Laser Linewidth Measurement in the Presence of RIN and Using the Recirculating Self Heterodyne Method

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Abstract

Laser linewidth measurements using the recirculating self heterodyne method are analysed. Both the relative intensity noise (RIN) and the phase noise are considered. It is assumed that the delay time of the optical feedback loop is considerable larger than the laser's coherence time, a condition to be met for unperturbed reconstruction of the laser spectrum. It is shown that the laser's phase noise spectrum is superimposed on its RIN spectrum.

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Chapter 1: Introduction

The Faculty of Electrical Engineering of the Eindhoven University of Technology is currently engaged in a research program on coherent optical communications. This involves the design and implementation of complete coherent systems. The optical wavelength used in these systems is centred around 1.55 \( \mu \)m. In this wavelength region, fiber attenuation is minimized to about 0.2 dB/km. However, due to dispersion in this particular wavelength region (about 11 ps/km/nm), narrow linewidth single mode light sources are needed in order to achieve high bitrates.

This paper describes a method of measuring the linewidth of a semiconductor laser diode using a recirculating self-heterodyne measurement set-up [1]. The set-up used is depicted in Fig.1:

![Fig.1: Recirculating self-heterodyne set-up for linewidth measurements.](image)

AOM: Acousto-Optic Modulator \( f = 80 \) MHz.

SA : Spectrum Analyzer.

In this paper the measured spectrum is analyzed. We examine the contribution of the amplitude noise as well as the contribution of the phase noise on the spectrum.

Several assumptions are made [2]:
1. The amplitude noise is statistically independent of the phase noise.
2. The amplitude noise spectrum has a LP-filter-like characteristic and a bandwidth of 1-10 GHz, yielding a time constant \( t_{ca} = 10^{-10} - 10^{-9} \) s.
3. \( \Delta \omega \) is defined as Full Width Half Maximum (FWHM) of the Lorentzian line width; \( t_{cf} = 2\pi/\Delta \omega \) is smaller then the delay \( T: 6 \) km x 5 ns/m = 30 \( \mu \)s.
Chapter 2: Description of devices used

2.1. The laser diode

A key assumption in the analysis of the laser spectrum is that \( t_{cf} \) (10\(^{-9}\) - 10\(^{-10}\) s) is much smaller than the delay \( T \) (30 \( \cdot \) 10\(^{-6}\) s). In other words: the amplitude noise bandwidth of the laser diode is assumed to be much larger than the modulation frequency \( f_m \).

At the frequency span 0-200 MHz the amplitude noise spectrum is a flat spectrum, having a nonzero average.

2.2. The acousto-optic modulator

The frequency of the optical wave is shifted by the acousto-optic modulator by 80 MHz. The reason for this is twofold:

1. The measurement span of the spectrum analyzer is from 100 kHz - 3 GHz. If the AOM is not used, the centre frequency of the spectrum will be around \( f=0 \), which is outside the range of the spectrum analyzer. This measurement technique is called a self-homodyne measurement technique. If we use the AOM, the spectrum will be shifted from \( f=0 \) to \( f=f_m=80 \) MHz. At this centre frequency the spectrum can be analyzed. This technique is called the recirculating self-heterodyne technique.

2. The linewidth of the semiconductor laser may be smaller than 100 kHz. In this case the AOM is necessary. Linewidths larger than 100 kHz can be analyzed using the self-homodyne set-up. Frequencies < 100 kHz can not be analyzed (measuring equipment limited).

The acousto-optic modulator itself is made from a piezoelectric material. High-frequent standing acoustic waves are brought about by an external source. The acoustic waves cause periodic refractive index changes in the material. These periodic refractive index changes form a grating which deflects the optical wave and shifts the optical frequency by \( f_m \) (80 MHz) [3].

2.3. The coupler

The coupler used in the set-up is a so-called 3-dB coupler. It is a linear optical device with two symmetric input and output ports. The input ports are marked by 1 and 2, the output ports by 3 and 4 (see Fig.2).
The linear polarized input light is written as:

\[ E_1(t) = E_0 e^{j(\omega_1 t + \phi_1(t))} \]  
(2.1)

\[ E_2(t) = E_0 e^{j(\omega_2 t + \phi_2(t))} \]  
(2.2)

Then the electric fields at the output ports can be written as:

\[ E_3(t) = \frac{E_1(t)}{\sqrt{2}} + \frac{E_2(t)}{\sqrt{2}} e^{j\frac{\pi}{2}} \]  
(2.3)

\[ E_4(t) = \frac{E_1(t)}{\sqrt{2}} e^{j\frac{\pi}{2}} + \frac{E_2(t)}{\sqrt{2}} \]  
(2.4)

Notice the relative phase shift of 90° of the cross-coupled field with respect to the straight-coupled field. Eqs. (2.3) and (2.4) can be applied to a 3-dB coupler as well as to a semi-transparent mirror.

In Fig.1, \( E_1(t) \) and \( E_2(t) \) are not independent of each other. The electric fields \( E_2(t) \), \( E_3(t) \) and \( E_4(t) \) can be expressed in terms of \( E_1(t) \). There are recursive relations between the electric fields:

\[ E_3(t) = \frac{E_1(t)}{\sqrt{2}} + \frac{E_2(t)}{\sqrt{2}} e^{j\frac{\pi}{2}} \]  
(2.5)

\[ E_4(t) = \frac{E_1(t)}{\sqrt{2}} e^{j\frac{\pi}{2}} + \frac{E_2(t)}{\sqrt{2}} \]  
(2.6)

\[ E_2(t) = E_4(t-T) e^{j\omega_m(t-T)} \]  
(2.7)
Substitution of (2.7) into (2.5) gives:

\[ E_3(t) = \frac{E_1(t)}{v_2^2} + \frac{E_4(t-T)}{v_2^2} e^{j\omega_m(t-T)} e^{j\frac{\pi}{2}} \]  

(2.8)

Substitution of (2.6) into (2.8) gives:

\[ E_3(t) = \frac{E_1(t)}{v_2^2} + \left\{ \frac{E_1(t-T)}{2} e^{j\frac{\pi}{2}} + \frac{E_2(t-T)}{2} \right\} e^{j\frac{\pi}{2}} e^{j\omega_m(t-T)} \]  

(2.9)

Substitution of (2.7) into (2.9) gives:

\[ E_3(t) = \frac{E_1(t)}{v_2^2} - \frac{E_1(t-T)}{2} e^{j\omega_m(t-T)} + \frac{E_4(t-2T)}{2} e^{j\omega_m(2t-3T)} e^{j\frac{\pi}{2}} \]  

(2.10)

Finally, multiple substitutions give:

\[ E_3(t) = \frac{E_1(t)}{v_2^2} - \frac{1}{v_2^2} \sum_{k=1}^{\infty} \frac{E_1(t-kT)}{v_2^k} e^{j\omega_m(t - \frac{k+1}{2}T)} \]  

(2.11)

2.4. The photodiode

The photodiode is a linear element which converts optical power into electric current. The photocurrent is linearly proportional to the electric field times its complex conjugate.

\[ I(t) \propto E_3(t)E_3^*(t) \]  

(2.12)
Chapter 3: Calculation of the power spectrum $S_{II}(\omega)$

3.1. Description of the photocurrent $I(t)$

The electric field at input port 1 can be written as:

$$E_1(t) = E(t) e^{j(\omega t + \phi(t))}$$

(3.1)

The amplitude $E(t)$ is a Gaussian process with average 0. The complex envelope is an ergodic and stationary process [4]. Then an autocorrelation function can be defined for $I(t)$.

The electric field at output port 3 can be written as:

$$E_3(t) = \frac{E_1(t)}{\sqrt{2}} - \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \frac{E_1(t-kT) e^{j\kappa (t - \frac{k+1}{2} T)}}{\sqrt{2}^k}$$

(3.2)

The photocurrent is linearly proportional to the product of $E_3(t)$ and $E_3^*(t)$, where $^*$ denotes the complex conjugate.

$$I(t) \propto E_3(t) E_3^*(t)$$

$$\propto \left\{ \frac{E_1(t)}{\sqrt{2}} - \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \frac{E_1(t-kT) e^{j\kappa (t - \frac{k+1}{2} T)}}{\sqrt{2}^k} \right\} \times$$

$$\left\{ \frac{E_1^*(t)}{\sqrt{2}} - \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \frac{E_1^*(t-kT) e^{-j\kappa (t - \frac{k+1}{2} T)}}{\sqrt{2}^k} \right\}$$

(3.3)

$$I^*(t) \propto E_3^*(t) E_3(t)$$

$$\propto 1/2 \left\{ |E_1(t)|^2 + \sum_{k=1}^{\infty} \frac{|E_1(t-kT)|^2}{\sqrt{2}^k} ight.$$  

$$- \sum_{k=1}^{\infty} \frac{E_1(t) E_1^*(t-kT) e^{-j\kappa (t - \frac{k+1}{2} T)}}{\sqrt{2}^k}$$

$$- \sum_{k=1}^{\infty} \frac{E_1(t) E_1(t-kT) e^{j\kappa (t - \frac{k+1}{2} T)}}{\sqrt{2}^k}$$

$$+ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{E_1^*(t-kT) E_1(t-lT) e^{-j(k-1)\omega_m (t - \frac{k+1+1}{2} T)}}{\sqrt{2}^{k+l}} \right\}$$

(3.4)
The autocorrelation function of the photocurrent \( R_{II}(\tau) \) is defined as 
\[ E[ I(t)I(t+\tau) ] \], where \( E \) denotes the statistical expectation.

3.2. Calculation of \( E[ I^*(t)I(t+\tau)] \)

The product of \( I^*(t)I(t+\tau) \) is written down for those terms which have an expectation unequal to 0. These are those product terms which are time independent. The first and second terms of \( I^*(t) \) contribute only in combination with the first and second terms of \( I(t+\tau) \), the third, fourth and fifth terms of \( I^*(t) \) only in combination with the third, fourth and fifth terms of \( I(t+\tau) \), for \( k=p, k=-p, k=p-q, k=-(p-q), k=1=p, k-1=-p \) and \( k-1=p-q \), respectively. The product terms which are \( t \) dependent, contain among others \( E[ e^{j\omega_m t} ] \) which equals 0.

\[
E[ I^*(t)I(t+\tau) ] = \frac{1}{4} \times \\
( E[ |E_1(t)|^2|E_1(t+\tau)|^2 ] + \\
E[ \sum_{p=1}^{\infty} \frac{|E_1(t)|^2|E_1(t+\tau-pT)|^2}{2^p} ] + \\
E[ \sum_{k=1}^{\infty} \frac{|E_1(t+\tau)|^2|E_1(t-kT)|^2}{2^k} ] + \\
E[ \sum_{p=1}^{\infty} \frac{E_1(t+\tau-pT)E_1(t+\tau+T) e^{j(p-q)\omega m(t+\tau - \frac{p+q+1}{2} T)}}{\sqrt{2}^p} ] \\
\] (3.5)
\begin{align*}
\mathbb{E}\left[ \sum_{k=1}^{\infty} \frac{|E_1(t-kT)|^2 |E_1(t+kT)|^2}{4^k} \right] + \\
\mathbb{E}\left[ \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} \frac{|E_1(t-kT)|^2 |E_1(t+kT)|^2}{2^{k+p}} \right] + \\
\mathbb{E}\left[ \sum_{k=1}^{\infty} \frac{E_1(t)E_1(t-kT)E_1(t+kT) e^{jk\omega_m \tau}}{2^k} \right] - \\
\mathbb{E}\left[ \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{E_1(t)E_1(t-kT)E_1(t+kT) E_1(t+\tau) x}{\sqrt{2^{k+p+q}}} e^{jk\omega_m \tau} e^{jk\omega_m \frac{k+1}{2} \frac{p+q+1}{2} T} \right] + \\
\mathbb{E}\left[ \sum_{k=1}^{\infty} \frac{E_1(t)E_1(t-kT)E_1(t+kT) E_1(t+\tau) x}{\sqrt{2^k}} e^{-jk\omega_m \tau} \right] - \\
\mathbb{E}\left[ \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{E_1(t)E_1(t-kT)E_1(t+kT) E_1(t+\tau) x}{\sqrt{2^{k+p+q}}} e^{-jk\omega_m \tau} e^{-jk\omega_m \frac{k+1}{2} \frac{p+q+1}{2} T} \right] - \\
\mathbb{E}\left[ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{p=1}^{\infty} \frac{E_1(t)E_1(t-kT)E_1(t+kT) E_1(t+\tau) x}{\sqrt{2^{k+l+p}}} e^{j\omega_m \tau} e^{j\omega_m \frac{k+1+1}{2} \frac{p+1}{2} T} \right] + \\
\mathbb{E}\left[ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{E_1(t)E_1(t-kT)E_1(t+kT) E_1(t+\tau) x}{\sqrt{2^{k+l+p+q}}} e^{j(p-q)\omega_m \tau} e^{-j(p-q)\omega_m \frac{p+q+1}{2} \frac{k+1+1}{2} T} \right]\quad (3.6)
\end{align*}
3.2.1. Definitions of amplitude related autocorrelation functions

The complex electric field is written as:

\[ E_1(t) = E(t) e^{j(\omega t + \phi(t))} \]  

(3.7)

We assume that the amplitude noise is statistically independent of the phase noise. Then the expectation of the complex electric field can be written as the product of the expectation of the amplitude noise and the expectation of the phase noise.

The expectation is written as:

\[ \mathbb{E}[ E_1(t) ] = \mathbb{E}[ E(t) ] \mathbb{E}[ e^{j(\omega t + \phi(t))} ] \]  

(3.8)

Introduction of the next definition:

\[ R_{EE}(\tau) = \mathbb{E}[ E(t)E(t+\tau) ] \]  

(3.9)

\( R_{EE}(\tau) \) is the electric field amplitude autocorrelation function.

\( E(t) \) is the real electric field amplitude. It can be written as:

\( E(t) = E_0 + E_n(t) \), where \( E_0 \) is the average and \( E_n(t) \) a time-dependent noise term. The average of the time-dependent noise term equals 0.

Introduction of the next definition:

\[ R_n(\tau) = \mathbb{E}[ E_n(t)E_n(t+\tau) ] \]  

(3.10)

\( R_n(\tau) \) is the autocorrelation function of \( E_n(t) \). The average of \( E_n(t) \) equals 0.

\( R_{EE}(\tau) \) can now be expressed in \( R_n(\tau) \):

\[ R_{EE}(\tau) = R_n(\tau) \]
\( R_{EE}(\tau) = \mathbb{E}[ E(t)E(t+\tau) ] = \)
\[ \mathbb{E}[ \{ E_0 + E_n(t)\} \{ E_0 + E_n(t+\tau)\} ] = \]
\[ \mathbb{E}[ E_0^2 + E_0 \{ E_n(t) + E_n(t+\tau)\} + E_n(t)E_n(t+\tau) ] = \]
\[ E_0^2 + R_n(\tau). \]

\( R_{EE}(\tau) = E_0^2 + R_n(\tau) \) \hspace{1cm} (3.12)

By definition the photocurrent amplitude autocorrelation function \( R_{AA}(\tau) \) is:

\[ R_{AA}(\tau) \equiv R_{EE}^2(\tau) = E_0^4 + 2E_0^2R_n(\tau) + R_n^2(\tau) \] \hspace{1cm} (3.13)

Define \( N_n(\tau) \) as:

\[ N_n(\tau) = 2E_0^2R_n(\tau) + R_n^2(\tau) \] \hspace{1cm} (3.14)

so the amplitude noise autocorrelation function can be written as:

\[ R_{AA}(\tau) = E_0^4 + N_n(\tau) \] \hspace{1cm} (3.15)

3.2.2. Calculation of photocurrent autocorrelation function

Resolution of the first term gives:

\[ \mathbb{E}[ |E_1(t)|^2 |E_1(t+\tau)|^2 ] = \]
\[ \mathbb{E}[ E_0^2(t) E_0^2(t+\tau) ] = \]
\[ \mathbb{E}[ \{ E_0^2 + 2E_0 E_n(t) + E_n^2(t)\} \{ E_0^2 + 2E_0 E_n(t+\tau) + E_n^2(t+\tau)\} ] = \]
\[
\mathbb{E}\left[ E_0^4 + 2E_0^3 \left( E_n(t) + E_n(t+\tau) \right) \right] + \\
E_0^2 \left( E_n^2(t) + E_n^2(t+\tau) + 4E_n(t)E_n(t+\tau) \right) + \\
2E_0 \left( E_n(t)E_n^2(t+\tau) + E_n^2(t)E_n(t+\tau) \right) + E_n^2(t)E_n^2(t+\tau) = \\
E_0^4 + 0 + E_0^2 \left( 2R_n(0) + 4R_n(\tau) \right) + \\
2E_0 \mathbb{E} \left[ \{E_n(t)E_n^2(t+\tau) + E_n^2(t)E_n(t+\tau) \} \right] + R_n^2(0) + 2R_n^2(\tau) = \\
E_0^4 + 2E_0^2 \left( R_n(0) + 2R_n(\tau) \right) + R_n^2(0) + 2R_n^2(\tau) = \\
E_0^4 + N_n(0) + 2N_n(\tau) = R_{AA}(0) + 2R_{AA}(\tau) - 2E_0. \tag{3.16}
\]

Intermezzo:

\[\mathbb{E}\left[ \{E_n(t)E_n^2(t+\tau) + E_n^2(t)E_n(t+\tau) \} \right] = 0\]

which can be proved using Price's theorem \cite{5}:

\[\mathbb{E}[X^kY^r] = k\int \mu \mathbb{E}[X^{k-1}Y^{r-1}] \, d\mu + \mathbb{E}[X^k]\mathbb{E}[Y^r] \]

where \(\mu = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\). With \(X = E_n(t)\), \(Y = E_n(t+\tau)\), \(k=2\) and \(r=1\) follows:

\[\mathbb{E}[X^2Y] = 2\int \mathbb{E}[X] \, d\mu + \mathbb{E}[X^2]\mathbb{E}[Y] \]

Because \(\mathbb{E}[X]\) and \(\mathbb{E}[Y]\) equal 0, \(\mathbb{E}[X^2Y]\) equals 0.

Resolution of the second term gives:

\[\mathbb{E}\left[ \sum_{p=1}^{\infty} \frac{\left| E_1(t) \right|^2 \left| E_1(t+\tau-pT) \right|^2}{2^p} \right] = \sum_{p=1}^{\infty} \mathbb{E}\left[ \frac{E_1^2(t)}{2^p} \right] \mathbb{E}\left[ \frac{E_1^2(t+\tau-pT)}{2^p} \right] = \\
\mathbb{E}^2\left[ E_1^2(t) \right] \sum_{p=1}^{\infty} \frac{1}{2^p} = R_{EE}(0) = R_{AA}(0) \tag{3.17}\]
Resolution of the third term gives:

\[
E[ \sum_{k=1}^{\infty} \frac{|E1(t+\tau)|^2 |E1(t-kT)|^2}{2^k} ] = \sum_{k=1}^{\infty} E[ E^2(t+\tau) ] E[ E^2(t-kT) ]
\]

\[
= E^2(E(t+\tau)) \sum_{k=1}^{\infty} \frac{1}{2^k} = R_{EE}(0) = R_{AA}(0).
\]

Resolution of the fourth term gives (equivalently to the first term):

\[
E[ \sum_{k=1}^{\infty} \frac{|E1(t-kT)|^2 |E1(t+\tau-kT)|^2}{4^k} ] = \frac{1}{3} ( R_{AA}(0) + 2 R_{AA}(\tau) - 2 E_0^4 )
\]

Intermezzo:

\[
\sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{3}, \text{ calculated using } \sum_{k=1}^{\infty} \frac{1}{a^k} = \frac{1}{a-1}
\]

Resolution of the fifth term gives:

\[
E[ \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} \frac{|E1(t-kT)|^2 |E1(t+\tau-pT)|^2}{2^{k+p}} ]
\]

\[
= E[ \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} \frac{E^2(t-kT) E^2(t+\tau-pT)}{2^{k+p}} ]
\]

\[
= R_{EE}(0) \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} \frac{1}{2^{k+p}} = \frac{2}{3} R_{EE}(0) = \frac{2}{3} R_{AA}(0)
\]

Intermezzo:

\[
\sum_{k=1}^{\infty} \frac{1}{2^{k+p}} = \sum_{k=1}^{\infty} \frac{1}{2^k} \left(1 - \frac{1}{2^k}\right) = 1 - \frac{1}{3} = \frac{2}{3}
\]
Resolution of the sixth term gives:

\[ E[ \sum_{k=1}^{\infty} E_1(t)E_1^*(t-kT)E_1^*(t+\tau-kT)e^{jk\omega_m \tau} ] = \]

\[ \frac{1}{2^k} \sum_{k=1}^{\infty} \{ e^{jk\omega_m \tau} E[ E(t)E(t-kT)E(t+\tau)E(t+\tau-kT) ] \} \times \]

\[ E[ e^{j(\omega t+\phi(t)-\omega(t-kT)-\phi(t-kT)-\omega(t+\tau)-\phi(t+\tau)+\omega(t+\tau-kT)+\phi(t+\tau-kT))} ] \}

\[ \sum_{k=1}^{\infty} \frac{e^{jk\omega_m \tau}}{2^k} R_{AA}(\tau) E[ e^{-j[\Delta \phi_1(\tau)-\Delta \phi_2(\tau)]} ] \]

\[ R_{AA}(\tau) E^2[ e^{j\Delta \phi(\tau)} ] \sum_{k=1}^{\infty} \frac{e^{jk\omega_m \tau}}{2^k} = (3.21) \]

---

**Intermezzo:**

\[ \Delta \phi_1(\tau) = \phi(t+\tau)-\phi(t) \text{ and } \Delta \phi_2(\tau) = \phi(t+\tau-kT)-\phi(t-kT), \]
both have a Gaussian probability density function and are mutually independent because the time delay \( T \) is larger than the coherence time \( t_c \).

\[ E[ e^{-j[\Delta \phi_1(\tau)-\Delta \phi_2(\tau)]} ] = E[ e^{-j\Delta \phi_1(\tau)} ] E[ e^{j\Delta \phi_2(\tau)} ] = E^2[ e^{j\Delta \phi(\tau)} ] \]

The electric field amplitude bandwidth is about 1-10 GHz, yielding a time constant between \( 10^{-9} \) and \( 10^{-10} \) s. This time constant is much smaller than the time delay \( T \) (30 μs), hence the amplitudes \( E(t) \) en \( E(t-kT) \) are mutually independent.

\[ E[ E(t)E(t+\tau)E(t-\tau-1T)E(t+\tau-1T) ] = E[ E(t)E(t+\tau) ] E[ E(t-\tau)E(t+\tau-1T) ] \]

\[ = R_{EE}(\tau) = R_{AA}(\tau) \]
Resolution of the seventh term gives:

\[
\mathbb{E}\left[ \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{E(t)E^*(t-kT)E(t+\tau-pT)E^*(t+\tau-qT)}{\sqrt{2}^{k+p+q}} \exp\left( jk\omega_m \frac{1}{2} - \frac{p+q+1}{2} \right) T \right] =
\]

\[
\sum_{k=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \left\{ e^{jk\omega_m \frac{1}{2}} \mathbb{E}\left[ E(t)E(t-kT)E(t+\tau-pT)E(t+\tau-qT) \right] \frac{1}{\sqrt{2}^{k+p+q}} \right\}
\]

The expectation of the phase term equals 0 (the phases are mutually independent), so term 7 equals 0. For the same reason terms 9, 10 and 11 equal 0.

Resolution of term 8 gives (analogous to the resolution of term 6):

\[
\mathbb{E}\left[ \sum_{k=1}^{\infty} E_1^*(t)E_1(t-kT)E_1^*(t+\tau)E_1(t+\tau-kT) e^{-jk\omega_m \tau} \right] =
\]

\[
R_{AA}(\tau) \sum_{k=1}^{\infty} \frac{e^{-jk\omega_m \tau}}{2^k} \mathbb{E}\left[ e^{j(\Delta \phi_1(t) - \Delta \phi_2(t))} \right] =
\]

\[
R_{AA}(\tau) \mathbb{E}^2\left[ e^{j\Delta \phi(t)} \right] \sum_{k=1}^{\infty} \frac{e^{-jk\omega_m \tau}}{2^k}
\]

(3.23)

Term 12 is unequal to 0 only if \(k=p\) and \(q=1\):

\[
\mathbb{E}\left[ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{E_1^*(t-kT)E_1(t-lT)E_1^*(t+\tau-pT)E_1(t+\tau-qT)}{\sqrt{2}^{k+l+p+q}} \exp\left( j(p-q)\omega_m \frac{1}{2} - \frac{p+q+1}{2} \right) T \right] =
\]
\[
\begin{align*}
\mathbb{E} \left[ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \mathbb{E} \left[ E(t-kT)E(t+\tau-kT)E(t+\tau-1T)E(t+\tau-1T) \right] \times e^{j(k-1)\omega_m t} \right] &= \\
\frac{1}{3} R_{AA}(\tau) \mathbb{E}^2 \left[ e^{j\Delta \phi(\tau)} \right] \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{e^{j(k-1)\omega_m t}}{2^{k+1}} = \\
&= \frac{1}{3} R_{AA}(\tau) \mathbb{E}^2 \left[ e^{j\Delta \phi(\tau)} \right] \sum_{k=1}^{\infty} \frac{e^{jkw_\tau} + e^{-jkw_\tau}}{2^k} 
\end{align*}
\]

Now, the autocorrelation function \( R_{II}(\tau) \) can be written as:

\[
R_{II}(\tau) = R_{AA}(0) + \frac{2}{3} \left( R_{AA}(\tau) - Eo^4 \right) + \\
\frac{1}{3} R_{AA}(\tau) \mathbb{E}^2 \left[ e^{j\Delta \phi(\tau)} \right] \sum_{k=1}^{\infty} \frac{e^{jkw_\tau} + e^{-jkw_\tau}}{2^k} (3.25)
\]

\( \mathbb{E}^2 \left[ e^{j\Delta \phi(\tau)} \right] \) should be interpreted as the phase noise autocorrelation function \( R_{FF}(\tau) \).

By definition:

\[
R_{FF}(\tau) \equiv \mathbb{E}^2 \left[ e^{j\Delta \phi(\tau)} \right] (3.26)
\]

Substitution of (3.26) into (3.25) gives:

\[
R_{II}(\tau) = R_{AA}(0) + \frac{2}{3} \left( R_{AA}(\tau) - Eo^4 \right) + \\
\frac{1}{3} R_{AA}(\tau) R_{FF}(\tau) \sum_{k=1}^{\infty} \frac{e^{jkw_\tau} + e^{-jkw_\tau}}{2^k} (3.27)
\]
Putting \( R_{FFV}(\tau) \) the phase noise autocorrelation function shifted by \( k\omega_m \), where \( \omega_m = 2\pi f_m \) and \( f_m \) is the modulation frequency, gives:

\[
R_{FFV}(\tau) \equiv R_{FF}(\tau) \sum_{k=1}^{\infty} \left( e^{j\omega_m \tau} + e^{-j\omega_m \tau} \right) \frac{e^{-jkw \tau}}{2^k} \tag{3.28}
\]

### 3.3. Calculation of \( S_{II}(\omega) \)

The autocorrelation function \( R_{II}(\tau) \) (eq. 3.26) consists of the amplitude noise autocorrelation function \( R_{AA}(\tau) \) and the phase noise autocorrelation function \( R_{FFV}(\tau) \). The spectrum \( S_{II}(\omega) \) can be expressed in \( S_{AA}(\omega) \) and \( S_{FFV}(\omega) \).

\[
R_{II}(\tau) = R_{AA}(0) + 2 \left( R_{AA}(\tau) - E_0^4 \right) + \frac{1}{3} R_{AA}(\tau) R_{FFV}(\tau) \tag{3.29}
\]

The spectrum \( S_{II}(\omega) \) is:

\[
S_{II}(\omega) = 2\pi \delta(\omega) \left( R_{AA}(0) - \frac{2}{3} E_0^4 \right) + \frac{2}{3} S_{AA}(\omega) + \frac{1}{3} \frac{1}{2\pi} S_{AA}(\omega) \ast S_{FFV}(\omega) \tag{3.30}
\]

where \( \ast \) is the convolution of two spectra.

#### 3.3.1. Calculation of \( S_{AA}(\omega) \)

By definition \( S_{nn}(\omega) = \tilde{\gamma} [ N_n(\tau) ] \), where \( \tilde{\gamma} \) denotes Fourier transformation.

\[
S_{nn}(\omega) = \int_{-\infty}^{\infty} N_n(\tau) e^{-j\omega \tau} d\tau \tag{3.31}
\]

The amplitude noise spectrum can be written as:
\[ S_{AA}(\omega) = 2\pi \delta(\omega) E_0^4 + S_{nn}(\omega) \]  \hspace{1cm} (3.32)

3.3.2. Calculation of \( S_{FFv}(\omega) \)

\[ \mathbb{E}[ e^{j\Delta \phi(\tau)} ] = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} e^{j\Delta \phi(\tau)} e^{-(\Delta \phi^2(\tau))/2\sigma^2} d(\Delta \phi(\tau)) = e^{-|\tau|\Delta \omega/2}, \text{ where } \sigma = \sqrt{\tau \Delta \omega}. \]  \hspace{1cm} (3.33)

\( \Delta \omega \) is the FWHM of the Lorentzian linewidth.

\[ R_{FF}(\tau) = \mathbb{E}[ e^{j\Delta \phi(\tau)} ] = e^{-|\tau|\Delta \omega} \]  \hspace{1cm} (3.34)

Intermezzo:

\[ \int_{-\infty}^{\infty} e^{-a^2x^2+bx} \, dx = \frac{\sqrt{\pi} a}{e^{b^2/4a^2}} \]

\[ S_{FF}(\omega) = \int_{-\infty}^{\infty} R_{FF}(\tau) e^{-j\omega \tau} \, d\tau = \int_{-\infty}^{\infty} e^{-|\tau|\Delta \omega} e^{-j\omega \tau} \, d\tau = \frac{2\Delta \omega}{\Delta \omega^2 + \omega^2} \]  \hspace{1cm} (3.35)

\[ S_{FFv}(\omega) = \int_{-\infty}^{\infty} e^{-|\tau|\Delta \omega} \sum_{k=1}^{\infty} \left( e^{jkw} + e^{-jkw} \right)^k \frac{2\Delta \omega}{\Delta \omega^2 + (\omega-kw)^2} \, d\tau \]  \hspace{1cm} (3.36)

\[ S_{FFv}(\omega) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{2\Delta \omega}{\Delta \omega^2 + (\omega-kw)^2} \]  \hspace{1cm} (3.37)

Substitution of eqs. (3.32) and (3.37) into (3.30) gives:
Resolution of the fourth term of (3.39) gives:

\[
\frac{1}{2\pi} S_{nn}(\omega) \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{2\Delta \omega}{\Delta \omega^2 + (\omega-k\omega_m)^2} =
\]

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{nn}(\nu-\omega) \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{2\Delta \omega}{\Delta \omega^2 + (\nu-k\omega_m)^2} \, d\nu =
\]

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{nn}(\nu-\omega) \frac{1}{1 + \left(\frac{\nu-k\omega_m}{\Delta \omega}\right)^2} \, d\nu =
\]

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{nn}(\omega) \frac{1}{1 + \left(\frac{\nu-k\omega_m}{\Delta \omega}\right)^2} \, d\nu =
\]

\[
S_{nn}(\omega) \cdot \frac{1}{2\pi} 2 \sum_{k=1}^{\infty} \frac{1}{2^k} \left[ \arctan \left(\frac{\nu-k\omega_m}{\Delta \omega_m}\right) \right]_{-\infty}^{\infty} = S_{nn}(\omega).
\]
3.4. Interpretation of the calculated spectrum

Assume the amplitude noise spectrum is white, which is correct because the laser noise bandwidth (1-10 GHz) is much larger than the modulation frequency \( f_m \) (80 MHz), then the second as well as the fourth terms in eq. (3.38) give a constant noise level.

The photocurrent spectrum is:

\[
S_{II}(\omega) = S_{nn}(\omega) + 2\pi R_{AA}(0) \delta(\omega) + \frac{1}{3} \frac{1}{E_o} \sum_{k=1}^{\infty} \frac{1}{k^2} \frac{2\Delta \omega}{\Delta \omega^2 + (\omega - k\omega_m)^2}
\]  

(3.41)
Chapter 4: Conclusions

The measured power spectrum consists of an amplitude noise spectrum and a phase noise spectrum. The phase spectrum exhibits a Lorentzian lineshape which can be found around integer multiples of \( f_m \). The amplitude noise spectrum is a flat noise spectrum. The phase noise spectrum is superposed on the amplitude noise spectrum.

The laser linewidth \( \Delta \omega \), as measured by the proposed set-up, is twice the actual laser linewidth.

This set-up can be used for lasers with very small linewidths. The phase noise spectrum can be found around integer multiples of the modulation frequency. Lasers having linewidths in the order of magnitude of this modulation frequency can not be analyzed accurately, due to the overlap of the spectra.

Furthermore, only if the time delay \( T \) is much larger than the reciprocal amplitude bandwidth \( t_{ca} \), are there uncorrelated phases.
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