Eindhoven University of Technology  
Department of Mathematics and Computing Science

MASTER’S THESIS

NUMERICAL INVESTIGATION  
OF PRESSURE PERTURBATIONS  
IN ACOUSTIC LINERS

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Eindhoven, Augustus 2005
Abstract

In this report we try to make a mathematical model of an acoustic liner. With this model we are able to perform numerical calculations on the rate of absorption of noise generated by the engine of an airplane.

First, the iris problem is discussed to highlight and show the problems we will encounter when trying to make such a model. Here, we suggest to use mode-theory and matching as a means to circumvent these problems.

Secondly, in chapter three we derive the equations that govern flow in our model of an acoustic liner. We use mode-theory and matching to express these as a set of equations. Next, we implement these equations numerically into a computer program to obtain a pressure profile. We observe that our model accurately describes the situation in an acoustic liner. The rate of absorption and the influence of the number of modes chosen can be observed.

When we add a uniform mean flow to our model, we can again use the same techniques to obtain a pressure and velocity profile. Again a computer program is used to discuss the situation in an acoustic liner.
Preface

Before you lies my Master’s thesis; my final work for the master Industrial and Applied Mathematics at the Eindhoven University of Technology. This project lasted little over six months and was performed as a combined assignment from both the physics and the mathematics and computer science department.

In the first chapter, a short introduction to acoustics, and more particular aero-acoustics, is given. One of the most important applications of aero-acoustics is sound reduction. Reduction of noise generated by airplanes is a very current topic. The last decades great improvements have been made in this field of research. In this report we focus on acoustic liners that absorb a large portion of the noise generated by the engines of an airplane. We try to search for models of an acoustical liner to allow numerical calculations.

The next chapter discusses the iris problem. This problem is discussed to understand more about the ill-posed boundary conditions of our problem. As a solution mode- theory and matching is suggested and we show the non-uniqueness of the solution by discussing the edge condition.

The third chapter deals with our situation again and we use the knowledge obtained from the iris problem to derive a mathematical model that describes the flow of sound through a schematic version of an acoustic liner. We, again, use mode- theory and matching to find a set of equations that describe flow and pressure perturbations. These equations are numerically implemented in a computer program and the results from simulations will be discussed.

The following chapter introduces a uniform mean flow in our liner. We again use the same theory and implement this into another program. Some simulations will be done and the results will be discussed.

Finally, we summarize what has been done in all chapters. Then we draw conclusions on what has been accomplished in this report and finally a number of recommendations are given for future research.

During the six months I worked on this thesis, I have had a lot of help from people around me. I would hereby like to thank professor Hirschberg and Gerben Kooijman from the physics department for showing me a different point of view to my problem and for their continuing interest in my work. In addition, I would like to thank my colleague students Wendy Versteeg and Dan Roozemond for many corrections, suggestions, and help with a large variety of questions. A
special thanks to Jaap Bastiaansen for a lot of feedback on my work and the many discussions we had on aero-acoustics in general. Finally I would like to thank the members of my exam committee. First of all professor van Dongen. Secondly, professor Mattheij, who has been my supervisor for three years now. And lastly a special thanks to dr. Rienstra for his supervision on my thesis.

I wish you a pleasant read.

Wouter van der Horst, bijgenaamd Linders
Eindhoven, August 12, 2005.
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Chapter 1

Introduction

An important archaeological maya-site can be found near the famous Mexi-
can tourist area of Cancun. It is the former political, military, and religious
capital of the Maya Indians between 800 and 1200 AD: Chichen Itza. Several
archaeologists have stumbled upon some interesting acoustical effects [2].

One of these interesting effects can be found on a sport field surrounded by a
wall. A coach can communicate with the other coach over a distance of hundred
meters and yet cannot be overheard by the players on the field. This is due to
the specific structure of the walls that transmit the sound of one coach to the
other with hardly any reflection.

Interesting acoustical effects can also be observed inside the El Castillo pyra-
mid. An observer seated on the lowest step of the pyramid hears the sound of
raindrops falling in a water bucket instead of footstep sounds when people, sit-
uated higher up the pyramid, climb the stairs. In addition, in response to a
handclap, the pyramid produces an echo that sounds like a chirp of the Quetzal
bird; a coloured bird from Central America that was considered holy by the
Maya’s.

These mysterious acoustical effects have been around for centuries. A good
reason for Declercq et al. from the University of Gent to perform extensive
research on this topic. Research pointed out that a mathematical sound pulse
causd some whistling, but not the specific sound that was observed in the pyra-
mid [2]. To obtain this result, Declercq had to simulate the handclapping in the
model for it contains the low frequencies needed to obtain the chirp. Hence, the
specific sound observed depends not only on the structure of the pyramid, but
also of the specific sound source.

New and quite complicated numerical models that cost a lot of computational
time can explain both the Quetzal sound as well as the sound of raindrops
falling. The question remains if the Maya’s have built this pyramid to cause
these acoustical effects or if it happened by accident. Perhaps they were also
performing a lot of research on this topic. Quite impressive for they did not
have the means that we have nowadays. We can at any rate conclude that the
Maya’s were already very much engaged in a field of science that is now called acoustics.

1.1 Acoustics and aero-acoustics

Acoustics is the science of sound, including its production, transmission, and effects \(^1\). Nowadays, sound is not only the phenomenon that can be perceived by our ears, but it stretches to all phenomena that are governed by the same physical principles. One can distinguish several forms of acoustics: sound in air, underwater sound, sound in solids, or structure-borne sound [7].

But as not all acoustics necessarily have applications, a significant part of this field of science has nowadays. The interest in the broad scope of acoustics can be ascribed to several reasons. The first reason is the mechanical radiation generated by natural or artificial sources. In addition, we have the ability to hear sound or communicate via sound along with a variety of psychological influences sound can have on observers. Sound is a means of transmitting information (whether we can hear it or not) and the variation of the media through which it passes and the objects positioned in the media, can be used to gather information.

As can be read before, acoustics has been a topic of interest for quite a long time. It started as early as with Greek philosophers and continued via Newton until the computer-age. A more extended history on acoustics can be read in Pierce [7].

Whenever a relative motion exists between two fluids or a fluid and a surface, sound is produced. In almost all occurrences, the source for the production of noise is some form of flow unsteadiness. An example of a study in this field of science is research concerning the physical principle for the generation of flow-induced pulsations in flexible risers. The discipline studying the principles and its related phenomena is called aero-acoustics [3].

1.2 Noise Reduction

One of the most important applications of aero-acoustics is that part where the reduction of noise is studied. For many companies, the reduction of noise has never been as important as it is today. Due to environmental reasons, economic reasons and the growing discontent among the population, political interest in noise reduction has risen. Therefore budgets have increased to study possible solutions to decrease the amount of noise generated.

Particular political interest has arisen concerning the sound generated by airplanes. Due to the amount of noise generated by an airplane, a social debate has begun concerning the rate of inconvenience for neighbouring citizens. Noise

\(^1\)This definition is conform to ANSI S1.1960 (R1976) American National Standard Acoustical Terminology, American National Standards Institute, Inc. (ANSI), New York, 1976
can not only be acknowledged as annoying, but it can also lead to damage of the health of citizens. Some of these health damages are disturbance of sleep, stress-related diseases, like coronary and vascular diseases, and influence on performances and concentration [9].

Research has pointed out that because of the nuisance of noise, 10% of the citizens living near an airport have serious forms of sleep disturbances, 2% of them wakes up more than 3 times each night, and that 0.5% of them use heavy sleeping pills because of the noise [9]. The disturbance of sleep due to aircraft is four times the disturbance from the entire railway and half the disturbance from the entire Dutch car-traffic.

On June 30, 2005, the Advisory Council for Transport, Public Works, and Water Management published their advice concerning the future of aviation in the Netherlands. In this article, the council states that, despite the health problems, Schiphol (Amsterdam Airport) must be able to develop even further. The importance of international connections for the Dutch economy is too large to prevent growth [9]. Even more reasons to perform extensive research on the possibilities of sound reduction on airplanes.

In order to know what leads to the reduction of sound, we need methods to measure the acoustic output of airplanes. We can roughly distinguish two types of methods to determine the sound level at certain distances from an airplane. We can use devices to measure the decibels or we can use computer simulations to (numerically) calculate the sound levels at specific distances. Research on this topic has blossomed the last 50 years. An example of a large company that spends a large portion of their budget on research regarding the reduction of noise is NASA (National Aeronautics & Space Administration). In the Netherlands most research is done by the NLR (National Aerospace Laboratory), Boeing, Rolls Royce, and Airbus. Most of this research is done for KLM (Royal Dutch Airlines).

1. The NASA has been performing research on sound reduction since the 50s. More recent: NASA has begun her Advanced Subsonic Transport (AST) Noise Reduction program in 1994 to develop technology to reduce jet transport noise [8]. The program focuses especially on technical aspects. The sources of sound are considered and revised in order to decrease the noise generated.

2. The KLM (Royal Dutch Airlines) is continually paying research laboratories and airplane manufacturers to work on sound reduction. To achieve this they have introduced new flight procedures. In addition, they have replaced airplanes for passenger and cargo flights. This has already lead to a significant (11 dB and 30% compared to 10 years before, respectively) reduction in sound level. They focus on technical measures to reduce the level of noise, nowadays [4].
1.3 Technical solutions

In this report we will focus on specific technical solutions. The last 50 years a lot of progress has been made in this field of science. Technical solutions to reduce noise are diverse, but a very important one is the development of new types of engines. The research for engines that generate less sound and the production of these engines has resulted in a reduction of noise. For an overview of the different type of the engines of the aircraft over the years and the amount of noise they produce, see figure (1.1).

![Diagram of noise levels from the 50s onwards]

Figure 1.1: Noise levels from the 50s onwards

We notice that the introduction of new types of engines and the implementation of these into both the passenger- as well as cargo-flights has resulted in a drop of sound generated. However, a minimum amount of noise generated seems to have been reached. This could imply that newer engines will not generate less sound in the future.

Since the number of both passenger as well as cargo flights still increases, a further noise level drop is needed to guarantee environmental and economic interests. Still a lot of environmental organizations plea to limit the number of flights. On the other hand, the Advisory Council for Transport, Public Works, and Water Management urges for lesser sound restrictions in order for Schiphol to increase the number of flights. A smaller production of noise around the airplane yields more flights which again yields in a larger profit. In addition, the environmental restrictions must not be violated. This directly implies that a lot can be gained by doing research on how to absorb a part of the sound...
generated by the jet engine of an airplane, since they are mainly responsible for the noise generated by an airplane.

A schematic picture of a jet engine is shown in figure (1.2).

The noise generated by a jet engine is mainly caused by the jet exhaust and the fan. Secondary sources, due to the internal combustor and turbine stages, typically produce less sound power and contribute little to the overall radiated noise [8]. In order to suppress the noise, acoustic damping material, protected by perforated plates, can be placed at the walls. Often the damping material itself is omitted and the space between the perforate plates and the backing wall is filled with honeycomb structure [5]. This is called an acoustic liner. Sometimes one or two additional layers of these holes are used to obtain a double Degree Of Freedom (DOF) or triple DOF liner, respectively.

Acoustic liners are positioned around the jet engine to absorb a part of the sound passing by. These liners are responsible for absorption of a significant part of the sound generated by the jet engine. We consider acoustic liners surrounding the engine in more detail, as shown in figure (1.3).
We observe that the sound flow, generated by the engine and the fan, passes the acoustic liners and that a part of this sound flows through an outlet pipe. Here, the sound is partially absorbed and a part is released again and flows back around the engine. Consider many of these outlet pipes next to each other. The purpose of these pipes is mainly to absorb a portion of the sound.

The acoustic liners can be recognized as plates with a honeycomb structure with a perforated plate on top. The specific structure accounts for the absorption of sound as shown in figure (1.4).
Now we want to analyze these liners in more detail. The liners contain small holes, which are positioned next to each other. The sound flow passes these holes one after another. At each particular one, a part of the flow enters the hole. Some of the flow will be reflected and continues along the others with the rest of the sound flow, while the other will be absorbed and will so-to-speak "be left behind". The holes can considered to be Helmholtz resonators and the liner as an array of resonators. They can also be considered to represent boundary conditions with impedance in the frequency domain (for low enough frequencies) as shown in figure (1.5).
Figure 1.5: Schematic picture of a DOF and a double DOF

We consider one Helmholtz resonator. We can make a mathematical model of this resonator step by step. See figure (1.6).

Figure 1.6: The design of a schematic picture of a Helmholtz resonator
We can model it as a duct with a small hole in the top when we consider
the resonator to be wide. Now we widen the hole more in order to highlight the
edge. In addition, we consider "outside the duct" bounded as well. Finally we
remove the second edge and use a model of a resonator with only one edge.

The advantage of the last picture is that we can write the solution as a sum of
modes. It will be discussed in the next chapters what is meant with this. For
now, the schematic picture of a resonator has been derived.

1.4 Target of this report

Currently, research on acoustic liners has been done mostly through performing
measurements. The objective of this project is to perform analytical and nu-
merical calculations on a mathematical model of an actual acoustic liner. This
is done to have a better insight into the rate of absorption and to calculate
pressure and velocity profiles. These calculations involve certain difficulties.
Since perforated plates are placed on the holes, we obtain a sharp edge at the
end. This edge is responsible for several physical properties. But even more, it
results, without further restrictions, mathematically in a non-unique solution.

In the second chapter we will show what is meant by that by considering
the iris problem treated in literature. This is a nice example of why we don’t
obtain a set of equations with nice initial and boundary conditions. Here, we
also suggest to use mode theory to finally solve the problem and, in addition,
say something about the influence of the edge.

In the third chapter we deal with our own problem again and use the mode
theory, as discussed in chapter two. Then we match the acoustical modes in the
different regions and numerically implement the system we then obtain. Finally
the results of the simulation will be discussed.

In chapter four we will again consider our problem, but this time we in-
troduce a uniform mean flow. We derive the equations that govern flow in our
system and again use mode theory and matching to derive a system of equations.
A numerically implementation and a short discussion of the results concludes
this chapter.

We end this report by summarizing the methods used in this report to per-
form calculations on flow in a DOF. Furthermore, we state that we are able to
calculate the portion of absorption of sound, determine the pressure and veloc-
ity profile, and discuss the dependence of the number of modes chosen in order
to state something more about the role of the edge in our model.
Chapter 2

The iris problem

In order to introduce our own model with its difficulties, we will first describe a familiar problem, better known as the iris problem. The problem is a good starting point for it contains the same edge condition as our own problem (in fact: this problem contains two edges). Therefore we shortly discuss our method of solving the problem and showing some results. This gives us some better insight in our acoustic liner problem.

2.1 Introduction

The exhaust ducting of furnaces and gasification installations transfers a mixture of sometimes highly pressurized gases from the high temperature furnace chamber to an area of lower temperatures. Sticky particles of ashes, tar, and soot are carried away with this gas mixture. Designers of furnace have raised the question whether the particles would be deposited on the wall. It would therefore be useful to measure the amount of contamination that accumulates at the same location in the pipe, since it could eventually block the pipe [10].

Due to the hostile environment inside the pipe, it is not possible to measure the amount of pollution directly. Therefore, it is suggested to measure it acoustically; i.e. we sent in an acoustic wave and measure the reflection by the pollution.

We consider a cylindrical hard-walled pipe, in cylindrical coordinates \((x,r,\theta)\) given by \(r = a\). At \(r = a\) the normal velocity vanishes \((u, n) \sim \partial p / \partial r = 0\). At \(x = 0\) a source generates a sound field of circular frequency \(\omega\). At \(x = L\) the pipe is connected via a flanged opening to the half space \(x > L\). We assume an annular hard walled iris at \(x = D\) between \(r = h\) and \(r = a\). This is the mathematically modelled pollution.

Rienstra [10] derived, assuming that the speed of sound is independent of \(x\), the equations governing the velocity in the pipe.
\[
\frac{1}{k^2} \nabla^2 p + p = 0, \quad (2.1)
\]
\[
i \omega p_0 \nabla v + \nabla p = 0, \quad (2.2)
\]

where \( k = \omega / c_0 \) and \( p(x) \) and \( v(x) \) are complex functions. In the pipe, the boundary conditions were easily deduced. In the duct, however, we have two sharp edges where we do not have proper boundary conditions. We are still able to tell something about the pressure, but we will need a different method.

Mode matching is a frequently employed method for boundary-value problems in piecewise constant geometries [6]. This technique is useful when the geometry of the structure can be identified as a junction of two or more regions, each belonging to a separable coordinate system. The first step in the mode matching procedure entails the expansion of unknown fields in the individual regions in terms of their respective normal modes. Once the functional form of the normal modes is known, the problem reduces to that of determining the set of modal coefficients (or modal amplitudes) associated with the field expansions in various regions. The modal representation is followed by the application of continuity conditions for the fields at the interfaces in the junction regions. This procedure, sometimes in conjunction with the orthogonality property of the normal modes, eventually leads to an infinite set of linear simultaneous equations for the unknown modal coefficients. From this, solutions can be deduced.

### 2.2 Mode matching

With separation of variables solutions \( p(x, r, \theta) = F(x)G(r)H(\theta) \) exist, satisfying a uniform boundary condition at the coordinate surface \( r = a \). These solutions are called modes. Mathematically, these modes are interesting because they form a complete basis by which any other solution can be represented by a so-called modal expansion. Physically, these modes are interesting because the usually complicated behaviour of a general field is easier understood via the simpler properties of its modal elements.

The modes of the hard walled infinite duct are the products of Bessel function and exponential functions [10]

\[
d_{\mu}^{\pm}(x, r, \theta) = J_0(\alpha_{\mu} r)e^{\mp ik_{\mu}x}, \quad (2.3)
\]

where \( J_0 \) is the zeroth order ordinary Bessel function of the first kind, \( \alpha_{\mu} a \) is the \( \mu \)-th nonnegative nontrivial zero of \( J_0' \), \( \gamma_{\mu} = k_{\mu} = \sqrt{k^2 - \alpha_{\mu}^2} \) is the axial wave number, and \( \mu \in \mathbb{N} \).

A general solution can now be built from the modes \( d_{\mu}^{\pm} \) as the sum
\[ p(x, r, \theta) = \sum_{\mu=1}^{\infty} (A_\mu d^+_\mu + B_\mu d^-_\mu), \quad (2.4) \]

where \( A_\mu \) and \( B_\mu \) are amplitudes, which can easily be determined. At the right side of any source we have only right-travelling waves \( (B_\mu = 0) \) and at the left side only left-travelling waves \( (A_\mu = 0) \).

For a given set of modes incident from \( x < D \) (A-modes) and modes incident from \( x > D \) (D-modes), we want to the reflected and transmitted modes generated in \( x < D \) (B-modes) and in \( x > D \) (C-modes). We use a reflection matrix \( R \) and a transmission matrix \( T \).

\[
\begin{align*}
R &= R A + T D, \\
C &= T A + R D.
\end{align*}
\quad (2.5)
\quad (2.6)
\]

A natural method to determine \( R \) and \( T \) is the technique of mode matching. By projecting the conditions of continuity along an interface to a suitable model basis, the problem may be reduced to one of linear algebra. For the numerical realization, we will work with truncated series and assume that the solution will be represented by \( N \) modes.

We use the notation \( \psi_\mu(r) = J_0(\alpha_\mu r) \) with boundary condition at \( r = a \) and \( \hat{\psi}_\mu(r) = J_0(\alpha_\mu r) \) with boundary condition at \( r = h \). Furthermore, \( (\psi_\mu, \psi_\nu) = \int_0^a \psi_\mu \psi_\nu r k^{-2} dr \). Since the problem is linear it is sufficient to determine the scattered field of a single mode. Now we have for \( \epsilon > 0 \) small and for \( n = 1, \ldots, N, \)

\[
x = D - \epsilon \quad p = \sum_{\mu=1}^{N} (\delta_{\mu n} + R_{\mu n}) \psi_\mu, \quad (2.7)
\]

\[
 x = D \quad ip_x = \sum_{\mu=1}^{N} (\delta_{\mu n} - R_{\mu n}) \gamma_\mu \psi_\mu, \quad (2.8)
\]

\[
x = D \quad ip_x = \sum_{\mu=1}^{Q} U_{\mu n} \psi_\mu, \quad \text{on } 0 \leq r \leq h, \quad (2.9)
\]

\[
 x = D \quad ip_x = 0, \quad \text{on } h \leq r \leq a, \quad (2.10)
\]

\[
x = D + \epsilon \quad p = \sum_{\mu=1}^{N} T_{\mu n} \psi_\mu, \quad (2.11)
\]

\[
 x = D + \epsilon \quad ip_x = \sum_{\mu=1}^{N} T_{\mu n} \gamma_\mu \psi_\mu, \quad (2.12)
\]

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where \( Q \) is the number of modes in the duct and where we introduced an auxiliary matrix \( \mathbf{U} \) with dimension \( Q \times N \). This matrix represents part of the flow inside the duct itself. Here \( \delta_{\mu n} \) is the wave entering the pipe on the left. Since \( p_x \) is continuous on both sides of the duct, we have that

\[
\sum_{\mu=1}^{N} (\delta_{\mu n} - R_{\mu n}) = \sum_{\mu=1}^{N} T_{\mu n},
\]

(2.13)

which is also an intuitive result. In addition, \( p \) is continuous before and after the duct. Hence

\[
\sum_{\mu=1}^{N} (\delta_{\mu n} + R_{\mu n})\psi_{\mu} = \sum_{\mu=1}^{N} T_{\mu n}\psi_{\mu}.
\]

(2.14)

From equation (2.13) we can derive

\[
\sum_{\mu=1}^{N} R_{\mu n}\psi_{\mu} = \sum_{\mu=1}^{N} (\delta_{\mu n} - T_{\mu n})\psi_{\mu}.
\]

(2.15)

Hence we substitute this into equation (2.14) and obtain

\[
\sum_{\mu=1}^{N} \delta_{\mu n}\psi_{\mu} = \sum_{\mu=1}^{N} T_{\mu n}\psi_{\mu}.
\]

(2.16)

If we multiply the left- and right-hand side with \( \hat{\psi}_\nu r/k^2 \) and integrate, we obtain

\[
(\hat{\psi}_\nu, \psi_n) = \sum_{\mu=1}^{N} (\hat{\psi}_\nu, \psi_{\mu}) T_{\mu n}.
\]

(2.17)

We introduce the auxiliary matrix \( S_{\mu \nu} = (\hat{\psi}_\mu, \psi_{\nu}) \) and obtain

\[
S_{\nu n} = S_{\nu \mu} T_{\mu n}.
\]

(2.18)

The continuity of \( p_x \) from passed the duct and inside the duct gives us

\[
\sum_{\mu=1}^{N} T_{\mu n}\gamma_{\mu} \psi_{\mu} = \sum_{\mu=1}^{Q} U_{\mu n} \hat{\psi}_{\mu}, \quad \text{on } 0 \leq r \leq h,
\]

(2.19)

\[
= 0 \quad \text{on } h < r \leq a.
\]

(2.20)

We multiply the left- and the right-hand side with \( \psi_{\nu} r/k^2 \), integrate, and use the orthogonality property in order to obtain, for \( 0 \leq r \leq h \),

\[
T_{\nu n}\gamma_{\nu} (\psi_{\nu}, \psi_{\nu}) = \sum_{\mu=1}^{Q} (\psi_{\nu}, \hat{\psi}_{\mu}) U_{\mu n}
\]

(2.21)
Now we introduce the auxiliary matrix $M$ as follows

$$M_{\nu \mu} = \frac{\langle \psi_\nu, \hat{\psi}_\mu \rangle}{\gamma_\nu \langle \psi_\nu, \psi_\nu \rangle}. \quad (2.22)$$

Then we have

$$T_{\nu n} = M_{\nu \mu} U_{\mu n} = M_{\nu \mu} M^{-1}_{\mu \nu} T_{\nu n}. \quad (2.23)$$

We eliminate $U$ by substituting the $T$ on the right-hand side of equation (2.23) with equation (2.18) and obtain, as a final result

$$T_{\nu n} = M_{\nu \mu} (S_{\mu \nu} M_{\nu \mu})^{-1} S_{\mu n}. \quad (2.24)$$

### 2.3 Numerical results and conclusions

Note that earlier we considered our solution to be represented by $N$ modes. In the iris we suggested a solution with a representation of $Q$ modes. It is reasonable to take the number of modes in the iris smaller than the number in the pipe, since the iris has a smaller radius than the pipe. Since the number of zeros of the $\mu$-th mode is $\mu - 1$, the typical radial wavelength of $\psi_\mu$ is $(\mu - 1)/2a$ and of $\hat{\psi}_\mu$ is $(\mu - 1)/2h$.

Therefore a balance between the number of modes in which the solution is represented requires the same smallest wave lengths. Therefore it seems logical to choose

$$\frac{Q}{N} \sim \frac{h}{a}. \quad (2.25)$$

Since we are working with truncated series, we have the fastest convergence to the physical solution for $N \to \infty$.

We know that this problem has a non-unique solution, so the question rises what will happen when we choose $\frac{Q}{N}$ very much different from $\frac{h}{a}$. We may converge to another solution, as can be seen in figure (2.1).
Figure 2.1: The real part of the first transmitted mode for different number of modes $N$ and $Q$ with $h/a = .5$

We see that we converge to a solution for $N/Q \geq 2$ and for increasing $N$. When $N/Q < 2$ then the mode jumps to another converging point. This states the non-uniqueness of our solution. In order to determine the correct one, we need to correct our situation by considering the edge. The integral of the energy over a small volume near the edge must be finite. Therefore it is wise to choose $N/Q \sim a/h$.

We observe in this chapter that a problem that contains a sharp edge can be numerically solved with mode theory and mode matching. However, we obtain a non-unique solution. When we consider the edge, we can select the right combination of the number of modes to determine the physical solution.
Chapter 3

Model of an acoustic liner

In the introduction we have given a description of a jet engine with its corresponding acoustic liners. We now want to create models of acoustic liners. Therefore, we will begin by stating the schematic representation of the liners. Furthermore, the equations for the model are derived. In the second section, we will take a closer look at the singularity. After that, we will consider mode theory to solve the problem, for the same reasons as in the previous chapter. In addition, we will match the modes in order to obtain one matrix equation. In the fourth section we will numerically implement our equations and discuss the results observed from the simulations. Finally, we draw conclusions.

3.1 Mathematical model

We want to determine the equations that govern the flow inside the tubes. We know that sound has to satisfy the conservation laws of fluid motion. Therefore we consider the equations of mass-, momentum- and energy balance.

Balance of mass

The change per timestep in density in a tube (volume $V$) is equal to the amount of mass entering through a surface $S$. In integral formulation, this gives

$$\frac{\partial}{\partial t} \iiint_V \rho \, d\tau = -\iint_S \rho (\vec{v}, \vec{n}) \, d\sigma,$$

where $t$ is the time, $\rho$ is the fluid density, $d\tau$ is a volume element, $\vec{v}$ is the velocity of the fluid, $\vec{n}$ is the outer normal, and $d\sigma$ is a surface element. With Gauss’ theorem we can rewrite this into

$$\iiint_V \left[ \frac{\partial \rho}{\partial t} + (\nabla \cdot (\rho \vec{v})) \right] \, d\tau = 0.$$

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Since this holds for any $V$, the integrand itself must be zero. Hence we obtain the mass balance in differential form:

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot (\rho \mathbf{v})) = 0,$$

(3.3)

When we write out the divergence term, then we can write down the balance of mass in terms of the material derivative of the density. Hence, this equation can be rewritten into

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0,$$

(3.4)

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v}, \nabla)$ is both the time derivative as well as the convective term.

**Balance of momentum**

We obtain the integral form of the balance of momentum by replacing the mass in the integral form of the balance of mass by the momentum ($\rho \mathbf{v}$). On the right hand side we need to add forces that influence the momentum. This yields

$$\frac{\partial}{\partial t} \int \int \int_V \rho \mathbf{v} \, d\tau = - \int_s \int (\rho \mathbf{v} \cdot \mathbf{n}) + (\mathbf{P} \cdot \mathbf{n}) \, d\sigma + \int \int \int_V \mathbf{f} \, d\tau,$$

(3.5)

where $\mathbf{P} = p\mathbf{I} - \mathbf{t}$ is called the pressure tensor, $p$ is the hydrostatic pressure, $\mathbf{I}$ is the identity tensor, $\mathbf{t}$ is the viscous stress tensor, and $\mathbf{f}$ an external force. From now on, we consider this force to be absent ($\mathbf{f} = 0$). Note that $\rho \mathbf{v}(\mathbf{v}, \mathbf{n}) = (\rho \mathbf{v}^T, \mathbf{n})$. Again with the use of Gauss’ theorem and by considering that the integral over any volume leads to zero implies that the integrand must be zero, gives us

$$\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (p\mathbf{I} - \mathbf{t} + \rho \mathbf{v}^T) = 0.$$

(3.6)

For this equation we can also write out the divergence term as well as the time derivative. This leads to

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) + \mathbf{v} \cdot \left(\frac{\partial \rho}{\partial t} + \nabla \rho + \rho \nabla \mathbf{v}\right) + \nabla p = 0.$$

(3.7)

Note that the first term is nothing more than the density multiplied by the material derivative of the flow velocity. In the second term on the left hand side we recognize the inproduct of the velocity with the balance of mass. Using equation (3.3) then gives us

$$\rho \frac{D\mathbf{v}}{Dt} + \nabla p = 0.$$

(3.8)
Balance of energy

Since we still have more unknowns than equations, we want to derive an equation for the balance of energy. We have to consider the different types of energy. We have kinetic energy of the flow and internal energy. Per mass unit, we have \( \frac{1}{2} V^2 + \epsilon \), where \( \epsilon \) represents the internal energy. This energy can flow through the surface \( S \) and change due to volume forces (per mass unit: \( f \)), surface sources (per square meter: \( P \)), molecular heat transport (with density \( \phi \) per cubic meters), and heat conduction \( q \). This results in

\[
\frac{\partial}{\partial t} \iiint_V (\rho \epsilon + \frac{1}{2} \rho (v, v)) \, d\tau = - \int_S (\rho \epsilon v + \frac{1}{2} \rho v (v, v) + (p I - \tau) v + q) \, n \, d\sigma \\
+ \iiint_V \phi \, d\tau.
\] (3.9)

Let us neglect the molecular heat transport, viscous dissipation and the heat transfer for acoustic perturbations (\( \phi = 0 \), \( \tau \) is the zero-matrix and \( q = 0 \), respectively) since their influence is small [3]. Then we have

\[
\frac{\partial}{\partial t} \iiint_V (\rho \epsilon + \frac{1}{2} \rho (v, v)) \, d\tau = - \int_S (\rho \epsilon v + \frac{1}{2} \rho v (v, v) + p I v) \, n \, d\sigma.
\] (3.10)

With Gauss’ theorem and the same reasoning as before we obtain the equation of conservation of energy in differential form

\[
\frac{\partial}{\partial t} (\rho \epsilon + \frac{1}{2} \rho (v, v)) + \nabla \cdot (\rho \epsilon v + \frac{1}{2} \rho (v, v)v) = -\nabla \cdot (p I v),
\] (3.11)

Note that we place all netto forces on the righthand side and all others on the left. This equation can be rewritten by taking the time derivative of the terms inside the brackets. This becomes

\[
\rho \frac{\partial \epsilon}{\partial t} + \epsilon \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t} (\frac{1}{2} \rho (v, v)) + \rho \epsilon \cdot \nabla \epsilon + \epsilon \nabla \cdot (\rho \epsilon v) + \nabla \cdot (\frac{1}{2} \rho (v, v)v)
= -\nabla \cdot (p I v),
\] (3.12)

Now we collect terms in order to use the notation of the material derivative of the internal energy. We reschedule terms and obtain

\[
\rho \frac{D \epsilon}{D t} + \rho \left[ \frac{\partial p}{\partial t} + \nabla \cdot (\rho \epsilon v) \right] + \frac{\partial}{\partial t} (\frac{1}{2} \rho (v, v)) + \nabla \cdot (\frac{1}{2} \rho (v, v)v)
= -p (\nabla \cdot v) - v \cdot \nabla p,
\] (3.13)
We recognize the equation of conservation of mass. Therefore we can use equation (3.3). In addition, we can reschedule all terms that have an inner product with the velocity. We write out the derivatives to time and place and obtain

\[
\rho \frac{\partial \epsilon}{\partial t} + p \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \left[ \nabla p + \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) \right] + \frac{1}{2} \mathbf{v} \cdot \left( \nabla p + \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) \right) + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} = 0.
\]

(3.14)

We recognize in this equation the equations of conservation of mass and momentum. With the use of equations (3.3) and (3.8), we can therefore reduce our equation for the conservation of energy into something simpler. Finally, we obtain

\[
\rho \frac{\partial \epsilon}{\partial t} + p \nabla \cdot \mathbf{v} = 0.
\]

(3.15)

We now want to eliminate energy terms from our equations. In order to do this, we need some thermodynamics. We consider its second law with respect to the change per time. It states that

\[
T \frac{Ds}{Dt} = \frac{De}{Dt} + p \frac{D\rho^{-1}}{Dt},
\]

(3.16)

where \( T \) is the temperature and \( s \) the entropy. We assume that the fluid is locally in thermodynamic equilibrium (Stokes’ Hypothesis) such that the pressure, the density, and the entropy are related with each other according to the equation of state

\[
p = p(\rho, s).
\]

(3.17)

Considering equation (3.15) and using our version of the second law of thermodynamics, gives

\[
\rho \left[ T \frac{Ds}{Dt} - p \frac{D\rho^{-1}}{Dt} \right] + p \nabla \cdot \mathbf{v} = 0.
\]

(3.18)

Now we reschedule terms such that we obtain an expression for the entropy. We write out the material derivative and obtain

\[
\frac{Ds}{Dt} = \frac{p}{\rho^2 T} \left[ \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} \right].
\]

(3.19)

Again, we recognize the equation of balance of mass in this equation. Therefore, we use (3.4). Since the remaining term can not be zero, the material derivative of the entropy must be zero, given the previous assumptions. Now we use the equation of state, (3.17)
where \( c \) is the speed of sound. This is our new equation that represents balance of energy.

**Homogeneous wave equation**

We have now derived a system of equations that represent balance of mass, momentum and energy. These equations are

\[
\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0, \quad (3.21)
\]
\[
\rho \frac{D\mathbf{v}}{Dt} + \nabla p = 0, \quad (3.22)
\]
\[
\frac{\partial p}{\partial \rho} \bigg|_{s = \text{const.}} = c^2. \quad (3.23)
\]

We now want to linearize this equation in order to separate a homogeneous, steady part and a perturbed part. Pressure fluctuations \( p_0 \) are small compared to the atmospheric pressure \( (p_0/p_{\text{atm}} = O(10^{-4})) \) which justifies the linearization. Analogously, we can therefore write

\[
p(x, t) = p_0(x) + p'(x, t), \quad (3.24)
\]
\[
\rho(x, t) = \rho_0(x) + \rho'(x, t), \quad (3.25)
\]
\[
\mathbf{v}(x, t) = \mathbf{v}_0(x) + \mathbf{v}'(x, t), \quad (3.26)
\]
\[
c(x, t) = c_0(x) + c'(x, t). \quad (3.27)
\]

We have \( \mathbf{v}_0 = 0 \) and assume the medium to be uniform. Therefore we have that \( \nabla \rho_0 = 0 \). We linearize (3.21), (3.22), and (3.23) and obtain

\[
\frac{\partial \rho'}{\partial t} = -\rho_0(\nabla \cdot \mathbf{v}'), \quad (3.28)
\]
\[
\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p', \quad (3.29)
\]
\[
\frac{\partial p'}{\partial t} = c_0^2 \frac{\partial \rho'}{\partial t}. \quad (3.30)
\]

Now we want to derive one equation. Therefore, we subtract the time derivative of equation (3.28) from the divergence of equation (3.29). This yields

\[
\nabla^2 p' - \frac{\partial^2 \rho'}{\partial t^2} = 0. \quad (3.31)
\]
We like to find an equation with one unknown. The pressure and density are linked to each other with the derived equation for the conservation of energy (3.30). Substituting this in the equation gives us

$$\nabla^2 p' - \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} = 0.$$ 

(3.32)

So the equation that describes the flow of sound in a tube, is, in our situation, a homogeneous wave equation. With smooth boundary conditions, the equation can be solved analytically with the use of Green functions.

**Geometry of the model**

Consider two hard-walled horizontal tubes, tube 1 and 2, separated by an infinitely thin solid wall. See figure (3.1).

![Diagram of an acoustic liner in Cartesian coordinates](image)

Figure 3.1: A schematic picture of an acoustic liner in Cartesian coordinates

From now on we will use Cartesian coordinates. Both tubes emerge in one larger tube at $x = 0$ (called tube 3). All tubes have a fixed height $h_1$, $h_2$, and $h_3$ for tube 1, 2, and 3, respectively. In addition, we have $h_1 + h_2 = h_3$ (height of the walls is zero).

Note that the $x$-direction is horizontal and the $y$-direction is vertical. At $x = 0$ tubes 1 and 2 - separated by the plane $y = 0$ - emerge in the larger tube. The area and volume of tube $i$ ($i = 1, 2, 3$) is denoted by $S_i$ and $V_i$, respectively. We denote the amplitudes of the waves travelling to the right in region $i$ with $A_i$ and the waves travelling to the left with $B_i$. For our model we therefore assume $A_1, A_2,$ and $B_3$ to be input variables and $B_1, B_2,$ and $A_3$ as output variables. In this chapter, we consider no mean flow.

Let us consider the boundary conditions of our problem. First consider region 1. The boundary conditions here are that the pressure gradient is zero at the boundary. The pressure gradient at the upper end of the tube is directed
downwards and at the lower end upwards. However, in the origin, we have no unique direction. Because of the edge, it is not possible to describe one boundary condition there. The same holds for region 2. This means that we obtain a solution that is non-unique [1]. We can only make this unique by defining the edge singularity. For this, we consider the singularity in more detail.

### 3.2 Singularity analysis

Research on singularities at sharp edges has been done before by, for example, Bouwkamp [1]. For small $k$ we can replace the dominating equation by the Laplace equation

$$\nabla^2 \phi \simeq 0 \quad (3.33)$$

For the geometry, see figure (3.2).

Figure 3.2: A schematic picture of an acoustic liner in Cartesian coordinates

Consider the angle of $y = 0$ to be $\alpha$. The picture is then drawn for $\alpha = 2\pi m$ for $m \in \mathbb{Z}$. Consider as a solution

$$\phi \sim r^n \cos(n\theta) + r^n \sin(n\theta), \quad (3.34)$$

with $r = \sqrt{x^2 + y^2}$ and where $\theta = \arccos(x) = \arcsin(y)$. The boundary conditions give us that
\[
\frac{\partial}{\partial \theta} \cos(n\pi) \bigg|_{\theta=0} = 0, \quad (3.35)
\]
\[
\frac{\partial}{\partial \theta} \cos(n\pi) \bigg|_{\theta=2\pi-\alpha} = 0, \quad (3.36)
\]

and therefore we know that
\[
\phi \sim r^n \cos(n\theta). \quad (3.37)
\]

The second boundary condition gives us that
\[
-n \sin(2\pi n - \alpha n) = 0,
2\pi n - \alpha n = m\pi,
\]
\[
n = \frac{m\pi}{2\pi - \alpha}. \quad (3.38)
\]

We see that when we have a straight angle (\(\alpha = \frac{\pi}{2}\)), then our singularity is \(\phi \sim r^{\frac{m}{2}} \cos(n\theta)\), where \(m\) is an integer. For an edge we have \(\alpha = 0\) and therefore \(\phi \sim r^{\frac{n}{2}} \cos(n\theta)\).

This solution is not unique, but there is also another physical restriction that needs to be added. Around the singularity there must be a finite amount of energy (the edge cannot be a source). In formula
\[
\int |\nabla \phi|^2 \, dx < \infty. \quad (3.39)
\]

For a sharp edge we have, \(\psi \approx r^{m/2}\), and therefore
\[
\int |\nabla \phi|^2 \, dx = \int_0^{2\pi} \int_0^\epsilon |r^{(m-2)/2}r| \, dr \, d\theta,
= \int_0^{2\pi} \int_0^\epsilon r^{(2m-2)/2} \, dr \, d\theta,
< \infty, \quad (3.40)
\]

if \(m \leq 1\). Hence \(m = 1\) yields the physical solution.

An example of a technique where the edge singularity is explicitly stated is the Wiener-Hopf method [11].

In the previous chapter, we have seen that this physical restriction is not directly included in the mode matching. However, it is connected with the number of modes chosen. When we want the right (physical) convergence to the solution,
we need to choose a certain amount of modes. This is a way to impose a physical restriction on the edge with the theory of mode matching. It will therefore be useful to derive the equations governing flow with mode theory and then perform simulations for different number of modes.

3.3 Mode- Theory & Matching

In this chapter, we will again employ mode theory and matching to find a solution for our problem, like in the previous chapter. We consider again the wave equation obtained in the previous section. We have

\[ \nabla^2 p' - \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} = 0 \]  

(3.41)

In order to construct solutions of our equation, we use the method of separation of variables. We consider our solutions to be time-harmonic and therefore we consider a solution of the form

\[ p'(x,y,t) = \text{Re}(X(x)Y(y)e^{i\omega t}). \]

(3.42)

This results in the equation that we will be considering from now on in this chapter

\[ \nabla^2 p' + \frac{\omega^2}{c_0^2} p' = 0 \]

(3.43)

In this equation we can use \( p = X(x)Y(y) \) and obtain

\[ \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{\omega^2}{c_0^2} = 0. \]

(3.44)

The first term is only dependent on \( x \) and the second term only on \( y \). Hence the first term and the second term must be constant. In the solution of \( X(x) \) we distinguish waves propagating to the right and to the left

\[ X(x) = e^{-ik_nx} + e^{ik_nx}, \]

(3.45)

where \( k_n \) is the \( n \)-th wave number, since we can consider equation (3.44) to be a Sturm-Liouville eigenvalue problem. Now we omit the apostrophe and consider

\[ p^+ = Y_n(y)e^{i(\omega t-k_nx)}, \]

(3.46)

\[ p^- = Y_n(y)e^{i(\omega t+k_nx)}. \]

(3.47)

We will find an expression for \( k_n \) later. When we substitute these into equation (3.41) then we obtain an expression for \( Y_n(y) \) as well. Note that since
we take the second derivative in the $x$-direction, we obtain the same result for $p^+$ and $p^-$. Therefore, the wave numbers are equal as well. We have

$$(k^2 - k_n^2)p^\pm + Y_n''(y)e^{i(\omega \pm \omega k_n x)} = 0,$$

$$k^2 - k_n^2 + \frac{Y_n''(y)}{Y_n(y)} = 0,$$

(3.48)

where we used that $k = \omega/c_0$. From this we can see that

$$Y_n(y) = C_1 \cos(\alpha_n y) + C_2 \sin(\alpha_n y),$$

(3.49)

where $\alpha_n = \sqrt{k^2 - k_n^2}$. We can now see that in order to determine $k_n$, we need to find $\alpha_n$. From the boundary condition

$$\frac{\partial p}{\partial y} \bigg|_{y=0} = 0,$$

(3.50)

we get that $C_2 = 0$. Note that the direction of the pressure gradient in the origin is undetermined. The other boundary condition states

$$\frac{\partial p}{\partial y} \bigg|_{y=h} = 0,$$

(3.51)

and this gives us that

$$C_1 \sin(\sqrt{k^2 - k_n^2} h) = 0,$$

$$\sqrt{k^2 - k_n^2} h = n\pi,$$

$$k^2 - k_n^2 = \left(\frac{n\pi}{h}\right)^2.$$

(3.52)

Hence $k_n = \sqrt{k^2 - \frac{n^2\pi^2}{h^2}}$ such that $\text{Im}(k_n) \leq 0$. Note that then equation (3.46) is indeed a decaying wave for increasing $x$. From this we have derived an expression for the $n$-th wave number. The coefficient $C_1$ can be determined if we normalize the mode as follows: the integral over $Y_n = \psi_n$ squared must yield one (orthogonality principle). Hence
\[
\int_{0}^{h} \psi_n \cdot \psi_n dy = \\
\int_{0}^{h} C_1^2 \cos^2(\frac{n\pi}{h}y)dy = \\
\int_{0}^{h} \frac{1}{2} C_1^2 + \frac{1}{2} C_1^2 \cos(\frac{2n\pi}{h}y)dy = \\
C_1^2 \left[ \frac{1}{2} y + \frac{h}{4n\pi} \sin(\frac{2n\pi}{h}y) \right]_0^h = \\
\frac{1}{2} h C_1^2 = 1. \quad (3.53)
\]

Hence: \( C_1 = \sqrt{\frac{2}{h}} \) for \( n \neq 0 \). When \( n = 0 \) we get \( C_1 = \sqrt{\frac{1}{h}} \). We will need to distinguish the case \( n = 0 \) and \( n \neq 0 \) in the future in order not to divide by zero.

As can be seen in figure (3.1), our situation is split up into three different regions. Then we have two waves (one travelling to the left and one to the right) in each region. We consider a general solution built from modes:

\[
p_1 = \sum_{n=0}^{\infty} A_{1,n} \psi_{1,n}(y)e^{-ik_{1,n}x} + B_{1,n} \phi_{1,n}(y)e^{i\mu_{1,n}x}, \quad (3.54)
\]

\[
p_2 = \sum_{n=0}^{\infty} A_{2,n} \psi_{2,n}(y)e^{-ik_{2,n}x} + B_{2,n} \phi_{2,n}(y)e^{i\mu_{2,n}x}, \quad (3.55)
\]

\[
p_3 = \sum_{n=0}^{\infty} A_{3,n} \psi_{3,n}(y)e^{-ik_{3,n}x} + B_{3,n} \phi_{3,n}(y)e^{i\mu_{3,n}x}. \quad (3.56)
\]

Note that \( \psi_n \) and \( k_n \) correspond to the wave travelling to the right and that \( \phi_n \) and \( \mu_n \) with the one travelling to the left. Without a mean flow, we obtain that \( \phi_{j,n} = \psi_{j,n} \) and \( \mu_{j,n} = k_{j,n} \) for all \( j \) and \( n \). In these equations the \( \psi_{j,n} \)'s can be determined as done in the previous paragraph. For the specific regions this leads to
\[ \psi_{1,n} = \phi_{1,n} = \sqrt{\frac{1}{h_1}}, \quad n = 0, \quad (3.57) \]
\[ \psi_{1,n} = \phi_{1,n} = \sqrt{\frac{2}{h_1}} \cos(n\pi \frac{y}{h_1}), \quad n > 0, \quad (3.58) \]
\[ \psi_{2,n} = \phi_{2,n} = \sqrt{\frac{1}{h_2}}, \quad n = 0, \quad (3.59) \]
\[ \psi_{2,n} = \phi_{2,n} = \sqrt{\frac{2}{h_2}} \cos(n\pi \frac{y}{h_2}), \quad n > 0, \quad (3.60) \]
\[ \psi_{3,n} = \phi_{3,n} = \sqrt{\frac{1}{h_3}}, \quad n = 0, \quad (3.61) \]
\[ \psi_{3,n} = \phi_{3,n} = \sqrt{\frac{2}{h_3}} \cos(n\pi \frac{y + h_2}{h_3}), \quad n > 0. \quad (3.62) \]

We can also find equations for the velocity using (3.29). This yields

\[ u_1 = \frac{1}{\rho_0 \omega} \sum_{n=0}^{\infty} k_{1,n} A_{1,n} \psi_{1,n} e^{-ik_{1,n}x} - k_{1,n} B_{1,n} \psi_{1,n} e^{ik_{1,n}x}, \quad (3.63) \]
\[ u_2 = \frac{1}{\rho_0 \omega} \sum_{n=0}^{\infty} k_{2,n} A_{2,n} \psi_{2,n} e^{-ik_{2,n}x} - k_{2,n} B_{2,n} \psi_{2,n} e^{ik_{2,n}x}, \quad (3.64) \]
\[ u_3 = \frac{1}{\rho_0 \omega} \sum_{n=0}^{\infty} k_{3,n} A_{3,n} \psi_{3,n} e^{-ik_{3,n}x} - k_{3,n} B_{3,n} \psi_{3,n} e^{ik_{3,n}x}. \quad (3.65) \]

The fields that we are using must satisfy continuity when passing from one region to another. This means that we have to match the modes at the boundaries of the regions. When we again check picture (3.1) then we can denote the following equations

\[ x = 0 : \quad p_1 = p_3, \quad 0 \leq y \leq h_1, \quad (3.66) \]
\[ v_1 = v_3, \quad 0 \leq y \leq h_1, \quad (3.67) \]
\[ p_2 = p_3, \quad -h_2 \leq y \leq 0, \quad (3.68) \]
\[ v_2 = v_3, \quad -h_2 \leq y \leq 0, \quad (3.69) \]

We now truncate the series we have written down for the pressure in order to obtain a finite number of equations. This is needed for a numerical implementation later. This becomes in terms of the finite modal representation in approximation
for the domain $N_1$, $N_2$, and $N_3$, respectively. Note that we start the summation for $n = 0$ for a zeroth mode exists. We do not assume that the number of modes in one region is equal to the number in another region. When we would match the number of modes on the left and on the right, we would rule out the possibility of an overdetermined system. Suppose, we project this on a basis $\{\chi_m\}$

\[
\sum_{n=0}^{N_1-1} \psi_{1,n}(y)(A_{1,n} + B_{1,n}) \simeq \sum_{n=0}^{N_3-1} \psi_{3,n}(y)(A_{3,n} + B_{3,n}), \quad 0 < y < h_1,
\]

\[
\sum_{n=0}^{N_1-1} k_{1,n} \psi_{1,n}(y)(A_{1,n} - B_{1,n}) \simeq \sum_{n=0}^{N_3-1} k_{3,n} \psi_{3,n}(y)(A_{3,n} - B_{3,n}), \quad 0 < y < h_1,
\]

\[
\sum_{n=0}^{N_2-1} \psi_{2,n}(y)(A_{2,n} + B_{2,n}) \simeq \sum_{n=0}^{N_3-1} \psi_{3,n}(y)(A_{3,n} + B_{3,n}), \quad -h_2 < y < 0,
\]

\[
\sum_{n=0}^{N_2-1} k_{2,n} \psi_{2,n}(y)(A_{2,n} - B_{2,n}) \simeq \sum_{n=0}^{N_3-1} k_{3,n} \psi_{3,n}(y)(A_{3,n} - B_{3,n}), \quad -h_2 < y < 0.
\]

We denote the number of modes in region 1, 2, and 3 with $N_1$, $N_2$, and $N_3$, respectively. Note that we start the summation for $n = 0$ for a zeroth mode exists. We do not assume that the number of modes in one region is equal to the number in another region. When we would match the number of modes on the left and on the right, we would rule out the possibility of an overdetermined system. Suppose, we project this on a basis $\{\chi_m\}$

\[
\sum_{n=0}^{N_1-1} (A_{1,n} + B_{1,n}) \int_{0}^{h_1} \psi_{1,n}(y) \chi_m \, dy \simeq \sum_{n=0}^{N_3-1} (A_{3,n} + B_{3,n}) \int_{0}^{h_1} \psi_{3,n}(y) \chi_m \, dy,
\]

\[
\sum_{n=0}^{N_1-1} (A_{1,n} - B_{1,n}) k_{1,n} \int_{0}^{h_1} \psi_{1,n}(y) \chi_m \, dy \simeq \sum_{n=0}^{N_3-1} (A_{3,n} - B_{3,n}) k_{3,n} \int_{0}^{h_1} \psi_{3,n}(y) \chi_m \, dy,
\]

\[
\sum_{n=0}^{N_2-1} (A_{2,n} + B_{2,n}) \int_{0}^{h_2} \psi_{2,n}(y) \chi_m \, dy \simeq \sum_{n=0}^{N_3-1} (A_{3,n} + B_{3,n}) \int_{0}^{h_2} \psi_{3,n}(y) \chi_m \, dy,
\]

\[
\sum_{n=0}^{N_2-1} (A_{2,n} - B_{2,n}) k_{2,n} \int_{0}^{h_2} \psi_{2,n}(y) \chi_m \, dy \simeq \sum_{n=0}^{N_3-1} (A_{3,n} - B_{3,n}) k_{3,n} \int_{0}^{h_2} \psi_{3,n}(y) \chi_m \, dy,
\]

We now write $<>_1$ for the integration over the domain $0 < y < h_1$ and $<>_2$ for the domain $-h_2 < y < 0$. For notation, we now define
\[ M_1 = \{ <\chi_m, \psi_{1,n}> \}, \quad M_2 = \{ <\chi_m, \psi_{2,n}> \}, \quad P_1 = \{ <\chi_m, \psi_{3,n}> \}, \quad P_2 = \{ <\chi_m, \psi_{3,n}> \}. \]

Note that these matrices have dimension \( M \times N_1 \). In addition, note that we have now assumed uniform convergence on \( 0 < y < h_1 \) and on \( -h_2 < y < 0 \) in order to exchange the summation and the integration. Furthermore,

\[
k_j = \begin{pmatrix} k_{j,0} & 0 & \ldots & 0 \\ 0 & k_{j,1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & k_{j,n-1} \end{pmatrix}.
\]

Now we have

\[
\begin{align*}
M_1 A_1 + M_1 B_1 &\simeq P_1 A_3 + P_1 B_3, \quad (3.78) \\
M_2 A_2 + M_2 B_2 &\simeq P_2 A_3 + P_2 B_3, \quad (3.79) \\
M_1 k_{1} A_1 - M_1 k_{1} B_1 &\simeq P_1 k_{3} A_3 - P_1 k_{3} B_3, \quad (3.80) \\
M_2 k_{2} A_2 - M_2 k_{2} B_2 &\simeq P_2 k_{3} A_3 - P_2 k_{3} B_3. \quad (3.81)
\end{align*}
\]

We consider \( A_1, A_2, \) and \( B_3 \) as input parameters and \( B_1, B_2, \) and \( A_3 \) as output parameters. We therefore reshuffle terms and obtain

\[
\begin{pmatrix} M_1 & 0 & -P_1 \\ 0 & M_2 & -P_2 \\ M_1 k_{1} & 0 & P_{1} k_{3} \end{pmatrix}
\begin{pmatrix} B_1 \\ B_2 \\ A_3 \end{pmatrix}
\simeq
\begin{pmatrix} -M_1 & 0 & P_1 \\ 0 & -M_2 & P_2 \\ M_1 k_{1} & 0 & P_{1} k_{3} \end{pmatrix}
\begin{pmatrix} A_1 \\ A_2 \\ B_3 \end{pmatrix}.
\]

Note that the matrices have dimension \( 4M \times (N_1 + N_2 + N_3), (N_1 + N_2 + N_3) \times 1, 4M \times (N_1 + N_2 + N_3), \) and \( (N_1 + N_2 + N_3) \times 1, \) from left to right. As stated before, \( A_1, A_2, \) and \( B_3 \) are input parameters and therefore known.

This system of equations can now be solved. Since the possibility of a large magnitude of equations and unknowns, we implement this into a computer program. Note that this is a system of \( 4M \) equations and \((N_1+N_2+N_3)\) unknowns. When we take \( 4M \geq (N_1 + N_2 + N_3) \) then we have more equations than unknowns. For the case \( 4M > (N_1+N2+N3) \) we use the Complex Least Squared Method. The equation has the form

\[
\begin{align*}
A x &= b, \\
A^T A x &= A^T b.
\end{align*}
\]
where \( \mathbf{A} \) and \( \mathbf{b} \) are known tensors and \( \mathbf{x} \) is the answer looked for. We can write this as

\[
\mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}.
\]  

(3.82)

With the Complex Least Squares Method, we can find a solution for our matrix equation. This implies that \( \mathbf{A}^T \) is Hermite conjugated. As input parameters we assume only an input flow in region 1. This means that we assume a harmonic pressure perturbation

\[
p_{in} = \cos(n\pi \frac{y}{h_1}) e^{-ik_n x} \quad \text{in region 1,}
\]

(3.83)

\[
p_{in} = 0 \quad \text{in region 2,}
\]

(3.84)

\[
p_{in} = 0 \quad \text{in region 3.}
\]

(3.85)

In words: we assume a single mode emanating from region 1 and no other input.

In order to numerically evaluate this equation, we have to determine \( M_1, M_2, P_1, \) and \( P_2 \). We will derive this in appendix A.
3.4 Numerical results

We implement our system of equations in Matlab to perform simulations for different numbers of modes. Note that from now on when we talk about the pressure, we imply the pressure perturbation (check derivation earlier). First we check whether we indeed observe continuity of pressure between the different tube regions. In symbols, we check whether

\[ x = 0 : \quad p_1 \simeq p_3, \quad 0 \leq y \leq h_1, \quad (3.86) \]
\[ p_2 \simeq p_3, \quad -h_2 \leq y \leq 0. \quad (3.87) \]

For reasons of accuracy, we choose a large number of modes: \( N_1 = 100, \ N_2 = 100, \ N_3 = 200, \) and \( M = 1000. \) Until further notice, we choose \( N_3 \) to be \( N_1 + N_2 \) since \( h_3 = h_1 + h_2 \) (20 = 10 + 10). Furthermore, we choose \( M \) such that we have an overdetermined system.

We take a cross-section at the point where all the tubes connect. We plot the real and imaginary part of the pressure in region 1, 2, and 3 and check whether they indeed align. See figure (3.3).
Between $y = 0$ and $y = h_1 = 10$ we have built a solution for the pressure from modes in region 1 and, between $y = -h_2 = -10$ and $y = 0$ for region 2 and between $y = -h_2$ and $y = h_1$ for region 3. The pressure in region 3 is plotted with diamonds. The solid line indicates the pressure in region 1 and aligns for the better part with the diamonds on the interval $y \in [0, 10]$, both for the real as the imaginary part of the pressure. The same holds for the dash dotted line, indicating the pressure in region 2, on the interval $y \in [-10, 0]$.

We hardly see any difference between the pressures in the cross section of the different tubes. We see that there are indeed two different pressures plotted. However, the approximation used by building a solution from modes is good. Note that in the origin, the largest deviations exist. This is due to the singularity placed there.

In addition, note that we observe the root singularity as discusses in section two.

Now that we have shown that continuity exists in the plane where the modes corresponding to the pressure are matched, we will show a pressure profile in
the tube. See figure (3.4) for the real part of the pressure in our system.

![Figure 3.4: Real part of the pressure throughout the tube](image)

We observe a standing wave pattern in region 1 and we notice continuity between the different regions.

We now like to determine the effect of the number of modes on the numerical results. Therefore we look at the dependence on the number of modes in region 1, \( N_1 \). We check the case \( N_1 = 2, 4, 8, \) and 256 for increasing \( N_3 \) and plot the real value of the amplitude of the second mode of the wave travelling to the right in region 3. We could have chosen a different mode and the same results would have been obtained. See figure (3.5).
If we start by looking at the solid line, i.e. the case of 2 modes in region 1, then we notice that for a small number of modes in region 3 the value of the amplitude changes rapidly. The value converges to a value of a bit above $-0.38$. This value is already attained for approximately 50 modes. For the case of 4 modes in region 1 we observe a similar behaviour. Except this time the convergence speed is lower and the value eventually attained seems to be $-0.36$. The same holds for the case of 8 modes in region 1 but this line has a more asymptotic behaviour. If we finally look at a much larger number of modes in region 1, then we observe again the same qualitative aspects. After approximately 50 modes the value seems to be at its limit already. It converges to a bit over $-0.36$. This seems to be a good approximation for the real part of the value of the amplitude of this mode. Note that when we take $N_1/N_3$ very differently from $h_1/h_3$ we observe different results as when matched, but this time we do not see the line converging to another solution. Due to symmetry, we obtain the same results for different number of modes in region 2.

Now, we take a closer look at the modes in region 3. Therefore we consider
the amplitude of the mode of the wave travelling to the right in region 3. We scale this amplitude by its index and plot it on logarithmic paper. We consider $N_1 = N_2 = 10$, $N_3 = 600$, and $M = 1000$. The results are plotted in figure (3.6).

![Figure 3.6: Scaled value of the amplitude of all modes (2 until N3-1) of A3](image)

We observe that the amplitude of the modes that represent the wave that travels to the right in region 3 keeps jumping between two regions. We distinguish lower and upper modes. The modes seem to fall upon two different lines. Hence: we notice two types of modes. The first group of modes -which will be called the upper modes- seems to converge to a certain point and the second group of modes, the lower modes, seems to converge to one point as well, yet another. Notice that two different points of convergence also imply two different solutions.

Rienstra [10] and Mittra & Lee [6] discuss two types of answers for the Iris problem. The first is where the number of modes used in the different regions is proportional to the height of the different regions. The other is where this relation is ignored and where the number of modes in one region is significantly
more than the relative difference in heights. Then we converge to another solution. This is not a drawback of the method, but this problem indeed also has a non-unique solution. This is due to the fact that we do not have a unique boundary condition at the end of the tube.

To acquire more information on these possible different answers, we simulate some more. We consider the same picture with the same values, except now with \( N_1 = N_2 = 100, N_3 = 200, \) and \( M = 1000. \) This only changes the relative difference between the modes. The results are plotted in figure (3.7).

![Figure 3.7](image)

**Figure 3.7:** Scaled value of the amplitude of all modes (2 until N3-1) of the wave travelling to the right in region 3

We do not observe the difference as in the case of the Iris problem. We still observe the convergence to two different points. What is different are some small oscillations for high number of modes. They do not appear to be a hazard of the method, but the oscillations grow when \( h_1/h_3 \) gets closer to \( N_1/N_3. \)

We now try to analyze the relative difference between \( N_1 \) and \( N_2. \) We consider the previous picture as our first test for taking them equal to each other.
Next, we try to vary them. We choose $N_1 = 10$ and $N_2 = 100$ and the other numbers we keep the same. Results our plotted in figure (3.8).

![Figure 3.8: Scaled value of the amplitude of all modes (2 until N3-1) of A3](image)

Here, we obtain a totally different result. We observe that we can hardly talk about two solutions anymore. We see one line of modes and this one seems to correspond the upper line from before. The lower line seems to have disappeared. The relative difference between the number of modes in region 1 and region 2 seem to have a great influence one the modes.

When we compare this picture with (3.7) then we see that dropping the number of modes in region 1 results in losing one of the two lines. When we compare it with (3.6) then we can also conclude that not the fact that we choose only $N_1 = 10$ modes is the result that we lose one line of modes, but the relative difference of $N_1$ and $N_2$ is solely responsible for this.

The question rises whether this behaviour is symmetric. In other words, if
we choose more modes in region 1 than in region 2, do we observe similar behaviour? Therefore, we choose $N_1 = 100$ and $N_2 = 10$ and $N_3$ and $M$ as before. See figure (3.9) for the results.

![Figure 3.9: Scaled value of the amplitude of all modes (2 until N3-1) of $A_3$](image)

We observe a symmetry between the number of modes in region 1 and region 2. This means that for this setting we can simply interchange the number of modes in region 1 and 2 and it does not make a difference for the behaviour of the modes on the transmission in region 3.

Now we consider what part of the flow is reflected and what part is transmitted. Since we have a standing wave pattern in region 1 we consider the wave reflecting in region 1 and 2 and the part that is transmitted to region 3. We choose $M = 1000$. First we take $N_1 = N_2 = 10$ and $N_3 = 20$. We then calculate that 45.0% is reflected. For $N_1 = N_2 = 10$ and $N_3 = 200$, we have 45.6% reflected. For $N_1 = N_2 = 100$ and $N_3 = 200$ we have 47.2% reflected. Again we consider the physical solution where the relative height difference is equal to
the relative different in number of modes. So a larger number of modes yields better results.

3.5 Conclusions

In this chapter, we made a model of an acoustic liner, without mean flow. We have used mode theory and mode matching to find a set of $4M$ equations and $N_1 + N_2 + N_3$ unknowns. With the use of the Complex Least Squared Method we can find a solution for this problem. With the use of a numerical implementation we have proven that our method gives accurate results for the flow in a tube. In addition, we noticed that the number of modes were important for the convergence of the solution and we could notice two different solutions for the modes in region 3. However, we do not observe a difference for different number of modes in region 1 and 2 compared with modes from region 3. The relative difference between the number of modes in region 1 and 2, however, is very important for it cancels one solution. When we choose our modes such that it is approximately equal to the difference in heights, then we observe two types of modes, upper and lower. Finally, when we choose sufficient modes, then we can also accurately calculate that part of the wave that is reflected.
Chapter 4

Model of an acoustic liner with uniform flow

In the previous chapter we have considered the pressure propagation in our model. Now we introduce a uniform mean flow, for there are many instances where such a flow is present. We want to be able to perform calculations in that situation as well.

4.1 Mathematical Model

In order to derive a system of equations for the flow through a tube with a subsonic uniform mean flow, we consider the equations that describe the motion of a perfect and isentropic gas, equations (3.21), (3.22), and (3.23),

\[
\begin{align*}
\rho \frac{Dv}{Dt} &= -\nabla p, \\
\frac{1}{\rho} \frac{D\rho}{Dt} &= -\nabla \cdot v, \\
c_0^2 \frac{D\rho}{Dt} &= \frac{Dp}{Dt},
\end{align*}
\]

where \( \frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla \) is the material derivative. We, again, assume \( p' \ll p_0 \) etc. to allow the linearization

\[
\begin{align*}
p &= p_0 + p' \\
v &= (u_0(y) + u')e_x + v'e_y,
\end{align*}
\]

with \( |u_0| < c_0 \). Note that \( u_0 = u_0(y)e_x \), \( p_0 = \text{const.} \), and \( \rho_0 = \text{const.} \) is a solution of the stationary equation. When we substitute this into the previous system of equations and neglect the higher-order terms, we obtain
\[
\rho_0 \left( \frac{\partial u'}{\partial t} + u_0(y) \frac{\partial u'}{\partial x} + v' \frac{\partial u_0}{\partial y} \right) = - \frac{\partial p'}{\partial x} \tag{4.6}
\]
\[
\rho_0 \left( \frac{\partial v'}{\partial t} + u_0(y) \frac{\partial v'}{\partial x} \right) = - \frac{\partial p'}{\partial y} \tag{4.7}
\]
\[
\frac{1}{\rho_0 c_0^2} \left( \frac{\partial p}{\partial t} + u_0(y) \frac{\partial p'}{\partial x} \right) = - \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right). \tag{4.8}
\]

We can derive an inhomogeneous wave equation by calculating \( \frac{\partial}{\partial x} (4.6) + \frac{\partial}{\partial y} (4.7) - \rho_0 \frac{D_0}{D_0 t} (4.8) \), where \( \frac{D_0}{D_0 t} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \). Note that we have done the same thing in the previous chapter. This results in
\[
\frac{\partial}{\partial x} (4.6) = \rho_0 \frac{\partial^2 u'}{\partial x \partial t} + \rho_0 u_0 \frac{\partial^2 u'}{\partial x^2} + \rho \frac{\partial u_0}{\partial y} \frac{\partial v'}{\partial x} + \frac{\partial^2 p'}{\partial x^2},
\]
\[
= \rho_0 \frac{\partial}{\partial x} \frac{D_0 u'}{D_0 t} + \rho_0 \frac{\partial u_0}{\partial y} \frac{\partial v'}{\partial x} + \frac{\partial^2 p'}{\partial x^2} = 0,
\]
\[
\frac{\partial}{\partial y} (4.7) = \rho_0 \frac{\partial^2 v'}{\partial y \partial t} + \rho_0 \frac{\partial u_0}{\partial y} \frac{\partial v'}{\partial x} + \rho u_0 \frac{\partial v'}{\partial x \partial y} + \frac{\partial^2 p'}{\partial y^2},
\]
\[
= \rho_0 \frac{\partial}{\partial y} \frac{D_0 v'}{D_0 t} + \rho_0 \frac{\partial u_0}{\partial y} \frac{\partial v'}{\partial x} + \frac{\partial^2 p'}{\partial y^2} = 0,
\]
\[
\rho_0 \frac{D_0}{D_0 t} (4.8) = \frac{1}{c_0^2} \frac{D_3}{D_0 t^2} + \rho_0 \frac{D_0}{D_0 t} \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) = 0.
\]

Combined this results in the inhomogeneous wave equation
\[
\frac{1}{c_0^2} \frac{D_3}{D_0 t^3} p' - \nabla^2 p' = 2 \rho_0 \frac{\partial u_0}{\partial y} \frac{\partial v'}{\partial x}. \tag{4.9}
\]

Note that we can rewrite the left hand side as the wave operator on \( p' \). This equation is not sufficient to solve the problem, since we have both the velocity and pressure as unknowns. One of the possibilities is to take again the material time derivative on both the left and right hand side. With equation (4.7) we can then derive
\[
\frac{1}{c_0^2} \frac{D_3}{D_0 t^3} p' - \frac{D_0}{D_0 t} \nabla^2 p' = 2 \rho_0 \frac{\partial u_0}{\partial y} \frac{\partial v'}{\partial x}
\]
\[
= -2 \rho_0 \frac{\partial^2}{\partial x \partial y} p'. \tag{4.10}
\]

This is similar to the Pridmore-Brown equation. We can also add an equation to the system. This is the method we will use. Consider the system (4.6) and (4.9).

We want to make this system dimensionless and introduce
\[ p^* = \frac{1}{\rho_0 c_0^2} p', u^* = \frac{1}{c_0} u', v^* = \frac{1}{c_0} v', \]
\[ M^* = \frac{1}{c_0} u_0, x^* = x/c_0, y^* = y/c_0. \]  (4.11)

We substitute this in equation (4.6) and obtain
\[ \frac{\partial u^*}{\partial t} + M^* \frac{\partial u^*}{\partial x^*} + v^* \frac{dM^*}{dy^*} = -\frac{\partial p^*}{\partial x^*}. \]  (4.12)

We omit the asterisk and obtain
\[ Du \frac{Dt}{Dt} + v \frac{\partial M}{\partial y} + \frac{\partial p}{\partial x} = 0, \]  (4.13)
with \( \frac{D}{Dt} = \frac{\partial}{\partial t} + M \frac{\partial}{\partial x} \). For equation (4.9), we obtain
\[ \left[ \frac{\partial}{\partial t} + M^* \frac{\partial}{\partial x^*} \right] p^* - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) p^* = 2 \frac{dM^*}{dy^*} \frac{\partial v^*}{\partial x^*}. \]  (4.14)

where \( \nabla^* = \frac{\partial}{\partial x^*} + \frac{\partial}{\partial y^*} \). We omit the asterisk and obtain the dimensionless wave equation
\[ \frac{D^2 p}{Dt^2} - \nabla^2 p = 2 \frac{\partial M}{\partial y} \frac{\partial v}{\partial x}. \]  (4.15)

### 4.2 Mode- Theory & Matching

For the case of a subsonic mean flow present, we still are able to use mode theory. Hence, again we write the pressure and velocity in terms of modes. Here, we will discuss the case of a uniform mean flow.

When we assume that the mean flow is uniform, then \( v \) and \( M \) are not depending on \( y \) except for possibly in the singularity. Hence, we obtain the wave equation in homogeneous form. We try separation of variables and assume that
\[ p^\pm(x, y, t) = Y_p(y)e^{i(\omega t + k_n x)}. \]  (4.16)

When we substitute this into equation (4.15) without the source term on the right hand side, then we obtain
\[
0 = \left[ \frac{\partial}{\partial t} + M \frac{\partial}{\partial x} \right]^2 p - \frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2},
\]
\[
= \left[ \frac{\partial^2}{\partial t^2} + 2M \frac{\partial}{\partial t} \frac{\partial}{\partial x} + M^2 \frac{\partial^2}{\partial x^2} \right] p - \frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2},
\]
\[
= -\omega^2 \pm 2M\omega k_n - (M^2 - 1)k_n^2 \frac{Y''_p(y)}{Y_p(y)}. \quad (4.17)
\]

When we bring all terms depending on \( y \) to the left we obtain
\[
\frac{Y''_p(y)}{Y_p(y)} = -\left[ \omega^2 \mp 2M\omega k_n + (M^2 - 1)k_n^2 \right],
\]
\[
=: -\alpha_n^2. \quad (4.18)
\]

Hence
\[
Y_p(y) = C_1 \cos(\alpha_n y) + C_2 \sin(\alpha_n y). \quad (4.19)
\]

We can determine \( C_1, C_2, \) and \( \alpha_n \) with the boundary conditions. The pressure gradient at \( y = 0 \) and \( y = h \) is zero, hence from
\[
\left. \frac{\partial p}{\partial y} \right|_{y=0} = 0, \quad (4.20)
\]
we get again that \( C_2 = 0 \). The other boundary condition states
\[
\left. \frac{\partial p}{\partial y} \right|_{y=h} = 0, \quad (4.21)
\]
and this gives us that
\[
C_1 \sin(\alpha_n h) = 0,
\]
\[
\alpha_n h = n\pi,
\]
\[
\alpha_n^2 = \left( \frac{n\pi}{h} \right)^2, \quad (4.22)
\]
for \( n \in \mathbb{Z} \). So
\[
(M^2 - 1)k_n^2 \mp 2M\omega k_n + \omega^2 - \left( \frac{n\pi}{h} \right)^2 = 0. \quad (4.23)
\]
This gives us an expression for \( k_n \)
\[
k_n^\pm = \frac{\pm M\omega \pm \sqrt{\omega^2 + \left( \frac{n\pi}{h} \right)^2(M^2 - 1)}}{M^2 - 1}, \quad \text{and} \quad (4.24)
\]
\[
k_n^\pm = \frac{\pm M\omega - \sqrt{\omega^2 + \left( \frac{n\pi}{h} \right)^2(M^2 - 1)}}{M^2 - 1}. \quad (4.25)
\]
As in the previous chapter we can determine $C_1$ by normalizing $Y_p$ and obtain that for $n = 0$ we have $C_1 = \sqrt{\frac{1}{h}}$ and for $n \neq 0$ we have that $C_1 = \sqrt{\frac{2}{h}}$.

We can again consider a solution built from modes and again omit the time dependence

$$p_1 = \sum_{n=0}^{N_1-1} A_{1,n} Y_{p,(1,n)}(y) e^{-i k_{1,n}^+ x} + B_{1,n} Y_{p,(1,n)}(y) e^{i k_{1,n}^- x}, \quad (4.26)$$

$$p_2 = \sum_{n=0}^{N_2-1} A_{2,n} Y_{p,(2,n)}(y) e^{-i k_{2,n}^+ x} + B_{2,n} Y_{p,(2,n)}(y) e^{i k_{2,n}^- x}, \quad (4.27)$$

$$p_3 = \sum_{n=0}^{N_3-1} A_{3,n} Y_{p,(3,n)}(y) e^{-i k_{3,n}^+ x} + B_{3,n} Y_{p,(3,n)}(y) e^{i k_{3,n}^- x}, \quad (4.28)$$

(4.29)

with

$$k_{j,n}^\pm = \pm M \omega + \frac{\sqrt{\omega^2 + (\frac{n\pi}{h})^2(M^2 - 1)}}{M^2 - 1}, \quad \text{and} \quad (4.30)$$

$$k_{j,n}^\pm = \pm M \omega - \frac{\sqrt{\omega^2 + (\frac{n\pi}{h})^2(M^2 - 1)}}{M^2 - 1}. \quad (4.31)$$

and

$$Y_{p,(j,n)} = \psi_{j,n}, \quad (4.32)$$

for $j = 1, 2, 3$ and $n \in \mathbb{N}$. Note that when we match the modes in $x = 0$ that the equations for the pressure are the same as for the case of no mean flow. Hence this yields that after projection on the basis $\chi_m$, we obtain $2M$ equations that we also had earlier.

We can build a solution for the velocity from modes as well. We have that $M \neq M(y)$ so using equation (4.13) we obtain

$$\frac{Du}{Dt} + \frac{\partial p}{\partial x} = 0. \quad (4.33)$$

This gives us that
\[
\begin{align*}
\mathbf{u}_1 &= \sum_{n=0}^{N_1-1} \left( \frac{k_{1,n}^+}{w-M*k_{1,n}^+}A_{1,n}\psi_{1,n} e^{-ik_{1,n}^-x} - \frac{k_{1,n}^-}{w+M*k_{1,n}^-}B_{1,n}\psi_{1,n} e^{ik_{1,n}^+x} \right), \\
\mathbf{u}_2 &= \sum_{n=0}^{N_2-1} \left( \frac{k_{2,n}^+}{w-M*k_{2,n}^+}A_{2,n}\psi_{2,n} e^{-ik_{2,n}^-x} - \frac{k_{2,n}^-}{w+M*k_{2,n}^-}B_{2,n}\psi_{2,n} e^{ik_{2,n}^+x} \right), \\
\mathbf{u}_3 &= \sum_{n=0}^{N_3-1} \left( \frac{k_{3,n}^+}{w-M*k_{3,n}^+}A_{3,n}\psi_{3,n} e^{-ik_{3,n}^-x} - \frac{k_{3,n}^-}{w+M*k_{3,n}^-}B_{3,n}\psi_{3,n} e^{ik_{3,n}^+x} \right).
\end{align*}
\]
(4.34) (4.35) (4.36)

When we match the speed as in the previous chapter then we obtain, after projection the following matrix equation

\[
\begin{pmatrix}
\mathbf{M}_1 & 0 & 0 \\
0 & \mathbf{M}_2 & 0 \\
0 & 0 & \mathbf{M}_{k_1^+ - k_1^-} \\
\mathbf{M}_{k_2^+ - k_2^-} & 0 & \mathbf{M}_{k_3^+ - k_3^-} \\
\mathbf{P} & \mathbf{P} & \mathbf{P} \\
\end{pmatrix}
\begin{pmatrix}
\mathbf{B}_1 \\
\mathbf{B}_2 \\
\mathbf{A}_3 \\
\end{pmatrix}
= \begin{pmatrix}
\mathbf{M}_1 & 0 & 0 \\
0 & \mathbf{M}_2 & 0 \\
0 & 0 & \mathbf{M}_{k_1^+ - k_1^-} \\
\mathbf{M}_{k_2^+ - k_2^-} & 0 & \mathbf{M}_{k_3^+ - k_3^-} \\
\mathbf{P} & \mathbf{P} & \mathbf{P} \\
\end{pmatrix}
\begin{pmatrix}
\mathbf{A}_1 \\
\mathbf{A}_2 \\
\mathbf{A}_3 \\
\end{pmatrix},
\]

where,

\[
k_{j}^{\pm} = \begin{pmatrix}
\frac{k_{j,0}^+}{w+M*k_{j,0}^+} & 0 & \ldots & 0 \\
0 & \frac{k_{j,1}^+}{w+M*k_{j,1}^+} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & \frac{k_{j,n-1}^+}{w+M*k_{j,n-1}^+}
\end{pmatrix}.
\]

This set of equations can be solved analogously with the same method as used in the previous chapter.
4.3 Numerical results

We implement our system of equations in Matlab to perform simulations for different numbers of modes. Note again that when we discuss pressure, we imply the pressure perturbation. First we check whether we indeed observe continuity of pressure between the different tube regions.

For reasons of accuracy, we choose a large number of modes: \( N_1 = 100, \ N_2 = 100, \ N_3 = 200, \) and \( M = 1000. \) We take a cross-section at the point where all the tubes connect. We plot the real and imaginary part of the pressure in region 1, 2, and 3 and check whether they indeed align. For the real part of the pressure see figure (4.1).

![Figure 4.1: Continuity of the real part of the pressure between the different tubes.](image)

For the imaginary part of the pressure we observe something similar.

Again, between \( y = 0 \) and \( y = h_1 \) we have built a solution for the pressure from modes in region 1 and, between \( y = -h_2 \) and \( y = 0 \) for region 2 and between \( y = -h_2 \) and \( y = h_1 \) for region 3. The pressure in region 3 is plotted...
with diamonds. The solid line indicates the pressure in region 1 and aligns for the better part with the diamonds on the interval $y \in [0, h_1]$, both for the real as the imaginary part of the pressure. The same holds for the dash dotted line, indicating the pressure in region 2, on the interval $y \in [-h_2, 0]$. The approximation used by building a solution from modes is good. Note that in the origin, the largest deviations exist. This is due to the singularity placed there.

Now that we have shown that continuity exists in the plane where the modes corresponding to the pressure are matched, we will also check if there exists continuity of velocity (note that we again mean the perturbations). See figure (4.2) for the real part of the velocity.

![Figure 4.2: Continuity of the real part of the velocity between the different tubes](image)
For the imaginary part of the velocity we observe something similar. We note continuity here as well and therefore we can conclude that our model is fit to perform pressure calculations and velocity calculations.

To give an idea what has changed by introducing a mean flow, we will discuss the pressure profile in our system. We choose a Mach-number of .5. This implies that we consider a speed of approximately 170m/s entering region 1 of our system. See figure (4.3)

![Figure 4.3: Real part of the pressure throughout the tube with mean flow (M ≈ .5)](image)

We observe a somewhat perturbed pressure profile over the tube and again note the continuity. The standing wave we had in the previous chapter is now perturbed due to the mean flow present.

We now like to determine the effect that the number of modes has on the numerical results of the flow. Therefore we look at the dependence on the number of modes in region 3, \( N_3 \). We check the case \( N_1 = N_2 = 2, 4, 6, \ldots 40 \) for increasing \( N_3 \) and plot the real value of the amplitude of the second mode of
the wave travelling to the right in region 3. See figure (4.4).

Figure 4.4: Real part of the amplitude of the second mode of the wave travelling to the right in region 3 for different values of N1 and N3 in the case of mean flow ($M \approx .5$)

We note here that for increasing values of $N_1$ and $N_2$ we observe convergence. If we look for instance at $N_3 = 8$, then we see that the amplitude is decreasing till $N_1 = 4$, then increasing and reaching its convergence point $N_1 = 6$. For $N_3 = 12$ we also a decreasing part at first followed by a converging and rising part. The amplitude has converged at approximately $N_1 = 14$. Hence we see that for increasing $N_3$ the converging occurs for larger $N_1$. For larger values of $N_3$ we observe something similar.

We observe that the length of the decreasing part of the amplitude does not seem to be highly dependent on the number of modes in region 3. However, the decrease is larger for smaller numbers of $N_3$. Moreover, for larger values of $N_3$ the point to which the amplitude converges is larger. For increasing $N_3$ the converging point seems to go to a converging point of his own. We also observe that when $N_1 \approx N_3$ that the amplitude has nearly reached its limit.
Now, we take a closer look at the modes in region 3. Therefore we consider the amplitude of the mode of the wave travelling to the right in region 3. We scale this amplitude by its index and plot it on logarithmic paper. We consider $N_1 = N_2 = 10$, $N_3 = 200$, and $M = 1000$. The results are plotted in figure (4.5).

![Figure 4.5: Scaled value of the amplitude of all modes (2 until N3-1) of A3](image)

We observe that the amplitude of the modes that represent the wave that travels to the right in region 3 again keeps jumping between two regions. We distinguish lower and upper modes. The first group of modes -which will be called the upper modes- seems to converge to a certain point and the second group of modes, the lower modes, seems to converge to one point as well, yet another. Notice that two different points of convergence also imply two different solutions.

To acquire more information on these possible different answers, we simulate some more. We consider the same picture with the same values, except now
with $N_1 = N_2 = 100$, $N_3 = 200$, and $M = 1000$. This only changes the relative difference between the modes. The results are plotted in figure (4.6).

Figure 4.6: Scaled value of the amplitude of all modes (2 until N3-1) of $A_3$

We hardly see any difference between this case and the one where $N_1 = N_2 = 10$, aside again for some small oscillations for the later modes. Hence, we can conclude that the relative difference between $N_1$ and $N_2$ on one hand and $N_3$ on the other is not relevant for the modes. We now try to analyze the relative difference between $N_1$ and $N_2$. We consider the previous picture as our first test for taking them equal to each other. Next, we try to vary them. We choose $N_1 = 10$ and $N_2 = 100$ and the other numbers we keep the same. Results our plotted in figure (4.7).
Here, we obtain a totally different result. We observe that we can hardly talk about two solutions anymore. We see one line of modes and this one seems to correspond the upper line from before. The lower line seems to have disappeared. The relative difference between the number of modes in region 1 and region 2 seem to have a great influence one the modes.

When we compare this picture with (4.6) then we see that dropping the number of modes in region 1 results in losing one of the two lines. When we compare it with (4.5) (which is similar except that we have now increased the number of modes in region 2) then we can also conclude that not the fact that we choose only $N_1 = 10$ modes is the result that we lose one line of modes, but the relative difference of $N_1$ and $N_2$ is solely responsible for this.

The question rises whether this behaviour is symmetric. In other words, if we choose more modes in region 1 than in region 2, do we observe similar be-
haviour? Therefore, we choose $N_1 = 100$ and $N_2 = 10$ and $N_3$ and $M$ as before. See figure (4.8) for the results.

![Graph showing amplitude of modes](image)

**Figure 4.8**: Scaled value of the amplitude of all modes (2 until N3-1) of A3

We observe a symmetry between the number of modes in region 1 and region 2. This means that for this setting we can simply interchange the number of modes in region 1 and 2 and it does not make a difference for the behaviour of the modes on the transmission in region 3.

Now we consider what part of the flow is reflected and what part is transmitted. Since we have a standing wave pattern in region 1 and a uniform mean flow we consider the wave reflecting in region 1 and 2 and the part that is transmitted to region 3. We choose $M = 1000$. First we take $N_1 = N_2 = 10$ and $N_3 = 20$. We then calculate that 25.6 % is reflected and the rest is transmitted. For $N_1 = N_2 = 10$ and $N_3 = 200$, we have 19.6% reflected. For $N_1 = N_2 = 100$ and $N_3 = 200$ we have 39.4% reflected. We notice that the part reflected depends highly on the number of modes chosen, especially in region 1 and 2.
The more modes we choose, the better our solution is, so we take $N_1 = N_2 = 200$ and $N_3 = 400$ and calculate the reflection again to be more precise. We obtain a 42.9% reflection.

4.4 Conclusions

In this chapter, we made a model of an acoustic liner, including a uniform mean flow. We have used mode theory and mode matching to find a set of $4M$ equations and $N_1 + N_2 + N_3$ unknowns. With the use of the Complex Least Squared Method we can find a solution for this problem. With the use of a numerical implementation we have proven that our method gives accurate results for the flow in a tube. In addition, we noticed that the number of modes were important for the convergence of the solution and we could notice two different solutions for the modes in region 3. However, we do not observe a difference for different number of modes in region 1 and 2 compared with modes from region 3. The relative difference between the number of modes in region 1 and 2, however, is very important for it cancels one solution. When we choose our modes such that it is approximately equal to the difference in heights, then we observe two types of modes, upper and lower. Finally, when we choose sufficient modes, then we can also accurately calculate that part of the wave that is reflected.
Chapter 5

Conclusions and Recommendations

In this chapter we first summarize what we have done so far. During our summary we draw conclusions per topic. Afterwards we combine our conclusions and draw a main conclusion. Finally, we end with some recommendations for further research.

5.1 Conclusions of this report

In the introduction of the report we have stated the importance of research on noise reduction with regard to airplanes. Most noise generated by an airplane is due to the engine and the fan. Part of this noise is absorbed by the acoustic liners positioned around the engine. An accurate mathematical model of such a liner can be used to perform calculations on pressure perturbations and velocity profiles. In addition, one could calculate which part of the sound flow is absorbed.

In the second chapter, the iris problem is discussed to give an introduction to the difficulties we face when trying to make a mathematical model of an acoustic liner. We, shortly, discuss mode theory and matching as a possible method to circumvent problems.

In chapter three we have derived and implemented a mathematical model of an acoustic liner. The simulations performed on the program, based on this mathematical model, show that this program gives an accurate image of the pressure profile in our liner. In addition, we show that the solution is not dependent on relative difference between the number of modes in region 1 and 2 and the number of modes in region 3. We do observe a dependence on the relative difference between the number of modes in region 1 and the number of modes in region 2. With our program, we can also calculate what part of the wave is reflected and what part is transmitted.
The proceeding chapter discusses a similar model, except this time with a uniform mean flow present. The simulations done on this program show equivalent results. We do obtain a different pressure and velocity profile, but the mode dependence can be found again. The effect that the number of modes has on the reflection coefficient is this time much larger than in the case without mean flow.

5.2 Recommendations for future research

In this report, two models have been numerically implemented in computer programs. These can be used to give accurate descriptions of the pressure and velocity profiles in an acoustic liner.

It is recommended that -in the future- a new mathematical model is derived and implemented. Again with a mean flow, but this time not necessarily with a uniform flow, but for any velocity profile. It is difficult to determine the direct use of such a model, but it is a more complete model than the one with uniform model.

In addition, analytical results could be gained as well. One can suggest to use the Wiener-Hopf technique and use complex function theory to derive qualitative solutions for the equations derived in order to compare these with the results in this report.
Bibliography


Appendix A

Analytical calculations on several matrices

The system of equations derived in chapter three, cannot be implemented before determining the sub-matrices there. We have to determine $M_1$, $M_2$, $P_1$, and $P_2$.

A.1 Determination of $M_1$

For $m = n = 0$ we have

$$M_1 = \int_0^{h_1} \psi_{1,n} \psi_{3,m} \, dy,$$

$$= \sqrt{h_1}. \quad (A.1)$$

For $m \neq 0$ and $n = 0$ we obtain

$$M_1 = \int_0^{h_1} \psi_{1,n} \psi_{3,m} \, dy,$$

$$= \int_0^{h_1} \sqrt{\frac{2}{h_1}} \cos(m\pi \frac{y + h_2}{h_3}) \, dy,$$

$$= \sqrt{\frac{2}{h_1 h_3}} \int_{h_2}^{h_3} \cos(m\pi \frac{z}{h_3}) \, dz,$$

$$= \sqrt{\frac{2}{h_1 h_3}} \left[ \frac{h_3}{m\pi} \sin(m\pi \frac{z}{h_3}) \right]_{h_2}^{h_3},$$

$$= -\frac{2h_3}{h_1} \frac{1}{m\pi} \sin(m\pi \frac{h_2}{h_3}). \quad (A.2)$$
For $n \neq 0$ and $m = 0$ we obtain

$$M_1 = \int_0^{h_1} \psi_{1,n,3,m} dy,$$

$$= \int_0^{h_1} \sqrt{\frac{1}{h_3}} \sqrt{\frac{2}{h_1}} \cos(n\pi \frac{y}{h_1}) dy,$$

$$= \sqrt{\frac{2}{h_1 h_3}} \left[ \frac{h_1}{n\pi} \sin(n\pi \frac{y}{h_1}) \right]_0^{h_1},$$

$$= 0.$$  \hspace{1cm} (A.3)

For $n \neq 0$ and $m \neq 0$ we obtain

$$M_1 = \int_0^{h_1} \psi_{1,n,3,m} dy,$$

$$= \sqrt{\frac{2}{h_1 h_3}} \int_0^{h_1} \cos(n\pi \frac{y}{h_1}) \cos(m\pi \frac{y + h_2}{h_3}) dy.$$  \hspace{1cm} (A.4)

Now if $\frac{n}{h_1} = \frac{m}{h_3}$ then

$$M_1 = \sqrt{h_1 h_3} \left[ \cos(m\pi \frac{h_2}{h_3}) - \frac{1}{2\pi n} \sin(m\pi \frac{h_2}{h_3}) \right].$$  \hspace{1cm} (A.5)

Else

$$M_1 = \sqrt{h_1 h_3} \left[ \frac{1}{\pi(n - m \frac{h_3}{h_1})} - \frac{1}{\pi(n + m \frac{h_3}{h_1})} \right] \sin(m\pi \frac{h_2}{h_3}).$$  \hspace{1cm} (A.6)

**A.2 Determination of $M_2$**

For $m = n = 0$ we have

$$M_2 = \int_{-h_2}^0 \psi_{2,n,3,m} dy,$$

$$= \sqrt{\frac{h_2}{h_3}}.$$  \hspace{1cm} (A.7)

For $m \neq 0$ and $n = 0$ we obtain
\[ M_2 = \int_{-h_2}^{0} \psi_{2,n}\psi_{3,m}dy, \]
\[ = \int_{-h_2}^{0} \sqrt{\frac{1}{h_2}} \sqrt{\frac{2}{h_3}} \cos(m\pi \frac{y + h_2}{h_3})dy, \]
\[ = \sqrt{\frac{2}{h_2h_3}} \int_{0}^{h_2} \cos(m\pi \frac{z}{h_3})dz, \]
\[ = \sqrt{\frac{2}{h_2h_3}} \left[ \frac{h_3}{m\pi} \sin(m\pi \frac{z}{h_3}) \right]_{0}^{h_2}, \]
\[ = \sqrt{\frac{2h_3}{h_2}} \frac{1}{m\pi} \sin(m\pi \frac{h_2}{h_3}). \quad (A.8) \]

For \( n \neq 0 \) and \( m = 0 \) we obtain

\[ M_2 = \int_{-h_2}^{0} \psi_{2,n}\psi_{3,m}dy, \]
\[ = \int_{-h_2}^{0} \sqrt{\frac{1}{h_3}} \sqrt{\frac{2}{h_2}} \cos(n\pi \frac{y}{h_2})dy, \]
\[ = \sqrt{\frac{2}{h_2h_3}} \left[ \frac{h_2}{n\pi} \sin(n\pi \frac{y}{h_2}) \right]_{-h_2}^{0}, \]
\[ = 0. \quad (A.9) \]

For \( n \neq 0 \) and \( m \neq 0 \) and \( \frac{n}{h_2} = \frac{m}{h_3} \) we obtain

\[ M_2 = \int_{-h_2}^{0} \psi_{2,n}\psi_{3,m}dy, \]
\[ = (-1)^n \sqrt{\frac{h_2}{h_3}} \quad (A.10) \]

For \( n \neq 0 \) and \( m \neq 0 \) and \( \frac{n}{h_2} \neq \frac{m}{h_3} \) we obtain

\[ M_2 = \int_{-h_2}^{0} \psi_{2,n}\psi_{3,m}dy, \]
\[ = \sqrt{\frac{h_2}{h_3}} \left[ \frac{1}{(n + m\frac{h_2}{h_3})\pi} - \frac{1}{(n - m\frac{h_2}{h_3})\pi} \right] \sin(m\pi \frac{h_2}{h_3}). \quad (A.11) \]

**A.3 Determination of \( P_1 \)**

For \( m = n = 0 \) we have

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\[ P_1 = \int_0^{h_1} \psi_{3,n} \psi_{3,m} dy, \]
\[ = \frac{h_1}{h_3}. \quad (A.12) \]

For \( m \neq 0 \) and \( n = 0 \) we obtain

\[ P_1 = \int_0^{h_1} \psi_{3,n} \psi_{3,m} dy, \]
\[ = \int_0^{h_1} \sqrt{\frac{1}{h_3}} \sqrt{\frac{2}{h_3}} \cos(m \pi \frac{y + h_2}{h_3}) dy, \]
\[ = \frac{\sqrt{2}}{h_3} \left[ \frac{h_3}{m \pi} \sin(m \pi \frac{z}{h_3}) \right]_{h_2}^{h_3}, \]
\[ = -\frac{\sqrt{2}}{m \pi} \sin(m \pi h_2). \quad (A.13) \]

For \( n \neq 0 \) and \( m = 0 \) we obtain

\[ P_1 = \int_0^{h_1} \psi_{3,n} \psi_{3,m} dy, \]
\[ = -\frac{\sqrt{2}}{n \pi} \sin(n \pi h_2). \quad (A.14) \]

For \( n \neq 0 \) and \( m \neq 0 \) and \( m = n \) we obtain

\[ P_1 = \int_0^{h_1} \psi_{3,n} \psi_{3,m} dy, \]
\[ = \frac{h_1}{h_3} - \frac{1}{2m \pi} \sin(2m \pi \frac{h_2}{h_3}). \quad (A.15) \]

Now if \( m \neq n \)

\[ P_1 = \int_0^{h_1} \psi_{3,n} \psi_{3,m} dy, \]
\[ = \frac{-(m + n) \sin \left( \frac{(m - n) \pi h_2}{h_3} \right) + (-m + n) \sin \left( \frac{(m + n) \pi h_2}{h_3} \right)}{(m - n)(m + n) \pi}. \quad (A.16) \]

### A.4 Determination of \( P_2 \)

For \( m = n = 0 \) we have
\[ P_2 = \int_{-h_2}^{0} \psi_{3,n} \psi_{3,m} dy, \]
\[ = \frac{h_2}{h_3}. \quad (A.17) \]

For \( m \neq 0 \) and \( n = 0 \) we obtain
\[ P_2 = \int_{-h_2}^{0} \psi_{3,n} \psi_{3,m} dy, \]
\[ = \int_{-h_2}^{0} \sqrt{\frac{1}{h_3}} \sqrt{\frac{h_3}{2}} \cos(m\pi \frac{y + h_2}{h_3}) dy, \]
\[ = \frac{\sqrt{2}}{h_3} \left[ \frac{h_3}{m\pi} \sin(m\pi \frac{z}{h_3}) \right]_{h_2}^{0}, \]
\[ = \frac{\sqrt{2}}{m\pi} \sin(m\pi \frac{h_2}{h_3}). \quad (A.18) \]

For \( n \neq 0 \) and \( m = 0 \) we obtain
\[ P_2 = \int_{-h_2}^{0} \psi_{3,n} \psi_{3,m} dy, \]
\[ = \frac{\sqrt{2}}{n\pi} \sin(n\pi \frac{h_2}{h_3}). \quad (A.19) \]

For \( n \neq 0 \) and \( m \neq 0 \) and \( m = n \) we obtain
\[ P_2 = \int_{-h_2}^{0} \psi_{3,n} \psi_{3,m} dy, \]
\[ = \frac{h_2}{h_3} \frac{\sin(2m\pi \frac{h_2}{h_3})}{2m\pi}. \quad (A.20) \]

Now if \( m \neq n \)
\[ P_2 = \int_{-h_2}^{0} \psi_{3,n} \psi_{3,m} dy, \]
\[ = \frac{\sin((m - n)\pi \frac{h_2}{h_3})}{\pi(m - n)} + \frac{\sin((m + n)\pi \frac{h_2}{h_3})}{\pi(m + n)} \quad (A.21) \]