Master’s Thesis:
INSURANCE RISK AND QUEUEING MODELS WITH THRESHOLD DEPENDENCE

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To My Lord Jesus-Christ
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Abstract

In this master thesis we consider a dependent setting of the classical queueing model. We assume that the distribution of the service time depends on the inter-arrival time between customers according to some threshold. An exact solution for the workload distribution (virtual waiting time) is obtained in the queueing system where the threshold is deterministic and the service time is set to zero, if the inter-arrival time is less than the threshold. The result is compared with the stationary ruin probability under a threshold of an insurance risk model. For the case of an exponential threshold, the Laplace-Stieltjes transform of the sojourn time of a customer is derived. The effect of the threshold dependence on the mean performance measures is illustrated by several numerical examples.
Keywords: Ruin Probability, Threshold, Dependence, Laplace-Stieltjes transform, Risk theory, Queueing model with dependencies, Duality.
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Chapter 1

Introduction

Nowadays Queueing theory and Risk theory are well developed branches of applied probability theory. A large body of literature, which is still growing rapidly, exists on these subjects. It is Erlang (Cf. Brockmeyer, Halström, Jensen [1948]), a Danish mathematician, who may be considered as the founder of Queueing theory. His studies, in the period 1909-1920, are now classical queueing theory. His classical queueing models rely on the assumption of independence between service time and inter-arrival times of the customers. This independence assumption is also made for the classical Cramér-Lundberg model to describe the surplus process of an insurance portfolio. A general connection between ruin models and queueing models is the equality between the survival probability and the stationary workload distribution of the queueing model, under independence assumption ([10] pages 161-162). In contrast to the classical assumption, we suppose that the claim sizes and the inter-occurrence times are dependent. Recently various results have been obtained concerning the asymptotic behavior of the probability of ruin with dependent claims size. Hansjörg Albrecher and Onno J. Boxma (2004) investigated the following dependence structure: The time between two claim occurrences depends on the previous claim size. They derived an exact solution for the probability of survival by means of Laplace-Stieltjes transforms. Albrecher and Boxma (2005) studied the Gerber-Shiu function in the model with dependence between claim size and inter-arrival time. Isaac Kwan and Hailiang Yang (2007) studied the reverse dependence structure of the model developed by Albrecher and Boxma (2004); a closed form for the ruin probability was found for the special case where the claim size distribution is exponential. Isaac Kwan and Hailiang Yang considered also a model similar to that in Albrecher and Boxma (2004), but with a fixed threshold. By solving a system of delay differential equations they found that when the claim size follows an exponential distribution, an explicit solution for the ruin probability can be obtained.
Motivated by a related model in risk theory (Albrecher and Boxma 2004, Kwan and Yang 2007), we consider in this master thesis a generalization of the classical M/G/1 and G/M/1 queue with some dependence structure.

The following notations are used to describe the various queueing and risk models in this thesis. In general $a/b/1$ denotes a queueing model with arrival distribution $a$ and service distribution $b$. The symbols $a$ and $b$ take values from the set $\{M, G\}$ denoting as usual either an exponential distribution or a general distribution. This is the well known Kendall notation, where e.g. $M/G/1$ denotes a Markovian queue with exponential inter-arrival times and general service times. All studied queues work under the FIFO regime, i.e. customers are served according to their arrival time. For the risk models we use a similar notation, namely $a/b$ for a risk model with inter-arrival time distribution $a$ and claim size distribution $b$.

To describe the various dependencies, we use arrow symbols. For instance $a/\leftarrow b/1$ denotes a $a/b/1$ queue, where the service time of the $n$th customer depends on the inter-arrival time between the $(n-1)$th and the $n$th customer or $a/\rightarrow b$ symbolizes a risk model where the inter-arrival time of the $n$th claim and the $(n+1)$th claim depends on the next claim size.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Interpretation for queueing model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a/\leftarrow b/1$</td>
<td>Service times depend on the previous inter-arrival times</td>
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<tr>
<td>$a/\rightarrow b/1$</td>
<td>Service times depend on the next inter-arrival times</td>
</tr>
<tr>
<td>$\leftarrow a/b/1$</td>
<td>Inter-arrival times depend on the next service times</td>
</tr>
<tr>
<td>$\rightarrow a/b/1$</td>
<td>Inter-arrival times depend on the previous service times</td>
</tr>
</tbody>
</table>

For the risk models a similar convention holds.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Interpretation for risk model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a/\leftarrow b$</td>
<td>Claim sizes depend on the previous inter-arrival times</td>
</tr>
<tr>
<td>$a/\rightarrow b$</td>
<td>Claim sizes depend on the next inter-arrival times</td>
</tr>
<tr>
<td>$\leftarrow a/b$</td>
<td>Inter-arrival times depend on the next claim sizes</td>
</tr>
<tr>
<td>$\rightarrow a/b$</td>
<td>Inter-arrival times depend on the previous claim sizes</td>
</tr>
</tbody>
</table>

Note that "dependence" does not refer to dependence in the probabilistic sense, but should be interpreted as a dependency of the distribution of one quantity on the outcome of the other. For example in the $a/\leftarrow b/1$ queue, the service time distribution
of the $n$th customer is completely determined by the previous inter-arrival time, while in the $\alpha/b/1$ model the inter-arrival distribution between customers $n$ and $(n+1)$ is determined by the service time of the $(n+1)$th customer.

In case of a dependency on a threshold, the type of threshold is denoted by a subscript. Basically two types of threshold dependence appear in this thesis, namely dependence on a deterministic threshold (with symbol $a_d$) and dependence on an exponential threshold (with symbol $a_M$). For example the symbol $M/G_d/1$ denotes a queue, where the service time distribution of the $n$th customer depends on his inter-arrival time in a way such that it is equal to some distribution $F_1$ whenever the inter-arrival time is below a deterministic threshold, and is equal to a different distribution $F_2$ otherwise.

We will also explore two different types of dualities between queueing models and risk models in this thesis. Both dualities, named $D_1$ and $D_2$, map a specific queueing model to a specific risk model, thereby acting in a characteristic way on the present distributions and dependencies.

![Figure 1.1: The duality $D_1$ between a risk model (below) and a queueing model (above)](image)

Duality one is defined as follows. Starting with a queueing model $a/b/1$ with inter-arrival times $A_1, A_2, \ldots$ ($A_1$ denoting the time between the first and the second customer) and service times $B_1, B_2, \ldots$, a new risk model $a/b$ is obtained by the following principle: we fix some $N \in \{1, 2, 3, \ldots \}$ and let the inter-arrival times of the claims be $A_N, A_{N-1}, A_{N-2}, \ldots$. The claim sizes of the dual risk model are given by
It turns out that this duality maps $a/b/1$ models to $a/b$ models, but flips the dependencies, since the process $(A_1, B_1), (A_2, B_2), \ldots$ is mapped to its time reversed version $(A_N, B_N), (A_{N-1}, B_{N-1}), \ldots$. To indicate these duality relations we write

$$
\begin{align*}
\frac{a}{b}/1 & \xrightarrow{D_1} \frac{a}{b}, \\
\frac{a}{b}/1 & \xrightarrow{D_1} \frac{a}{b}, \\
\frac{a}{b}/1 & \xrightarrow{D_1} \frac{a}{b}, \\
\frac{a}{b}/1 & \xrightarrow{D_1} \frac{a}{b}.
\end{align*}
$$

Figure 1.2: The duality $D_2$ between a risk model (below) and a queueing model (above)

The second duality is defined in a simple way by letting the inter-arrival times and the claim sizes of the dual risk model be given by the service and inter-arrival times of the original queueing model, that is, the queueing model $a/b/1$ is mapped to the risk model $b/a$. Here the dependencies do not change their direction, so that we can
write
\[
\begin{align*}
\overset{\rightarrow}{a}/b/1 & \overset{D_2}{\rightarrow} \overset{\leftarrow}{b}/a, \\
\overset{\rightarrow}{a}/b/1 & \overset{D_2}{\rightarrow} \overset{\leftarrow}{b}/a, \\
a/\overset{\rightarrow}{b}/1 & \overset{D_2}{\rightarrow} \overset{\leftarrow}{b}/a, \\
a/\overset{\rightarrow}{b}/1 & \overset{D_2}{\rightarrow} \overset{\leftarrow}{b}/a,
\end{align*}
\]
(1.2)

The studied queueing model in this thesis mainly originates from the risk model \(M_\overset{\rightarrow}{G}_d\), so we will actually apply only the dualities (1.1) and (1.2).

The thesis is organized as follows.

- In the introductory chapter we describe the classical Cramér-Lundberg model of risk theory and show Beekman’s convolution formula. We also show how formulas for the ruin probabilities can be found.

- Chapter 3 deals with several queueing and risk models related to the \(M_\overset{\rightarrow}{G}_d\) model.
  - In Section 3.1 we introduce the risk model \(M_\overset{\rightarrow}{G}_d\) described by Kwan and Yang (2007), which is of great importance for this thesis.
  - Section 3.2 deals with the special case \(M_\overset{\leftarrow}{M}_d\).
  - In Section 3.3 we investigate another special case of the \(M_\overset{\rightarrow}{G}_d\) model, namely the case where the claim sizes are either zero, if the inter-occurrence time is smaller than a threshold \(a\), or exponential, if not.
  - In Section 3.4 the duality \(D_1\) is applied to the \(M_\overset{\rightarrow}{M}_d\) ruin model. It turns out that the resulting \(M_\overset{\leftarrow}{M}_d/1\) queuing model is equivalent to a standard \(G/M/1\) queue.
  - We apply duality \(D_2\) to the model \(M_\overset{\rightarrow}{M}_d\) in Section 3.5 and obtain a \(\overset{\leftarrow}{M}_d/M/1\) queuing model. This queue is in fact equivalent to a standard \(M/G/1\) queue.
  - In Section 3.6 we generalize the duality argument of Section 3.4 to the \(M_\overset{\rightarrow}{G}_d\) ruin model and the dual model \(M_\overset{\rightarrow}{G}_d/1\).

- Chapter 4 is entirely devoted to an \(M_\overset{\rightarrow}{G}_M/1\), numerical illustrations are given and the effect of dependence is investigated.
Chapter 2

Ruin Theory

2.1 Introduction

In this chapter we introduce ruin theory. We consider the development in time of the capital $U(t)$ of an insurer. When the capital becomes negative, we say that ruin occurs. Let $\psi(u)$ denote the probability that this happens, provided that the annual premium and the claims process remain unchanged; here $u = U(0)$ is the initial capital. Our interest throughout this chapter is to compute this probability for two types of claim distributions:

1. Exponential distributions and sums, mixtures and combinations.
2. Distribution with only a finite number of values.

In addition we wish to derive an exponential bound: $\psi(u) \leq e^{-Ru}$, for some $R > 0$.

We define the surplus process or risk process as follows:

$$U(t) = u + ct - S(t), \quad t \geq 0,$$

(2.1)

where

$U(t)$ = the insurer’s capital at time $t$;
$u = U(0)$ = the initial capital;
$c$ = the (constant) premium income per unit of time;
$S(t)$ = $X_1 + X_2 + \ldots + X_{N(t)}$, the aggregate claim amount

with

$N(t)$ = the number of claims up to time $t$, assumed to follow a Poisson process with rate $\lambda$, and
$X_i$ = the size of the $i^{th}$ claim, assumed to be non-negative.
Let $T$ denote the point in time at which the ruin occurs. So,
\[
T = \begin{cases} 
\min\{t | t \geq 0\} & \text{if } U(t) < 0 \text{ for some } t, \\
\infty & \text{if } U(t) \geq 0 \text{ for all } t.
\end{cases}
\]

The probability that $T$ is finite is called the ruin probability. It is written as follows:
\[
\psi(u) = P[T < \infty | U(0) = u].
\]

Let
\[
P(x) = P[X \leq x]; \quad \mu_j = E[X^j], \quad j = 1, 2, \ldots
\]
denote the distribution function and the $j$th moment of the claim sizes. We define the so-called safety loading $\theta$ by
\[
\theta = \frac{c}{\mu_1} - 1
\]
and the adjustment coefficient $R$ as the positive solution of the following equation in $r$
\[
1 + (1 + \theta)\mu_1 r = m_X(r), \quad (2.3)
\]
where $m_X(r) = E[e^{rX}]$ is the moment generating function of the claim size distribution.

**Theorem 2.1.1** (Lundberg’s exponential bound for the ruin probability). For a compound Poisson risk process, we have the following inequality for the ruin probability
\[
\psi(u) \leq e^{-Ru}.
\]

**Proof.** Let $\psi_k(u)$, with $-\infty < u < \infty$ and $k = 0, 1, 2, \ldots$ be the probability that ruin occurs at or before the $k^{th}$ claim. We will prove that $\psi_k(u) \leq e^{-Ru}$ and take the limit as $k \to \infty$. We proceed by induction on $k$. If $k = 0$ the inequality holds, because $\psi_0(u) = 1$ if $u < 0$, and $\psi_0(u) = 0$ if $u > 0$. If we assume that the first claim happens at time $t$ with size $x$, then the capital at that moment is $u + ct - x$. We derive a recurrence relation between the probability of ruin before the $k^{th}$ claim and the probability of ruin before the $(k - 1)^{th}$ claim. Starting with the capital $u$ and knowing that the $k^{th}$ claim occurs with the probability $\psi_k(u)$ is the same as starting in $u + ct - x$ with probability $\psi_{(k-1)}(u + ct - x)$. Now integrating over all possible values of $t$ and $x$ gives the following equation:
\[
\psi_k(u) = \int_0^\infty \int_0^\infty \psi_{(k-1)}(u + ct - x) dP(x) \lambda e^{-\lambda t} dt. \quad (2.4)
\]
We assume that \( \psi_{(k-1)}(u) \leq e^{-Ru} \). By (2.4) we get:

\[
\psi_k(u) \leq \int_0^\infty \int_0^\infty \exp\{-R(u + ct - x)\}dP(x)\lambda e^{-\lambda t} dt
= e^{-Ru} \int_0^\infty \lambda \exp\{-t(\lambda + Rc)\}dt \int_0^\infty e^{Rx}dP(x)
= e^{-Ru} \frac{\lambda}{\lambda + cR}m_x(R)
= e^{-Ru},
\]

where the last equality follows from (2.3). Now if we take the limit when \( k \to \infty \) we get the desired inequality. \( \square \)

**Theorem 2.1.2.** The ruin probability for \( u \geq 0 \) satisfies

\[
\psi(u) = \frac{e^{-Ru}}{E[e^{-RU(T)|T < \infty}]}.
\]

For the proof see ([1], page 89).

### 2.2 Beekman’s convolution formula

In this section we show that the survival probability can be written as a compound geometric distribution function. We denote by \( L \) the maximal difference between the payments and the earned premium up to time \( t \):

\[
L = \max\{S(t) - ct|t \geq 0\}.
\]

The event \( L > u \) occurs if and only if a finite point in time \( t \) exists such that \( U(t) \leq 0 \). This means that \( L > u \) and \( T < \infty \) are equivalent, as a consequence we have \( \psi(u) = 1 - F_L(u) \) where \( F_L(u) \) is the distribution function of \( L \). Let the random variables \( L_j, j = 1, 2, \ldots \) denote the amounts by which the \( j^{th} \) record low is less than the \( (j - 1)^{th} \) one then we have \( L = L_1 + L_2 + \ldots + L_M \), where \( M \) is a random number of new records. From the memoryless property of the Poisson process, the probability that a particular record low is the last one is the same every time. This implies that \( M \) follows a geometric distribution. The parameter of \( M \) is \( (1 - \psi(0)) \), that is the probability of non-ruin starting with a capital of \( u = 0 \). The amounts of improvements \( L_1, L_2, \ldots \) are independent and identically distributed. So \( L \) has a compound geometric distribution.
Theorem 2.2.1 (Distribution of the capital at time of ruin). If the initial capital $u$ equals 0, then for all $y > 0$ we have

$$P[U(T) \in (-y - dy, -y), T < \infty] = \frac{\lambda}{\mu} (1 - P(y)) dy. \quad (2.7)$$

The proof of this theorem can be found in [1] page 97.

- As a consequence of this theorem, the ruin probability at 0 depends only on the safety loading. In fact, integrating the expression in (2.7) for $y \in (0, \infty)$ gives $\psi(0) = \frac{1}{1 + \theta}$.

- Assuming that there is at least one new record low, $L_1$ has the same distribution as the deficit at ruin starting from $u = 0$ if ruin occurs. Thus

$$P[L_1 > y | T < \infty] = \frac{\lambda}{\mu} \int_{y}^{\infty} (1 - P(u)) \frac{1}{\psi(0)} du. \quad (2.8)$$

It imply that the density of $f_{L_1}$ is given by

$$f_{L_1}(y) = \frac{1 - P(y)}{\mu_1}. \quad (2.9)$$

- The mgf (moment generating function) of the maximal aggregate loss $L$ is given by

$$m_L(r) = \frac{\theta}{1 + \theta} + \frac{\theta(m_X(r) - 1)}{1 + \theta (1 + \theta) \mu_1 r - m_X(r)}. \quad (2.10)$$

This last assertion is proved as follows. Since $M \sim \text{geometric}(p)$ for $p = 1 - \psi(0)$ it follows that:

$$m_L(r) = m_M(\log m_{L_1}(r)).$$

But

$$m_M(t) = \frac{p}{1 - (1 - p)e^t},$$

thus $m_L(r) = \frac{p}{1 - (1 - p)e^{m_{L_1}(e^r)}}$. Now it suffices to compute $m_{L_1}(r)$ to obtain a closed form of $m_L(r)$. From the density of $m_{L_1}(r)$ we have

$$m_{L_1}(r) = \frac{1}{\mu_1} \int_{0}^{\infty} e^{rx} (1 - P(x)) dx$$

$$= \frac{1}{\mu_1} \left\{ \frac{1}{r} (e^{rx} - 1)(1 - P(x)) \bigg|_{0}^{\infty} + \int_{0}^{\infty} \frac{1}{r} (e^{rx} - 1) dP(x) \right\}$$

$$= \frac{1}{\mu_1 r} (m_X(r) - 1).$$
Recursive formula for ruin probability  The ruin probability in \( u \) can be expressed in term of the ruin probabilities at smaller initial capitals as follows:

\[
\psi(u) = \frac{\lambda}{c} \int_0^u (1 - P(y)) \psi(u - y) dy + \frac{\lambda}{c} \int_u^\infty (1 - P(y)) dy \tag{2.11}
\]

Proof. We have

\[
\psi(u) = P[T < \infty] = P[T < \infty \& M > 0] = P[T < \infty | M > 0] P[M > 0],
\]

but

\[
P[M > 0] = 1 - P[M = 0] = 1 - p = \frac{1}{1 + \theta}\tag{2.13}
\]

and the density of \( F_{L_1} \) is given by

\[
f_{L_1}(y) = \frac{1 - P(y)}{\mu_1}, y > 0,
\]

so

\[
\psi(u) = \frac{1}{1 + \theta} \int_0^\infty P[T < \infty | L_1 = y] f_{L_1}(y) dy
\]

\[
= \frac{\lambda}{c} \left( \int_0^u \psi(u - y)(1 - P(y)) dy + \int_u^\infty (1 - P(y)) dy \right). \tag{2.15}
\]

\[
\square
\]

2.3 Explicit expressions for the ruin probabilities

We are going to compute the mgf of the maximum aggregate loss \( L \):

\[
m_L(r) = \int_0^\infty e^{ru} d[1 - \psi(u)]
\]

\[
= 1 - \psi(0) + \int_0^\infty e^{ru} (-\psi'(u)) du.
\]

It follows from the previous theorem that the 'mgf' for \(-\psi'(u)\) is equal to

\[
\frac{1}{1 + \theta} \frac{\theta(m_X(r) - 1)}{1 + (1 + \theta)\mu_1 r - m_X(r)}. \tag{2.16}
\]

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Example 2.3.1. Ruin probability for exponentially distributed claim size. Let the claims distribution be exponential(1) then

\[ \int_0^\infty e^{ru}(-\psi'(u))du = \frac{1}{1 + \theta} \left( \frac{\theta(\frac{1}{1-r} - 1)}{(1 + (1 + \theta)r - \frac{1}{1-r})} \right) \]

\[ = \frac{\theta}{(1 + \theta)[\theta - (1 + \theta)r]} \]

\[ = \frac{\delta \gamma}{\gamma - r}, \]

where \( \gamma = \frac{\theta}{1 + \theta} = 1 - \delta \). We see that except for the constant \( \delta \) this is a mgf of an exponential distribution with parameter \( \gamma \).

Remark 2.3.2. Kaas et al. ([1] page 91) present a discrete-time model counterpart of the continuous-time model. In fact they consider a more general risk model, now only on the discrete time points 0, 1, 2, ... , and let \( U_n, n = 0, 1, 2, \ldots \) denote the surplus at time \( n \). Letting \( G_n \) denote the profit between time point \( n - 1 \) and \( n \), we have

\[ U_n = u + G_1 + G_2 + \ldots + G_n, \quad n = 0, 1, 2, \ldots \]

The authors define a discrete-time version of the ruin probability \( \tilde{\psi}(u) \) and the ruin time \( \tilde{T} \) as follows: \( \tilde{T} = \min\{n : U_n < 0\} \); \( \tilde{\psi}(u) = P[\tilde{T} < \infty|U(0) = u] \).

We end this chapter by giving the ruin probability for a discrete distribution. If the claim size \( X \) can have only a finite number of positive values \( x_1, x_2, \ldots, x_n \), with probabilities \( p_1, p_2, \ldots, p_n \), the ruin probability equals

\[ \psi(u) = 1 - \frac{\theta}{1 - \theta} \sum_{k_1, k_2, \ldots, k_n} (-z)^{k_1 \ldots k_n} e^{z} \prod_{j=1}^{m} \frac{p_j^{k_j}}{k_j!}, \tag{2.17} \]

where \( z = \frac{\lambda}{c}(u - k_1 x_1 - \ldots - k_n x_n) \_ \). For the proof see [9].
Chapter 3

Ruin probability in an insurance risk model with threshold

3.1 Introduction

In this chapter we study the ruin probability in an insurance risk model of type $(M/Gd)$ where the claim size is dependent on the previous inter-arrival claim time. A similar model with dependence was first introduced by Albrecher and Boxma (2004). But in their model, the inter-arrival time depends on the previous claim size through a threshold. It was possible to get the ruin probability as the inversion of a Laplace transform, but it was not possible to obtain a closed form solution. In this chapter we consider a model developed by Isaac K. M. Kwan and Hailiang Yang (2007). We also study the case where all claims are rejected if the inter-arrival time between two successive claims is less than some deterministic threshold $\alpha$.

Let $T_i$ be the time between the $(i-1)$th claim and the $i$th claim. We assume that the number of claims follows a Poisson process with parameter $\lambda$, thus the $T_i$ are exponentially distributed with parameter $\lambda$. As in the previous chapter we define $T_u = \inf_{t \geq 0} \{ t | U(t) \leq 0 \}$ to be the time of ruin. We also define $\phi(u) = 1 - \psi(u)$ as the survival probability. In the following, we assume that if $T_i \leq a$ then $B_i$ follows a distribution with cumulative distribution function (cdf) $F(x)$ and pdf $f(x)$ and if $T_i > a$, $B_i$ follows a distribution with cdf $G(x)$ and pdf $g(x)$. We define also $\overline{F}(x)$ (resp. $\overline{G}(x)$) to be the tail probability of $F(x)$ (resp. of $G(x)$). We assume the following net profit condition is satisfied:

They should be a $\theta > 0$ such that

$$ ct = (1 + \theta)E[S_t], \theta > 0, $$

so by Wald’s Identity we have

$$ ct = (1 + \theta)\lambda t((1 - e^{-\lambda a})\mu_F + e^{-\lambda a}\mu_G). $$
where $\theta$ is the safety load, $\mu_F$ and $\mu_G$ are respectively the mean of $F$ and $G$. Let us start with the computation of $\phi(u)$. By conditioning on the time of occurrence $T_1$ of the first claim, we obtain

$$\phi(u) = P[T = \infty | u(0) = u]$$

$$= \int_0^\infty P[T = \infty | u(0) = u, T_1 = t] \lambda e^{-\lambda t} dt + \int_a^\infty P[T = \infty | u(0) = u, T_1 = t] \lambda e^{-\lambda t} dt$$

$$= \int_0^\infty \int_0^{u+ct} P[T = \infty | X = x, u(0) = u, T_1 = t] f(x) \lambda e^{-\lambda x} dx dt + \int_a^\infty \int_0^{u+ct} \phi(u + ct - x) \lambda e^{-\lambda x} dx dt$$

We substitute $t = \frac{v-u}{c}$. Then

$$c\phi(u) = \int_u^{u+ca} \int_0^v \phi(v-x) f(x) \lambda e^{-\lambda \frac{v-u}{c}} dx dv + \int_u^{\infty} \int_0^{u+ca} \phi(v-x) g(x) \lambda e^{-\lambda \frac{v-u}{c}} dx dv$$

We differentiate the left and right hand side with respect to $u$:

The first part gives

$$\frac{d}{du} \int_u^{u+ca} \int_0^v \phi(v-x) f(x) \lambda e^{-\lambda \frac{v-u}{c}} dx dv$$

$$= \int_0^{u+ca} \phi(u + ca - x) \lambda e^{-\lambda a} f(x) dx - \int_0^u \phi(u - x) \lambda f(x) dx$$

$$+ \frac{\lambda}{c} \int_0^{u+ca} \int_0^v \lambda \phi(v-x) e^{-\lambda \frac{v-u}{c}} f(x) dx dv$$

and the second part gives

$$\frac{d}{du} \int_u^{\infty} \int_0^v \phi(v-x) g(x) \lambda e^{-\lambda \frac{v-u}{c}} dx dv$$

$$= - \int_0^{u+ca} \phi(u + ca - x) \lambda e^{-\lambda a} g(x) dx + \frac{\lambda}{c} \int_{u+ca}^\infty \int_0^v \lambda \phi(v-x) e^{-\lambda \frac{v-u}{c}} g(x) dx dv$$
We finally obtain

\[ c \frac{d\phi(u)}{du} = \int_0^{u+ca} \phi(u + ca - x) \lambda e^{-\lambda a} f(x) dx - \int_0^u \phi(u - x) \lambda f(x) dx + \frac{\lambda}{c} \int_0^{u+ca} \int_0^v \phi(v - x) e^{-\lambda \frac{v-u}{c}} f(x) dv dx + \int_0^{u+ca} \phi(u + ca - x) \lambda e^{-\lambda a} g(x) dx + \frac{\lambda}{c} \int_0^{u+ca} \int_0^v \phi(v - x) e^{-\lambda \frac{v-u}{c}} g(x) dv dx \]

\[ = \lambda \left( \frac{\lambda}{c} \int_0^{u+ca} \int_0^v \phi(v - x) e^{-\lambda \frac{v-u}{c}} f(x) dv dx + \int_0^{u+ca} \int_0^v \phi(v - x) e^{-\lambda \frac{v-u}{c}} g(x) dv dx \right) + \lambda e^{-\lambda a} \int_0^{u+ca} \phi(u + ca - x) (f(x) - g(x)) dx - \lambda \int_0^u \phi(u - x) f(x) dx. \]

We observe that the first part of this last equation is just \( \lambda \phi(u) \). It follows that we have proved: (see also [4])

**Theorem 3.1.1 (Survival Probability).** The survival probability satisfies the integro-differential equation

\[ c \frac{d\phi(u)}{du} - \lambda \phi(u) = \lambda e^{-\lambda a} \int_0^{u+ca} \phi(u + ca - x) (f(x) - g(x)) dx - \lambda \int_0^u \phi(u - x) f(x) dx. \]  

### (3.1)

#### 3.2 Exponential claim size distribution

Let \( F(x) = 1 - e^{-\mu_1 x} \) and \( G(x) = 1 - e^{-\mu_2 x} \) be the cumulative distributions of the claim sizes, i.e. we consider the model \( M/\tilde{M}_d \). Then the ruin probability satisfies the third order differential equation [4]:

\[ c \frac{d^3 \psi(u)}{du^3} + (c \mu_1 + c \mu_2 - \lambda) \frac{d^2 \psi(u)}{du^2} + \mu_2 (c \mu_1 - \lambda) \frac{d\psi(u)}{du} - \lambda e^{-\lambda a} (\mu_1 - \mu_2) \frac{d\psi(u + ca)}{du} = 0. \]  

### (3.2)

**Proof.** By (3.1) we get

\[ c \frac{d\phi(u)}{du} - \lambda \phi(u) = \lambda e^{-\lambda a} \int_0^{u+ca} \phi(u+ca-x)(\mu_1 e^{-\mu_1 x} - \mu_2 e^{-\mu_2 x}) dx - \lambda \int_0^u \phi(u-x) \mu_1 e^{-\mu_1 x} dx. \]  

### (3.3)
We differentiate both sides of (3.3). To simplify the differentiation we set \( v = u + ca - x \);

\[
\frac{d^2 \phi(u)}{du^2} - \lambda \frac{d\phi(u)}{du} = \lambda e^{-\lambda a} \left[ \phi(u + ca)(\mu_1 - \mu_2) - \mu_1^2 \int_0^{u+ca} \phi(u + ca - x) e^{-\mu_1 x} dx \right] 
+ \lambda e^{-\lambda a} \mu_2 \int_0^{u+ca} \phi(u + ca - x) e^{-\mu_2 x} dx 
- \lambda \mu_1 \phi(u) + \lambda \mu_1 \mu_1 \int_0^{u} \phi(u - x) e^{-\mu_1 x} dx 
= \lambda e^{-\lambda a} \phi(u + ca)(\mu_1 - \mu_2) + \lambda e^{-\lambda a} \mu_2 \int_0^{u+ca} \phi(u + ca - x) e^{-\mu_2 x} dx 
- \mu_1 \left[ c \frac{d\phi(u)}{du} - \lambda \phi(u) + \lambda e^{-\lambda a} \mu_2 \int_0^{u+ca} \phi(u + ca - x) e^{-\mu_2 x} dx \right] - \lambda \mu_1 \phi(u) 
= \lambda e^{-\lambda a} \phi(u + ca)(\mu_1 - \mu_2) - c \mu_1 \frac{d\phi(u)}{du} 
+ \lambda \mu_2 e^{-\lambda a} (\mu_2 - \mu_1) \int_0^{u+ca} \phi(u + ca - x) e^{-\mu_2 x} dx.
\]

Rearranging the terms,

\[
\frac{d^2 \phi(u)}{du^2} = (c \mu_1 - \lambda) \frac{d\phi(u)}{du} - \lambda e^{-\lambda a} (\mu_1 - \mu_2) \phi(u + ca) 
= \lambda e^{-\lambda a} \mu_2 (\mu_2 - \mu_1) \int_0^{u+ca} \phi(u + ca - x) e^{-\mu_2 x} dx.
\]

Now we differentiate this last equation with respect to \(u\). Let

\[
t(u) = \lambda e^{-\lambda a} \mu_2 (\mu_2 - \mu_1) \int_0^{u+ca} \phi(v) e^{-\mu_2 (u+ca-v)} dv.
\]

Then,

\[
\frac{dt(u)}{du} + \mu_2 t(u) = \lambda e^{-\lambda a} \mu_2 (\mu_2 - \mu_1) \phi(u + ca).
\]

Now we substitute

\[
t(u) = c \frac{d^2 \phi(u)}{du^2} + (c \mu_1 - \lambda) \frac{d\phi(u)}{du} - \lambda e^{-\lambda a} (\mu_1 - \mu_2) \phi(u + ca)
\]

in Equation (3.5) and obtain

\[
\frac{d^3 \phi(u)}{du^3} + ((\mu_1 + \mu_2) c - \lambda) \frac{d^2 \phi(u)}{du^2} + \mu_2 (\mu_1 c - \lambda) \frac{d\phi(u)}{du} - (\mu_1 - \mu_2) \lambda e^{-\lambda a} \frac{d\phi(u + ca)}{du} = 0.
\]

By using the relation \( \psi(u) = 1 - \phi(u) \) we obtain (3.2) and this completes the proof. \( \square \)

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3.3 The case where no claim is accepted before some deterministic threshold of the inter-arrival time

Let the claim size \( B \) equal zero if the inter-occurrence time between claims is less than some threshold \( a \) and let \( B \) have a pdf \( g(x) \) if the claim inter-occurrence times is more than \( a \). This means that \( F(x) = 1 \) for all \( x \geq 0 \). By an integration of (3.1) from 0 to \( u \) and using the fact that \( \phi(u) \to 1 \) as \( u \to \infty \) to find \( \phi(0) \), we obtain the following integral equation:

\[
 c\psi(u) = \lambda e^{-\lambda a} \left[ \int_{u+ca}^{\infty} G(x)dx + \int_0^{u+ca} \psi(u+ca-x)G(x)dx \right]. \tag{3.6}
\]

Let \( G(x) = e^{-\mu x} \), then (3.6) becomes

\[
 c\psi(u) = \lambda e^{-\lambda a} \left[ \int_{u+ca}^{\infty} e^{-\mu x}dx + \int_0^{u+ca} \psi(u+ca-x)e^{-\mu x}dx \right] \tag{3.7}
\]

\[
 = \frac{\lambda}{\mu} e^{-\lambda a} e^{-\mu(u+ca)} + \lambda e^{-\lambda a} \int_0^{u+ca} \psi(u+ca-x)e^{-\mu x}dx. \tag{3.8}
\]

Now we differentiate (3.8) with respect to \( u \).

We have

\[
 \frac{c}{\mu} \frac{d\psi(u)}{du} = - \frac{\lambda}{\mu} e^{-\lambda a} e^{-\mu(u+ca)} + \lambda e^{-\lambda a} \left[ \frac{d}{du} \int_0^{u+ca} \psi(v)e^{-\mu(u+ca-v)}dv \right]
\]

\[
 = - \frac{\lambda}{\mu} e^{-\lambda a} e^{-\mu(u+ca)} + \lambda e^{-\lambda a} \left[ \psi(u+ca) - \int_0^{u+ca} \mu \psi(v)e^{\mu(u+ca-v)}dv \right]
\]

\[
 = - \mu c\psi(u) + \lambda e^{-\lambda a} \psi(u+ca). \tag{3.8}
\]

Bringing all terms to the left side we obtain

\[
 \frac{c}{\mu} \frac{d\psi(u)}{du} + \mu c\psi(u) - \lambda e^{-\lambda a} \psi(u+ca) = 0. \tag{3.9}
\]

This is in accordance with (3.2) because if we divide the left and the right side of (3.2) by \( \mu_1 \) and let \( \mu_1 \) tend to \( \infty \) we obtain (3.9). By Taylor’s Theorem we have

\[
 \psi(u+ca) = \sum_{k=0}^{\infty} \frac{(ca)^k \psi^{(k)}(u)}{k!}. \tag{3.8}
\]
Thus we have just proved that for this simple case the ruin probability satisfies the following differential equation:

\[
\frac{d\psi(u)}{du} + \mu c \psi(u) - \lambda e^{-\lambda a} \sum_{k=0}^{\infty} \frac{(ca)^k \psi^{(k)}(u)}{k!} = 0.
\]

Suppose that equation (3.10) has as solution

\[
\psi(u) = Ae^{Ru}.
\]

By substituting this solution into (3.10) we obtain the following characteristic equation

\[
Ra - \lambda e^{-\lambda a} e^{Rca} + \mu c = 0.
\]

The non-linearity of this equation makes it difficult to solve. So even for this simple case where the threshold is deterministic, the ruin probability is still difficult to obtain.

Is it possible to derive, as in the classical model when the claim size is exponentially distributed, an explicit expression for the ruin probability? This question can be answered if we can find a negative solution for the above equation. In fact equation (3.12) has exactly one negative solution \( R \), when \( a > 0, \mu > 0, \lambda > 0, c > 0 \) and \( \mu > \lambda e^{-\lambda a}/c \).

Proof. We write \( \kappa = \lambda e^{-\lambda a}/c \). Note that by the net profit condition \( \mu > \kappa > 0 \). To prove that (3.12) has exactly one negative solution, we first state Rouché’s Theorem which will be useful (See [5] page 652).

**Theorem 3.3.1** (Rouché’s Theorem). If \( f(z) \) and \( g(z) \) are analytic inside and on a closed contour \( D \) and \( |f(z)| < |g(z)| \) on \( D \) then \( g(z) \) and \( f(z) + g(z) \) have the same number of zeros inside \( D \).

The functions \( g(z) = z + \mu \) and \( f(z) = -\kappa e^{\kappa cz} \) are analytic in the contour \( D \) delimited by the left-half ball with center at the origin and with radius \( r \) because they can be expressed as power series there. On the boundaries we have \( |f(z)| < |g(z)| \). Indeed for \( z = iy \) with \( y \) real, \( |f(z)| = |\kappa e^{\kappa cy}| = \kappa < \mu < \sqrt{\mu^2 + y^2} = |g(z)| \) and for \( z = re^{ix} \) we have \( |f(z)| = \kappa e^{\kappa Re(z)} < \mu < |\mu + z| \). So by [5], the equation \( g(z) + f(z) = 0 \) always has exactly one negative solution because \( g(z) \) has only one solution inside \( D \).

To find the constant \( A \) we substitute (3.11) into (3.6) and set \( u = 0 \). After some computations we obtain,

\[
A = \frac{\lambda e^{-\lambda a} e^{-\mu ca}}{\mu(c - \lambda e^{-\lambda a}(e^{Rca} - e^{-\mu ca})/R + \mu)} = \frac{\lambda e^{-\lambda a}}{\mu c} e^{Ra}.
\]
The last equality follows from (3.12). If we take \( a = 0 \) then \( A = \psi(0) = \frac{\lambda}{\mu} \) which gives the ruin probability of the classical model provided we start with a capital of 0. So our ruin probability has the form

\[
\psi(u) = \frac{\lambda e^{-\lambda a}}{\mu c} e^{R(u+a)}.
\] (3.13)

### 3.4 Duality \( D_1 \)

In this section we stress the connection between the risk model developed in Section 3.3 (\( M/\overline{M}_d \)) and the \( G/M/1 \). The model in Section 3.3 can be seen as a classical risk model where the inter-occurrence time between claims has a general distribution and the claim size is exponential. The classical result (see [10], page 162) is that the ruin probabilities for the \( G/M \) risk model with \( c = 1 \) (the inflow of premium per unit time) are related to the workload process \( \{V_t\} \) and to the waiting time of an initially empty \( G/M/1 \) queue by means of

\[
1 - \psi^{(0)}(u) = P[V \leq u] = \lim_{t \to \infty} P[V_t \leq u]
\] (3.14)

and

\[
1 - \psi(u) = P[W \leq u] = \lim_{n \to \infty} P[W_n \leq u].
\] (3.15)

Here \( V \) is the steady-state workload, \( \psi(u) \) is the zero-delayed ruin probability and \( \psi^{(0)}(u) \) is the stationary ruin probability which is in fact the ruin probability of the delayed risk process, where the first inter-arrival time has a delay distribution \( A_0 \) with density \( \frac{A(x)}{\mu} \). The notation \( M/\overline{M}_d/1 \) means that the service time is of the following Markovian type: if the inter-arrival time of the next customer is less than a threshold \( a \), then \( B_i = 0 \), otherwise it is exponential with rate \( \mu \), the \( d \) in the above notation stands for deterministic. The workload is sometimes called virtual waiting time because it is the time a hypothetical customer arriving just after time \( t \) has to wait. In other words, \( V_t \) is the time necessary for the server to clear the system provided no new customers arrive and that we are in FIFO (First In, First Out).

This \( M/\overline{M}_d/1 \) queueing model is the dual model of our above risk model, where no claim is accepted if the inter-occurrence time \( T_i \) of a claim is less than \( a \). This means that the claim size is set to zero for \( T_i \leq a \) and otherwise the claim size is exponential with rate \( \mu \).

The important observation is that the \( M/\overline{M}_d/1 \) is in fact equivalent to a standard \( G/M/1 \) queue, where the distribution of the inter-arrival time has the following Laplace-Stieltjes transform:
Figure 3.1: Duality $D_1$. Dotted lines represents rejected claims and service times of rejected customers.

**Theorem 3.4.1.** The Laplace-Stieltjes transform of the inter-arrival times is given by

$$\tilde{T}(s) = \frac{\lambda}{\lambda + se^{(s+\lambda)a}}.$$  \hspace{1cm} (3.16)

**Proof.** We denote by $X_s$ the exponential random variable of the small inter-arrival time of a customer and by $X_l$ the exponential random variable of the large one. The number $N$ of successive customers who receive zero service time is geometric with parameter $p = e^{-\lambda a}$. So the inter-arrival time of the customers receiving service is a compound geometric plus one extra large $X_l$. Let $F_s(t)$ (resp. $F_l(t)$) be the cdf of the small inter-arrival times (resp. the large inter-arrival times). Then we have

$$F_s(t) = P[T \leq t | T \leq a] = \frac{P[T \leq t, T \leq a]}{P[T \leq a]} = \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda a}}, \quad t \leq a.$$  \hspace{1cm} (3.17)

$$F_l(t) = P[T \leq t | T > a] = \frac{P[a < T \leq t]}{P[T > a]} = \frac{e^{-\lambda a} - e^{-\lambda t}}{e^{-\lambda a}} = 1 - e^{-\lambda(t-a)}, \quad t > a.$$  \hspace{1cm} (3.18)

Consequently the Laplace-Stieltjes transform of the inter-arrival time of customers
for which $T_i \leq a$ is given by

$$\tilde{X}_s(s) = E[e^{-s(\sum_{i=1}^{N} X_i)}] = \sum_{k=0}^{\infty} E[e^{-s(\sum_{i=1}^{N} X_i)}|N = k]P[N = k]$$

$$= \sum_{k=0}^{\infty} (E[e^{sX_i}])^k(1 - e^{-\lambda a})^k e^{-\lambda a}$$

$$= \sum_{k=0}^{\infty} \left(\int_0^a e^{-x_s s} \frac{\lambda e^{-\lambda x_s}}{1 - e^{-\lambda a}} dx_s\right)^k (1 - e^{-\lambda a})^k e^{-\lambda a}$$

$$= \frac{(s + \lambda)e^{-\lambda a}}{s + \lambda e^{-(s+\lambda)a}}.$$

The Laplace-Stieltjes transform of $X_t$ is given by

$$\frac{\lambda}{s + \lambda} e^{-s a}. \tag{3.20}$$

We know that the Laplace-Stieltjes transform of a convolution of two distributions is equal to the product of the Laplace-Stieltjes transforms. So the Laplace-Stieltjes transform of the inter-arrival time of those customers which receive service is:

$$\tilde{T}(s) = \tilde{X}_s(s) \cdot \tilde{X}_l(s) = \frac{\lambda}{\lambda + s e^{(s+\lambda)a}}. \tag{3.21}$$

We know that (see [6] page 279) if $\mu$ is the service intensity and $\mu_A$ the mean of the inter-arrival time of a $G/M/1$ queue, then in the steady state: (a) The distribution of the waiting time $W$ is a mixture of an atom at 0 and an exponential distribution with intensity $\eta$ on $(0, \infty)$ with weights $1 - \theta$, resp. $\theta$.

(b) The distribution of the workload $V$ is a mixture of an atom at 0 and an exponential distribution with intensity $\eta$ on $(0, \infty)$ with weights $1 - \rho$, resp. $\rho$. Here $\rho = (\mu \mu_A)^{-1} < 1$ and $\theta = 1 - \frac{\rho}{\mu}$ where $\eta$ is the positive solution of the equation $1 = \frac{\mu}{\mu - \eta} \int_{0}^{\infty} e^{-\eta x} A(dx)$, where $A(x)$ is the cdf of the inter-arrival times. Thus

$$P[V > t] = \rho e^{-\eta t}, \tag{3.22}$$

and

$$P[W > t] = \theta e^{-\eta t}. \tag{3.23}$$
In the above G/M/1 we have $\rho = \frac{\lambda e^{-\lambda a}}{\mu}$ and $\theta = \frac{\mu - \eta}{\mu}$. It follows that

\[
P[W > u] = \frac{\mu - \eta}{\mu} e^{-\eta u} \\
= \frac{\mu + R}{\mu} e^{R u} \\
= \frac{\lambda}{\mu} e^{-\lambda a} e^{(a+u)R} \\
= \psi(u),
\]

as required. The line before the last follows from (3.12).

Now we verify that the previous result of the workload is in compliance with Formula (3.14).

By ([10] page 153 (1.4)), the ruin probability for the stationary case can be expressed in terms of the one for the zero-delayed case by integrating the following:

\[
\psi_s(u) = G(u + s) + \int_0^{u+s} \psi(u + s - y) G(dy) \quad \text{w.r.t. } A_{eq},
\]

where $\psi_s(u)$ is the ruin probability for the delayed case with $T_1 = s$. Let now $A_{eq}(x) = \frac{\mu}{\lambda} (1 - A(u)) du$ be the equilibrium distribution and $\phi_{eq}$ be the Laplace-Stieltjes transform of the equilibrium distribution, then

\[
\phi_{eq}(s) = \int_0^\infty e^{-sx} \left(1 - A(x)\right) dA_{eq}(x).
\]

Hence we obtain

\[
\phi_{eq}(s) = \frac{1 - \phi(s)}{\mu A s} = \frac{\lambda}{\mu s e^{\lambda a}} \frac{se^{(\lambda + s)a}}{\lambda + se^{(\lambda + s)a}} = \phi(s) e^{sa}.
\]

This is the Laplace-Stieltjes transform of the difference $X - a$, so that $A_{eq}(x) = A(x + a)$, i.e. for this particular distribution, the equilibrium distribution is the distribution of a shift by $a$.

We now derive a relation between the stationary ruin probability and the zero delayed one. We have

\[
\psi^{(0)}(u) = \int_0^\infty (G(u + x) + \int_0^{u+x} \psi(u + x - y) G(dy)) dA_{eq}(x) \\
= \int_0^\infty (e^{-\mu(u+x)} + \int_0^{u+x} \psi(u + x - y) \mu e^{-\mu y} dy) dA_{eq}(x) \\
= \int_0^\infty (e^{-\mu(u-a+x)} + \int_0^{u-a+x} \psi(u - a + x - y) \mu e^{-\mu y} dy) dA(x) \\
= \int_0^\infty (e^{-\mu(u-a+x)} + \int_0^{u-a+x} \psi(u - a + x - y) \mu e^{-\mu y} dy) dA(x).
\]

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The last step is due to the fact that $dA(x) = 0$ for $x < a$ (the inter-arrival time is always larger than $a$). This can be recognized as $\psi(u - a)$, so $\psi(u(0)) = \psi(u - a) = \frac{\lambda e^{-\lambda a}}{\mu} e^{Ru}$. By setting $R = -\eta$, we obtain the same result as for the workload. So we conclude that this result is in compliance with Equation (3.14).

### 3.5 Duality $D_2$

In this section we construct another type of duality between the risk model that was introduced in Section 3.3 and the standard M/G/1 queue. Here we reverse the distribution of both models as follows: the claim size become the inter-arrival time of customers in the M/G/1 and the service time is just the claim inter-occurrence time of the risk model described in Section 3.3.

Let us consider an initially empty single server queueing system, where we have two types of customers: Exceptional customers and normal customers. Each customer brings an exponential amount of work with parameter $\lambda$. Normal customers have a service time less than some threshold $a$ and the exceptional customers have a service time larger than the threshold. If the service time is less than $a$ then the inter-arrival time is set to zero, otherwise the inter-arrival time is exponential with rate $\mu$. Exceptional customers enter the system immediately upon arrival and begin the service if the server is empty; otherwise they wait in the queue in front of the server. In other words, customers are served in groups composed of $n$ customers with successive service times less than the threshold plus one customer with large service time (the exceptional customer). So the number of customers in a group is a geometric number and the service time is a geometric sum of exponential random variables with parameter $\lambda$.

Let $A_i$ be a sequence of i.i.d. exponential random variables having rate $\mu$, representing the inter-arrival time between the $i$th and $(i+1)$th customer, where $A_i$ is set to zero if the service time of the customer $i$ is less than the threshold; let $B_i$ be a sequence of i.i.d. exponential random variables for the service time with rate $\lambda$. We have seen above that this dependent M/M/1 queue is equivalent to a normal G/M/1 queue. Because in this duality we take as inter-arrival time for a customer the size of the claims and for service time the inter-occurrence time between claims, the above described model is just an M/G/1.

Let $\mu$ be the inter-arrival intensity, $\mu_B$ the mean service time of a M/G/1, and $\rho = \mu \mu_B < 1$ the occupation rate. Then in steady-state the distribution of the workload $V$ and the waiting time $W$ are identical (By PASTA).
The Laplace-Stieltjes transform of the service time is given by see (3.21):

$$\tilde{B}(s) = \frac{\lambda}{\lambda + se^{(s+\lambda)a}}.$$  \hfill (3.29)

For a M/G/1 the Laplace-Stieltjes transform for the waiting time is given by ([7] page 66)

$$\tilde{W}(s) = \frac{(1 - \rho)s}{\mu \tilde{B}(s) + s - \mu} = \frac{(1 - \rho)s}{\mu(\frac{\lambda}{\lambda + se^{(s+\lambda)a}}) + s - \mu} = (1 - \rho) + \rho \frac{\lambda e^{-\lambda a}(1 - \rho)}{\lambda e^{-\lambda a}(e^{-sa} - \rho) + s}. \hfill (3.30)$$

**Theorem 3.5.1.** The stationary waiting time is zero with probability $1 - \rho$ and is equal to $W^+$ with probability $\rho$, where $W^+$ is a random variable with density

$$(1 - \rho)e^{\mu(x+a)}g(x+a), x \geq 0. \hfill (3.31)$$

Here $g$ is the density of a distribution $G$ with Laplace-Stieltjes transform

$$\psi_G(s) = \frac{\lambda'}{\lambda' + se^{(x+s)a}}$$

and $\lambda'$ is a solution of $\lambda' = \lambda e^{a(\lambda' - \lambda - \mu)}$.

For more properties of $G$ see [8] pages 141-163.

**Proof.** According to formula (3.30) the Laplace-Stieltjes transform of the stationary waiting time is given by

$$\tilde{W}(s) = (1 - \rho) + \rho e^{(s-\mu)a} \frac{\lambda (1 - \rho)}{\lambda e^{-\mu a} + (s - \mu)e^{(\lambda+(s-\mu)a)}}.$$

We multiply the numerator and the denominator of the fraction on the right hand side by $\frac{\lambda'}{\lambda} e^{\mu a}$ and obtain

$$\tilde{W}(s) = (1 - \rho) + \rho (1 - \rho) e^{sa} \frac{\lambda'}{\lambda' + (s - \mu)\frac{\lambda'}{\lambda} e^{a(\lambda + s)}} = (1 - \rho) + \rho (1 - \rho) e^{sa} \frac{\lambda'}{\lambda' + (s - \mu) e^{a(\lambda+s-\mu)}} \hfill (3.32)$$

$$= (1 - \rho) + \rho e^{sa} \frac{\psi_G(s-\mu)}{\psi_G(-\mu)}.$$
The line before the last one follows from the relation \( \lambda' = \lambda e^{a(\lambda' - \lambda - \mu)} \) and the last line from the fact that \( \psi_G(-\mu) = (1 - \rho)^{-1} \). So if \( G(x) \) is a distribution function with density \( g(x) \) and Laplace-Stieltjes transform \( \psi_G(s) \) then \( \frac{\psi_G(s-\mu)}{\psi_G(-\mu)} \) is the Laplace transform of the density \( (1 - \rho)e^{\mu x}g(x) \). It follows that \( e^{sa}\frac{\psi_G(s-\mu)}{\psi_G(-\mu)} \) is the Laplace-Stieltjes transform of the random variable with the density given in (3.31).

\[ \text{3.6 Sample path argument for duality } D_1 \]

Let \( W_n \) be the waiting time of the \( n \)th customer in a G/G/1-type queue with a possible dependency between \( U_n \), the service time of the \( n \) customer, and \( A_n \), the inter-arrival time between the \( n \)th customer and the \((n+1)\)th customer. Then \( W_1 = 0 \) and \( W_n \) fulfills Lindley’s equation, i.e.

\[ W_{n+1} = (W_n + Z_n)^+, \]

where \( Z_n = U_n - A_n \) (see Figure 3.2). Thus \( \{W_n\} \) evolves as a random walk with increments \( Z_1, Z_2, \ldots \) as long as the random walk only takes non-negative values, and it is reset to 0 once it hits 0.

Before we explain the relation between the G/G/1 queue and a dual risk process, we state the following well known result.

**Lemma 3.6.1.** (see [10] page 49) Let \( (Z_i)_{i=1,2,\ldots} \) be a sequence of identically distributed random variables (not necessarily independent) and let \( S_n = Z_1 + Z_2 + \cdots + Z_{n-1} \) with \( S_1 = 0 \). Define the Lindley process generated by \( Z_1, Z_2, \ldots \), by \( W_1 = 0 \) and \( W_{n+1} = (W_n + Z_n)^+ \). Then for every \( N \geq 1 \)

\[ W_N = S_N - \min_{n=1,\ldots,N} S_n. \quad (3.33) \]

and \( W_N \overset{d}{\sim} \max_{n=1,\ldots,N} S_n \) where \( \overset{d}{\sim} \) denote equality in distribution.

**Proof.** We are going to show that \( Q_N = S_N - \min_{n=1,\ldots,N} S_n \) fulfills Lindley’s equation. We have

\[
Q_{N+1} = \begin{cases} 
0 & \text{if } S_{N+1} < \min_{n=1,\ldots,N} S_n \\
S_{N+1} - \min_{n=1,\ldots,N} S_n & \text{else}
\end{cases} = \begin{cases} 
0 & \text{if } S_{N+1} - \min_{n=1,\ldots,N} S_n < 0 \\
Q_N + Z_N & \text{else}
\end{cases} = (Q_N + Z_N)^+.
\]
Since the distributions of \((S_{N} - S_1, S_{N} - S_2, \ldots, S_{N} - S_{N-1}, 0)\) and \((S_{N}, S_{N-1}, \ldots, S_1)\) coincide, we have that
\[
W_{N} = \max \{S_{N} - S_1, S_{N} - S_2, \ldots, S_{N} - S_{N-1}, 0\} \quad \sim \quad \max \{S_{N}, S_{N-1}, \ldots, S_1\}.
\]

Now consider the following risk process. We fix some \(N\) and let \(Y_k = A_{N-k} - U_{N-k}\), where \(A_k\) denotes the time between the \((k - 1)\)th and the \(k\)th claim and where \(U_k\) is the size of the \(k\)th claim. Then, assuming that \(R_0 = u\), the surplus after the \(k\)th claim is \(R_k := u + Y_1 + \ldots + Y_k\) and \(\zeta_u = \inf \{n : R_n < 0\}\) is equal to the number of claims that leads to ruin.

**Theorem 3.6.2** (see [10] page 49). The two events \(\{\zeta_u \leq N\}\) and \(\{W_N > u\}\) coincide.

**Proof.** Let \(S_1 = 0\) and \(S_n = Z_1 + Z_2 + \cdots + Z_{n-1} = -Y_1 - Y_2 - \cdots - Y_{n-1} = u - R_{n-1}\). By the previous lemma we have
\[
W_N > u \iff S_N - \min_{n=1}^{N} S_n > u \iff S_N - \min_{n=1}^{N-1} S_n > u
\]
\[
\iff S_N - S_m > u, \quad \text{for some } m \in \{1, 2, \ldots, N - 1\}
\]
\[
\iff Z_m + Z_{m+1} + \cdots + Z_{N-1} > u, \quad \text{for some } m \in \{1, 2, \ldots, N - 1\}
\]
\[
\iff -Y_{N-m} - Y_{N-m-1} - \cdots - Y_1 > u, \quad \text{for some } m \in \{1, 2, \ldots, N - 1\}
\]
\[
\iff u + Y_{N-m} + Y_{N-m-1} + \cdots + Y_1 < 0, \quad \text{for some } m \in \{1, 2, \ldots, N - 1\}
\]
\[
\iff R_{N-m} < 0, \quad \text{for some } m \in \{1, 2, \ldots, N - 1\}
\]
\[
\iff \zeta_u < N. \quad \square
\]

If \(P[W_N > u]\) tends to a limit \(P[W > u]\) as \(n\) tends to infinity (i.e. \(W_n \to W\) in distribution), then we have
\[
P[\zeta_u < \infty] = P[W > u]. \tag{3.34}
\]

In this case we can apply the results from Theorem 3.1.1. Recall that for the threshold model with Poisson inter-arrival times and claim size distributions \(F\) and \(G\), the survival probability fulfills (3.1). We then conclude from (3.34) that the same is true for the limiting distribution of the waiting times for a modified M/G/1 queue. Thus, letting \(H(x) = P[W \leq x]\), we have the equation
\[
\frac{d}{du} H(u) - \lambda H(u) = \lambda e^{-\lambda a} \int_0^{x+ca} H(u+ca-x)(f(x) - g(x))dx
\]
\[
-\lambda \int_0^{u} H(u-x)f(x)dx. \tag{3.35}
\]
See the figure below for the sample path giving the ruin time and the waiting time.

Figure 3.2: Queueing process

Figure 3.3: Associated risk process with \( N = 5 \). Here \( \zeta(u) = 2 \).

### 3.7 Conclusion

In this chapter, we studied the special case of the model developed by Kwan and Yang (2007), where they found a closed formula for the ruin probability in the case of exponential claim size distribution. In our case we assume that no claim is accepted if the claim inter-arrival time is below a certain deterministic threshold. We have studied duality between this risk model and two queueing models. We found the same duality result as in the classical setting.
Chapter 4

A queueing model in which the service time depends on the inter-arrival time

The motivation for this model comes from the work of Yang and Kwan (2007) in an insurance risk model. They assume that the claim size of an insurance business depends on the time between claims. By the duality principle between the stationary ruin probability and the workload process in the classical risk model, we investigate the corresponding queueing model where the service time of the current customer depends on the inter-arrival time between the previous customer and the current customer. We consider a $M/G_M/1$ queue where customers numbered $n = 0, 1, 2, \ldots$, arrive according to a Poisson process. The inter-arrival times $A_n, n = 1, 2, \ldots$, between the $n$th customer and $(n + 1)$th customer are assumed i.i.d. and exponentially distributed. We let $B_n, n = 1, 2, \ldots$, be a sequence of i.i.d. random variables, $B_n$ representing the service time of the $n$th customer. We assume further that if the inter-arrival time $A_i$ is less than a threshold $T_i$, then the service distribution is $G(x)$ otherwise the service distribution is $F(x)$. The $T_i$ are assumed to be i.i.d. random variables (independent from the inter-arrival and service times) with exponential distribution function $T$ having parameter $\tau$. We are interested in the relevant performance measures of this queueing model; in particular, in the mean performance measures such as the mean sojourn time, the mean number of customers and the mean waiting time. This model is applicable in supermarkets where customers who pick only a small number of items move directly to the server and receive service; in this case customers with short inter-arrival time receive a small amount of service time. We begin by investigating the sojourn time of a given customer by computing his Laplace-Stieltjes transform.
4.1 Laplace-Stieltjes transform of the sojourn time of a customer

We now turn to the computation of how long a customer spends in the system, which is known as sojourn time. The sojourn time $S_{n+1}$ of the $(n+1)$th arriving customer is evidently the service time of the $(n+1)$th customer if and only if he arrives after the departure of the $n$th arriving customer. Hence

$$S_{n+1} = \begin{cases} \quad B_{n+1} & \text{if } S_n - A_n \leq 0, \\ S_n - A_n + B_{n+1} & \text{if } S_n - A_n > 0. \end{cases}$$

For $n \to \infty$ the distribution function of $S_n$ has a limit. Let $S$ be this limit random variable, then we have $S \overset{d}{=} (S - A)^+ + B$. In the sequel we call $B_F$ the random variable of the service time with distribution $F$ and $B_G$ the random variable of the service time with distribution $G$.

What is exactly the stability condition of this queue? We know that the system will be stable if $E[A] > E[B]$, that is

$$\frac{1}{\lambda} > \frac{\lambda}{\lambda + \tau} E[B_G] + \frac{\tau}{\lambda + \tau} E[B_F].$$

**Theorem 4.1.1.** The Laplace-Stieltjes transform of the sojourn time of this $M/\overline{\text{G}}_M/1$ type queue is given by:

$$E[e^{-\gamma S}] = -\frac{C_1 \gamma (\lambda + \tau - \gamma)(\lambda + \tau)E[e^{-\gamma B_F}] - \lambda \gamma (\lambda - \gamma) (E[e^{-\gamma B_F}] - E[e^{-\gamma B_G}]) C_2}{(\lambda + \tau)(\lambda - \lambda E[e^{-\gamma B_F}]) - \gamma)(\lambda + \tau - \gamma) + \lambda (\lambda - \gamma) (E[e^{-\gamma B_F}] - E[e^{-\gamma B_G}])},$$

where $C_1 = E[e^{-\lambda S}]$, $C_2 = E[e^{-(\lambda+\tau)S}]$.

**Proof.** By conditioning successively on the inter-arrival time, the threshold and the service time of a customer we have

$$E[e^{-\gamma S}] = E[e^{-\gamma((S-A)^+ + B)}] = E[e^{-\gamma B \mathbf{1}_{S \leq A}}] + E[e^{-\gamma(S-A + B)} \mathbf{1}_{S > A}].$$
We compute the two expectations of the right hand side separately.

\[
E[e^{-\gamma B}1_{S\leq A}] = E[e^{-\gamma B}1_{S\leq A}1_{A\leq T}] + E[e^{-\gamma B}1_{S\leq A}1_{A>T}]
\]

\[
= E[e^{-\gamma B G} \int_0^\infty E[1_{S\leq A}1_{A\leq u}] \tau e^{-\tau u} du + E[e^{-\gamma B F} \int_0^\infty E[1_{S\leq A}1_{A>u}] \tau e^{-\tau u} du]
\]

\[
= E[e^{-\gamma B G} \int_0^u \int_0^u E[1_{S\leq t}] \lambda e^{-\lambda t} dt \tau e^{-\tau u} du + E[e^{-\gamma B F} \int_0^\infty \int_u^\infty E[1_{S\leq t}] \lambda e^{-\lambda t} dt \tau e^{-\tau u} du].
\] (4.2)

The double integrals in (4.2) can be reduced to a single integral by changing the order of integration. We obtain the following:

\[
E[e^{-\gamma B}1_{S\leq A}] = E[e^{-\gamma B G} \int_0^\infty E[1_{S\leq t}] \lambda e^{-(\lambda + \tau) t} dt + E[e^{-\gamma B F} \int_0^\infty E[1_{S\leq t}] \lambda (e^{-\lambda t} - e^{-(\lambda + \tau) t}) dt]
\]

\[
= E[e^{-\gamma B G} \int_0^\infty \lambda e^{-(\lambda + \tau) t} dt + E[e^{-\gamma B F} \int_0^\infty \lambda (e^{-\lambda t} - e^{-(\lambda + \tau) t}) dt]
\]

\[
= E[e^{-\gamma B F} E[e^{-\lambda S}] + \frac{\lambda}{\lambda + \tau} E[e^{-\gamma B G}][(E[e^{-\gamma B G}] - E[e^{-\gamma B F}])].
\] (4.3)

We also have

\[
E[e^{-\gamma (S-A+B)} 1_{S>A}] = E[e^{-\gamma (S-A+B)} 1_{S>A} 1_{A\leq T}] + E[e^{-\gamma (S-A+B)} 1_{S>A} 1_{A>T}]
\]
Now expression $E[e^{-\gamma(S-A+B)}1_{S>A}1_{A\leq T}]$ becomes

\[
E[e^{-\gamma(S-A+B)}1_{S>A}1_{A\leq T}] = E[e^{-\gamma B}G] \int_{0}^{\infty} E[e^{-\gamma(S-A)}1_{S>A}1_{A\leq u}] \tau e^{-\tau u} du
\]

\[
= E[e^{-\gamma B}G] \int_{0}^{\infty} \int_{0}^{u} E[e^{-\gamma(S-t)}1_{S>t}] \lambda e^{-\lambda t} dt \tau e^{-\tau u} du
\]

\[
= E[e^{-\gamma B}G] \int_{0}^{\infty} \left( \int_{t}^{\infty} e^{-\tau u} du \right) E[e^{-\gamma(S-t)}1_{S>t}] \lambda e^{-\lambda t} dt
\]

\[
= E[e^{-\gamma B}G] \lambda \int_{0}^{\infty} e^{-(\lambda+\tau) t} E[e^{-\gamma(S-t)}1_{S>t}] dt
\]

\[
= E[e^{-\gamma B}G] \lambda E \left[ \int_{0}^{\infty} e^{-(\lambda+\tau) t} e^{-\gamma(S-t)}1_{S>t} dt \right]
\]

\[
= E[e^{-\gamma B}G] \lambda E \left[ \int_{0}^{S} e^{-(\lambda+\tau-\gamma) t} e^{-\gamma S} dt \right]
\]

\[
= E[e^{-\gamma B}G] \frac{\lambda}{\lambda + \tau - \gamma} E \left[ e^{-\gamma S} - e^{-(\lambda+\gamma) S} \right]
\]

\[
= \frac{\lambda}{\lambda + \tau - \gamma} E[e^{-\gamma B}G] \left( E[e^{-\gamma S}] - E[e^{-(\lambda+\gamma) S}] \right). \quad (4.4)
\]
In a similar way we compute the expression $E[e^{-\gamma(S-A+B)}1_{S>A}1_{A>T}]$

\[
E[e^{-\gamma(S-A+B)}1_{S>A}1_{A>T}] = E[e^{-\gamma B_F}] \int_0^\infty \int_0^{\infty} E[e^{-\gamma(S-t)}1_{S>t}, \lambda e^{-\gamma t} du] \tau e^{-\gamma u} du
= E[e^{-\gamma B_F}] \int_0^\infty \int_0^{\infty} \left( \int_0^{\infty} \tau e^{-\gamma u} du \right) E[e^{-\gamma(S-t)}1_{S>t}, \lambda e^{-\gamma t} du] \tau e^{-\gamma u} du
= E[e^{-\gamma B_F}] \int_0^\infty \left( (1-e^{-\gamma t}) E[e^{-\gamma(S-t)}1_{S>t}, \lambda e^{-\gamma t} du] \right) \tau e^{-\gamma u} du
= E[e^{-\gamma B_F}] \int_0^\infty \lambda e^{-\gamma t} E[e^{-\gamma(S-t)}1_{S>t}] dt
- E[e^{-\gamma B_F}] \int_0^\infty \lambda e^{-(\lambda+\gamma)t} E[e^{-\gamma(S-t)}1_{S>t}] dt
= E[e^{-\gamma B_F}] \left[ E\left( \lambda \int_0^S e^{-(\gamma S+(\lambda-\gamma) t) dt} \right) + E\left( \lambda \int_0^S e^{-(\gamma S+(\lambda+\gamma) t) dt} \right) \right]
= E[e^{-\gamma B_F}] \frac{\lambda}{\lambda - \gamma} E\left( e^{-\gamma S} \left( 1 - e^{-(\lambda-\gamma) S} \right) \right)
- E[e^{-\gamma B_F}] \frac{\lambda}{\lambda + \tau - \gamma} E\left( e^{-\gamma S} \left( 1 - e^{-(\lambda+\gamma) S} \right) \right)
= \frac{\lambda}{\lambda - \gamma} E[e^{-\gamma B_F}] \left( E[e^{-\gamma S}] - E[e^{-\lambda S}] \right)
- \frac{\lambda}{\lambda + \tau - \gamma} E[e^{-\gamma B_F}] \left( E[e^{-\gamma S}] - E[e^{-(\lambda r) S}] \right).
\]

(4.5)

So we finally get

\[
E[e^{-\gamma S}] = E[e^{-\gamma B_F}] \left( E[e^{-\lambda S}] - \frac{\lambda}{\lambda + \tau} E[e^{-(\lambda+\gamma) S}] \right) + \frac{\lambda}{\lambda + \tau} E[e^{-\gamma B_G}] \left( E[e^{-(\lambda+\gamma) S}] \right)
+ E[e^{-\gamma B_F}] \frac{\lambda}{\lambda - \gamma} \left( E[e^{-\gamma S}] - E[e^{-\lambda S}] \right)
- \frac{\lambda}{(\lambda + \tau - \gamma)} E[e^{-\gamma B_F}] \left( E[e^{-\gamma S}] - E[e^{-(\lambda+\gamma) S}] \right)
+ E[e^{-\gamma B_G}] \frac{\lambda}{\lambda + \tau - \gamma} \left( E[e^{-\gamma S}] - E[e^{-(\lambda+\gamma) S}] \right).
\]

We collect all terms with $E[e^{-\gamma S}]$ and obtain
\[ E[e^{-\gamma S}] = \frac{\gamma - \lambda}{\gamma - \lambda} E[e^{-\gamma B_F}] E[e^{-\lambda S}] + \frac{\lambda}{\lambda + \tau - \gamma} (E[e^{-\gamma B_F}] - E[e^{-\gamma B_G}]) E[e^{-(\lambda + \tau)S}] \]

\[ = - \frac{C_1 \gamma (\lambda + \tau - \gamma) (\lambda + \tau) E[e^{-\gamma B_F}] - \lambda \gamma (\lambda - \gamma) (E[e^{-\gamma B_F}] - E[e^{-\gamma B_G}]) C_2}{(\lambda + \tau) [(\lambda - \lambda E[e^{-\gamma B_F}] - \gamma)(\lambda + \tau - \gamma) + \lambda (\lambda - \gamma) (E[e^{-\gamma B_F}] - E[e^{-\gamma B_G}])]}. \]

This completes the proof of Theorem 4.1.1.

This Laplace-Stieltjes transform has two unknown parameters, namely \( C_1 \) and \( C_2 \). Let \( h(\gamma) = E[e^{-\gamma S}] \). To determine \( C_1 \) we use the fact that \( \lim_{\gamma \to 0} \frac{1 - E[e^{-\gamma B_F}]}{\gamma} = E[B_F] \), \( \lim_{\gamma \to 0} \frac{1 - E[e^{-\gamma B_G}]}{\gamma} = E[B_G] \) and take the limit as \( \gamma \to 0 \) of \( h(\gamma) \); this leads to:

\[ C_1 = 1 - \frac{\lambda \tau E[B_F] + \lambda E[B_G]}{\lambda + \tau} = 1 - \rho, \]

where \( E[B_F] \) and \( E[B_G] \) are respectively the expectation of the random variable \( B_F \) and \( B_G \) and \( \rho = \frac{\lambda}{(\lambda + \tau)}(\tau E[B_F] + \lambda E[B_G]) \).

We realize that

\[ E[e^{-\lambda S}] = \int_0^\infty e^{-\lambda x} f_s(x) dx = \int_0^\infty P[A > x] f_s(x) dx = P[A > S]. \]

It follows that \( E[e^{-\lambda S}] \) is the probability that the system is empty on arrival. By the stability condition found above, \( C_1 > 0 \).

### 4.2 Laplace-Stieltjes transform of the waiting time

In the preceding section we found the Laplace-Stieltjes transform for the sojourn time, which is useful for this section. The sample path of the process allows us to find another relation between the sojourn time and the waiting time; this is given by:

\[ W \overset{d}{=} (S - A)^+. \]  

(4.6)
By conditioning on the sojourn time we get:

\[
E[e^{-\gamma W}] = E[e^{-\gamma(S-A)}]
\]
\[
= P[A > S] + E[e^{-\gamma(S-A)}1_{S > A}]
\]
\[
= E[e^{-\lambda S}] + \int_0^\infty E[e^{-\gamma(S-A)}1_{S > A}|S = u]dS(u)
\]
\[
= C_1 + E\left[ \int_A^\infty e^{\gamma u}e^{-\gamma u}dS(u) \right]
\]
\[
= C_1 + \int_0^\infty \int_0^\infty e^{\gamma t}e^{-\gamma u}dS(u)\lambda e^{-\lambda t}dt
\]
\[
= C_1 + \frac{\lambda}{\gamma - \lambda} \int_0^\infty ((e^{(\gamma - \lambda)u} - 1)e^{-\gamma u}dS(u))
\]
\[
= C_1 + \frac{\lambda}{\gamma - \lambda} (C_1 - E[e^{-\gamma S}])
\]
\[
= \frac{1}{\gamma - \lambda} (\gamma C_1 - \lambda E[e^{-\gamma S}]).
\] (4.7)

**Example 4.2.1.** \((M/M/1)\) Suppose that the inter-arrival time is exponentially distributed with parameter \(\lambda\) and the service time is exponentially distributed with mean \(\frac{1}{\mu}\). Then \(C_1 = (1 - \frac{\lambda}{\mu}) = 1 - \rho\).

Thus

\[
E[e^{-\gamma W}] = \frac{1}{\gamma - \lambda} \left( \gamma(1 - \rho) - \frac{\lambda\mu(1 - \rho)}{\mu(1 - \rho) + \gamma} \right)
\]
\[
= (1 - \rho) + \rho \frac{\mu(1 - \rho)}{\mu(1 - \rho) + \gamma}.
\] (4.8)

By the inversion of the Laplace-Stieltjes transform we obtain, \(F_W(t) = P(W \leq t) = (1 - \rho) + \rho(1 - e^{-\mu(1-\rho)t}), t \geq 0\). Hence, with probability \(1 - \rho\) the waiting time is zero (i.e., the system is empty on arrival) and, given that the waiting time is positive (i.e. the system is not empty on arrival), the waiting time is exponentially distributed with parameter \(\mu(1 - \rho)\).

The Laplace-Stieltjes transforms found in the previous sections allow us to compute the mean performance measures of the model.

### 4.3 Performance measures

We denote by \(\mu_G, \mu_G^{(2)}\) resp. \(\mu_F, \mu_F^{(2)}\) the first and second moment of \(B_G\) and \(B_F\) respectively.
4.3.1 Mean sojourn time

Using the Taylor series expansion of $E[e^{-\gamma B^G}]$ and $E[e^{-\gamma B^F}]$ in Theorem 4.1.1 we obtain

$$h(\gamma) = \frac{f(\gamma)}{g(\gamma)},$$

where $f(\gamma)$ and $g(\gamma)$ are given by

$$f(\gamma) = -C_1[(\lambda + \tau) - \mu_F(\lambda + \tau)\gamma + \frac{1}{2}(\lambda + \tau)\mu_F^{(2)}\gamma^2 - \gamma + \mu_F\gamma^2 + O(\gamma^3)]$$

$$+ [\lambda(\mu_G - \mu_F)\gamma + \frac{1}{2}\lambda(\mu_F^{(2)} - \mu_G^{(2)})\gamma^2 - (\mu_G - \mu_F)\gamma^2 + O(\gamma^3)]C_2;$$

and

$$g(\gamma) = [(\lambda + \tau)(\lambda\mu_F - 1) - \frac{\lambda(\lambda + \tau)}{2}\mu_F^{(2)}\gamma - (\lambda\mu_F - 1)\gamma + \frac{\lambda}{2}\mu_F^{(2)}\gamma^2 + O(\gamma^3)]$$

$$+ (\mu_G - \mu_F)\lambda^2 + \frac{1}{2}\lambda^2(\mu_F^{(2)} - \mu_G^{(2)})\gamma - \lambda(\mu_G - \mu_F)\gamma - \frac{1}{2}\lambda(\mu_F^{(2)} - \mu_G^{(2)})\gamma^2 + O(\gamma^3).$$

We know that $E[S] = -h'(0)$; it follows that

$$E[S] = \left(\frac{g'(0)}{g(0)} - \frac{f'(0)}{f(0)}\right)h(0)$$

$$= \frac{(\lambda + \tau)[\lambda(\lambda\mu_G^{(2)} + \tau\mu_G^{(2)}) - 2(1 - \lambda\mu_G)] + 2[C_1(\lambda + \tau)(1 + \mu_F(\lambda + \tau)) + \lambda^2(\mu_G - \mu_F)C_2]}{2C_1(\lambda + \tau)^2}.$$

4.3.2 Mean number of customers in the system

Let $N_d$ denote the number of customers which arrive during the sojourn time of a customer, $N_a$ the number of customers in the system just before an arrival and $N$ the number of customers in the system in the steady state.

By the well-known step argument we get $N_a \overset{d}{=} N_d$ and by PASTA we also have $N_a \overset{d}{=} N$.

The generating function of $N_d$ is given by:

$$E[z^{N_d}] = \int_0^\infty \sum_{n_d=0}^\infty z^{n_d}e^{-\lambda t}(\lambda t)^{n_d}(n_d)!dS(t)$$

$$= \int_0^\infty e^{-(1-z)\lambda t}dS(t)$$

$$= E[e^{-(1-z)\lambda S}],$$

(4.10)
This implies that
\[ E[N_d] = \left. \frac{d(E[z^{N_d}])}{dz} \right|_{z=1} = \lambda E[S]. \] (4.11)

Thus we have \( E[N_d] = \lambda E[S] = E[N] \) by Little’s Law.

In the sequel we suppose that \( B_G \sim \exp(1/\mu_G) \) and \( B_F \sim \exp(1/\mu_F) \). Our Laplace-Stieltjes transform is now
\[
E[e^{-\gamma S}] = -\frac{C_1(\lambda + \tau)(\lambda + \tau - \gamma)(1 + \mu_G \gamma) - \lambda \gamma(\lambda - \gamma)(\mu_G - \mu_F)C_2}{(\lambda + \tau)g(\gamma)}
\]
\[ = \frac{f(\gamma)}{g(\gamma)}, \] (4.12)

where \( g(\gamma) \) is given by
\[
g(\gamma) = \mu_G \mu_F \gamma^3 + (\mu_G + \mu_F - 2\lambda \mu_G \mu_F - \tau \mu_G \mu_F)\gamma^2 \\
+ (1 - 2\lambda \mu_G + \mu_G \mu_F(\lambda^2 + \lambda \tau) - \lambda \mu_F - \tau(\mu_G + \mu_F))\gamma \\
- (\lambda + \tau - \lambda \tau \mu_F - \lambda^2 \mu_G).
\]

**Lemma 4.3.1.** Under the stability condition, the denominator \( g(\gamma) \) of the above Laplace-Stieltjes transform (4.12) has exactly one positive real zero \( \theta \) and two negative real zeros.

**Proof.** Since \( \mu_G \mu_F > 0 \) is the leading coefficient of \( g(\gamma) \) it follows that \( g(\gamma) \to \infty \) as \( \gamma \to \infty \) and \( g(\gamma) \to -\infty \) as \( \gamma \to -\infty \). Furthermore we have \( g(0) = -C_1 < 0 \); this implies that there is at least one positive zero of \( g(\gamma) \).

Next we show that we have two negative zeros. We suppose first that \( \mu_G < \mu_F \). Then
\[ g\left(-\frac{1}{\mu_F}\right) = \lambda \tau (\mu_F - \mu_G) > 0. \]

On the other hand if \( \mu_G > \mu_F \) then
\[ g\left(-\frac{1}{\mu_G}\right) = \lambda \tau (\mu_G - \mu_F) \frac{1 + \lambda \mu_G}{\mu_G} > 0. \]

If \( \mu_G = \mu_F \) then
\[ g\left(-\frac{1}{\mu_F}\right) = g(\lambda - \frac{1}{\mu_F}) = 0. \]

So in any of these cases \( g \) has two negative zeros. It follows that \( g(\gamma) \) has exactly one positive zero since \( g(\gamma) \) is a polynomial of degree 3. \( \square \)
Since $h(\gamma)$ is an analytic function in the right half-plane, the positive zero of the denominator of $h(\gamma)$ is also a zero of the numerator of $h(\gamma)$; this allows us to compute the constant $C_2$. Let $\theta$ be the positive zero of $g(\gamma)$ then by the argument stated above we have

$$C_1(\lambda + \tau - \theta)(\tau + \lambda)E[e^{-\theta B_F}] - \lambda(\lambda - \theta)(E[e^{-\theta B_F}] - E[e^{-\theta B_G}])C_2 = 0.$$ 

This implies that

$$C_2 = \frac{C_1(\tau + \lambda - \theta)(\tau + \lambda)E[e^{-\theta B_F}]}{\lambda(\lambda - \theta)(E[e^{-\theta B_F}] - E[e^{-\theta B_G}])},$$

where the last equality follows by replacing $C_1$ by its value found above.

**Example 4.3.2.** For the special case where $B_F = B_G \overset{d}{\sim} \exp(\mu)$ we obtain $C_1 = 1 - \frac{\lambda}{\mu}$, and $E[e^{-\gamma S}] = \frac{\mu(1-\phi)}{\mu(1-\theta)+\gamma}$ which is the well-known formula for the Laplace-Stieltjes transform for the sojourn time of the standard $M/M/1$ queue.

### 4.3.3 Numerical illustrations

For the numerical result we compare our model with the standard $M/G/1$ where the service distribution is a mixture of two exponentials distributions. So the service distribution is

$$B = \frac{\lambda}{\lambda + \tau} B_G + \frac{\tau}{\lambda + \tau} B_F. \quad (4.13)$$

In the first example we take $\tau$ very small, so that the threshold becomes very large in the mean. This means that $B_G \overset{d}{\sim} \exp(\frac{1}{\mu_G})$ is more probable to occur and we approximately have an $M/M/1$ queue.

**Example 4.3.3.** Let $T \overset{d}{\sim} \exp(0.01), A \overset{d}{\sim} \exp(1), F(x) = 1 - e^{-100x}, G(x) = 1 - e^{-10x}$.

Then the constants are given by: $C_1 = \frac{2099}{100100}$ and $C_2 = 0.206211$ and the expected sojourn time is $E[S] = 3.8647$.

In Figure 4.1 below, the Laplace-Stieltje transform of the sojourn time of the approximated $M/M/1$ queue and the one with a mixture of service time distributions are shown.

In the second example we take $\tau$ large and this means that $B_F \overset{d}{\sim} \exp(\frac{1}{\mu_F})$ is more probable to occur and again we approximately have an $M/M/1$ queue.
Example 4.3.4. Let $T \sim \exp(10)$, $A \sim \exp(2)$, $F(x) = 1 - e^{-\frac{26}{100}x}$, $G(x) = 1 - e^{-x}$.
Then the constants are given by: $C_1 = 0.0353535$ and $C_2 = 0.00715273$ and the expected sojourn time is $E[S] = 19.0154$.

Figure 4.2 below is the Laplace-Stieltje transform of the approximated M/M/1 queue and the one with a mixture of service time distribution.

Example 4.3.5. Let now $T \sim \exp(4)$, $A \sim \exp(2)$, $F(x) = 1 - e^{-10x}$, $G(x) = 1 - e^{-x}$.
Then the constants are given by: $C_1 = \frac{1}{5}$ and $C_2 = 0.12058$ and the expected sojourn time is $E[S] = 4.56029$.

Here below are the Laplace-Stieltje transforms of this M/M/1 type queue and the one with mixture of service time distributions.
Figure 4.2: Laplace-Stieltjes transform of sojourn time.

Figure 4.3: Laplace-Stieltjes transform of sojourn time.
In the following numerical illustration, values of the mean sojourn time are obtained for both the dependence and independence case. Their values are compared and the following measures are shown in the tables below: $ES_{dep}$ = dependent mean sojourn time, $ES_{ind}$ = independent mean sojourn time, $EW_{dep}$ = dependent mean waiting time, $EN_{dep}$ = dependent mean number of customers, $EN_{ind}$ = independent mean number of customers, ABS(difference) = absolute difference; Reld = absolute difference/$ES_{dep}$. For the numerical illustration, we have chosen: $\tau = 6$, $\mu_1 = 1$, $\mu_2 = 1/4$ in the first table, $\tau = 4$, $\mu_1 = 1$, $\lambda = 2$ in the second table and $\lambda = 2$, $\mu_1 = 1/4$, $\mu_2 = 5/8$ in the third table.

Figure 4.4: Difference between dependent sojourn time and independent sojourn time as function of $\lambda$. 

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Figure 4.4: Difference between dependent sojourn time and independent sojourn time as function of $\lambda$. 

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Table 4.1: Effect of the dependence when $\tau$, $\mu_1$, $\mu_2$ are fixed.

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Table 4.2: Effect of the dependence when $\lambda$, $\tau$, $\mu_1$ are fixed.

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Table 4.3: Effect of the dependence when $\lambda$, $\mu_2$, $\mu_1$ are fixed.

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4.4 Conclusion

The computation of the Laplace-Stieltjes transform of this $M/\tilde{G}_M/1$ allows us to find the mean performance measures of this queueing model for the case where the service time distribution is exponential. It would be nice to extend this to other service time distributions such as the Erlang, Phase-type or Hyper-exponential distribution. The result of these mean performance measures was compared with a mixture of exponential service time distributions. In heavy traffic both mean sojourn times differ up to a constant multiplicative factor.
Chapter 5

Conclusions and Recommendations

5.1 Conclusions

Risk and Queueing models are two classes of storage models which are closely connected. In this master thesis a dependent risk model was considered and the queueing model counterpart was investigated. According to the classical Cramér-Lundberg risk model, a queueing model with the FIFO discipline has a straightforward interpretation in a risk model: The zero delayed ruin probability and steady-state waiting time distribution coincide and the stationary ruin probability is also equal to the steady-state distribution of the workload process. We saw in chapter 3 that for a special case of dependence this result remains valid. In chapter 4 the Laplace-Stieltjes transform of the sojourn time distribution for the M/G/1 queue with dependence was found. According to the numerical results it seems that compared to a natural M/G/1 queue, in heavy traffic their mean queue lengths (and mean sojourn times) differ by a constant multiplicative factor.

5.2 Recommendations

Queueing Systems with dependence remains a large area of research. In our analysis, we concentrated on dependence of claim sizes or service times on inter-arrival times. It would be interesting to study the case where the distribution of the inter-arrival time between two customers depends on the next service time in general and to compare the result with the work of H. Albrecher and O. Boxma (2004) who derive exact solutions for the probability of ruin by means of Laplace-Stieltjes transforms. Also the study of the queueing model $M/G_d$ remains open. In Chapter 4, we were not able
to prove the existence of a positive root of the denominator of the Laplace-Stieltjes transform for the sojourn time when the service time has a general distribution. We recommend to investigate the proof by means of Rouché’s theorem and to investigate the sojourn time distribution of this dependence queue. It would be also interesting to study the heavy traffic behavior of this queueing with dependent structure.
Bibliography


