On the Robust Stability of MIMO linear Feedback Systems

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ABSTRACT

This work studies the stability of feedback systems where process models are subject to errors, i.e., the robust stability is considered. The multi-input multi-output (MIMO) process is given by a transfer matrix as its nominal model, and by an upper bound matrix as a "structured" description of the model uncertainty (modelling errors). Robust stability analysis will be studied, and several robust stability criteria will be compared. Based on the analysis, a procedure for maximizing the robust stability of the feedback system will be proposed. The weighting functions selection for the sensitivity minimization will be highlighted.

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1. INTRODUCTION

When a process (plant) model is subject to errors, it is more realistic to take the model uncertainty (modelling errors) into account during system analysis and design steps. A so called robust control theory has been developed recently for this purpose (see, e.g. Vidyasagar, 1985, Francis, 1987 and Curtain, 1987).

One of the basic approaches for robustness analysis of MIMO systems is the singular value analysis (Doyle and Stein, 1981). This method describes the process by a nominal model and a perturbation (model uncertainty) which is norm-bounded. The model uncertainty is described by a scalar, hence it is called unstructured model uncertainty meaning that the matrix structure of the uncertainty is lacking in the description. This method is mathematically simple, but it can not use the structural information about model uncertainty which is often available in practice. Therefore it may lead to conservative conclusions.

One way of representing structured model uncertainty of MIMO processes is that the additive model error matrix is bounded by an upper bound matrix $\bar{A}$ where the entries of $\bar{A}(\omega)$ are real positive functions of the frequency $\omega$.

Cloud and Kouvaritakis (1986) proposed such a bound matrix for the black-box finite-impulse-response models, where process output disturbances are assumed to be white noises. Zhu (1987a,b,c) derived such a bound matrix for more general situations, namely, if a model and a (properly generated) input-output data sequence of a linear process is given, the bound matrix can always be estimated: the output disturbances need not be white noises.

Several researchers have studied the problem of robust stability for uncertain systems subject to the structured model errors. Owens and Chotai (1984) and Lunze (1984) proposed to use the spectral radius technique for assessing the stability, which is based on the properties of positive matrices. But their method is not necessarily better than the singular value analysis, in the sense that the method can give more conservative result than the singular value method, and vice versa. Kouvaritakis and Latchman (1985a,b) developed a method
for robust stability analysis, which uses a non-similarity scaling technique. They have derived necessary and sufficient stability conditions, hence the result is not conservative.

In section 2, the methods for robust stability analysis will be introduced and compared; based on the analysis, in section 3, the procedure for maximally robust controller design will be proposed. In section 4, it will be shown how to use the information of model uncertainty to determine the weighting matrix for $H_\infty$-optimization. Section 5 gives conclusions.

**Notation**

- $A$: Matrix with complex elements
- $\bar{\sigma}(A)$: Maximum singular value of $A$
- $\sigma(A)$: Minimum singular value of $A$
- $A^T$: Transpose of $A$
- $A^+$: A matrix with all elements replaced by the absolute values of $A$
- $\rho(A)$: Spectral radius of square matrix $A$
- $P_0(s)$: The true transfer function matrix of the process, dimension $p \times m$, real rational; for convenience the argument will often be dropped
- $P(s)$: The nominal model of $P_0(s)$, real rational, dimension $p \times m$
- $\Delta(s)$: Modelling error: $P_0(s) - P(s)$
- $\bar{A}(\omega)$: Upper bound matrix, with real positive entries, not necessarily rational
- $u$: Input vector of the process, dimension $m$
- $y$: Output vector of the process $P_0(s)$, dimension $p$
- $K(s)$: Feedback controller matrix, dimension $m \times p$, real rational.

**2. ROBUST STABILITY ANALYSIS**

When a controller $K$ has been designed for some purposes, e.g. sensitivity reduction, input tracking, based on the process model $P$, $K$ must stabilize $P$ at least. But this is not enough, $K$ must stabilize the real process $P_0$, which is not completely known. By robust stability analysis we mean checking if this condition has been fulfilled.
2.1 The Class of Perturbations

In this work, the process is described by the model $P$, and the upper bound matrix $\bar{A}$, such that

$$P_0(s) = P(s) + \Delta(s)$$

$$|\Delta_{ij}(j\omega)| \leq \bar{A}_{ij}(\omega) \quad \forall i, j \quad \forall \omega$$

where $P_0(s)$ is the true causal transfer function matrix of the process, which is real rational; $P(s)$ is the nominal model of $P(s)$, real rational and proper; $P(s)$ and $P_0(s)$ have the same number of unstable poles; $\Delta(s)$ is the unknown perturbation matrix, also real rational; and $\bar{A}(\omega)$ is the bound matrix, its entries take the real positive values which are functions of the frequency $\omega$, need not to be rational.

The description of model uncertainty in (2.1) is called structured perturbation, because it keeps the multivariable nature of the problem: the amplitude of the error of each transfer function is bounded by the entries of $\bar{A}$ in the frequency domain, the only missing information is the phase angles of $\Delta(j\omega)$.

Earlier description of the model uncertainty is given by an upper bound on its maximum singular value (norm), $\bar{\sigma}(\Delta(i\omega))$. This class of model uncertainty is called unstructured perturbation, because the bound is a scalar, cannot use any knowledge about the structure of $\Delta(j\omega)$.

One might ask what is the relation between the unstructured uncertainty and structured one given in (2.1). It can be shown (Kouvaritakis and Latchmill, 1985, Zhu 1987c).

$$\bar{\sigma}(\Delta(j\omega)) \leq \bar{\sigma}(\Delta^+(j\omega)) \leq \bar{\sigma}(\bar{\Delta}(\omega))$$

where $\Delta^+ := \{|A_{ij}^+|\}$

This means that if $\bar{\Delta}(\omega)$ is known, one can calculate a bound for the maximum singular value of $\Delta$ from it.

Denote the class of structured perturbations
and the class of unstructured perturbation

\[ D_u := \{ \Delta : \sigma(\Delta) \leq \sigma(\bar{\Delta}) \quad \forall \omega \} \quad (2.4) \]

Then (2.2) implies that the set of \( D_s \) is a subset of the set of unstructured perturbation \( D_u \) (see Kouvaritakis and Latchman, 1985a, for a formal proof). This means that the singular value analysis may be used to determine only sufficient stability conditions, for the perturbations in the class \( D_s \). Thus, the result of the analysis can be conservative.

2.2 Robust Stability Criteria

The block diagram of the feedback system is shown in Fig. 2.1, where \( K \) is assumed to be real rational and \( P \) and \( \Delta \) are defined in (2.1).

![Fig. 2.1 The process and the controller in closed-loop.](image)

The problem is: suppose \( K \) stabilizes \( P \), check if \( K \) stabilizes \( P_0 = P + \Delta \), based on the knowledge of \( P, \Delta \) and \( K \).

The robust stability criteria of feedback systems are based on the MIMO generalization of the Nyquist criterion, which can be formulated as (see e.g. Vidyasagar, 1985).
Lemma 2.1.

Suppose \( P_0(s) \) and \( K(s) \) has \( n_p, n_k \) poles in the open right-half-
plane, counted according to McMillan degree, and non on the \( j\omega \)-axis. Then \( K(s) \) stabilizes \( P_0(s) \) if and only if the plot of 
\[
\det(I+P_0(j\omega)K(j\omega))
\]
as \( \omega \) decreases from \( \infty \) to \( -\infty \) does not pass through 
the origin of the complex plane and circles the origin \( n_p+n_k \) times 
in the clockwise sense.

It is obvious that we cannot use this criterion to test the stability 
of the system in Fig. 2.1, because \( P_0 \) is not exactly known in (2.1).
What we can check is the number of encirclements (n.o.e.) of 
\[
\det(I+P(j\omega)K(j\omega)).
\]

According to our assumption, \( P(s)K(s) \) and \( P_0(s)K(s) \) have the same 
number of unstable poles. Hence the system is robustly stable if and only if 
\[
\{ \text{n.o.e. } \det[I+(P+\Delta)K] = \text{n.o.e. } \det[I+PK], \quad \forall \Delta \in D_s \}.
\]

This is assured if and only if \( \det[I+(P+\Delta)K] \) remains non-zero as \( P \) is 
warped continuously towards \( (P+\Delta) \), i.e.
\[
\det[I+(P(j\omega)+\epsilon\Delta(j\omega))K(j\omega)] \neq 0 \quad \forall \epsilon \in [0,1] \quad \forall \omega \quad \forall \Delta \in D_s \quad (2.5)
\]

But
\[
\det[I+(P+\epsilon\Delta)K] \\
= \det[I+PK] + \epsilon \Delta K \\
= \det[I+PK] \det[I+\epsilon \Delta K(I+PK)^{-1}] 
\]

Because \( K \) stabilizes \( P \), we have
\[
\det[I+PK] \neq 0 \quad \forall \omega.
\]
Thus (2.5) is assured if and only if
\[
\det[I+\epsilon \Delta K(I+PK)^{-1}] \neq 0 \quad \forall \epsilon \in [0,1] \quad \forall \omega \quad \forall \Delta \in D_s \quad (2.6)
\]

Denote \( \rho(A) \) as the spectral radius of matrix \( A \):
\( p(A) := \max_i |\lambda_i(A)| \), and \( \lambda_i(A) \) is the \( i \)-th eigenvalue of \( A \), then we have

**Lemma 2.2.**
The system in Fig. 2.1 is stable if and only if

\[
p[\Delta K(I+PK)^{-1}] < 1 \quad \forall \omega \quad \forall \Delta \in \mathbb{D}_s \tag{2.7}
\]

**Proof.** It will be shown that (2.6) and (2.7) are equivalent.

"If" part. Suppose that

\[
\det [I+\varepsilon \Delta K(I+PK)^{-1}] = 0,
\]
then \(-1\) is an eigenvalue of \( \varepsilon \Delta K(I+PK)^{-1} \). But

\[
p[\varepsilon \Delta K(I+PK)^{-1}] \leq p[\Delta K(I+PK)^{-1}] \quad \forall \varepsilon \in [0,1]
\]

\[
p[\Delta K(I+PK)^{-1}] \geq 1
\]
This contradicts (2.7).

"Only if" part. It is obvious that

\[
p[\varepsilon \Delta K(I+PK)^{-1}] \neq 1 \quad \forall \varepsilon \in [0,1], \forall \Delta \in \mathbb{D}_s
\]
is a necessary condition of (2.6). Now suppose that \( \exists \Delta \in \mathbb{D}_s \), such that

\[
p[\Delta K(I+PK)^{-1}] > 1
\]
then \( \exists \varepsilon \in [0,1] \) such that, for the same \( \Delta \in \mathbb{D}_s \)

\[
p[\varepsilon \Delta K(I+PK)^{-1}] = 1
\]
and (2.6) is violated. This ends the proof.

Again, this can not be used directly, because we only know the upper bound matrix \( \bar{\Delta} \). But (2.7) is the starting point for deriving other applicable stability criteria.

We know that the spectral radius of a matrix is bounded by its maximum singular value (Doyle and Stein, 1981). Then for each \( \omega \):

\[
p[\Delta K(I+PK)^{-1}]
\]
\[ \sigma(A) \leq \sigma[\Delta R(I+PR)^{-1}] \]
\[ \sigma(A) \text{ is a norm of } A \]
\[ \sigma(\Delta) \sigma[K(I+PR)^{-1}] \leq \sigma[\Delta R(I+PR)^{-1}] \]

Denote \( T := K(I+PR)^{-1} \), we get the well known singular value analysis criterion:

**Theorem 2.1.**
The system in Fig. 2.1 is stable if
\[ \sigma(\Delta) \sigma(T) < 1 \quad \forall \omega \]  \hspace{1cm} (2.8)

As discussed before, this criterion does not use the structural information of \( A \), it only gives sufficient stability condition and can be conservative. The advantages of this method is its numerical simplicity and reliability. Therefore it can be used as a rough assessment of the stability.

The bound matrix \( \Delta \) is a positive matrix, one can think of using the theory of non-negative matrices for robust stability test. From this theory (Berman and Plemmos, 1979), we have
\[ \rho(\Delta T) \leq \rho[(\Delta T)^+] \leq \rho(\Delta^+ T^+) \leq \rho(\Delta^+ T^+) \]  \hspace{1cm} (2.9)

Thus, (2.7) and (2.9) give another stability condition (Owens and Chotai, 1984; Lunze, 1984), which is called spectral radius method:

**Theorem 2.2.**
The system in Fig. 2.1 is stable if
\[ \rho(\Delta T^+) < 1 \quad \forall \omega \]  \hspace{1cm} (2.10)

This method uses the structural information of \( \Delta \), but it does not fully use the information of \( T \) - it ignores the phase information in matrix \( T \). In general \( T \) is a complex matrix, hence this method can
also be conservative. The advantage of this method is the same as the singular value analysis—it is numerically simple and reliable.

Lunze (1984) claimed that the spectral radius method is superior to the singular value analysis, because

$$\rho(\tilde{A}T^+) \leq \sigma(\tilde{A})\sigma(T). \quad (2.11)$$

But the following is a counter-example of (2.11).

Let $$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

then

$$\rho(\tilde{A}T^+) = 2 > \sigma(\tilde{A})\sigma(T) = \sqrt{2}$$

This implies that (2.11) does not hold in general, and the spectral radius method is not necessarily better than the singular value analysis. At the end of this section, we will see an example, where (2.11) does hold. So, one can not say that singular value analysis is better than the spectral radius method either.

If we cannot find a better method, now the best we can do is to use:

**Corollary to Theorem 2.1 and 2.2.**

The system in Fig. 2.1 is stable if

$$\min \{\sigma(\tilde{A})\sigma(T), \rho(\tilde{A}T^+)\} < 1 \quad \forall \omega \quad (2.12)$$

Fortunately, there is a better way. Let L and R be the diagonal, positive, non-singular matrices, then

$$\rho(\tilde{A}T) = \rho(L\tilde{A}R \, R^{-1}T L^{-1}) \quad \text{(Similarity transformation)}$$

$$\leq \sigma(L\tilde{A}R^{-1}T L^{-1})$$

$$\leq \sigma(L\tilde{A}R) \, \sigma(R^{-1}T L^{-1})$$

We know that (from (2.2)) $$\sigma(L\tilde{A}R) \leq \sigma(L\tilde{A}R)$$, hence
\[ \rho(\Delta T) \leq \bar{\sigma}(L\bar{A}R) \bar{\sigma}(R^{-1}TL^{-1}) \] 

where \( L \) and \( R \) are scaling matrices, introduced by Kouvaritakis and Latchman (1984a,b). They suggested choosing \( L \) and \( R \) such that

\[ \bar{\sigma}(L\bar{A}R)\bar{\sigma}(R^{-1}TL^{-1}) \]

is minimized, and developed the following important result:

**Theorem 2.3.**
The system in Fig. 2.1 is stable if and only if

\[ k_\circ(\bar{A},T) := \min_{L,R} [\bar{\sigma}(L\bar{A}R)\bar{\sigma}(R^{-1}TL^{-1})] < 1 \quad \forall \omega \] 

This is called non-similarity scaling method. The "if" part of the theorem is proved by combining (2.7) and (2.13).

The remarkable feature of this stability condition is that (2.14) is also a necessary condition, meaning that if (2.14) does not hold, there exists at least one modelling error in the class of structured uncertainty, \( \Delta D_s \), such that the system in Fig. 2.1 is unstable, see Kouvaritakis and Latchman (1985b) for the proof. Because (2.14) is a necessary and sufficient stability condition, this method makes the best use of the structural information given in \( \bar{A} \).

The optimal scaling matrices \( L \) and \( R \), which make \( \bar{\sigma}(L\bar{A}R)\bar{\sigma}(R^{-1}TL^{-1}) \)
minimal, must satisfy (Kouvaritakis and Latchman, 1985b)

\[ \sigma(LT^{-1}R) \frac{\partial}{\partial x} [\bar{\sigma}(L\bar{A}R)] = \bar{\sigma}(LTR) \frac{\partial}{\partial x} [\sigma(LT^{-1}R)] \] 

where \( L = \text{diag} \{ l_1, l_2, \ldots, l_m \} \), \( R = \text{diag} \{ r_1, r_2, \ldots, r_p \} \)
and \( x^T = [l_1, l_2, \ldots, l_m, r_1, r_2, \ldots, r_p] \); \( \sigma(A) \) is the minimum singular value of \( A \).

Explicit formulae for the derivatives of \( \bar{\sigma}(L\bar{A}R) \) and \( \sigma(L^{-1}TR) \) with respect to the elements of \( L \) and \( R \) are given in (Kouvaritakis and Latchman, 1985a), which form the basis of an efficient numerical algorithm for computing the optimal \( L \) and \( R \). They have tried numerous examples, the algorithm reliably converged to the optimal solution.
2.3. Comparing the Three Criteria

The non-similarity scaling method in Theorem 2.3 gives a necessary and sufficient robust stability condition, under the structured perturbation, and the stability criterion is not conservative. The singular value analysis method in Theorem 2.1 and the spectral radius method in Theorem 2.2 only give sufficient conditions for the stability, they are conservative for the structured perturbation. Here we will give some more explicit comparison of the three criteria.

In (2.14), if we take \( L = I, R = I \), we get the singular value method as in Theorem 2.1. But \( L = I, R = I \) in general are not the optimal scaling matrices, hence

\[
\sigma(\tilde{A}) \sigma(T) \geq \min_{L,R} \sigma(L\tilde{A}R) \sigma(R^{-1}TL^{-1})
\]

This means that the optimal scaling always give a tighter bound on \( \rho(\Delta T) \).

Because \( \Delta T^+ \) is a positive square matrix, then according to matrix theory, \( \Delta T^+ \) has a positive eigenvalue, equal to its spectral radius; and there is a positive eigenvector associated with this eigenvalue which is called Perron eigenvector. The same follows for \( T^+\Delta^+ \).

Let \( x \) and \( y \) be the right and left Perron eigenvectors of \( \Delta T^+ \) respectively, and \( u \) and \( v \) be the equivalent Perron eigenvectors of \( T^+\Delta^+ \).

Define the Perron scaling to be

\[
L_* = \text{diag} \left\{ \sqrt{y_i/x_i} \right\} \text{ and } R_* = \text{diag} \left\{ \sqrt{u_j/v_j} \right\}
\]

where \( x_i, y_i, i=1,2,\ldots, p \), \( u_j, v_j, j=1,2,\ldots, m \), are the \( i \)-th and \( j \)-th element of \( x, y, u \) and \( v \) respectively. According to Bauer (see Kouvaritakis and Latchman, 1985b), we have

\[
\min_{L,R} \sigma(L\tilde{A}R) \sigma(R^{-1}TL^{-1})
\]

\[
= \sigma(L_*\tilde{A}R_*) \sigma(R_*^{-1}T^+L_*^{-1})
\]

\[
= \rho(\Delta T^+)
\]

Apply (2.2) to \( R_*^{-1}TL_*^{-1} \), it follows
Combining this with (2.17) we obtain:

**Theorem 2.4** (Kouvaritakis and Latchman, 1985b)

\[
\sigma(R_*^{-1}TL_*^{-1}) \leq \sigma \left[ (R_*^{-1} TL_*^{-1})^+ \right] = \sigma(R_*^{-1} T^+ L_*^{-1})
\]

This theorem provides a suboptimal solution to the non-similarity scaling problem, which is an explicit improvement over the spectral radius method. The Perron scaling matrices \(L_\ast\) and \(R_\ast\) are easy to obtain. The suboptimal solution gives in general only sufficient stability conditions, and can be conservative.

**Example 2.1** (Kouvaritakis and Latchman, 1985b)

Suppose at some frequency we have

\[
\begin{bmatrix}
  0.5 & 0.6 & 1 & 1 \\
  2 & 0.7 & 0.3 & 0.3 \\
  1 & 0.4 & 0.1 & 0.8 \\
  0.25 & 0.2 & 0.2 & 0.2
\end{bmatrix},
\quad
\begin{bmatrix}
  1+j1 & -1+j2 & 3-j5 & -3+j2 \\
  4-j5 & 0+j2 & -1+j0 & 1+j0 \\
  3-j1 & 2+j4 & 1-j1 & 0+j3 \\
  0-j4 & 1+j0 & 2+j0 & 1-j1
\end{bmatrix}
\]

The results are in the following table

| \(\bar{\sigma}(\bar{\Delta})\bar{\sigma}(T)\) | 28.2021 |
| \(\rho(\bar{\Delta}T^+)\) | 27.144 |
| \(\min_{L_\ast R_\ast} \bar{\sigma}(L_\ast R_\ast) \bar{\sigma}(R_\ast^{-1} TL_\ast^{-1})\) | 23.75124 |
| \(\bar{\sigma}(L_\ast\bar{\Delta}R_\ast)\bar{\sigma}(R_\ast^{-1} TL_\ast^{-1})\) | 24.2535 |

We see that the optimal scaling gives the best result; the suboptimal scaling gives better result than the singular value result for this example; note that here the spectral radius result is better than the singular value result.
3. **ROBUST STABILITY OPTIMIZATION**

In the previous section we studied robust stability analysis, i.e., given $P$, $K$, and $\Delta$, check if the system in Fig. 2.1 is stable. Here, based on the analysis, we will study the problem of robust stabilizing, i.e., given $P$ and $\Delta$, find $K$ such that the closed-loop system in Fig. 2.1 is robustly stable; if possible, find the maximally robust controller.

From lemma 2.2 we know that
\[
\rho(\Delta T) < 1 \quad \forall \omega \quad \forall \Delta \in \Delta D_S
\]
is the necessary and sufficient stability condition. So, the maximally robust controller is the one which
\[
\min_{K} \sup_{\Delta \in \Delta D_S} \sup_{\omega} \rho(\Delta(j\omega)T(j\omega)) \tag{3.1}
\]
When the process is stable, then $\Delta$ and $P$ are stable, $K=0$ belongs to the set of stabilizing controller, (3.1) tells us that $K=0$ is also the maximally robust controller, because
\[
\rho(\Delta(j\omega)T(j\omega) = 0 \quad \text{for } K=0 \quad \forall \omega \quad \forall \Delta \in \Delta D_S \tag{3.2}
\]
Note that $K=0$ means no feedback control, and the result tells us that the best stabilizing controller for a stable process is "no control". This is true in practice: no one asks you to stabilize an already stable process; feedback can stabilize an unstable process, it can destabilize a stable process as well.

So, we assume for the rest of this section that the process is unstable. In this situation, we cannot do much with (3.1), because the minimization in (3.1) is mathematically difficult. From theorem 2.3, it is clear that (3.1) is equivalent to
\[
\min_{L,R,K} \sup_{\omega} \sigma(LAR)\sigma(R^{-1}TL^{-1}) \tag{3.3}
\]
This is a very complicated minimization problem, with $L(\omega)$, $R(\omega)$ being diagonal, invertable, non-rational, and $K(j\omega)$ being real-rational.
We propose the following algorithm for solving (3.3).

Algorithm 3.1

(0) Put $L_0 = I$, $R_0 = I$ and $i = 1$

i-th iteration

(1) Determine $K_i$ by

$$
\min_{K_i} \sup_{\omega} \sigma(L_{i-1} \bar{A} R_{i-1}) \| R_{i-1} T_i L_{i-1} \|_{\infty}
$$

that is

$$
\min_{K_i} \sigma(L_{i-1} \bar{A} R_{i-1}) \| R_{i-1} T_i L_{i-1} \|_{\infty}
$$

(2) Determine $L_i(\omega)$, $R_i(\omega)$ by Theorem 2.3

$$
\min_{L_i, R_i} \sigma(L_i \bar{A} R_i) \sigma(R_i T_i L_i) \quad \forall \omega
$$

(3) Set $i = i + 1$

Goto (1).

Define

$$(RC)_i := \sup_{\omega} \sigma(L_i \bar{A} R_i) \sigma(R_i T_i L_i)$$

Then, it is obvious that for the i-th iteration in the algorithm,

$$(RC)_i \leq (RC)_{i-1}.$$

But $(RC)_i \geq 0 \quad \forall i$. Hence, we can say

Theorem 3.1.
Algorithm 3.1 converges.

This is a "principle algorithm", because the optimization in step (1) is not solvable when $R_i^{-1}$ and $L_i^{-1}$ are not rational. The applicable algorithm which approximates the "principle algorithm" is
the following:

**Algorithm 3.2**

(0) Put \( \bar{L}_0 = I, \bar{R}_0 = I \)

*i-th iteration*

(1) Determine \( K_i \) by

\[
\min_{K_i} \| w_{i-1}(j\omega)\bar{R}_{i-1} \cdot T_i \cdot \bar{L}_{i-1} \|_\infty
\]

(2) Determine \( L_i(\omega), R_i(\omega) \) by

\[
\min_{L_i, R_i} \sigma(L_i\bar{A}R_i)\sigma(R_i^{-1}T_i^{-1}) \quad \forall \omega
\]

(3) Approximate \( \sigma(L_i\bar{A}R_i), L_i^{-1}(\omega) \) and \( R_i^{-1}(\omega) \) by the real-rational

\( w_i(j\omega), \bar{L}_i(j\omega) \) and \( \bar{R}_i(j\omega) \), which are stable and minimum-phase.

Goto (1)

(4) Stop when \( (RC)_i > (RC)_{i-1} \). This can happen because step (3) causes errors.

We see that each iteration of Algorithm 3.2 includes (1) \( H_\infty \)-optimization, (2) determination of the optimal scaling matrices and (3) model reduction which finds the stable real-rational functions to approximate the given frequency response. The \( H_\infty \)-optimization deals with real-rational matrices, so, it is parametrical; finding the optimal scaling matrices is done at each frequency, it is not parametrical; and they are connected by a special kind of model reduction.

If one, for the simplicity, decides to work with the unstructured perturbation \( D_u = \{\Delta: \sigma(\Delta) \leq \sigma(\bar{\Delta}) \quad \forall \omega\} \), then from Theorem 2.1, maximally robust controller with respect to \( D_u \) is determined by

\[
\min_{K} \sup_{\omega} \sigma(\bar{\Delta}) \sigma(T) \quad \text{(3.4)}
\]
But

\[ \tilde{\sigma}(\Delta) \tilde{\sigma}(T) = \tilde{\sigma} [\tilde{\sigma}(\Delta) T] \]

and

\[ \sup \tilde{\sigma} [\tilde{\sigma}(\Delta) T] = : \| \tilde{\sigma}(\Delta) T \|_\infty \]

Thus the problem (3.4) becomes

\[ \min \| \tilde{\sigma}(\Delta) T \|_\infty \] \hspace{1cm} (3.5)

But in practice \( \tilde{\sigma}(\Delta(\omega)) \) is a non-parametrized data sequence, not rational, so (3.5) is not yet solvable.

Denote \( w(j\omega) \) as the real rational approximation of \( \tilde{\sigma}(\Delta) \):

\[ |w(j\omega)| = \tilde{\sigma}(\Delta) \] \hspace{1cm} \forall \omega

and \( w(j\omega) \) is stable and minimum-phase. Replacing \( \tilde{\sigma}(\Delta) \) by \( w(j\omega) \) in (3.5), we get the nearly optimal solution \( K \) by

\[ \min_K \| w(j\omega) T (j\omega) \|_\infty \] \hspace{1cm} (3.6)

This is a standard \( H_\infty \)-control problem, which can be solved either via state-space realizations (Francis, 1987) or by polynomial method (Kwakernaak, 1987).

But this solution is not optimal with respect to the structured model uncertainty \( D_s \).

Remark:

Neither Algorithm 3.2 nor (3.6) will guarantee to deliver a \( K \) which is robustly stabilizing for all \( \Delta \in D_s \), but this can be checked by the robust stability test.

4. WEIGHTING FUNCTION SELECTION FOR SENSITIVITY MINIMIZATION

In previous sections, only robust stability was considered.

In the real engineering control system design, however, one should consider other design aspects. For a large class of industrial process control, the most important requirements are disturbance attenuation (sensitivity reduction), control signal power and bandwidth limitation and robust stability. We will show here, how these problems can be transferred to the standard \( H_\infty \)-optimization problem, by
proper choices of weighting functions.

Considering the feedback system in Fig. 2.1 again, and assume that the process disturbance acts at system output, see Fig. 4.1.

\[ u = y 
\]

\[ \Delta \]

\[ P \]

\[ K \]

\[ y \]

\[ V_0 \]

\[ d \]

\[ v \]

\[ \Phi_{yy} = S_0 V_0^* S_0^* \quad \text{and} \quad \Phi_{uu} = T_0 V_0^* S_0^* \]

where \( S_0 := (I+P_0 K)^{-1} \) is called sensitivity matrix, and

\( T_0 := K S_0 = K(I+P_0 K)^{-1} \) is called power transform matrix, and

\[ s_0^*(-j\omega) = S_0^T(-j\omega) \]

Then, following Grimble (1986), the disturbance attenuation and control signal power and bandwidth limitation can be done by solving the
\( H_\infty \)-optimal control problem

\[
\min \| W_s \Phi_{yy} + W_T \Phi_{uu} \|_\infty
\]

(4.1)

where \( W_s(s) \) and \( W_T(s) \) are weighting matrices, reflecting the designers requirements on the control system. Compared to the LQG problem, the physical significance of \( H_\infty \)-control problem in (4.1) is that a signal power in certain frequency range is more important than the total energy in the signal.

If we work with the models of \( P_o \) and \( V_o \), (4.1) becomes

\[
\min \| W_s SVV^*S^* + W_T TVV^*T^* \|_\infty
\]

(4.2)

where \( S := (I+PK)^{-1} \) and \( T := KS = K(I+PK)^{-1} \).

It was shown in previous section that robust stability optimization with respect to unstructured model perturbation is done by

\[
\min \| w^T \|_\infty
\]

\( K \)

It is easy to see that this is equivalent to

\[
\min \| w \|^2 \| T^* \|_\infty
\]

(4.3)

Finally, we see that the natural way to combine disturbance attenuation, power limitation and robust stability, is to minimize the following criterion:

\[
\| X \|_\infty := \| W_s SVV^*S^* + \alpha W_T TVV^*T^* + \beta \| w \|^2 T^* \|_\infty
\]

(4.4)

This is a mixed sensitivity \( H_\infty \)-optimization problem, where \( \alpha \) and \( \beta \) are positive real constants, which are used to adjust the relative importance of each term in (4.4).

5. CONCLUSIONS

Robust stability analysis and design have been studied. Process model uncertainty is described as structured perturbation. Three robust stability analysis methods have been discussed and compared. The singular value analysis method and spectral radius method are both simple, but the results are conservative. The nonsimilarity scaling method is not conservative. But it needs some optimization procedure
to find the optimal scaling matrices. The suboptimal solution to the problem, however, is easy to obtain, and gives better result than the spectral radius method.

Based on the analysis, the procedures for finding the maximally robust controller are proposed. It becomes a $H_\infty$-optimization when using the unstructured perturbation. When the structured perturbation is used, one needs more complicated iteration procedure, where $H_\infty$-optimization is part of the iteration.

Finally, robust stability is considered together with disturbance attenuation and control signal power limitation, and it becomes a mixed sensitivity $H_\infty$-optimization problem; and only unstructured model perturbation is used.

It should be clear that when the process is single-input single-output (SISO), the class of unstructured perturbation is the same as the class of structured one; $D_u=D_g$. The singular value analysis method will give necessary and sufficient stability conditions and non-similarity scaling method in section 2 and the iteration procedure in section 3 are not necessary.

All the results given in this work, are computable, therefore, they form part of a basis for computer aided control system analysis and design.

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