is almost impossible to reason about more sophisticated and generalizing tasks starting from those four rules. Just the recognition of the structure of a mathematical concept provides for deeper insight. Although recognition of structure is a form of abstraction, it must not be confused with the use of abstract mathematical language.

Instead of a focus on finding structures, we saw a tendency to discern recognizable situation that correspond with a case specific procedure that can be automated. In this manner, mathematics is reduced to the recognition of (an almost insurmountable number of) situations and the execution of the accompanying strategy to solve that particular situation.

**Coherence**

Mathematics is by nature highly interrelated. There is coherence in two directions. That is, since mathematics has a strong hierarchical structure there is coherence in the curriculum between topics that follow one another. There is also coherence between adjacent strands of mathematics. In RME for example, the strong relations between ratio, proportion, percentages and fractions are stressed (Van Galen et al., 2008). Coherences may however also lie at a higher plane and relate to more general structures. Big ideas such as efficiency and optimization, relative versus absolute, critical path, translation and transformation, inverse, and principles of proof are examples of such large structure that connect many topics in mathematics.

We believe that students should at least experience the tight connections between the topics that they study in mathematics education. This is in contrast with the current practice in (secondary) education, where each chapter seems to introduce an entirely new isolated mathematical topic. We reason that attention for the structure and coherence of mathematics, would make mathematics more logic to students and may offer an independent basis for understanding.
In curriculum design we also plead for more attention for coherence. The first step would be to improve the transition between primary and secondary education. However, in addition we suggest for a broader focus than just on prior knowledge. That is, we believe that attention to how a certain topic is prior knowledge for another topic later in the curriculum may improve mathematics education. This concerns also the preparation for reaching a level of "relations between relations" (Van Hiele, 1986).

Ambiguity

Although the main concern about the results of the grade 9 students may be the mastery of general ways of solving tasks, we want to draw attention to another characteristic of mathematics. We claim that ambiguity, or the notion that in mathematics, ideas, concepts and symbols can be at the same time differently interpreted from different perspectives (e.g. Byers, 2007), is essential. The difficulty of the application of fraction knowledge, for instance, can be explained by the ambiguity of fractions and that a similar argument can be made for mathematics in general. The strength and beauty of mathematics lies in transcending ‘the conflict’ between these perspectives and to flexibly go back and forth between these interpretations (Meester, 2009). Or in the words of Byers (2007): “... I maintain that what characterizes important ideas is precisely that they can be understood in multiple ways; this is the way to measure the richness of the idea”.

Even though this might represent an advanced level of understanding, which might not be reached by all students, it is important for those students who proceed towards algebra. In order to address ambiguity at a later stage in learning, we argue that some preparation is already required in the initial stages of learning. There are two aspects of ambiguity that we want to address here: the many-sidedness of mathematical concepts and how these many sides are inextricably bound up with each other.

Many-sidedness

The first aspect of ambiguity is that in general mathematical ideas can be considered from many sides. Given that mathematics is a science of structures and that the same structure can be found in many different contexts, all these contextual meanings of the idea are part of that idea, even though they don’t apply for all situations. The many-sidedness of mathematical constructs could be explained by the convergent processes in the field of mathematics (e.g. Timothy Gowers [ref opnieuw zoeken]). The link between previously unrelated mathematical fields has often meant an enormous breakthrough, like for example the connection between algebra and geometry. A simple example is division. As we discussed in Chapter 2, often partitive and measurement division are distinguished. To develop full understanding of the concept of division, at least these two sides are to be addressed. A one-sided approach to division would not lead to this full understanding and would limit the usefulness of the concept of division. The fact that these sides, and the endless others you could think of, are in fact the same (structure), makes this among other things such a powerful concept. The strength of mathematics lies among others in its generality, but for students this is not evident. Rather, this is something that they must acquire.
The way the students solved the tasks showed that the notion of a fraction as a division was hardly developed (Chapter 4). Although one could argue that full ambiguity is out of reach for these students, we stress that the mere fact that these items—in which fractions must be considered as a division—have such high difficulties is an indication of an imbalance in the curriculum. We propose that the education in fractions can be improved if also this aspect of fractions is addressed in an earlier stage.

**Inseparability**

Ambiguity concerns more than just many-sidedness. Becoming familiar with the many sides of a mathematical ideas is just the first step in using and understanding the ambiguity of that idea. In the learning process of fractions, the understanding of fractions is to be enriched and deepened by familiarizing students with the different aspects of fractions. Examples of such aspects are the subconstructs (e.g. Streefland, 1991). Kieren (e.g. 1976) already argued that a complete understanding of rational numbers requires an understanding of how the subconstructs interrelate, rather than only understanding each of them separately. The subconstructs can be considered distinguishable, yet inseparable aspects of a mathematical construct, that constitutes its ambiguity. Key in understanding fractions is to realize how these aspects interrelate and that these are by principle inseparable. The wide range of aspects of ambiguity involved in the fraction domain can be considered as a source of difficulty.

Another general example of ambiguity is what Sfard (1991) calls the ‘dual nature of mathematics’ and Gray and Tall (1994) name ‘procept’. Mathematical signs can both be interpreted as a process and an object. Both interpretations (process and object) are considered to be necessary for a deep understanding of mathematics. A well known example is the ‘–’ sign. It can both be interpreted as a ‘command’ to calculate the result (compare with the ‘=’ sign on a calculator) and as a symbol for equality. In the context of fractions, ratio-rate duality can be regarded as an example of this dual nature. The example of ratio-rate illustrates that ambiguity is more than a change of perspective and using one interpretation in the one situation and another in a different situation. In such contexts the fraction represents both the relation between two quantities and at the same time an entity in and of its own (Chapter 4). Examples are speed, density and sine. These examples illustrate, that it is not a matter of choice of just one appropriate perspective. The essence of ambiguity is that these perspectives are there at the same moment. The sine for example is at the same time a value that represents a property of a specific angle, the y-coordinate of a point on the unit circle and the ratio between the opposite and adjacent side in any right triangle with that angle. This is what makes ambiguity so difficult to discuss. The moment you try to describe one of the sides, ambiguity is lost. Furthermore, ambiguity implies flexibility in interpreting mathematical symbols.

To conclude, we argue that even though a full understanding of the ambiguity of mathematical ideas may be a level that will not be reached by all students, this aspect needs to be addressed in the initial learning of that idea. The first pre-requisite is that the many sides of the idea have to be addressed. Thus a balanced introduction of the idea is needed that touches upon its most important aspects, what Treffers (1987) calls a phenomenological exploration. In a later stage, deeper understanding is to be reached, that is, these many sides
are to be linked to each other. We argue that already in the introduction of the idea, obstacles for reaching this deeper understanding and this ambiguous notion, are to be tackled.

7.5 Recommendations

In the prior we discussed our concern about the gap between mathematics as a discipline and mathematics education. In general we plead for more attention for the nature of mathematics in three themes of structure, coherence and ambiguity. We’ve also expressed our concern about some adverse factors that impair the initial ideas of reform mathematics. Those were the orientation on answers rather than on the mathematics at hand (proceduralization), the effect of local analysis, the two-sided role of contexts, and the role of teachers.

These are general recommendations for reform mathematics based curricula. In this section we return to the Dutch context and discuss some additional recommendations. Our research has shown that the proficiency level of students leaves much to be desired. Students are not well prepared for the mathematics courses in upper secondary education. We have also shown that this can be attributed to many causes, making the proficiency problem in the Netherlands complex. Consequently, many areas of the curriculum can be improved. We discuss here the textbooks, transition to secondary education, and transition to higher education.

7.5.1 Textbooks

Our study revealed the incoherence between the textbooks of primary education and secondary education as well as compartmentalization of fraction multiplication. Consequently, we recommend to reconsider the content of textbooks and align the primary and secondary education curriculum. Our study shows that in secondary education, expectations of the exit level of primary education do not meet the actual level of students in grade 6. The textbook analysis showed that formal fraction arithmetic is merely “freshened up” in grade 7. In later years, fractions are used in various situations, but are not given explicit attention. Thus, in secondary education proficiency in fractions is not further developed. It is even so that neither in primary education, nor in secondary education division with fractions is explicitly addressed in the textbooks. Moreover, the pedagogical ideas and the views on mathematics are not in keeping with the ideas and views in primary education. Thus, in the alignment of the curricula there is a need to consider different levels. To bridge the differences between the educational systems for example, explicit attention is to be given to the meaning of the shared artifacts in both systems.

Furthermore, we argue that in the introduction of fractions, “fraction as a division” needs to be addressed to meet the pre-requisite for later development of an ambiguous notion of fractions. Moreover, we argue that generalization and formalization of informal strategies is needed to reach general rules for arithmetic. However, we suggest to go further than only knowledge of these rules. We argue that deeper understanding, that can be characterized as the higher Van Hiele (1986) level of “relations between relations”, is needed to give students a solid basis for algebra. Additionally an ambiguous notion of fractions has to be developed.
Secondly we suggest for an analysis of the textbooks with respect to compartmentalization. Our study revealed the compartmentalization of fraction multiplication. However we have reason to believe that this compartmentalization is not restricted to the fraction domain. Also in secondary education, we’ve seen examples of compartmentalization.

7.5.2 Transition primary to secondary education

In line with the previous, we suggest for a different approach to solving transition issues between primary education and secondary education. In our study we found that the difference in meanings that are attributed to the same artifacts was an aspect that made the gap between these two educational systems too large for students. We have also shown that this gap can be characterized as the transition from a procedural use of informal strategies to a more formal understanding of the underlying mathematics. Most initiatives that have been developed to remedy the gap, overlook this aspect. We argue that there is a risk that only symptoms of the problems are controlled, while the true problem is not addressed. In these initiatives, there is still a focus on the mastery of rules for arithmetic rather than attention for deeper understanding. We argued in the previous sections, that this deeper understanding is needed in a later stage, as a solid basis for algebra.

7.5.3 Transition to higher education

Concerning the transition to higher education we suggest to take the level of understanding that students already developed more into consideration. We argue for structuring and generalizing their prior experiences rather than starting all over again following a different pedagogical approach. We argue that there is a special role for the institutes for teacher training. They experience the same transition problems as other types of higher education, in that they have to start with courses to bring students to a higher skill level. However, they are also in part responsible for the proficiency level in both primary and secondary education. This asks for another approach in teacher training regarding the basic mathematical skills. We propose more attention to the deeper understanding of basic mathematical proficiency and a higher Van Hiele (1986) level.
Bibliography


American chapter of the international group for the psychology of mathematics education, pages 421–426, Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.


### List of test items

#### A.1 Part-whole (P)

<table>
<thead>
<tr>
<th>tasks</th>
<th>difficulty (logits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P19: Welk deel van de figuren is grijs gekleurd? <em>see figure A.1</em></td>
<td>1.17</td>
</tr>
<tr>
<td>P18: 3 kinderen verdelen 5 chocoladerepen. Hoeveel krijgt elk kind? <em>see figure A.3</em></td>
<td>0.01</td>
</tr>
<tr>
<td>P17: Welk deel van de figuren is grijs gekleurd? <em>see figure A.1</em></td>
<td>-0.33</td>
</tr>
<tr>
<td>P16: 6 kinderen verdelen 4 pannekoeken. Elk kind krijgt __ pannekoek.</td>
<td>-0.42</td>
</tr>
<tr>
<td>P15: Welk deel van de figuren is grijs gekleurd? <em>see figure A.2</em></td>
<td>-0.82</td>
</tr>
<tr>
<td>P14: 4 kinderen verdelen 7 pizza's. Hoeveel krijgt elk kind? <em>see figure A.4</em></td>
<td>-1.12</td>
</tr>
<tr>
<td>P13: 6 kinderen verdelen 2 chocoladerepen. Ieder krijgt __ chocoladereep.</td>
<td>-1.55</td>
</tr>
<tr>
<td>P12: Teken hoe je de taart kunt snijden. Kleur dan het stuk dat bij de taart staat. <em>see figure A.5</em></td>
<td>-1.83</td>
</tr>
<tr>
<td>P11: Teken hoe je de taart kunt snijden. Kleur dan het stuk dat bij de taart staat. <em>see figure A.6</em></td>
<td>-1.83</td>
</tr>
<tr>
<td>P10: Vier kinderen verdelen drie pizza's. Teken hoe je de pizza's snijdt. Kleur het stuk dat één kind krijgt. <em>see figure A.7</em></td>
<td>-2.27</td>
</tr>
<tr>
<td>P9 : Welk deel van de potloden heeft een gummetje? <em>see figure A.8</em></td>
<td>-2.81</td>
</tr>
<tr>
<td>P8 : Welk deel van de voetballen heeft zwarte vlakken? <em>see figure A.9</em></td>
<td>-2.95</td>
</tr>
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<td>P7 : Teken hoe je de taart kunt snijden. Kleur dan het stuk dat bij de taart staat. <em>see figure A.6</em></td>
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</tr>
<tr>
<td>P6 : Teken hoe je de taart kunt snijden. Kleur dan het stuk dat bij de taart staat. <em>see figure A.5</em></td>
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<td>P5 : Welk deel van de figuren is grijs gekleurd? <em>see figure A.2</em></td>
<td>-4.20</td>
</tr>
<tr>
<td>P4 : Welk stuk bestellen deze mensen: <em>see figure A.10</em></td>
<td>-4.26</td>
</tr>
<tr>
<td>P3 : Welk stuk bestellen deze mensen: <em>see figure A.10</em></td>
<td>-4.51</td>
</tr>
<tr>
<td>P2 : Welk stuk bestellen deze mensen: <em>see figure A.11</em></td>
<td>-4.64</td>
</tr>
<tr>
<td>P1 : Welk stuk bestellen deze mensen: <em>see figure A.11</em></td>
<td>-4.64</td>
</tr>
</tbody>
</table>
Figure A.1: Item P19 (left) and P17 (right)  
Figure A.2: Item P5 (left) and P15 (right)  
Figure A.3: Item P18  
Figure A.4: Item P14  
Figure A.5: Item P6 (left) and P12 (right)  
Figure A.6: Item P11 (left) and P7 (right)
Figure A.7: Item P10

Figure A.8: Item P9

Figure A.9: Item P8

Figure A.10: Item P4 (left) and P3 (right)

Figure A.11: Item P2 (left) and P1 (right)
### A.2 Order (O)

<table>
<thead>
<tr>
<th>tasks</th>
<th>difficulty (logits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>O25: Zet een cirkel om de grootste van de twee breuken. Als de breuken even groot zijn, zet je er een '=' tussen. $\frac{20}{21}$ of $\frac{9}{10}$</td>
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<td>O24: Zet een cirkel om de grootste van de twee breuken. Als de breuken even groot zijn, zet je er een '=' tussen. $\frac{2}{5}$ of $\frac{3}{4}$</td>
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<tr>
<td>O23: Zet een cirkel om de grootste van de twee breuken. Als de breuken even groot zijn, zet je er een '=' tussen. $\frac{5}{7}$ of $\frac{10}{11}$</td>
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</tr>
<tr>
<td>O22: Vul in &lt;, &gt;, =: $\frac{2}{5}$.. $\frac{4}{5}$</td>
<td>-0.49</td>
</tr>
<tr>
<td>O21: Zet een cirkel om de grootste van de twee breuken. Als de breuken even groot zijn, zet je er een '=' tussen. $\frac{4}{9}$ of $\frac{7}{12}$</td>
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<tr>
<td>O20: Zet de breuken van klein naar groot: $\frac{2}{5}$.. $\frac{8}{11}$</td>
<td>-0.72</td>
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<tr>
<td>O19: Vul in &lt;, &gt;, =: $\frac{5}{9}$.. $\frac{7}{12}$</td>
<td>-0.93</td>
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<td>O18: Je ziet telkens twee breuken. Omcirkel de grootste van de twee of zet er een '=' teken tussen als ze even groot zijn.</td>
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<tr>
<td>O17: Zet een cirkel om de grootste van de twee breuken. Als de breuken even groot zijn, zet je er een '=' tussen. $\frac{3}{7}$ of $\frac{4}{5}$</td>
<td>-1.51</td>
</tr>
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<td>O16: Vul in &lt;, &gt;, =: $\frac{1}{5}$.. $\frac{1}{3}$</td>
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<td>O15: Zet een cirkel om de grootste van de twee breuken. Als de breuken even groot zijn, zet je er een '=' tussen. $\frac{6}{11}$ of $\frac{8}{13}$</td>
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<td>O14: Zet een cirkel om de grootste van de twee breuken. Als de breuken even groot zijn, zet je er een '=' tussen. $\frac{1}{4}$ of $\frac{2}{5}$</td>
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<td>O13: Zet een cirkel om de grootste van de twee breuken. Als de breuken even groot zijn, zet je er een '=' tussen. $\frac{3}{6}$ or $\frac{5}{13}$</td>
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</tr>
<tr>
<td>O9 : Teken een rondje om de grootste van de twee breuken. $\frac{7}{9}$ or $\frac{4}{7}$</td>
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</tr>
<tr>
<td>O8 : Teken een rondje om de grootste van de twee breuken. $\frac{1}{2}$ or $\frac{1}{7}$</td>
<td>-3.42</td>
</tr>
<tr>
<td>O7 : Teken een rondje om de grootste van de twee breuken. $\frac{3}{5}$ or $\frac{1}{7}$</td>
<td>-3.47</td>
</tr>
<tr>
<td>O6 : Teken een rondje om de grootste van de twee breuken. $\frac{3}{5}$ or $\frac{1}{7}$</td>
<td>-3.47</td>
</tr>
<tr>
<td>O5 : Teken een rondje om de grootste van de twee breuken. $\frac{3}{5}$ or $\frac{1}{7}$</td>
<td>-3.52</td>
</tr>
<tr>
<td>O4 : Teken een rondje om de grootste van de twee breuken. $\frac{3}{5}$ or $\frac{1}{7}$</td>
<td>-3.57</td>
</tr>
<tr>
<td>O3 : Teken een rondje om de grootste van de twee breuken. $\frac{3}{5}$ or $\frac{1}{7}$</td>
<td>-3.68</td>
</tr>
<tr>
<td>O2 : Teken een rondje om de grootste van de twee breuken. $\frac{3}{5}$ or $\frac{1}{7}$</td>
<td>-3.68</td>
</tr>
<tr>
<td>O1 : Teken een rondje om de grootste van de twee breuken. $\frac{3}{5}$ or $\frac{1}{7}$</td>
<td>-3.89</td>
</tr>
</tbody>
</table>
A.3 Reduce and complicate (E)

<table>
<thead>
<tr>
<th>tasks</th>
<th>difficulty (logits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E8 : ( \frac{30}{44} ) is hetzelfde als ......</td>
<td>-0.93</td>
</tr>
<tr>
<td>E7 : ( \frac{42}{60} ) is hetzelfde als ( \frac{7}{10} )</td>
<td>-1.11</td>
</tr>
<tr>
<td>E6 : ( \frac{21}{35} ) is hetzelfde als ( \frac{3}{5} )</td>
<td>-1.11</td>
</tr>
<tr>
<td>E5 : ( \frac{30}{54} ) is hetzelfde als ( \frac{5}{9} )</td>
<td>-1.35</td>
</tr>
<tr>
<td>E4 : ( \frac{42}{60} ) is hetzelfde als ( \frac{7}{10} )</td>
<td>-1.43</td>
</tr>
<tr>
<td>E3 : ( \frac{30}{54} ) is hetzelfde als ( \frac{5}{9} )</td>
<td>-1.57</td>
</tr>
<tr>
<td>E2 : ( \frac{20}{30} ) is hetzelfde als ( \frac{4}{6} )</td>
<td>-1.75</td>
</tr>
<tr>
<td>E1 : ( \frac{42}{60} ) is hetzelfde als ......</td>
<td>-2.42</td>
</tr>
</tbody>
</table>

A.4 Improper fractions and mixed numbers (I)

<table>
<thead>
<tr>
<th>tasks</th>
<th>difficulty (logits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I4 : Haal de hele getallen eruit en schrijf de restbreuk op. ( \frac{53}{8} = )</td>
<td>-0.38</td>
</tr>
<tr>
<td>I3 : Vul in ( \frac{58}{9} = )</td>
<td>-0.41</td>
</tr>
<tr>
<td>I2 : Vul in ( \frac{8}{5} = )</td>
<td>-0.97</td>
</tr>
<tr>
<td>I1 : Haal de hele getallen eruit en schrijf de restbreuk op. ( \frac{10}{7} = )</td>
<td>-1.00</td>
</tr>
</tbody>
</table>
A.5 Numberline (N)

<table>
<thead>
<tr>
<th>tasks</th>
<th>difficulty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schrijf de juiste breuken bij de vraagtekens - 1  ( \frac{1}{2} )</td>
<td>0.95</td>
</tr>
<tr>
<td>Schrijf de juiste breuken bij de vraagtekens - 1  ( \frac{1}{6} )</td>
<td>0.63</td>
</tr>
<tr>
<td>Schrijf de juiste breuken bij de vraagtekens - 5  ( \frac{1}{6} )</td>
<td>0.24</td>
</tr>
<tr>
<td>Schrijf de juiste breuken bij de vraagtekens - 2  ( \frac{1}{6} )</td>
<td>0.19</td>
</tr>
<tr>
<td>Schrijf de juiste breuken bij de vraagtekens - 5  ( \frac{1}{8} )</td>
<td>0.10</td>
</tr>
<tr>
<td>Schrijf de juiste breuken bij de vraagtekens - 2  ( \frac{1}{8} )</td>
<td>-0.07</td>
</tr>
</tbody>
</table>

Figure A.12: Item N1 (top), N3 (left) and N6 (right)

Figure A.13: Item N2 (top), N4 (left) and N5 (right)
## A.6 Addition and subtraction (A)

<table>
<thead>
<tr>
<th>tasks</th>
<th>difficulty (logits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A17: Schrijf als één breuk: $\frac{a}{b} + \frac{c}{d} =$</td>
<td>6.36</td>
</tr>
<tr>
<td>A16: $\frac{4}{1} - \frac{3}{2} =$</td>
<td>2.03</td>
</tr>
<tr>
<td>A15: $\frac{5}{6} - \frac{2}{3} =$</td>
<td>1.95</td>
</tr>
<tr>
<td>A14: $\frac{3}{4} + \frac{4}{5} =$</td>
<td>1.23</td>
</tr>
<tr>
<td>A13: $\frac{2}{3} + \frac{1}{2} =$</td>
<td>1.17</td>
</tr>
<tr>
<td>A12: $\frac{2}{3} + \frac{5}{6} =$</td>
<td>0.83</td>
</tr>
<tr>
<td>A11: $\frac{6}{5} + \frac{3}{2} =$</td>
<td>0.55</td>
</tr>
<tr>
<td>A10: $\frac{5}{4} + \frac{3}{6} =$</td>
<td>0.49</td>
</tr>
<tr>
<td>A9 : $\frac{7}{5} - \frac{3}{4} =$</td>
<td>0.31</td>
</tr>
<tr>
<td>A8 : $\frac{5}{6} - \frac{2}{3} =$</td>
<td>-0.14</td>
</tr>
<tr>
<td>A7 : $\frac{8}{1} - \frac{4}{3} =$</td>
<td>-0.15</td>
</tr>
<tr>
<td>A6 : $\frac{5}{4} - 2 \frac{1}{2} =$</td>
<td>-0.47</td>
</tr>
<tr>
<td>A5 : $\frac{3}{2} + 2 \frac{1}{2} =$</td>
<td>-0.56</td>
</tr>
<tr>
<td>A4 : $\frac{4}{3} + 2 \frac{1}{4} =$</td>
<td>-0.56</td>
</tr>
<tr>
<td>A3 : $\frac{5}{1} + \frac{4}{3} =$</td>
<td>-1.22</td>
</tr>
<tr>
<td>A2 : $\frac{3}{4} + \frac{5}{6} =$</td>
<td>-1.28</td>
</tr>
<tr>
<td>A1 : $\frac{7}{4} + \frac{4}{5} =$</td>
<td>-2.59</td>
</tr>
</tbody>
</table>
A.7 Multiplication (M)

<table>
<thead>
<tr>
<th>tasks</th>
<th>difficulty (logits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>M48: ( \frac{3}{2} \times \frac{3}{2} = )</td>
<td>4.00</td>
</tr>
<tr>
<td>M47: ( \frac{a}{b} \times \frac{a}{b} = )</td>
<td>3.95</td>
</tr>
<tr>
<td>M46: ( 2\frac{1}{2} \times 1\frac{1}{2} = )</td>
<td>3.42</td>
</tr>
<tr>
<td>M45: ( 1\frac{1}{2} \times 2\frac{1}{2} = \text{see figure A.14} )</td>
<td>2.90</td>
</tr>
<tr>
<td>M42: ( \frac{3}{4} \times \ldots = 1 )</td>
<td>1.69</td>
</tr>
<tr>
<td>M41: de helft van ( \frac{3}{8} ) taart is \ldots</td>
<td>1.66</td>
</tr>
<tr>
<td>M40: ( \frac{4}{3} \times 81 = )</td>
<td>1.56</td>
</tr>
<tr>
<td>M39: ( \frac{1}{3} ) van ( \frac{6}{7} ) taart is \ldots</td>
<td>1.51</td>
</tr>
<tr>
<td>M38: ( \frac{1}{7} ) van ( \frac{3}{8} ) cake is \ldots</td>
<td>1.41</td>
</tr>
<tr>
<td>M36: ( \frac{3}{5} \times \frac{4}{5} = )</td>
<td>1.28</td>
</tr>
<tr>
<td>M35: ( 3\frac{1}{11} \times \frac{7}{11} = )</td>
<td>1.28</td>
</tr>
<tr>
<td>M34: ( \frac{7}{4} \times 49 = )</td>
<td>1.14</td>
</tr>
<tr>
<td>M33: ( 12\frac{1}{2} \times 3 \times 4 = )</td>
<td>1.00</td>
</tr>
<tr>
<td>M32: ( \frac{1}{2} ) van ( \frac{3}{4} ) cake is \ldots</td>
<td>0.94</td>
</tr>
<tr>
<td>M30: ( 4 \times 3 \times 12\frac{1}{2} = )</td>
<td>0.92</td>
</tr>
<tr>
<td>M29: een derde ( \frac{5}{7} ) reepkoek is \ldots</td>
<td>0.86</td>
</tr>
</tbody>
</table>

Figure A.14: Item M45
<table>
<thead>
<tr>
<th>tasks</th>
<th>difficulty (logits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>M28: $8 \times \ldots = \frac{1}{2}$</td>
<td>0.76</td>
</tr>
<tr>
<td>M27: $\frac{3}{7} \times \frac{14}{7} =</td>
<td>0.68</td>
</tr>
<tr>
<td>M26: $\frac{1}{3} \times \frac{8}{3} =</td>
<td>0.62</td>
</tr>
<tr>
<td>M25: $\frac{3}{6} \times \frac{2}{3} =</td>
<td>0.54</td>
</tr>
<tr>
<td>M24: $\frac{1}{2} \times \frac{3}{3} =</td>
<td>0.48</td>
</tr>
<tr>
<td>M23: $\frac{3}{4} \times 4 =</td>
<td>0.47</td>
</tr>
<tr>
<td>M22: $4 \times \frac{2}{7} =</td>
<td>0.40</td>
</tr>
<tr>
<td>M21: Wat kost 0.743 kg ongeveer als één kilo 8 euro kost?</td>
<td>0.35</td>
</tr>
<tr>
<td>M20: $\frac{3}{7} \times \frac{6}{6} =</td>
<td>0.21</td>
</tr>
<tr>
<td>M19: $4 \times 3 \frac{1}{4} =</td>
<td>0.16</td>
</tr>
<tr>
<td>M18: $8 \times \frac{3}{4} =</td>
<td>0.02</td>
</tr>
<tr>
<td>M17: $\frac{1}{2} \text{ van } \frac{4}{5} \text{ cake is } \ldots</td>
<td>0.01</td>
</tr>
<tr>
<td>M16: $10 \times \frac{5}{5} =</td>
<td>-0.07</td>
</tr>
<tr>
<td>M15: $4 \frac{1}{4} \times 3 =</td>
<td>-0.34</td>
</tr>
<tr>
<td>M14: Wat kost 0.329 kg ongeveer als één kilo 6 euro kost?</td>
<td>-0.58</td>
</tr>
<tr>
<td>M13: $5 \times 4 \frac{1}{5} =</td>
<td>-0.69</td>
</tr>
<tr>
<td>M12: Marij doet 72 kaarten op de post. $\frac{5}{2}$ deel gaat naar een adres in Brabant. Hoeveel kaarten zijn dat?</td>
<td>-1.71</td>
</tr>
<tr>
<td>M11: Eén kg drop kost 6 euro. Hoeveel kost $\frac{1}{2}$ kg drop?</td>
<td>-1.87</td>
</tr>
<tr>
<td>M10: $4 \frac{1}{2} \text{ deel van } 35 \text{ euro is } \ldots</td>
<td>-1.98</td>
</tr>
<tr>
<td>M9: De hele taart € 4. Hoeveel kost $\frac{1}{5}$ stuk taart?</td>
<td>-2.16</td>
</tr>
<tr>
<td>M8: $\frac{4}{7} \text{ deel van } 35 \text{ euro is } \ldots</td>
<td>-2.41</td>
</tr>
<tr>
<td>M7: Er staan 81 mensen in de rij voor de achtbaan. $\frac{4}{5}$ deel is jongen. Hoeveel jongens staan in de rij?</td>
<td>-2.88</td>
</tr>
<tr>
<td>M6: $\frac{3}{7} \text{ deel van } 49 \text{ euro is } \ldots</td>
<td>-1.71</td>
</tr>
</tbody>
</table>
A.8 Division (D)

<table>
<thead>
<tr>
<th>tasks</th>
<th>difficulty (logits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>D24: $34 \div 2\frac{1}{2}$</td>
<td>3.76</td>
</tr>
<tr>
<td>D23: $3\frac{1}{2} \div 6$</td>
<td>3.67</td>
</tr>
<tr>
<td>D22: Reken uit op jouw manier: $3\frac{1}{8} \div 6 = \text{see figure A.15}$</td>
<td>3.26</td>
</tr>
<tr>
<td>D21: $4\frac{7}{8} \div 8$</td>
<td>3.08</td>
</tr>
<tr>
<td>D20: $\frac{8}{9} \div \frac{5}{6}$</td>
<td>3.00</td>
</tr>
<tr>
<td>D19: $32 \div 2\frac{1}{2}$</td>
<td>2.76</td>
</tr>
<tr>
<td>D18: $\frac{4}{5} \div \frac{2}{3} = \text{see figure A.16}$</td>
<td>2.72</td>
</tr>
<tr>
<td>D17: $6 \div \frac{2}{3}$</td>
<td>2.69</td>
</tr>
<tr>
<td>D16: $\frac{7}{9} \div \frac{3}{4}$</td>
<td>2.20</td>
</tr>
<tr>
<td>D15: $6\frac{2}{3} \div 2$</td>
<td>2.00</td>
</tr>
<tr>
<td>D14: $\frac{5}{9} \div \frac{3}{2}$</td>
<td>1.99</td>
</tr>
<tr>
<td>D13: $9 \div \frac{5}{6}$</td>
<td>1.58</td>
</tr>
<tr>
<td>D12: $\frac{8}{11} \div \frac{2}{7}$</td>
<td>1.46</td>
</tr>
<tr>
<td>D11: $4 \div \frac{1}{7}$</td>
<td>1.37</td>
</tr>
<tr>
<td>D10: $6\frac{2}{3} \div \frac{3}{4}$</td>
<td>1.36</td>
</tr>
<tr>
<td>D9: $\frac{8}{9} \div \frac{2}{3}$</td>
<td>1.24</td>
</tr>
<tr>
<td>D8: $\ldots \div \frac{3}{8} = 2$</td>
<td>1.20</td>
</tr>
<tr>
<td>D7: $3 \div \frac{1}{2}$</td>
<td>0.89</td>
</tr>
<tr>
<td>D6: Gijs heeft een stuk touw van $6\frac{3}{4}$ meter. Hoeveel stukken van $\frac{3}{4}$ meter kan hij daarvan knippen?</td>
<td>0.37</td>
</tr>
<tr>
<td>D5: In een vat zit 19 liter ranja. Hoeveel kannen van $\frac{1}{8}$ liter kun je daarmee vullen?</td>
<td>0.01</td>
</tr>
<tr>
<td>D4: $\frac{7}{9} \div \frac{2}{3}$</td>
<td>-0.08</td>
</tr>
<tr>
<td>D3: In een tank zit 21 liter olie. Hoeveel blikken van $\frac{1}{4}$ liter kun je daarmee vullen?</td>
<td>-0.58</td>
</tr>
<tr>
<td>D2: $\frac{1}{3}$ deel van de bezoekers aan de speeltuin heeft zelf drinken meegenomen. Dat zijn er 90. Hoeveel bezoekers zijn er?</td>
<td>-2.26</td>
</tr>
<tr>
<td>D1: $\frac{1}{5}$ deel van de kinderen in een klas voetbalt. Dat zijn er 4.Hoeveel kinderen zijn er in de klas?</td>
<td>-2.93</td>
</tr>
</tbody>
</table>

Figure A.15: Item D22
Figure A.16: Item D18
## A.9 Application of fraction knowledge (T)

<table>
<thead>
<tr>
<th>task</th>
<th>difficulty (logits)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T17: Een fabrikant van chocoladerepen heeft een slimme manier bedacht om de winst te vergroten: ze maken de repen kleiner maar verlagen de prijs niet. Je krijgt nu $\frac{1}{2}$ minder voor hetzelfde geld. OP hoeveel procent prijsverhoging komt dat neer?</td>
<td>6.66</td>
</tr>
<tr>
<td>T16: Bij natuurkunde wordt bij het berekenen van vervangingsweerstanden de volgende formule gebruikt als er 3 weerstanden parallel worden geschakeld, $\frac{1}{R_v} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. Bereken de waarde van de vervangingsweerstand $R_v$ als $R_1 = 3$, $R_2 = 4$ en $R_3 = 2$. $R_v =$</td>
<td>5.51</td>
</tr>
<tr>
<td>T15: Los op: $\frac{2}{3}x - 1 = x - \frac{3}{4}$</td>
<td>4.61</td>
</tr>
<tr>
<td>T14: Los op: $\frac{3}{x} + 4 = 0$</td>
<td>4.22</td>
</tr>
<tr>
<td>T13: Schrijf zo eenvoudig mogelijk: $\frac{x}{1-x} + \frac{x}{x-1}$</td>
<td>3.30</td>
</tr>
<tr>
<td>T12: Bij een volleybalwedstrijd was in de speelzaal $\frac{2}{3}$ deel supporter en de rest speler. Wat was de verhouding tussen het aantal supporters en spelers?</td>
<td>3.11</td>
</tr>
<tr>
<td>T11: $\frac{1}{2 + \frac{1}{3}} =$</td>
<td>2.91</td>
</tr>
<tr>
<td>T10: Van een armband is gegeven dat de massa 45 gram is en de dichtheid 12 g/cm$^3$. Bereken het volume van deze armband.</td>
<td>2.72</td>
</tr>
<tr>
<td>T9: Gegeven is de dichtheid van alcohol. Deze is 0,80 $\frac{g}{ml}$. Hoeveel gram weegt 55 ml alcohol?</td>
<td>2.24</td>
</tr>
<tr>
<td>T8: Marie stopt in een grote pot rode en zwarte ballen in een verhouding 2 staat tot 3. In totaal zitten er 30 ballen in de pot. Hoeveel zwarte ballen zitten er in deze pot?</td>
<td>2.14</td>
</tr>
<tr>
<td>T7: Marie stopt in een grote pot rode en zwarte ballen in een verhouding 4 staat tot 6. In totaal zitten er 60 ballen in de pot. Hoeveel zwarte ballen zitten er in deze pot?</td>
<td>2.04</td>
</tr>
<tr>
<td>T6: Aan elk snoer zitten 300 kralen. Je kunt het niet helemaal zien. Hoeveel witte kralen zitten er aan elk snoer, denk je? see figure A.17</td>
<td>1.75</td>
</tr>
<tr>
<td>T5: Aan elk snoer zitten 30 kralen. Je kunt het niet helemaal zien. Hoeveel witte kralen zitten er aan elk snoer, denk je? see figure A.17</td>
<td>1.59</td>
</tr>
<tr>
<td>T4: Aan elk snoer zitten 30 kralen. Je kunt het niet helemaal zien. Hoeveel witte kralen zitten er aan elk snoer, denk je? see figure A.17</td>
<td>1.37</td>
</tr>
<tr>
<td>T3: In de uitverkoop wordt een dvd-speler met 25 % korting verkocht. De uitverkoop prijs is dan 120 euro. Hoeveel euro korting heeft de klant gekregen?</td>
<td>1.28</td>
</tr>
<tr>
<td>T2: Aan elk snoer zitten 300 kralen. Je kunt het niet helemaal zien. Hoeveel witte kralen zitten er aan elk snoer, denk je? see figure A.17</td>
<td>1.16</td>
</tr>
<tr>
<td>T1: In een grote pot zitten rode en zwarte ballen. Er zijn drie keer zoveel rode als zwarte ballen. In totaal zitten er 36 ballen in de pot. Hoeveel rode ballen zitten er in deze pot?</td>
<td>-0.04</td>
</tr>
</tbody>
</table>
Figure A.17: Top figure for item T4 and T2 and bottom figure for item T5 and T6
Summary

The development of proficiency in the fraction domain, affordances and constraints in the curriculum

Proficiency in basic mathematical skills is a topic of heated discussions in many countries. International comparative studies on mathematical skills such as PISA and TIMMS have lead to concerns about the mathematics curriculum especially in countries with a relatively low rating in the summary reports (the league-tables effect). In the Netherlands –as in many other countries– these discussions concentrate on the questions whether and to what extend there is reduced mastery, what causes this problem and how curriculum reform might remedy this. Given the lack of data to answer these last three questions, there is a need for further research. This doctoral dissertation therefore explores the nature and causes of problems with basic mathematical skills in the Dutch curriculum, taking the transition from primary to secondary education as its main focus. More concretely, this dissertation investigates the development of the students’ proficiency in the fraction domain from grade 4 through 9 and analyzes how textbooks support students in the transition from primary to secondary education and in reaching the formal understanding that is needed for the transition from arithmetic to algebra. In doing so, possible footholds for improvement of instruction and contributions to theory building are investigated.

This dissertation follows two lines of research. In the first part, the proficiency of students is studied using the test results of 1485 students from grades 4 through 9. The second part concentrates on textbooks and how the curriculum can account for the result we found in the first line of research. In the final chapter, these two perspectives on the curriculum are reflected upon.

We started our study with the development of a test to provide us with detailed empirical data on the proficiency of the students (Chapter 2). This test has been the basis of our analysis of student proficiency. The test is designed as a paper and pencil tests to allow for efficient assessment of a large number of students in grades 4 through 9. The theoretical framework of this test is based on literature on the learning of fractions and the nature of the fraction domain itself. We identify five so-called ‘big ideas’ that describe the domain at the level of underlying concepts: relative comparison, equivalence, reification, from natural to rational numbers and relation division-multiplication. A list of so-called ‘complexity factors’ describes the external characteristics of tasks that influence their difficulty. The systematic construction of the test, using our framework of big ideas and complexity factors, allows for an analysis of test results at two different levels. The first level of analysis involves an item per item analysis of the types of tasks the students can or cannot answer correctly. The second, more complex level of analysis involves the combination of tasks, aiming at insight in the students’ understanding of the concepts underlying fraction proficiency.
In Chapter 3 we exemplify how our test meets three pre-set criteria: (1) Is it possible to construct a single linear scale for fraction proficiency on which all our items can be ordered according to their difficulty?, (2) Can the development of proficiency from grade 4 to 9 be described with the results of this test?, and (3) Does this provide diagnostic data which can be used for improving instruction? A single linear scale for fraction proficiency is created employing a Rasch model on the data from the fraction test. This analysis and evaluation of the results regarding all 1485 students and 169 test items results in positive conclusions about the validity and the reliability of the test. The data fits the Rasch model and we conclude that our test measures one latent trait. The Rasch scale based on our test serves as scale for proficiency in fractions in the remainder of our studies. The examples of possible analyses focus on the proficiency of students in grade 6 regarding the addition of fractions and three of the big ideas, namely unit, equivalence and the development of rational number. These examples show how the development of proficiency can be described at the levels of tasks and items and that this provides footholds for improving instruction. The analyses show that the students were able to add and subtract fractions with a common denominator, but had not yet mastered the addition and subtraction with unlike denominators. Regarding the big idea of unit, most students were able to answer items on naming parts and representing fractions with part-whole models. Most of the students did not master conceptual mapping in a context or in relation to the numberline. Regarding the idea of equivalence, the students were able to reduce fractions to its lowest term, but had difficulty in recognizing fractions as equivalent when there was no whole number factor between numerators. Finally, regarding the development of rational number, the students mastered improper fractions on the numberline, but were not able to use improper fractions/mixed numbers in multiplication and division tasks. For the students, improper fractions appeared to have not become rational numbers that have the character of object like entities that can be used as a number. Overall, the analyses show that the students were capable to solve tasks that ask for reproduction and procedural use of symbols and operations. However, tasks that differ from standard and ask for more conceptual understanding surpassed the ability of most of the 6th grade students.

In Chapter 4 we analyse the results of secondary education. Our expectation was to find that students would have deepened their understanding thanks to their experience in using fractions in a variety of tasks. However, it shows that there was no significant progress in fraction proficiency in lower secondary education. This result mirrors the lack of explicit attention to fractions in textbooks.

We find problems in two different lines. On the item level we find that the grade 9 students had not developed generalized strategies for the basic operations. Instead these students developed number specific strategies, which they could not generalize over various situations. This finding corresponds with our findings on textbooks (Chapter 5 and Chapter 6). The analysis at concept level indicates that the students lacked the notion of a fraction as a division and the interpretation of a fraction as both two numbers and one, an interpretation that relates to the so-called ratio-rate duality.

We conclude that the proficiency level of most students is insufficient for the transition to algebra, since the students in grade 9 are still in the process of mastering general ways of solving fraction tasks that involve addition, subtraction, multiplication and division. That
is, for algebra, plain knowledge of these general rules for arithmetic as well as conceptual understanding of these operations and their relations with other concepts underlying the domain of fraction, are a prerequisite.

Although our results are based on the students of one school and one textbook series, we argue that the results are representative. Our analysis of textbooks shows that the two major textbook series (with a current estimated total market share of 90% do not differ much in respect to the explicit attention to fraction arithmetic and understanding (Chapter 5). Furthermore, the results are corroborated by data that we collected in tertiary education, where we tested 97 first year science students using a selection of our items.

In this dissertation not only students’ proficiency but also the formal curriculum is analyzed. In the analyses in part 2, we consider textbooks as representative for the curriculum. We analyze the texts and inscriptions on fractions in mathematics textbooks for grade 6 and 7 and study how these textbooks relate to prototypical instructional sequences and tasks produced in design research, and to the principles of realistic mathematics education (RME). We assume that aspects of an educational system are reflected in the textbooks. We further assume that at the same time textbooks determine instructional practices to a large extent. In the Netherlands this connection between educational practice and the content of the textbooks may even be tighter than in other countries. Analysis of Dutch textbook use in mathematics education shows a strong connection between the textbook and the practice of teaching, both in primary and secondary education. The link between textbook and educational practice is dialectic, in that teachers not only rely heavily on the textbook regarding content and pedagogy, textbooks can also be regarded as the product of the culture among teachers. Given this strong relationship between instructional practices and textbooks, we assume that actual incoherencies between practices in primary and secondary mathematics education can be observed in the textbooks.

In Chapter 5, we focus on the transition from primary to secondary education starting from a CHAT perspective. We hypothesize that part of the problems with fraction proficiency may be explained by the fact that the fraction curriculum stretches over a large number of years, in the Dutch curriculum even over primary and secondary education. Among others, the confusion of tongue in the public debate gave reason to believe that some of the problems stem from differences in the educational practices in the two educational systems. We hypothesize that the cultural and historically determined traditions of both primary and secondary education, will express themselves in the educational practices and the use of artifacts, leading to confusing differences in the epistemological messages expressed.

We have found that differences between these educational systems result in a gap between primary and secondary education textbooks with regard to fraction multiplication. We argue that this gap is difficult for students to overcome. Given the different traditions in both educational systems, we illustrate how similar artifacts get a different meaning in the context of each of these traditions. In secondary education textbooks, students are expected to use different strategies for solving tasks which require an understanding of fractions, which is more formal than can be expected on the basis of the primary school textbooks. Our analysis also points to differences between primary and secondary education implying that students develop an “operational conception” of fractions in primary school, whereas they are expected to reason with a “structural conception” of fractions in secondary school.

143
In neither systems students are supported in making the transition to a structural conception. Moreover, primary school textbooks aim at training number-specific solution procedures which are likely to become a barrier for the generalization and formalization that is required for coming to understand fractions at a more formal level. However, differences stay hidden because textbooks in primary and secondary school use similar artifacts, with different meanings. Furthermore, such differences are usually unnoticeable for participants in each of the educational systems, because these participants bring with them different frameworks of reference.

In Chapter 6 we address the content of the textbooks in primary school and its relation to prototypical work in design research on fractions and the basic principles of RME. It shows that the way this prototypical work has been incorporated in the textbooks leads to a number of very distinct procedures for multiplying fractions. That is, multiplication of fractions became a set of number specific procedures. We distinguish four compartmentalized interpretations of the multiplication sign. In the textbooks, tasks on a “whole number times proper fractions” are related to repeated addition. Tasks on “proper fraction times whole number” are to be interpreted as part of that whole number. Tasks on “proper fraction times proper fraction” are connected with multiplication as an area and finally “multiplication with mixed numbers” is related to splitting. The test results show that this compartmentalization is still observable in the performance of students in grade 6 and 7. From the outcomes of the analysis of the empirical data it follows that the success rate and type of errors are related to the order of operands and the abstraction of the problem formulation. We conclude that this compartmentalization may hinder the process towards more formal understanding of multiplication of fractions. Thus, the versatile approach of mathematical concepts is not yet elaborated in such a manner that it allows students to come to grips with the conventional mathematical method for multiplying fractions as intended. In reflection, we show that the use of contexts has two sides. On the one hand it both affords and supports students’ in building upon informal knowledge. On the other hand however, contexts can hinder mathematization when their inherent characteristics become constraints in the generalization process.

In conclusion, we find that problems in the fraction domain start already early in education and continue in later grades. Although the proficiency of the students developed considerably from grade 4, analysis of the students’ proficiency level at the end of grade 6 reveals a lack of deeper understanding of fractions. Corresponding to the textbooks, the teaching of fractions and fraction arithmetic in primary education is directed towards procedural problem solving of standard tasks. The students did not develop deeper understanding of underlying concepts such as unit, rational number and equivalence. Moreover, strategies that students did learn appeared to be number-specific. In the transition from primary to secondary education, problems arise because of an apparent misunderstanding of the intended level of proficiency at the end of primary school. From the results of both our test (Chapter 4) and the analysis of primary and secondary school textbooks (Chapter 5) it follows that secondary school teachers, and textbook authors do not realize that the fraction curriculum is not completed in primary school. In addition, insufficient attention is given to generalizing and formalizing in secondary education.

This dissertation provides new insights in the Dutch curriculum for fractions especially
regarding the transition from primary education, compartmentalization in primary education and preparation for basic algebraic skills. In Chapter 7 we reflect on these findings and discuss the broader significance of the results. We argue that our results represent more general problems in the mathematics curriculum. We discuss the influence of the educational setting by identifying problems in proceduralization, local analysis, contexts, and the role of teachers. Further, we discuss where mathematics education deviated from the nature of academic mathematics. We argue for more emphasis on structure, coherence and ambiguity. That is, we think that there is a need for more attention to the ambiguity of mathematical symbols and concepts, for more attention to mathematical structures and to the relation between concepts instead of an exclusive attention for algorithms to solve bare arithmetic tasks. Finally, we recommend changes for textbooks, the transition from primary to secondary education, and the transition to higher education.
Curriculum Vitae

Geeke Bruin-Muurling (MSc) studied applied mathematics at Delft University of Technology. In 2000 she graduated on the topic of ‘Fault-tolerant IC-Design’, a project performed with Philips Natlab which resulted in a patent. In the period 2000-2004 she worked as a cryptographer at the research department of SafeNet. Here she co-developed a ‘large number arithmetic unit’ for RSA and ECC applications. In 2004 she started to work in the field of education, first as a mathematics teacher in secondary school. She graduated at the teacher training institute at the TU Eindhoven and received the qualification for higher secondary education. She was also involved in the development of mathematics textbooks for talented students in primary education (Rekentijger, Zwijsen). In 2006 she started with her PhD project at the Eindhoven School of Education, a collaborative institute of the TU Eindhoven and Fontys University of Applied Science.
List of publications

Journal articles:


Book or book chapter:


Conference proceedings:


Oral presentations (without proceedings):


Patent:

ESoE dissertation series


De Bakker, G.M. (2010). Allocated online reciprocal peer support via instant messaging as a candidate for decreasing the tutoring load of teachers.


Bruin-Muurling, G. (2010). The development of proficiency in the fraction domain; affordances and constraints in the curriculum
Index

action, 66
activity, 66
ambiguity, 15, 84, 115
anti-didactical inversion, 86

BBL, see educational system
big ideas, 15, 26
division-multiplication, 16, 20, 54
equivalence, 16, 19, 40
natural to rational number, 16, 19, 40
reification, 16, 18
relative comparison, 16, 54

CHAT, 66
course-grained analysis, 71
cohesion, 114
compartmentalization, 92
complexity factors, 20
concept level analysis, 20, 38, 54
context, 69, 93
cross-sectional analysis, 47
cultural-historical activity theory, see CHAT
design research, 87
division-multiplication, see big ideas

educational system, 5, 68
emergent modeling, 69, 86
equivalence, see big ideas

fair sharing, 74
fine-grained analysis, 73
fit, 35

general complexity factors, 20
genres of inscriptions, 72
GL, see educational system
guided reinvention, 69, 110, 112
Gymnasium, see educational system

HAVO, see educational system

inscription, 71
item level analysis, 20, 37, 49

KBL, see educational system

local analysis, 110
logistic ogive, 34
longitudinal analysis, 48

MBO, see educational system
meaning, 68, 76
measure, see subconstruct
measurement division, 17, 110
model, 69
multiplicative operator, see subconstruct

natural to rational number, see big ideas

operation, 66
part-whole, see subconstruct
partitive division, 16, 110
probability of success, 34
proceduralization, 108
proficiency, 6, 14, 100

quotient, see subconstruct

Rasch model, 33
Rasch scale, 35
ratio number, see subconstruct
ratio-rate, 17, 54
realistic mathematics education, 5, 69, 85
reference, see meaning
reform mathematics, 84
reification, see big ideas
relative comparison, see big ideas
repeated addition, 73, 89
RME, see realistic mathematics education
role of teachers, 112

sense, see meaning
publications

splitting, 74
structure, 113
subconstructs, 16
subdomains, 20, 26
  application of fraction knowledge, 25
  basic operations, 24
    addition and subtraction, 24, 37, 49
    division, 25, 53
    multiplication, 24, 51, 73, 90
initial fraction concepts, 22
  mixed numbers, 23
  number line, 23
  order, 22
  part-whole, 22
  reduce and complicate, 24

TL, see educational system

unit, 18, 38

vinculum, 17
VMBO, see educational system
VWO, see educational system