Evolutions and Construction of Wavelet Transform on the Similitude Group

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Abstract

Enhancement of structures in noisy image data is relevant for many biomedical applications but most commonly used enhancement techniques fail at crossings in an image. Utilizing the inherent symmetry in images a generic invertible wavelet transform on the 2D Similitude group ($SIM(2)$) is constructed. This transforms an image into its scale orientation score (scale-OS) which allows to differentiate between different scales and orientations in an image. Unlike some wavelet transform based techniques this method allows stable reconstruction of an image from the corresponding wavelet transform. Appropriate contextual-enhancement operators on an image can be constructed by using the group structure of the scale-OS. This is done via left-invariant diffusions in the wavelet domain (unlike the usual wavelet thresholding). It is shown that left-invariant operators on the scale-OS are Euclidean and scaling invariant, i.e. the result of these operators does not change under the translation, rotation and scaling of the image. Hence a framework of well-posed operators on an image is provided.

Sharp Gaussian estimates for Green’s function of linear left-invariant diffusion on the $SIM(2)$ group are presented by using subcoercive operators and it is illustrated that even linear diffusion in combination with greyvalue transforms successfully handles crossing curves. By using the Cartan connection and a left-invariant metric on $SIM(2)$, general (non)linear diffusion equation in covariant form is presented and it is shown that diffusion in this case takes place only along exponential curves.

Finally, nonlinear adaptive diffusion defined on the Euclidean motion group ($SE(2)$) known as CED-OS, is extended to multiple orientation scores and the improvements on (medical) images are exhibited.
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“...all actions in their entirety culminate in knowledge”.

- The Bhagavad Gita
Chapter 1

Introduction

1.1 Background

Elongated structures such as fibres and blood vessels are commonly found in the human body and often require analysis for diagnostic purposes. To this end many medical imaging techniques exist to acquire an image of these structures. Often used techniques include magnetic resonance imaging (MRI), microscopy, X-ray flouroscopy, fundus imaging etc.

Many (bio)medical questions related to such images require the detection and tracking of the elongated structures. Usually the medical images acquired are noisy and therefore enhancement is an important preprocessing step for subsequent feature detection. Enhancement of noisy images has been a topic of successful research for several years. However most methods fail to consistently handle elongated structures that cross each other in an image, since it is usually assumed that at any position in the image there is either no elongated structure or a single elongated structure. The recent framework of "Invertible Orientation Scores" (OS), on which this study is based, developed by Duits [12] in collaboration with Almsick [39] and used for enhancement of elongated structures by Franken [21] successfully addresses this issue.

The OS framework does not include the notion of scales in an image. In this study we explicitly include the notion of scaling in the construction of OS and develop mathematical methods for enhancement based on this modified OS. Images in Figure 1.1 illustrate some typical medical images containing elongated crossing structures, suggesting that appropriate handling of crossings is relevant.

1.2 Literature Survey

In this section we briefly review a few recent and relevant techniques in image analysis, adaptive diffusion techniques by Perona and Malik [30] and Weickert [41], curvelets by Donoho and Candèes [9, 10] and the invertible orientation score framework by Duits [15, 17]. We conclude this section by explaining how our work relates to this existing literature.
1.2.1 PDEs in Image Denoising

Partial differential equations (PDEs) have led to an entire new field in image processing and computer vision as they offer several advantages:

- Deep mathematical results with respect to well-posedness are available, such that stable algorithms can be found.
- They allow a reinterpretation of several classical methods under a novel unifying framework. This includes many well-known techniques such as Gaussian convolution, median filtering, dilation or erosion.
- The PDE formulation is genuinely continuous. Thus, their approximations aim to be independent of the underlying grid and may reveal good rotational invariance, an often important requirement.

PDE-based image processing techniques are mainly used for smoothing and restoration purposes. Many evolution equations for restoring images can be derived as gradient descent methods for minimizing a suitable energy functional, and the restored image is given by the steady-state of this process. Typical PDE techniques for image smoothing regard the original image as initial state of a parabolic (diffusion-like) process, and extract filtered versions from its temporal evolution. This whole evolution is called a scale-space.
Mathematically, a scale space representation $u_f : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ of an image $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is obtained by solving an evolution equation on the additive group $(\mathbb{R}^d, +)$. The most common evolution equation, in image analysis, is the diffusion equation

$$
\begin{align*}
\frac{\partial u_f(x, s)}{\partial s} &= \nabla_x \cdot (C(u_f(\cdot, s))(x) \nabla_x u_f(x, s)) \\
\frac{\partial u_f(x, 0)}{\partial s} &= f(x),
\end{align*}
$$

where $C : L_2(\mathbb{R}^2) \cap C^2(\mathbb{R}^2) \rightarrow C^2(\mathbb{R}^2)$ is a function which takes care of adaptive conductivity; that is $C(u_f(\cdot, s))(x)$ models the conductivity depending on the differential structure at $(x, s, u_f(x, s))$. If $C = 1$ the solution is given by convolution $u_f(x, s) = (G_s * f)(x)$ with a Gaussian kernel $G_s(x) = \frac{1}{(4\pi s^2)^{d/2}} e^{-\frac{x^2}{4s^2}}$ with scale $s = \frac{1}{2}\sigma^2 > 0$.

A special class of parabolic equations called nonlinear diffusion filters bridge the gap between scale-space and restoration ideas. Methods of this type have been proposed for the first time by Perona and Malik [30]. In order to smooth an image and to simultaneously enhance important features such as edges, they apply a diffusion process whose diffusivity is steered by derivatives of the evolving image. They pointed out that nonlinear adaptive isotropic diffusion is achieved by replacing $C = 1$ by $C(u_f(\cdot, s))(x) = c(||\nabla_x u_f(x, s)||)$, where $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is some smooth strictly decaying positive function vanishing at infinity. The idea behind this choice is that diffusion should not occur when the (local) gradient value is large. By restricting to $c > 0$ one ensures that the diffusion is always forward and thus ill-posed backward diffusion is avoided. The common choices are,

$$
e(t) = e^{-\frac{e^t}{(\lambda \sigma)^2}}, \quad c(t) = \frac{1}{(t/\lambda)^{2\rho} + 1}, \quad t \geq 0. \tag{1.1}
$$

involving parameters $\rho > 1/2$, $\lambda > 0$. A further improvement of the Perona and Malik scheme is the “coherence-enhancing diffusion” (CED) proposed by Weickert [41], which also uses the direction of the gradient $\nabla_x u_f$. The diffusion constant $c$ is now replaced by a diffusion matrix:

$$
S(u_f(\cdot, s))(x) = (G_s * \nabla u_f(\cdot, s) \nabla u_f(\cdot, s)^T)(x), \quad \alpha I + (1 - \alpha) e^{-\frac{\lambda_x (S(u_f(\cdot, s))(x))}{\lambda_2 (S(u_f(\cdot, s))(x))} e_2(S(u_f(\cdot, s))(x)) e_2^T(S(u_f(\cdot, s))(x))},
$$

where $\alpha \in (0, 1)$, $c > 0$, $\sigma > 0$ are parameters and $S$ with eigenvalues $\{\lambda_i(S(u_f(\cdot, s))(x))\}_{i=1,2}$ is used to get a measure of local anisotropy $e^{-\frac{\lambda_x (S(u_f(\cdot, s))(x))}{\lambda_2 (S(u_f(\cdot, s))(x))}}$ together with the orientation estimate $e_2(S(u_f(\cdot, s))(x))$, which is the eigenvector of $S$ corresponding to smallest eigenvalue. Figure 1.2 illustrates the visually appealing results due to nonlinear diffusions. However this method fails in image analysis applications with crossing curves as the direction of gradient at crossing locations is ill-defined, [16].

![Figure 1.2: From left to right: input image $f$ of the famous Van Gogh self portrait; computed on comparable slices $u_f(\cdot, s)$ in a linear scale representation $C = 1$; Perona and Malik nonlinear scale space representation [30]; Weickert’s coherence-enhancing diffusion [41]](image-url)
Introduction

Figure 1.3: Illustration of curvelet transform construction in Fourier domain. From left to right: basic wavelet at a particular scale $a$, $\hat{\gamma}_{a00}$; support of basic wavelet; tiling of the frequency domain into wedges for curvelet construction. Figures extracted from [28].

1.2.2 Curvelets

The discrete wavelet transform is a widely used tool for Mathematical analysis and signal processing, but has the disadvantage of poor directionality, which undermines it’s usage in image denoising applications. In 1999, an anisotropic geometric wavelet transform, named ridgelet transform (also called first-generation curvelet transform), was proposed by Candès and Donoho [8]. Later, a considerably simpler second-generation curvelet transform based on a frequency partition technique was proposed by the same authors, see [9, 10]. The second-generation curvelet transform has been successfully used for many different applications, see [28] for more details. Here we briefly summarize the continuous curvelet transform (CCT) in $\mathbb{R}^2$ as presented in [9].

Let $x$ and $\xi$ denote the spatial and frequency variables respectively and $(\rho, \phi)$ the polar coordinates in the frequency domain. Consider a pair of windows $W(\rho)$ and $V(\phi)$ (radial and angular windows respectively) which are smooth, non-negative and real valued, with $W$ taking positive real arguments and supported on $\rho \in (-1/2, 2)$ and $V$ taking real arguments and supported on $\phi \in [-1, 1]$. These windows obey the admissibility conditions

$$\int_0^\infty W(a\rho^2)\frac{da}{a} = 1 \quad \forall \rho > 0 , \quad \int_{-1}^1 (V(u))^2 du = 1.$$  \hspace{1cm} (1.3)

The windows are used in the frequency domain to construct a family of analyzing elements with three parameters: scale $a > 0$, location $b \in \mathbb{R}^2$ and orientation $\theta \in [0, 2\pi)$. At scale $a$, the family is generated by translation and rotation of a basic element $\gamma_{a00}$:

$$\gamma_{ab0}(x) = \gamma_{a00}(R_\theta(x-b)) = \gamma_{100}(R_\theta(a^{-1}(x-b)))$$

where $R_\theta$ is the planar rotation matrix effecting rotation by $\theta$ radians. The generating element at scale $a$ is defined in terms of Fourier coordinates $(\rho, \phi)$ by setting,

$$\gamma_{a00}(\rho, \phi) = W(\rho) V(\phi/\sqrt{a}) a^{3/4}, \quad 0 < a < a_0.$$  \hspace{1cm} (1.4)

Therefore the support of each $\gamma_{a00}$ is a polar wedge defined by the support of $W$ and $V$ applied with scale-dependant window widths in each direction. Note that these curvelets are highly oriented. Figure 1.3 illustrates the idea of this construction graphically.

Equipped with this family of curvelets a continuous curvelet transform $\Gamma_f$, a function on scale/location/direction space, is defined by

$$\Gamma_f(a, b, \theta) = \langle \gamma_{a,b,\theta}, f \rangle, \quad a < a_0, b \in \mathbb{R}^2, \quad \theta \in [0, 2\pi).$$  \hspace{1cm} (1.5)
Let $f \in L_2(\mathbb{R}^2)$ have a Fourier transform vanishing for $\xi < 2/a_0$. Let the $V$ and $W$ obey the admissibility condition (1.3). For such high-frequency functions we have a reproducing formula,

$$f(x) = \int \Gamma_f(a,b,\theta)\gamma_{a,b,\theta}(x)\mu(da,db,d\theta),$$

and a Parseval formula

$$||f||_{L_2} = \int ||\Gamma_f(a,b,\theta)\gamma_{a,b,\theta}||^2 \mu(da,db,d\theta),$$

where in both cases, $\mu$ denotes the reference measure $d\mu = \frac{1}{\pi}dadbdb$. 

This transform can be extended to functions containing low frequencies using a single coarse-scale isotropic wavelet.

Note that the Curvelet transform samples $g \mapsto (U_g \psi, f)$ with $\psi = \gamma_{100}$ and $U_g \psi(x) = \psi(R_g(x - b))$ where $g = (x,a,\theta)$ and so the operators on the curvelet transform should be left-invariant. However due to the discrete construction it is not straightforward to guarantee left-invariance of the operators on the curvelet transform. Thereby it is not possible to ensure that the effective operator in the image domain are Euclidean invariant, whereas in our case this is ensured, see Section 4.1.

### 1.2.3 Invertible Orientation Score

The idea of orientations score (OS) is inspired by the early visual system of mammals, in which receptive fields exist that are tuned to various locations and orientations. Thereby a simple cell receptive field can be parametrized by its position and orientations. Figure 1.4 illustrates this phenomenon. In essence the primary visual cortex "transforms" the perceived image, which can be seen as a map of luminance values for each spatial positions $(x,y)$, to a three dimensional representations, i.e. a map of orientation confidence for each position and orientation $(x,y,\theta)$. This redundant representation of the image simplifies subsequent analysis in higher visual cortex regions. We briefly explain the concept of invertible OS developed in [12, 15]. The Euclidean motion group (i.e. the group of planar rotations and translations) $SE(2)$ is defined as $SE(2) = \mathbb{R}^2 \times SO(2)$ where $SO(2)$ is the group of planar rotations. An OS $U_f : SE(2) \rightarrow \mathbb{C}$ of an image $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is obtained by means of an anisotropic convolution kernel $\psi : \mathbb{R}^2 \rightarrow \mathbb{C}$ via

$$U_f(g) = \int_{\mathbb{R}^2} \psi(R_\theta^{-1}(y-x))f(y)dy, \ g = (x,\theta) \in SE(2),$$

where $\psi(-x) = \bar{\psi}(x)$ and $R_\theta \in SO(2)$ is the 2D counter-clockwise rotation matrix. Assume $\psi \in L_2(\mathbb{R}^2)$, then the transform $W_\psi$ which maps images $f \in L_2(\mathbb{R}^2)$ can be rewritten as

$$U_f(g) = (W_\psi f)(g) = (U_g \psi, f)_{L_2(\mathbb{R}^2)},$$

where $g \mapsto U_g$ is a unitary (group-)representation of the Euclidean motion group $SE(2)$ into $L_2(\mathbb{R}^2)$ given by $U_g f(y) = f(R_\theta^{-1}(y-x))$ for all $g = (x,\theta) \in SE(2)$ and for all $f \in L_2(\mathbb{R}^2)$. The general wavelet reconstruction results from [1, 23] are not applicable to the transform.
\[ \mathbf{f} \mapsto \mathbf{U}_f, \] since the representation \( \mathbf{U} \) is reducible. To accommodate reducible representations, in [11] a more general wavelet transform is proposed. With this wavelet transform the OS, \( \mathbf{U}_f : \text{SE}(2) \to \mathbb{C} \) is constructed by means of an admissible vector \( \psi \in L^2(\mathbb{R}^2) \) such that the transform \( \mathbf{W}_\psi \) is unitary onto the unique reproducing kernel Hilbert space \( \mathbb{C}^{\text{SE}(2)}_K \) of functions on \( \text{SE}(2) \) with reproducing kernel \( K(g, h) = (\mathbf{U}_g \psi, \mathbf{U}_h \psi) \), which is a closed vector subspace of \( L^2(\text{SE}(2)) \). For the abstract construction of the unique reproducing kernel space \( \mathbb{C}^{\text{SE}}_K \) on a set \( I \) see [5]. Note that,

\[
(W_\psi f)(x, \theta) = (\mathbf{U}_{\mathbf{R}_{\theta}^x} \psi, f), \quad f \in L^2(\mathbb{R}^2),
\]

where the rotation and translation operators on \( L^2(\mathbb{R}^2) \) are defined by \( \mathbf{R}_\theta f(y) = f(\mathbf{R}^{-1}_\theta y) \) and \( \mathbf{T}_x f(y) = f(y - x) \). This leads to the essential Plancherel formula,

\[
\| \mathbf{W}_\psi f \|_{\mathbb{C}^{\text{SE}(2)}_K}^2 = \int_{\mathbb{R}^2} \int_0^{2\pi} |(\mathbf{W}_\psi f)(\omega, \theta)|^2 \frac{1}{M_\psi(\omega)} d\omega d\theta
\]

where \( M_\psi \in C(\mathbb{R}^2, \mathbb{R}) \) is given by \( M_\psi(\omega) = \int_0^{2\pi} |\mathbf{F}_\psi(R^\omega_\theta \omega)|^2 d\theta \). If \( \psi \) is chosen such that \( M_\psi = 1 \) then we gain \( L^2 \) norm preservation. But this is not possible as \( \psi \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \) implies that \( M_\psi \) is a continuous function vanishing at infinity. In practice, however, because of finite grid sampling, \( \mathbf{U} \) is restricted to the space of disc limited images,

\[
L^2_\phi(\mathbb{R}^2) = \{ f \in L^2(\mathbb{R}^2) | \text{supp}(\mathbf{F} f) \subset B_{\phi, \omega} \}.
\]

Since the wavelet transform \( \mathbf{W}_\psi \) maps the space of images \( L^2(\mathbb{R}^2) \) unitarily onto the space
of orientation scores $C^{SE(2)}_K$ (provided $M_\psi > 0$) the original image $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ can be reconstructed from it’s orientation score $U_f : SE(2) \rightarrow \mathbb{C}$ by using the adjoint of the wavelet transform,

\[ f = \mathcal{W}_\psi^* \mathcal{W}_\psi[f] = \mathcal{F}^{-1} \left[ \omega \mapsto \int_0^{2\pi} \mathcal{F}[U_f(\cdot, \theta)](\omega)M_\psi^{-1}(\omega)d\theta \right]. \]

Remarks:

- With restriction to $L^2_\varrho(\mathbb{R}^2)$, $M_\psi = 1$ implies equality of $C^{SE(2)}_K$ and $L_2(SE(2))$ norms i.e. $\|U\|_{C^{SE(2)}_K} = \|U\|_{L_2(SE(2))}$ for all $U \in C^{SE(2)}_K$.
- With restriction to $L^2_\varrho(\mathbb{R}^2)$, $M_\psi = 1$ means that $\psi$ has equal length over each irreducible subspace of $L^2_\varrho(\mathbb{R}^2)$ so that each irreducible subspace of $L^2_\varrho(SE(2)) = \{u \in L^2_\varrho(SE(2)) | u(\cdot, \theta) \in L^2_\varrho(\mathbb{R}^2), \forall \theta \in [0, 2\pi]\}$ and the unitarity result is due to Plancherel formula on $SE(2)$. See [17, App.A] for more details.

For examples of wavelets $\psi$ for which $M_\psi = 1|_{B_{0,\infty}}$ and details on fast approximative reconstruction by integration over angles see [12]. For details on image processing (particularly enhancement and completion of crossing elongated structures) via orientation scores, see [13, 16, 17, 18]. An intuitive illustration of the relation between an elongated structure in a 2D image and the corresponding elongated structure in an orientation score is given in Figure 1.5.

1.2.4 In the context of related work

PDE techniques [30, 41], usually fail at crossings, [16], which are commonly encountered in medical images. The orientation score framework solves this problem by constructing evolutions on a Lie group $(SE(2))$ rather than the usual space of images, $\mathbb{R}^2$ [16, 17, 18]. Most of the noise in (medical) images exists at lower scales and this suggests the need for scale based enhancement. Further the utility of curvelets (which is a scale based technique), [9, 10, 28], for enhancement also leads to a similar conclusion. In this study we introduce the notion of scaling to the $SE(2)$-OS framework via continuous wavelet transform on the 2D Similitude
group \((SIM(2))\) and arrive at a scale-OS. Wavelet transform on \(SIM(2)\) group was first used by Antoine et al. for feature detection by construction of appropriate directional wavelets, \([2, 3, 4]\). Though similar in the basic idea we make use of a more general wavelet transform proposed in \([11]\) for our study, where we do not rely on the celebrated result by Grossman et al. \([23]\) and where the requirements for irreducible representations and the usual reconstruction via \(L_2\)-adjoint are dropped. The explicit construction of our transform is similar to the discrete-WT based construction utilized by Donoho et al. in \([9, 10]\). However it is important to note that our construction has a group structure which allows us to define Euclidean and scaling invariant operators on the scale-OS which is an improvement over all existing techniques. We extend the ideas of PDE based techniques by constructing evolutions on the \(SIM(2)\) group which like in the case of \(SE(2)\)-OS allows handling of crossing structures in an image.

1.3 Content and Structure of this Thesis

In Chapter 2, the concept of an abstract wavelet transform in a group theoretic setting is introduced. This is followed by a generalized version of the wavelet transform. We then apply this transform to our case of interest, the 2D Similitude group \((SIM(2))\) and conclude the chapter with the design of appropriate wavelets and graphically illustrate the practical utility of this design.

Chapter 3 deals with relevant differential-geometric properties of the \(SIM(2)\) group followed by an explicit formulation of the exponential and logarithmic curves in this case. The theoretical and numerical issues pertaining to \(SIM(2)\)-convolutions are also discussed.

In Chapter 4, we begin by showing that all “legal” operators on orientation scores are left invariant. We then present sharp Gaussian estimates for Green’s function of linear diffusion on \(SIM(2)\) group. Adaptive nonlinear evolutions on \(SIM(2)\) in a differential geometric setting is presented by making use of the Cartan connection and a method for curvature estimation is proposed. This is followed by an extension of adaptive diffusion on the \(SE(2)\) group to multiple orientation scores.

Finally in Chapter 5, we summarize this thesis and discuss possible improvements and future work.

The main results of this thesis are:

1. Utilizing the inherent symmetry in images a generic invertible wavelet transform on the Similitude group \((SIM(2))\) is presented, which transforms an image into it’s scale orientation score (scale-OS). This allows differentiating between different scales and angles in an image. This method allows stable reconstruction of the image from the corresponding wavelet transform.

2. We explicitly derive the exponential and logarithmic curves in the \(SIM(2)\)-group using left-invariant vector fields.

3. Using the wavelet transform mentioned above, we show that the left-invariant operators on scale-OS are invariant under the operations of translation, rotation and scaling of the original image. Thus we provide a framework for well-posed operators on scale-OS.

4. Sharp Gaussian estimates for the green’s function of the linear (non-adaptive) diffusion equation on the \(SIM(2)\)-group are presented by using subcoercive operators and we illustrate that in scale-OS framework even linear diffusion combined with greyvalue transformations leads to successful enhancement of crossing curves.
5. By choosing an appropriate metric on $SIM(2)$ and making use of the Cartan connection a covariant form of nonlinear left-invariant diffusion equation on $SIM(2)$ is presented and it is shown that the diffusion only takes place along the exponential curves. Based on these results we propose a method for extraction of spatial curvature from scale-OS.

6. Finally we extend the $SE(2)$ based nonlinear adaptive diffusion (CED-OS) to multiple orientation scores and show improvements on medical images.
Chapter 2

Generalized Wavelet Transform on the Similitude Group

We begin with a few preliminaries followed by a review of the construction of unitary maps on functional Hilbert spaces given in [11], which leads to the famous result by Grossman et al. [23] in a group-theoretic setting. Though the results in [23] are applicable to our context, we make use of a more general theory presented in [11, 12], which allows the construction of (admissible) wavelets suited to our application area of signal/image processing. Finally we present a practical design for a wavelet transformation. We emphasise the importance of a group structure as it will allow us to design appropriate (enhancement/contextual) operations on the wavelet transform of an image, see (2.14) and Section 4.1.

2.1 Preliminaries

• \( L^p(\Omega, \mu) \): The quotient space consisting of functions with finite \( L^p \) norm \((0 < p < \infty)\), i.e. \( \left( \int_\Omega \| f \|_p^p d\mu \right)^{1/p} < \infty \) on \( \Omega \) with respect to the nilspace of the \( L^2 \)-norm, which consists of all functions \( f \) on \( \Omega \) with zero measure support, i.e. \( f = 0 \) almost everywhere.

• Images are assumed to be within \( L^2(\mathbb{R}^2) \), unless explicitly stated otherwise.

• An image \( f \in L^2(\mathbb{R}^2) \) is called disc-limited if the support of it’s Fourier transform is bounded, by say a sphere of radius \( \rho \). The space of these images is given by \( L^\rho_2(\mathbb{R}^2) = \{ f \in L^2(\mathbb{R}^2) | \text{supp}(\mathcal{F}f) \subset B_{0,\rho} \} \), \( \rho > 0 \).

• A Hilbert space is a vector space equipped with an inner product such that every Cauchy sequence converges with respect to the norm induced by the inner product.

• \( \mathcal{B}(X, Y) \): The vector space consisting of continuous (i.e. bounded) linear operators from \( X \) onto \( Y \). In case, \( X = Y \), we write \( \mathcal{B}(X) \).

• The range of a linear operator \( A \) will be denoted by \( \mathcal{R}(A) \) and the nilspace will be denoted by \( \mathcal{N}(A) \).

• The Fourier transform \( \mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \), is almost everywhere defined by

\[
\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)e^{-i\omega \cdot x} dx.
\]
By Plancherel and Convolution theorem we have \( \|F(f)\|^2 = \|f\|^2 \) for all \( f \in L_2(\mathbb{R}^d) \) and that \( F[f \ast g] = (2\pi)^{d/2} F[f] F[g] \), for all \( f, g \in L_2(\mathbb{R}^d) \).

- We will use the short notation for the following groups:
  - \( \text{Aut}(\mathbb{R}^d) = \{ A : \mathbb{R}^d \to \mathbb{R}^d \mid A \text{ linear and } A^{-1} \text{ exists} \} \)
  - dilation group \( D(d) = \{ A \in \text{Aut}(\mathbb{R}^d) \mid A = aI, a > 0 \} \)
  - orthogonal group \( O(d) = \{ X \in \text{Aut}(\mathbb{R}^d) \mid X^T = X^{-1} \} \)
  - rotation group \( SO(d) = \{ R \in O(d) \mid \det(R) = 1 \} \)
  - circle group \( T = \{ z \in \mathbb{C} \mid |z| = 1 \} \)

Let \( b \in \mathbb{R}, g \in T \) and \( \tau : T \to SO(2) \subset \text{Aut}(\mathbb{R}^2) \):
\[
\tau(z) = R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},
\]
(2.1)

- Let \( T \) and \( S \) be locally compact groups and let \( \tau : T \to \text{Aut}(S) \) be a group homomorphism. The **semi-direct product** \( S^T \rtimes \tau T \) is defined to be the group (which is again locally compact) with underlying group \( \{(s, t) \mid s \in S, t \in T\} \) and group operation
\[
(s, t)(s', t') = (s\tau(t)s', tt').
\]
(2.2)

In this study we consider the following two groups:

- The group \( \mathbb{R}^2 \rtimes T \), where \( T \) is given by (2.1). The group product (2.2) is now given by
\[
\begin{align*}
  gg' &= (b, e^{i\theta})(b', e^{i\theta'}) = (b + R_\theta b', e^{i(\theta + \theta')}), \\
  g &= (b, e^{i\theta}), \\
  g' &= (b', e^{i\theta'}). 
\end{align*}
\]
This group is the **Euclidean motion group**, \( SE(2) \).

- The group \( \mathbb{R}^2 \times (T \times D(1)) \), where \( T \) is given by (2.1). The group product (2.2) is now given by
\[
\begin{align*}
  gg' &= (b, a, e^{i\theta})(b', a', e^{i\theta'}) = (b + aR_\theta b', a'e^{i(\theta + \theta')}), \\
  g &= (b, e^{i\theta}), \\
  g' &= (b', e^{i\theta'}). 
\end{align*}
\]
This group is the **Similitude group**, \( \text{SIM}(2) \).

- A representation \( \mathcal{R} \) of a group \( G \) into a Hilbert space \( H \) is a homomorphism \( \mathcal{R} \) between \( G \) and \( \text{B}(H) \), the space of bounded linear operators on \( H \). It satisfies \( \mathcal{R}_{gh} = \mathcal{R}_g \mathcal{R}_h \) for all \( g, h \in G \) and \( \mathcal{R}_e = I \). A representation \( \mathcal{R} \) is irreducible if the only invariant closed subspaces of \( H \) are \( \{0\} \) and \( H \), otherwise reducible. We consider unitary representations, i.e., \( \|U_g \psi\|_H = \|\psi\|_H \) for all \( g \in G \) and \( \psi \in H \), which will be denoted by \( \mathcal{U} \) rather than \( \mathcal{R} \). Within the class of unitary representations we consider the representations of \( \mathbb{R}^d \times T \) in \( L_2(\mathbb{R}^d) \) which are given by
\[
(U_g \psi)(x) = \frac{1}{\sqrt{\det(\tau(t))}} \psi(\tau^{-1}(t)(x - b)), \quad \text{with } g = (b, \tau(t)) \in \mathbb{R}^d \times T.
\]
(2.3)

These representations are called **left-regular actions** of \( G = \mathbb{R}^d \times T \) in \( L_2(\mathbb{R}^d) \).

- Let \( b \in \mathbb{R}^d, a > 0 \) and \( g \in G \) with corresponding \( \tau(g) \in \text{Aut}(\mathbb{R}^d) \). Then the unitary operators \( f \mapsto f, T_b, D_a \) and \( R_g \) on \( L_2(\mathbb{R}^d) \) are defined by
\[
\begin{align*}
  \hat{f}(x) &= f(-x) \\
  T_b \psi(x) &= \psi(x - b) \\
  D_a \psi(x) &= \frac{1}{a^\frac{d}{2}} \psi\left(\frac{x}{a}\right) \\
  R_g \psi(x) &= \frac{1}{\sqrt{\det(\tau(g))}} \psi(\tau(g) x^{-1}).
\end{align*}
\]
(2.4)

The mappings \( b \mapsto T_b, g \mapsto R_g, a \mapsto D_a \) are respectively left regular actions of \( \mathbb{R}^d, D(d), G \) into \( L_2 \mathbb{R}^d \).
• A functional Hilbert space\(^1\) is a Hilbert space consisting of complex valued functions on an index set \(\mathbb{I}\) on which the point evaluation \(\delta_a\), is a continuous/bounded linear functional for all \(a \in \mathbb{I}\). Consequently, it has a Riesz representant \(K_a \in H\)

\[
    f(a) = \langle \delta_a, f \rangle = (K_a, f)_H.
\]

The function \(K : \mathbb{I} \times \mathbb{I} \to \mathbb{C}\) given by \(K(a, b)_H = (K_a, K_b)_H = K_{ab}(a)\) is called reproducing kernel. Note that the spaces \(L^2(\mathbb{R}^d)\) are not functional Hilbert spaces.

• The \(d\)-dimensional Gaussian kernel \(G_s\) at scale \(s\) is given by

\[
    G_s(x) = \frac{1}{(4\pi s)^{d/2}} e^{-\frac{|x|^2}{4s}}. \tag{2.5}
\]

Occasionally, the Gaussian kernel will be parametrized by standard deviation \(\sigma\) and we write \(G_{\sigma} = G_s\), where we notice that \(s = \frac{\sigma^2}{2}\).

### 2.2 Unitary maps from a Hilbert space \(H\) to a functional Hilbert space \(C^\mathbb{I}_K\)

In this section we present an abstract wavelet transform given in [11]. Recall that a functional Hilbert space is a Hilbert space such that point evaluation is continuous, and therefore by Reisz representation theorem there exists a set \(\{K_m|m \in \mathbb{I}\}\) with

\[
    (K_m, f)_H = f(m), \text{ for all } m \in \mathbb{I} \text{ and } f \in H.
\]

Clearly the span of the set \(\{K_m|m \in \mathbb{I}\}\) is dense in \(C^\mathbb{I}_K\), for if \(f \in H\) is orthogonal to all \(K_m\) then \(f = 0\) on \(\mathbb{I}\).

Define \(K(m, m') = K_{mm'}(m) = (K_m, K_{m'})_H\) for all \(m, m' \in \mathbb{I}\). \(K\) is called reproducing kernel and it is a function of positive type on \(\mathbb{I}\), i.e.

\[
    \sum_{i=1}^{n} \sum_{j=1}^{n} K(m_i, m_j)c_i c_j \geq 0, \text{ for all } n \in \mathbb{N}, c_1, \ldots, c_n \in \mathbb{C}, m_1, \ldots, m_n \in \mathbb{I}.
\]

Therefore to every functional Hilbert space there belongs a reproducing kernel, which is a function of positive type. Conversely as mentioned in [5], a function \(K\) of positive type on set \(\mathbb{I}\), induces uniquely a functional Hilbert space consisting of functions on \(\mathbb{I}\) with reproducing kernel \(K\).

Let \(V = \{\phi_m|m \in \mathbb{I}\}\) be a subset of \(H\). Define function \(K : \mathbb{I} \times \mathbb{I} \to \mathbb{C}\) by \(K(m, m') = (\phi_m, \phi_{m'})\). From this function of positive type the space \(C^\mathbb{I}_K\) can be constructed. From [11] we have the following important result:

**Theorem 2.1 (Abstract Wavelet Theorem).** Define \(W : \overline{V} \to C^\mathbb{I}_K\) by

\[
    (Wf)(m) = (\phi_m, f)_H. \tag{2.6}
\]

Then \(W\) is a unitary mapping.

In our study we are interested in the case \(\overline{V} = H\). In this case, Theorem 2.1 becomes the following.

**Theorem 2.2.** If the span of \(V = \{\phi_m|m \in \mathbb{I}\}\) is dense in \(H\), then the transform \(W : H \to C^\mathbb{I}_K\) defined by

\[
    (W[f])(m) = (\phi_m, f)_H \tag{2.7}
\]

is a unitary mapping, i.e. \(\|W[f]\|_{C^\mathbb{I}_K} = \|f\|_H\).

\(^1\)Also known as reproducing kernel Hilbert space.
2.3 Functional Hilbert spaces on Groups

From now on we will assume $I$ to be a group $G$. Further assume the group to have a representation on $H$, i.e., a map $\mathcal{R} : G \to \mathcal{B}(H)$, which satisfies $\mathcal{R}_g \mathcal{R}_h = \mathcal{R}_{gh}$ and $\mathcal{R}_e = I$, $\forall g, h \in G$, where $e$ is the identity element in $G$.

Given $\psi \in H$ (the wavelet), let $g \mapsto \mathcal{R}_g$ be a representation of $G$ onto $H$ such that

$$V_\psi = \{ \mathcal{R}_g \psi | g \in G \}$$

is dense in $H$ and $K(g, g') = (\mathcal{R}_g \psi, \mathcal{R}_{g'} \psi)$. Starting from $V_\psi$ we can construct the functional Hilbert subspace $C^G_K$. Then as a consequence of Theorem 2.2 we have the following result.

**Theorem 2.3** (Wavelet Theorem for group representations). Let $\mathcal{R}$ be a representation of a group $G$ in a Hilbert space $H$. Define the function $K : G \times G \to \mathbb{C}$ of positive type by

$$K(g, g') = (\mathcal{R}_g \psi, \mathcal{R}_{g'} \psi)_{H}.$$  

(2.9)

Define the set $V_\psi = \{ \mathcal{R}_g \psi | g \in G \}$. Then the wavelet transformation $W_\psi : V_\psi \to C^G_K$ defined by

$$(W_\psi f)(g) = (\mathcal{R}_g \psi, f)_H.$$  

(2.10)

is a unitary mapping.

In case $V_\psi = H$, the wavelet transformation $W_\psi : H \to C^G_K$ is unitary.

2.3.1 Wavelet Transformations constructed by Unitary Irreducible representations of Locally Compact Groups

Recall that a representation $\mathcal{R}$ of a group $G$ in a Hilbert space $H$ is irreducible if the only closed subspaces of $H$ which are invariant under all $\mathcal{R}_g$ for all $g \in G$ are $H$ and $\{0\}$. Clearly for every nonzero $\psi \in H$ the subspace $V_\psi$, where $V_\psi = \{ \mathcal{R}_g \psi | g \in G \}$, is invariant under all $\mathcal{R}_g$ with $g \in G$ and since it is non-empty, $V_\psi = H$. Hence the wavelet transformation is a unitary mapping for all $\psi \in H$.

The representation $\mathcal{R}$ is called square integrable if there exists $\psi \in H$ with $\psi \neq 0$ and

$$C_\psi := \int_G \frac{|\mathcal{U}_g \psi, \psi|^2}{(\psi, \psi)_H} \, d\mu_G(g) < \infty.$$  

(2.11)

$\psi \in H$ with $\psi \neq 0$ such that $C_\psi < \infty$ is known as the admissible wavelet.\footnote{In literature, see [9], we also encounter an alternative but equivalent definition of $C_\psi$ given by $C_\psi = (2\pi)^2 \int_{\mathbb{R}^2} \frac{|\mathcal{F}_\psi(\omega)|^2}{|\omega|^2} \, d\omega$. See [27] for more details.} If the group representation is unitary, irreducible and square integrable, then the functional Hilbert space will always be a closed subspace of $L^2(G)$, whenever the wavelet satisfies (2.11). This was first shown by Grossmann et al. [23]. We now state an important theorem from [23].

**Theorem 2.4** (The Wavelet Reconstruction Theorem). Let $\mathcal{U}$ be an irreducible, unitary and square integrable representation of a locally compact group $G$ on a Hilbert space $H$. Let $\psi \in H$ such that (2.11) holds. Then the wavelet transform $W : H \to C^G_K$ defined by

$$(W[f])(m) = (\mathcal{U}_g \psi, f)_H$$
is a linear isometry (up to a constant) from the Hilbert space $H$ onto a closed subspace $C^G_{K_\psi}$ of $L_2(G, d\mu)$:

$$\|W_\psi f\|^2_{L_2(G)} = C_\psi \|f\|^2_H. \tag{2.12}$$

Here the space $C^G_{K_\psi}$ is the functional Hilbert space with reproducing kernel

$$K_\psi(g, g') = \frac{1}{C_\psi} (U_{g'} \psi, U_g \psi).$$

The corresponding orthogonal projection $P_\psi : L_2(G, d\mu) \rightarrow C^G_{K_\psi}$ is given by

$$(P_\psi \phi)(g) = \int_G K_\psi(g, g') \phi(g') d\mu(g') / \text{where} / \phi \in L_2(G, d\mu). \tag{2.13}$$

Furthermore, $W_\psi$ intertwines $U$ and the left regular representation $L$ (i.e. $L_g$ is given by $L_g(\phi) = (h \mapsto \phi(g^{-1} h))$) on $L_2(G)$

$$W_\psi U_g = L_g W_\psi. \tag{2.14}$$

**Consequences of Theorem 2.4**

- The inverse wavelet transform is the adjoint of the wavelet transform up to a constant: $W_\psi^{-1} = \frac{1}{C_\psi} (W_\psi)^*$, which is given by,

$$W_\psi^*(\phi) = \int_G (U_g \psi) \phi(g) d\mu_G(g). \tag{2.15}$$

This follows from a straightforward evaluation given in the appendix.

- Applying the Polarization identity to (2.12) it follows that $(W_\psi[f_1], W_\psi[f_2])_{L_2(G)} = C_\psi(f_1, f_2)_H$ for all $f_1, f_2 \in H$.

- In the case where $G = \mathbb{R} \times T$ and $U$ a left regular action of $G$ in $L_2(\mathbb{R}^d)$, given by (2.3), which is unitary and if it is also irreducible one has the reconstruction formula:

$$f(x) = \frac{1}{C_\psi} \int_{\tau(T)} \int_{\mathbb{R}^d} \frac{1}{\text{det}(\tau(t))} W_\psi[f](t, b) \psi((\tau^{-1}(t))(x - b)) db dt \tag{2.16}$$

for almost every $x \in \mathbb{R}^d$.

- Operators on (multiple scale) wavelet transforms must commute with $L_g$, $\forall g \in SIM(2)$, to ensure that the effective operator $W_\psi^* \circ \Phi \circ W_\psi$ on images commutes with $U_g$, $\forall g \in SIM(2)$.

Theorem 2.4 thus allows perfectly well-posed reconstruction of the image when the left regular action $U$ of the group $G$ in Hilbert space $H$, is an irreducible representation. The following lemma shows that we are allowed to use Theorem 2.4 for the $SIM(2)$ case.

**Lemma 2.5.** The left regular action $U$ of the $SIM(2)$ group in $L_2(\mathbb{R}^2)$, given by

$$(U_{b,e^{i\theta}} \psi)(x) = \frac{1}{a} \psi \left( R_{a^{-1}}^{-1}(x - b) \right), \quad a > 0, \quad \theta \in [0, 2\pi], \quad b \in \mathbb{R}^2, \tag{2.17}$$

is an irreducible unitary representation.
For proof, see [27, Pg. 51]. Often, this irreducibility condition is very strong and even simple (and often encountered) group representations do not satisfy it. The left regular action $\mathcal{V}$ of the Euclidean Motion group $SE(2)$ in $L^2(\mathbb{R}^2)$, given by

$$Y_{\theta,b,e} \psi(x) = \psi(R_{\theta}^{-1}(x - b)), \quad \theta \in [0, 2\pi], \ b \in \mathbb{R}^2,$$

is a reducible representation.

### 2.4 Generalized Wavelet Transformation on Locally Compact Groups

As mentioned above irreducibility is a strong condition. In [11], a generalized wavelet transform is introduced which does not have the condition of irreducibility. In this section we give a compilation of results from [11] which explicitly characterize $C_K^G$, when $G = \mathbb{R}^d \rtimes T$, $T$ is a locally compact group and $\mathcal{U}$ is the left regular action of $G$ into $L^2(\mathbb{R}^d)$ and then apply them to our case. At this point we mention that all locally compact groups have a (left/right) Haar measure defined on them. See appendix for the Haar measure for $SIM(2)$. For more details on locally compact groups see [20, Chap. 2].

$\psi \in L^2(\mathbb{R}^d)$ is called an **admissible wavelet** if

$$0 < M_\psi := (2\pi)^{d/2} \int_T \left| \frac{\mathcal{F}[R_t \psi]}{\sqrt{\text{det} T(t)}} \right|^2 d\mu_T(t) < \infty \text{ a.e. on } \mathbb{R}^d,$$

(2.18)

where $R_t$ is defined in (2.4) and $d\mu_T(t)$ is the left-invariant Haar measure of $T$.

We define $\tilde{\psi}$ almost everywhere on $\mathbb{R}^d$ by

$$\tilde{\psi}(x) = \int_T \frac{1}{\text{det} T(t)} (R_t \psi \ast R_t \psi)(x) d\mu_T(t),$$

(2.19)

where $\tilde{\psi}$ is defined in (2.4). Note that $\tilde{\psi}$ is the inverse Fourier transform of $M_\psi$.

The proof of the following lemma is very informative and is presented in the appendix.

**Lemma 2.6.** Let $\psi$ be an admissible wavelet. Then the span of $V_\psi = \{R_g \psi | g \in G\}$, is dense in $L^2(\mathbb{R}^d)$, i.e. $\langle V_\psi \rangle = L^2(\mathbb{R}^d)$.

**Corollary 2.7.** If the wavelet $\psi$ is admissible, then the corresponding wavelet transform $W_\psi : L^2(\mathbb{R}^d) \rightarrow C_K^G$ is unitary.

**Proof.** Follows from Theorem 2.2 and Lemma 2.6. \hfill \square

Define the linear operator $T_{M_\psi}$ on $C_K^G$ by

$$[T_{M_\psi}[\phi]](b, t) = F^{-1} \left[ \omega \mapsto (2\pi)^{-d/4} M_\psi^{-1/2}(\omega) \mathcal{F}[\phi(\cdot, t)](\omega) \right](b).$$

Operator $T_{M_\psi}$ is well defined since $M_\psi > 0$ a.e. on $\mathbb{R}^d$ and $M_\psi^{-1/2} \mathcal{F}[\phi(\cdot, t)] \in L^2(\mathbb{R}^d)$ for all $t \in T$. 
Theorem 2.8. Let $G = \mathbb{R}^d \rtimes T$. Let $\psi$ be an admissible wavelet. Then $T_{M, \psi} \in L_2(G, d\mu_G(g))$ for all $\psi \in C^G_K$. Therefore $(\cdot, \cdot)_{T_{M, \psi}} : C^G_K \times C^G_K \rightarrow \mathbb{C}$ defined by
\begin{equation}
(\Phi, \Psi)_{T_{M, \psi}} = (T_{M, \psi}[\Phi], T_{M, \psi}[\Psi])_{L_2(G)},
\end{equation}
is an explicit characterization of the inner product on $C^G_K$, which is the unique functional Hilbert space with reproducing kernel $K : G \times G \rightarrow \mathbb{C}$ given by
\begin{equation}
K(g, h) = \langle U_g \psi, U_h \psi \rangle_{L^2(\mathbb{R}^d)} = \langle U_{h^{-1}g} \psi, \psi \rangle_{L^2(\mathbb{R}^d)}, \quad g, h \in G.
\end{equation}

The wavelet transformation $\mathcal{W}_\psi$ defined by
\begin{equation}
\mathcal{W}_\psi[f](b, t) = \langle U_b \psi, f \rangle_{L^2(\mathbb{R}^d)}, \quad f \in L_2(\mathbb{R}^d), \quad g = (b, t) \in \mathbb{R}^d \rtimes T,
\end{equation}
is a unitary mapping from $L_2(\mathbb{R}^d)$ to $C^G_K$. The space $C^G_K$ is a closed subspace of the Hilbert space $H_\psi \otimes L_2(T; \frac{d\mu_T(t)}{|det(\tau(t))|})$, where $H_\psi = \{ f \in L_2(\mathbb{R}^d) | M_\psi^{-\frac{1}{2}} f \in L_2(\mathbb{R}^d) \}$ is equipped with the inner product
\begin{equation}
\langle f, g \rangle = \langle M_\psi^{-\frac{1}{2}} f, M_\psi^{-\frac{1}{2}} g \rangle_{L_2(\mathbb{R}^d; (2\pi)^{-d/2}dx)}.
\end{equation}
The orthogonal projection $\mathcal{P}_\psi$ of $H_\psi \otimes L_2(T; \frac{d\mu_T(t)}{|det(\tau(t))|})$ onto $C^G_K$ is given by $(\mathcal{P}_\psi[\phi])(g) = (K(\cdot, g, \Phi))_{M_\psi}$.

Proof. See [11, Pg. 27-30]. \hfill \square

Remarks:

1. Since $\mathcal{W}_\psi : L_2(\mathbb{R}^d) \rightarrow C^G_K$ is unitary, the inverse equals the adjoint and thus the image $f$ can be reconstructed from it’s orientation score $\mathcal{W}_\psi[f]$ by
\begin{equation}
f \ = \ \mathcal{W}_\psi^*[\mathcal{W}_\psi[f]] = \mathcal{F}^{-1} \left[ \frac{1}{(2\pi)^{d/2}} \int_T \mathcal{F}[\mathcal{W}_\psi[f](\omega)]d\mu_T(\omega) \mathcal{F}[\mathcal{R}_t\psi](\omega) \frac{d\mu_T}{|det(\tau(t))|} M^{-1}_\psi \right].
\end{equation}
First we prove the following lemma.

Lemma 2.9. Let $\psi \in L_2(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$. Then $(\mathcal{W}_\psi f)(\cdot, t) \in L_2(\mathbb{R}^d)$ for all $f \in L_2(\mathbb{R}^d)$ and $t \in T$.

Proof. Let $f \in L_2(\mathbb{R}^d)$ and $t \in T$. Then
\begin{equation}
(\mathcal{T}_x \mathcal{R}_t \psi, f)_{L_2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (\mathcal{R}_t \psi)(x' - x)f(x')dx' = \mathcal{R}_t \psi \ast_{\mathbb{R}^d} f,
\end{equation}
for all $x'$, where recall the definition of $\mathcal{R}_t \psi$ from (2.4). Thus we arrive at a convolution of a $L^1(\mathbb{R}^2)$ and a $L^2(\mathbb{R}^2)$ function, which is again a $L^2(\mathbb{R}^2)$ function. \hfill \square

This means that for all elements $\Phi \in C^G_K$ the function $\Phi(\cdot, t)$ belongs to $L^2(\mathbb{R}^2)$ for fixed $t \in T$. Hence the Fourier transform of $\Phi(\cdot, t)$ is well defined.

We prove the remark based on the following description of the adjoint map,
\begin{equation}
(\Phi, \mathcal{W}_\psi f)_{C^G_K} = (\mathcal{W}_\psi^* \Phi, f)_{L_2(\mathbb{R}^d)}.
\end{equation}
The wavelet transform can be written as,
\[
(W_\psi f)(x, t) = (T_x R_t \psi, f)_{L_2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} R_t \psi(x - x') f(x') dx'
\]
\[
= (R_t \psi *_{\mathbb{R}^2} f) = \mathcal{F}^{-1}(\mathcal{F} R_t \mathcal{F} f).
\]
(2.26)
Thus using the $M_\psi$ inner product defined earlier and (2.26) we have,
\[
(\Phi, W_\psi f)_{C_G^2} = \int_{\mathbb{R}^d} \int_{\mathbb{T}} \mathcal{F}[\Phi(t)][(\omega)] \mathcal{F}[W_\psi f(t)][(\omega)] (2\pi)^{-d/2} M_\psi^{-1}(\omega) \frac{d\mu_T(t)}{|\det(\tau(t))|} d\omega
\]
\[
= \int_{\mathbb{R}^d} \int_{\mathbb{T}} \mathcal{F}[\Phi(t)][(\omega)] \mathcal{F}[R_t \psi](\omega) \mathcal{F}[f(\omega)] (2\pi)^{-d/2} M_\psi^{-1}(\omega) \frac{d\mu_T(t)}{|\det(\tau(t))|} d\omega.
\]
(2.27)
Further expanding the R.H.S of (2.25) we have,
\[
(W_\psi^* \Phi, f)_{L_2(\mathbb{R}^d)} = (F[W_\psi^* \Phi], F f)_{L_2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \mathcal{F}[W_\psi^* \Phi](\omega) \mathcal{F}[f(\omega)] d\omega.
\]
(2.28)
Thus comparing (2.27) and (2.28), we arrive at the exact reconstruction formula,
\[
f = W_\psi^* [W_\psi[f]]
\]
\[
= \mathcal{F}^{-1} \left[ \omega \mapsto \frac{1}{(2\pi)^{d/2}} \int_T \mathcal{F}[W_\psi[f](t)][(\omega)] \mathcal{F}[R_t \psi](\omega) \frac{d\mu_T(t)}{|\det(\tau(t))|} M_\psi^{-1}(\omega) \right].
\]

2. If $M_\psi = 1$ on $\mathbb{R}^2$, then $C_G^2$ is a closed subspace of $L_2(G)$.

**Definition 2.10.** The inner product on $H_\psi \otimes L_2(T; \frac{d\mu_T(t)}{|\det(\tau(t))|})$ induces a norm $\| \|_{M_\psi} : H_\psi \otimes L_2(T; \frac{d\mu_T(t)}{|\det(\tau(t))|}) \to \mathbb{R}^+$, which is given by
\[
\| \Phi \|_{M_\psi} = \sqrt{(\Phi, \Phi)_{M_\psi}} = \int_{\mathbb{R}^d} \int_T |\mathcal{F}[\Phi(t)][(\omega)]|^2 (2\pi)^{-d/2} M_\psi^{-1}(\omega) \frac{d\mu_T(t)}{|\det(\tau(t))|} d\omega,
\]
which is called the $M_\psi$-norm.

In our specific case of $G = SIM(2)$ and $H = L_2(\mathbb{R}^2)$, where note that $\mathbb{R}^2$ is a locally compact abelian group. Also let $T := SO(2) \times \mathbb{R}^+$, which is a locally compact group. Recall that, $SIM(2) = \mathbb{R}^2 \rtimes_r (SO(2) \times \mathbb{R}^+)$ \footnote{We drop the subscript $r$ from the semi-direct product hereon for the ease of notation.} where the semi-direct product is defined to be the group with underlying set $\mathbb{R}^2 \times (SO(2) \times \mathbb{R}^+)$ and group operation,
\[
(x, a, \theta)(x', a', \theta') = (x + \tau(a, \theta)x', aa', \theta + \theta'), \quad \forall (x, a, \theta), (x', a', \theta') \in SIM(2),
\]
(2.29)
where $\tau(a, \theta) = aR_\theta$ \footnote{$R_\theta$ is the standard counter-clockwise rotation matrix.}.

Since both $\mathbb{R}^2$ and $SO(2) \times \mathbb{R}^+$ are locally compact groups, $SIM(2)$ is a locally compact group as well and thus has a left-invariant Haar measure defined on it. As mentioned in Lemma 2.5 there exists a unitary representation of $SIM(2)$ in $L_2(\mathbb{R}^2)$ given by,
\[
U_{y = (b, a, \theta)} \psi(x) = \frac{1}{a} \psi \left( \frac{R_\theta^{-1}(x - b)}{a} \right), \quad a > 0, \quad \theta \in [0, 2\pi], \quad b \in \mathbb{R}^2,
\]
(2.30)
We denote \( U : (x, t) = (x, a, \theta) \mapsto U(x, t) \) as
\[
U_{x,t}f = T_{x}R_{t}f, \quad t = (a, \theta)
\]  
(2.31)
where \((T_{x}f)(x') = f(x' - x)\) and \((R_{t}f)(x') = \frac{1}{2}f(\frac{1}{a}R_{\theta}x')\).

Given below is the interpretation of Theorem 2.8 in our context.

**Corollary 2.11.** The space of orientation scores is a reproducing kernel Hilbert space \( \mathbb{C}_{K}^{SIM(2)} \) which is a closed subspace of \( \mathbb{H}_{\psi} \otimes L_{2}(SO(2) \times \mathbb{R}^{+}) \) which is a vector subspace\(^5\) of \( L_{2}(G) \). The inner product on \( \mathbb{C}_{K}^{\mathbb{R}^{2} \times (SO(2) \times \mathbb{R}^{+})} \) is given by (2.20) and is explicitly characterized by means of the function \( M_{\psi} \) given by,
\[
M_{\psi}(\omega) = (2\pi) \int_{T=SO(2) \times \mathbb{R}^{+}} \left| \frac{\mathcal{F}[R_{\psi}](\omega)}{\sqrt{det(t)}} \right|^{2} dt.
\]  
(2.32)

The wavelet transform which maps an image \( f \in L_{2}(\mathbb{R}^{2}) \) onto its orientation score \( U_{f} \in \mathbb{C}_{K}^{SIM(2)} \) is a unitary mapping: \( \|f\|_{L_{2}(\mathbb{R}^{2})}^{2} = \|U_{f}\|_{M_{\psi}}^{2} \). Thus the image \( f \) can be reconstructed from its orientation score \( U_{f} = W_{\psi}[f] \) by means of the adjoint wavelet transformation \( W_{\psi}^{*} \):
\[
f = W_{\psi}^{*}[W_{\psi}[f]] = \mathcal{F}^{-1} \left[ \omega \mapsto \frac{1}{(2\pi)^{2}} \int_{\mathbb{R}^{+}} \int_{\mathbb{R}^{2}} \mathcal{F}[U_{f}(\cdot, a, e^{i\theta})](\omega) \mathcal{F}[R_{a,e^{i\theta}\psi}](\omega) d\theta \frac{da}{a} M_{\psi}^{-1}(\omega) \right].
\]  
(2.33)

Explicitly \( M_{\psi} \) as defined in (2.32) can be written as,
\[
M_{\psi}(\omega) = (2\pi) \int_{T} \left| \frac{\mathcal{F}[R_{\psi}](\omega)}{\sqrt{det(t)}} \right|^{2} dt = (2\pi) \int_{0}^{2\pi} \int_{\mathbb{R}^{+}} \left| \frac{\mathcal{F}[R_{(a, \theta)}](\omega)}{a} \right|^{2} da d\theta
\]  
(2.34)
\[
= (2\pi) \int_{0}^{2\pi} \int_{\mathbb{R}^{+}} \left| a^{2} \psi(aR_{\theta}^{-1}\omega) \right|^{2} da d\theta = 2\pi \int_{0}^{2\pi} \int_{\mathbb{R}^{+}} \left| \psi(aR_{\theta}^{-1}\omega) \right|^{2} da d\theta
\]  
(2.35)
for all \( t = (a, \theta) \in SO(2) \times \mathbb{R}^{+} \). Note that in the last equality we have made use of the substitution \( \tau = \log_{e}(a) \). Irrespective of the choice of wavelet \( \psi \), \( M_{\psi}|_{\omega=0} = 0 \) or \( \infty \), as the stabilizer of \( \omega = 0 \) is not compact. But this is an issue (dealt with in the next section) as the admissibility condition requires that \( M_{\psi} \) have a non-zero bounded value for almost every \( \omega \in \mathbb{R}^{2} \).

### 2.5 The Discrete Analogue

Hereon in this chapter we deal with the practical aspects of the implementation of the continuous wavelet transform discussed in the previous sections.

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\( ^{5} \)i.e. is a subspace as a vector space, but is equipped with a different norm
Recall that $SIM(2) = \mathbb{R}^2 \rtimes (SO(2) \times \mathbb{R}^+)$ where $T = (SO(2) \times \mathbb{R}^+)$ is a locally compact group, but not a compact group. We can choose $SO(2)$ to be finite rotation group, denoted by $\mathbb{T}_N$ (equipped with discrete topology) which is locally compact\(^6\) i.e.  
\[ T_N = \{ e^{i k \Delta} | k \in \{0, 1, \ldots, N - 1\}, \Delta_T = \frac{2 \pi}{N} \}, \text{ for } N \in \mathbb{N}. \]  
(2.36)

On the other hand the scaling group $\mathbb{R}^+$ cannot be written in terms of a finite scaling group due to the following well known result from group theory.

**Lemma 2.12.** Every finite subgroup of the multiplicative group of a field is a cyclic subgroup.

See [33] for proof. The only finite subgroups of the group $\mathbb{R}^+ = (\mathbb{R}/\{0\}, \ast)$ is the trivial subgroup $\{1\}$ and the 2nd order group generated by $-1, \{ -1, 1 \}$. The scaling group $\mathbb{R}^+$ is a subgroup of $\mathbb{R}^*$ and therefore it does not have any finite subgroups other than the trivial subgroup. So it is important to note that, we loose the inherent group structure in the discrete version, unlike in the case of orientation score over the $SE(2)$ group, see [12, Sec 4.4].

Consider the scaling group $\mathbb{R}^+$; this group consists of all positive reals greater than zero. In the discrete case we need to have a lower and an upper bound on the choice of the scales. We assume that $a \in [a^-, a^+]$ where $0 < a^- < a^+$. We have the following discretization for scales,

\[ \mathbb{D}_M = \left\{ e^{(\pi^+ - k \Delta_t)} | k \in \{0, 1, \ldots, M - 1\}, \Delta_D = \frac{\pi^+ - \pi^-}{M} \right\}, \text{ for } M \in \mathbb{N}, \]  
(2.37)

where $\pi^- = \log(a^-)$ and $\pi^+ = \log(a^+)$. Using the notation, $t_{kl} = (a_l, \theta_k), \ k \in \{0, 1, \ldots, N - 1\}, \ l \in \{0, 1, \ldots, M - 1\}$ where $a_l = \pi^+ - k \Delta_D$ and $\theta_k = k \Delta_T$, we write the discrete version of (2.31)

\[ U_f^{N,M}(b, a_l, \theta_k) = (T_b R_{(a_l, \theta_k)} \psi, f)_{L_2(\mathbb{R}^2)}, \]  
(2.38)

which is the discrete orientation score of an image $f \in L_2(\mathbb{R}^2)$. We emphasize that we do not consider a discrete subgroup of $SIM(2)$-group due to Lemma 2.12. The discrete version of $M_\psi$ is,

\[ M_\psi(\omega) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} \left| \frac{F(R_{(a_l, \theta_k)} \psi)(\omega)}{a_l} \right|^2. \]  
(2.39)

### 2.5.1 Stable reconstruction of image $f$ from its orientation score $U_f$

We assume that the support of the Fourier transform of our images is bounded by an annulus. The space of these images is a Hilbert space given by,

\[ L_2^{\theta^- \theta^+}(\mathbb{R}^2) = \{ f \in L_2(\mathbb{R}^2) | \text{ supp} (F[f]) \subset B_{0, \theta^+} \setminus B_{0, \theta^-} \}, \ \ \theta^+ > \theta^- > 0. \]  
(2.40)

The reason for the assumption of an upper bound $(\theta^+)$ on the support of the Fourier transform of the images is the Nyquist theorem, which states that every band-limited function is determined by its values on a discrete grid. For example if $u_B : \mathbb{R}^2 \to \mathbb{C}$ is band-limited on a square: $\text{ supp} (F[u_B]) \subset [-1/2, 1/2] \times [-1/2, 1/2]$, then

\[ u_B(x, y) = \sum_{(k_1, k_2) \in \mathbb{Z}^2} u_B(\frac{2 \pi k_1}{l}, \frac{2 \pi k_2}{l}) \text{sinc}(\frac{l x}{2} - k_1 \pi) \text{sinc}(\frac{l y}{2} - k_2 \pi), \]

where the cutoff frequency $\theta = l/2$ is called the Nyquist frequency.

\(^6\)Topological spaces with a discrete topology are locally compact.
Remark. From a practical point of view it does not make sense to store frequencies of order greater than 1/2 if the signal contains l samples, as Fourier transform of a discrete image \( f : \{1, \ldots, n\} \times \{1, \ldots, n\} \rightarrow \mathbb{C} \) is again of the form \( \mathcal{F} f : \{1, \ldots, n\} \times \{1, \ldots, n\} \rightarrow \mathbb{C} \). Thus a finite width puts an upper bound on the domain of the Fourier transform.

The assumption of the lower bound \( \varrho^- \) on the Fourier transform has a theoretical as well as a practical underpinning. Theoretically it allows us to avoid the origin where any choice of wavelet \( \psi \) fails to satisfy the admissibility condition. The value for \( \varrho^- \) directly relates to the coarsest scale we wish to detect in the spatial domain. Therefore the removal of extremely low frequencies from the image essentially corresponds to background removal in the image which is often an essential pre-processing step in medical image processing, see [6, 42].

We wish to construct a wavelet transform

\[
\mathcal{W}_\psi^\varrho^- : L_2^\varrho^-(\mathbb{R}^2) \rightarrow \mathbb{L}_2(SIM(2))
\]

which requires that,

\[
\mathcal{U}_{a,\varrho}(\psi) \in L_2^\varrho^-(\mathbb{R}^2), \quad a \in [a^-, a^+] \text{ and } \varrho \in [0, 2\pi].
\]

Therefore we choose

\[
\psi \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2) \text{ with } \text{supp}(\mathcal{F}[\psi]) \subset B_{0, \varrho^+} \setminus B_{0, \varrho^-}. \quad (2.42)
\]

and therefore we satisfy \( \text{supp}(\mathcal{F}[\mathcal{U}_{a,\varrho}(\psi)]) \subset B_{0, \varrho^+} \setminus B_{0, \varrho^-}, \) where \( a \in [a^-, a^+] \) and \( \varrho \in [0, 2\pi]. \)

Note that in our current context, \( \psi \in L_2^\varrho^-\varrho^+(\mathbb{R}^2) \) is called an admissible wavelet if

\[
0 < M_\psi = (2\pi) \int_0^{2\pi} a^+ \int_0^{2\pi} a^- \frac{\det(t)}{\mathcal{F}[\mathcal{R}_{a,\varrho}(\psi)]} \frac{d\varrho}{a} < \infty \text{ on } B_{0, \varrho^+} \setminus B_{0, \varrho^-}, \quad (2.43)
\]

where \( \mathcal{R}_t \) is given by Equation (2.4). Corresponding to (2.19) we can define \( \tilde{\psi} \) as,

\[
\tilde{\psi}(x) = \int_0^{2\pi} a^+ \int_0^{2\pi} a^- (\mathcal{R}_{a,\varrho}(\tilde{\psi})(x) \frac{d\varrho}{a} d\varrho. \quad (2.44)
\]

By the compactness of \( SO(2) \times [a^-, a^+] \) and since the convolution of two \( L_1(\mathbb{R}^2) \) functions is again in \( L_1(\mathbb{R}^2) \), we have that \( \psi \in L_1(\mathbb{R}^2) \), and so its Fourier transform is a continuous function on \( B_{0, \varrho^+} \setminus B_{0, \varrho^-} \).

Definition 2.13. Let \( \psi \) be an admissible wavelet in the sense of (2.43). Then the wavelet transform \( \mathcal{W}_\psi^\varrho^- : L_2^\varrho^-\varrho^+(\mathbb{R}^2) \rightarrow \mathbb{L}_2(SIM(2)) \) is given by

\[
(\mathcal{W}_\psi^\varrho^-[f])(g) = (\mathcal{U}_g \psi, f)_{L_2(\mathbb{R}^2)}, \quad f \in L_2^\varrho^-\varrho^+(\mathbb{R}^2),
\]

for almost every \( g = (x, a, \varrho) \in \mathbb{R}^2 \times (SO(2) \times [a^-, a^+]). \)

2.5.2 Quantification of Stability

The usual way to quantify well-posedness/stability of an invertible linear transformation \( A : V \rightarrow W \) from a normed space \( (V, \| \cdot \|_V) \) to a normed space \( (W, \| \cdot \|_W) \) is by means of the condition number

\[
\text{cond}(A) = \| A^{-1} \| \| A \| = \left( \sup_{x \in V} \frac{\| x \|_V}{\| Ax \|_W} \right) \left( \sup_{x \in V} \frac{\| Ax \|_W}{\| x \|_V} \right) \geq 1. \quad (2.45)
\]
The closer it approximates 1, the more stable the operator and its inverse is. Note that the condition number depends on the norms imposed on $V$ and $W$. We wish to apply this general concept to the wavelet transformation which maps image $f$ to its scale orientation score $U_f$.

Recall from the previous section that the wavelet transform is a unitary mapping from the space $L_2(\mathbb{R}^2)$ to the space $C_K^{SIM(2)}$ respectively equipped with the $L_2$-norm and the $M_\omega$-norm. Thus choosing these norms the condition number becomes 1. However from a practical point of view it is more appropriate to impose the $L_2(SIM(2))$-norm on the scale orientation score since it does not depend on the choice of the wavelet $\psi$ and since we also employ a $L_2$-norm on the space of images.

**Theorem 2.14.** Let $\psi$ be an admissible wavelet, with $M_\psi(\omega) > 0$ for all $\omega \in B_{0,\varrho^+} \setminus B_{0,\varrho^-}$. Then the condition number $\text{cond}(W_\psi^{\varrho^-\cdot\varrho^+})$ of the wavelet transformation $W_\psi^{\varrho^-\cdot\varrho^+} : \mathbb{R}^2 \to L_2$, $(G = SIM(2))^7$ is defined by

$$\text{cond}(W_\psi^{\varrho^-\cdot\varrho^+}) = \|(W_\psi^{\varrho^-\cdot\varrho^+})^{-1}\| \left( \sup_{f \in \mathbb{L}_2} \frac{\|f\|_{L_2}}{\|U_f\|_{L_2}} \right) \left( \sup_{f \in \mathbb{L}_2} \frac{\|U_f\|_{L_2}}{\|f\|_{L_2}} \right)$$

and satisfies

$$1 \leq \left( \text{cond}(W_\psi^{\varrho^-\cdot\varrho^+}) \right)^2 \leq \left( \sup_{\varrho^- \leq ||\omega|| \leq \varrho^+} M_\psi^{-1}(\omega) \right) \left( \sup_{\varrho^- \leq ||\omega|| \leq \varrho^+} M_\psi(\omega) \right).$$

**Proof.** Since $M_\psi > 0$ and is continuous on the compact set $B_{0,\varrho^+} \setminus B_{0,\varrho^-} = \{ \omega \in \mathbb{R}^2 | \varrho^- < ||\omega|| < \varrho^+ \}$, $\sup_{\varrho^- < ||\omega|| < \varrho^+} M_\psi(\omega) = \max_{\omega \in B_{0,\varrho^+} \setminus B_{0,\varrho^-}} M_\psi(\omega)$ do exist. The same holds for $M_\psi^{-1}$ as well.

Further for all $f \in \mathbb{L}_2$, the restriction of the corresponding orientation scores, to fixed orientations and scales also belong to the same space, i.e. $U_f(\cdot, \varrho^+, \varrho^-) \in \mathbb{L}_2$, where $\theta \in [0, 2\pi]$ and $a \in [a^-, a^+]$. This follows from (2.26) which gives, $\mathcal{F}[U_f](\omega) = \mathcal{F}[\mathcal{R}_{a,\varrho^+}\varrho^-)(\omega)\mathcal{F}[f](\omega)$. Recall from Corollary 2.11, that $\|f\|_{L_2} = \|U_f\|_{M_\psi}$ and thus we have,

$$(\text{cond}(W_\psi^{\varrho^-\cdot\varrho^+}))^2 \leq \left( \sup_{\varrho^- \leq ||\omega|| \leq \varrho^+} M_\psi^{-1}(\omega) \right) \left( \sup_{\varrho^- \leq ||\omega|| \leq \varrho^+} M_\psi(\omega) \right).$$

Further we have, $1 = \|(W_\psi^{\varrho^-\cdot\varrho^+})^{-1}(W_\psi^{\varrho^-\cdot\varrho^+})\| < \|(W_\psi^{\varrho^-\cdot\varrho^+})^{-1}\|(W_\psi^{\varrho^-\cdot\varrho^+})\|$. \hfill $\square$

**Corollary 2.15.** The stability of the (inverse) wavelet transformation $W_\psi^{\varrho^-\cdot\varrho^+} : \mathbb{L}_2^{\varrho^-\cdot\varrho^+} \to L_2(SIM(2))$ is optimal if $M_\psi(\omega) = \text{constant}$ for all $\omega \in \mathbb{R}$, with $\varrho^- \leq ||\omega|| \leq \varrho^+$.

Thus in general, the closer the function $M_\psi$ approximates the constant function, say $1_{B_{0,\varrho^+} \setminus B_{0,\varrho^-}}$, the better the $M_\psi$ norm on $C_K^{SIM(2)}$ approximates the $L_2(SIM(2))$ norm, the better the stability of reconstruction. In case of a good approximation, i.e. $M_\psi \approx 1_{B_{0,\varrho^+} \setminus B_{0,\varrho^-}}$, the reconstruction formula in Corollary 2.11 can be simplified to,

$$f \approx \mathcal{F}^{-1} \left[ \omega \mapsto \frac{1}{(2\pi)^2} \int_{0}^{2\pi} \int_{\varrho^-}^{\varrho^+} \mathcal{F}[U_f(\cdot, a, e^{i\theta})(\omega)] \mathcal{F}[\mathcal{R}_{a,\varrho^+}\varrho^-)(\omega)]d\theta\frac{d\omega}{\varrho} \right].$$

(2.46)

7Note that while the norm is $L_2(SIM(2))$-norm we only consider an interval in the scaling group.
2.6 Design of proper wavelets

In this sequel a wavelet \( \psi \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \) with \( M_\psi \) smoothly approximating \( 1_{B_{n,e^+}} \setminus B_{n,e^-} \), is called a proper wavelet. The entire class of proper wavelets allows for a lot of freedom in the choice of \( \psi \). In practice it is mostly sufficient to consider wavelets that are similar to the long elongated patch one would like to detect and orthogonal to structures of local patches which should not be detected, in other words employing the basic principle of template matching. We restrict the possible choices by listing below certain practical requirements to be fulfilled by our transform.

1. The wavelet transform should yield a finite number of orientations \( (N_0) \) and scales \( (M_0) \). This requirement is obvious from an implementation point of view.
2. The wavelet should be strongly directional, in order to obtain sharp responses on oriented structures.
3. The transformation should handle lines, contours and oriented patterns. Thus the wavelet should pick up edge, ridge and periodic profiles.
4. In order to pick up local structures, the wavelet should be localized in spatial domain.

To ensure that the wavelet is strongly directional, we require that the support of the wavelet be contained in a convex cone in the Fourier domain, \([2]\).

The following lemma gives a simple but practical approach to obtain proper wavelets \( \psi \), with \( M_\psi(\omega) = 1_{B_{0,e^+}} \setminus B_{0,e^-} \). Note that we make use of polar coordinates \( (\rho, \varphi) \), \( \rho = \| \omega \| \), \( \omega = (\rho \cos \varphi, \rho \cos \varphi) \).

**Lemma 2.16.** Let \( \tau^-, \tau^+ \) be chosen such that \( \tau^- = \log(a^-) \) and \( \tau^+ = \log(a^+) \), where \( 0 < a \in [a^-, a^+] \) is the finite interval of scaling. Let \( A : SO(2) \to \mathbb{C} \setminus \mathbb{R}^- \) and \( B : [\tau^-, \tau^+] \to \mathbb{C} \setminus \mathbb{R}^- \) such that

\[
2\pi \int_0^{\tau^+} |A(\rho)| \, d\rho = 1, \quad \int_{\tau^-}^{\tau^+} |B(\rho)| \, d\rho = 1, \tag{2.47}
\]

then the wavelet \( \psi = F^{-1}[\omega \to \sqrt{A(\varphi)B(\rho)}] \) has \( M_\psi(\omega) = 1 \) for all \( \omega \in B_{0,e^+} \setminus B_{0,e^-} \).

**Proof.** From (2.35) and (2.43), for all \( \omega \in B_{0,e^+} \setminus B_{0,e^-} \) we have,

\[
M_\psi(\omega) = 2\pi \int_0^{\tau^+} \int_0^e |\hat{\psi}(e^\tau R_{\varphi}^{-1})\omega|^2 \, d\tau \, d\theta = 2\pi \int_0^{\tau^+} \int_0^e |\sqrt{A(\varphi)}B(e^\tau \rho)|^2 \, d\tau \, d\theta = 1,
\]

for all \( \omega \in B_{0,e^+} \setminus B_{0,e^-} \). \( \square \)

Lemma 2.16 can be translated into discrete framework \( \mathbb{R}^2 \ltimes (\mathbb{T}_N \times \mathbb{D}_M) \), recall (2.36) and (2.37), where condition (2.47) is replaced respectively by,

\[
\frac{1}{N} \sum_{k=0}^{N-1} |A(\varphi - \theta_k)| = 1 \quad \text{and} \quad \frac{1}{M} \sum_{l=0}^{M-1} |B(e^{\alpha_l} \rho)| = 1. \tag{2.48}
\]

where we have made use of discrete notations introduced in (2.38).
If moreover \(2\pi \int_0^{2\pi} \sqrt{|A(\phi)|}d\phi \approx 1\) and \(\int |B(\rho)|dp \approx 1\), we have a fast and simple approximation for the reconstruction:

\[
\tilde{f}(x) = 2\pi \int_0^{2\pi} \int_{\tau}^{\tau^+} (W_\psi f)(x, \tau, \theta)d\tau d\theta \approx \mathcal{F}^{-1}[\omega \mapsto \sqrt{M_\psi * \mathcal{F}[f](\omega)}](x), \text{ for a.e. } \omega \in B_{0,e^+} \setminus B_{0,e^-}.
\]

Note that we have made use of the description of \((W_\psi f)\) given in (2.26). We need to fulfil the requirement \(M_\psi(\omega) \approx 1\) with an appropriate choice of kernel satisfying the condition given above, to achieve this simple reconstruction.

The idea is to “fill the cake by pieces of the cake” in the Fourier domain. In order to avoid high frequencies in the spatial domain, these pieces must be smooth and thereby must overlap. A choice of B-spline based functions in the angular and the radial direction is an appropriate choice for such a wavelet kernel.

The \(k\)th order B-spline denoted by \(B^k\) is defined as,

\[
B^k(x) = (B^{k-1} * B^0)(x), \quad B^0(x) = \begin{cases} 1 & \text{if } -1/2 < x < +1/2 \\ 0 & \text{otherwise} \end{cases}
\]

B-splines have the property that they add up to 1. For more details see [38]. Based on the requirements and considerations above we propose the following kernel

\[
\psi(x) = \mathcal{F}^{-1}_{\mathbb{R}^2}[\omega \mapsto \sqrt{A(\phi)B(\rho)}](x)G_\sigma(x),
\]

where \(G_\sigma\) is a Gaussian window that enforces spatial locality cf. requirement 5.

\[A : SO(2) \to \mathbb{R}^+, \text{ is defined by,}
\]

\[
A(\varphi) = B^k \left( \frac{(\varphi \mod 2\pi) - \pi/2}{s_{\varphi}} \right),
\]

where \(s_{\varphi} = \frac{2\pi}{N_\theta}\) (\(N_\theta\) denotes the number of orientations chosen) and \(B^k\) denotes the \(k\)th order B-spline.

\[B : [a^-, a^+] \to \mathbb{R}^+, \text{ is defined as,}
\]

\[
B(\rho) = \sum_{l=0}^{N_s-1} B^k \left( \frac{\log[\rho]}{s_\rho} - l \right),
\]

where \(s_\rho = (\log[a^+] - \log[a^-])/N_s\), with \(N_s\) equals the number of chosen scales and \(a^-\), \(a^+\) are predefined scales, based on \(a^-\), \(a^+\) respectively. The motivation for \(\log\) based spline construction \((B(\rho))\) for the scaling group will become clear in the next chapter. See Figure 2.1 for a plot of these B-splines. It is easily checked that,

\[
\sum_{l=0}^{N_s-1} B^k \left( \frac{\log[\rho]}{s_\rho} - l \right) = 1 \quad \text{and} \quad \sum_{l=0}^{N_s-1} \sqrt{B^k \left( \frac{\log[\rho]}{s_\rho} - l \right)} \approx 1.
\]

For the second approximation, see Figure 2.1. Similar results also follow for the radial kernel. Therefore for this choice of kernels we can make use of the simple reconstruction given in (2.49).
Design of proper wavelets

Figure 2.1: Plot for the B-splines given in (2.53). Values chosen: $a^- = \rho^- = 10^{-8}$, $a^+ = \rho^+ = 50$, $N_s = 7$. From Left to Right: Plot of each B-spline, $B^k\left(\frac{\log(\rho)}{s_\rho} - l\right)$. Observe that the B-splines are skewed which is due to non linear sampling in the scale dimension; Notice that $\sum_{l=0}^{N_s-1} B^k\left(\frac{\log(\rho)}{s_\rho} - l\right) = 1$; Plot of square root of the B-spline, i.e. $\sqrt{B^k\left(\frac{\log(\rho)}{s_\rho} - l\right)}$, $\sum_{l=0}^{N_s-1} \sqrt{B^k\left(\frac{\log(\rho)}{s_\rho} - l\right)} \approx 1$.

i.e. the image can be reconstructed from its scale-OS by adding over all scales and orientations. Moreover the real valued part of the kernel is appropriate for line detection and the imaginary part of the kernel is appropriate for edge detection. In the following pages we give various plots and figures for our choice of wavelet as well as the resulting Scale-OS of an image.

The structure of our kernels (Figure 2.2) in the Fourier domain looks very similar to curvelets in the Fourier domain as shown in Figure 1.3. Note that unlike the cake-kernels used in this study which are based on a group-theoretic approach, the curvelets are based on the discrete wavelet transform and do not involve a group theoretical structure, which we will exploit in designing enhancement operators later on, in Section 4.1.
Figure 2.2: Plots of the graph of $\psi = \mathcal{F}_{\mathbb{R}^2}^{-1}[\omega \rightarrow \sqrt{A(\varphi)B(\rho)}]$, with $A(\varphi)$, $B(\rho)$ given in (2.52) and (2.53) respectively determined by the discrete Fourier transform of $\omega \rightarrow \sqrt{A(\varphi)B(\rho)}$ sampled on a $100 \times 100$ equidistant grid. Values chosen: $N_s = 6$, $N_\theta = 12$. From top to bottom: Row1- (Spatial domain) Plots of the real part of kernels at a fixed orientation, but different scales. Row2- (Spatial domain) Plots of the real part of kernels at a fixed scale, but different orientations. Row3- (Spatial domain) Plots of the imaginary part of kernels at a fixed scale, but different orientations. Row4- (Frequency domain) Plots of the of kernels at a fixed orientation, but different scales. Row5- (Frequency domain) Plots of the of kernels at a fixed scale, but different orientations. Row6- (Frequency domain) Plots of sum over all orientations at a fixed scale.
Design of proper wavelets

Figure 2.3: From left to right: $M_\psi$; $\tilde{\psi} = F^{-1}[M_\psi]$; Sum of all rotated and translated kernels in Fourier domain; Sum of all rotated and translated kernels in Spatial domain.

Figure 2.4: Illustration of invertible nature of wavelet transform on a section of Retinal Fundus image (200 × 200 pixels). Left: Original image; Right: Result of approximative reconstruction by only adding over all scales ($N_s = 6$) and orientations ($N_\theta = 12$) of scale-OS (2.49). As expected the fast approximative reconstruction is very close to exact reconstruction.

Figure 2.5: Reconstruction per scale by summation over all orientations at a fixed scale; from left to right the first scale (low detail) to the last scale (high detail).

Figure 2.6: Scale-OS corresponding to different scales at a fixed orientation, i.e. following the same direction.
Figure 2.7: Scale-OS of the Retinal Fundus image. Rows: Fixed orientation; Columns: Fixed scale
Chapter 3

The Similitude Group

In this chapter we introduce the differential-geometric structure of the SIM(2) group which is essential for designing left-invariant enhancement-contextual operators on the scale-OS. This is followed by introducing SIM(2)-convolution which is required for linear operators on scale-OS.

3.1 Geometry of the SIM(2) group

To perform analysis on the local structure in scale orientation scores, for instance in detecting the presence of an oriented structure locally, differential geometry is an essential tool. In this section we establish the basics of differential geometry on SIM(2). We briefly introduce Lie groups and Lie Algebras followed by applying this general setting to the specific case of the SIM(2) group. Group properties as well as the concept of left-invariant vectors and vector fields is defined and the notion of exponential and logarithmic mapping is introduced. For an analogous study in the case of Euclidean motion group SE(2), see [12, 21].

3.1.1 Lie Groups and Lie Algebras

A Lie Group $G$ is an infinite group, such that the group elements are parametrized by a finite dimensional differentiable manifold endowed with a metric, i.e. there is a notion of distance between group elements. The fact that group elements form a manifold makes it possible to use differential calculus on Lie groups.

Formally, a (real/complex) Lie group is a set $G$ such that,

(a) $G$ is a group.
(b) $G$ is a finite dimensional (real/complex) smooth manifold.
(c) The group multiplication in (a) of the product manifold

$$\mu : G \times G \to G : (x, y) \to xy$$

and the group inversion operation in (a)

$$\iota : G \times G : x \to x^{-1}$$

are analytic functions relative to the structure in (b).
By looking at the differential properties at the unity element $e$ of $G$ we obtain the corresponding Lie Algebra denoted by $T_e(G)$, which is a vector space endowed with a binary operator called the Lie bracket or commutator, $[\cdot, \cdot] : T_e(G) \times T_e(G) \to T_e(G)$ with the following properties:


(b) Anticommutativity: $[X, Y] = -[Y, X]$ for all $X, Y \in T_e(G)$.

(c) Jacobi Identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in T_e(G)$.

The Lie group and the Lie algebra relate to each other by an exponential map, $exp : T_e(G) \to G$, which can be defined in following equivalent ways:

(a) If $G$ is a ($N$-dimensional) matrix Lie Group\(^1\), then both $g \in G$ and $X \in T_e(G)$ can be expressed as $N \times N$ square matrices $G \in \mathbb{C}^{N\times N}$ and $X \in \mathbb{C}^{N\times N}$ respectively, and the exponential map coincides with the matrix exponential and is given by the ordinary series expansion,

$$G = exp(X) = \text{I}_{N \times N} + \sum_{n=1}^{\infty} \frac{X^n}{n!},$$  \hspace{1cm} (3.1)

where $\text{I}_{N \times N}$ is the $N$-dimensional identity matrix.

(b) Let $X \in T_e(G)$. Then, $exp(X) = \gamma(1)$, where

$$\gamma : \mathbb{R} \to G$$  \hspace{1cm} (3.2)

is the unique one parameter subgroup of $G$ whose tangent vector at the identity is equal to $X$. It follows from the chain rule that, $exp(tX) = \gamma(t)$. The map $\gamma$ can be constructed as the integral curve of the left invariant vector fields associated with $X$. Note that, the integral curve exists for all real parameters follows from left translating the solution near zero.

The matrix representations of Lie groups are often employed, as they simplify computations. For instance, the Lie bracket is found by,

$$[X_i, X_j] = X_iX_j - X_jX_i.$$  \hspace{1cm} (3.3)

A set of Lie algebra elements $A_i$ (or the corresponding matrix representation $X_i$), $i \in \{1, 2, \ldots, N\}$, only form a valid $N$-dimensional basis for a Lie algebra if they are closed under the Lie bracket, i.e. one can write

$$[A_i, A_j] = \sum_{k=1}^{N} c_{ij}^k A_k, \ \forall \ j \in \{1, 2, \ldots, N\},$$  \hspace{1cm} (3.4)

where $c_{ij}^k$ are the structure constants. The table of Lie brackets $[A_i, A_j]$, for all $i, j \in \{1, 2, \ldots, N\}$ completely describes the structure of the Lie algebra. If the exponential mapping is surjective, it also completely captures the structure of the Lie group. Two Lie algebras are isomorphic if the table of Lie brackets is the same.

For more details on Lie groups and Lie algebras see [32]. In the rest of this chapter we consider the particular case of 2D similitude group, $SIM(2)$ and clarify the concepts mentioned in this section in the context of $SIM(2)$.

\(^1\)Ado’s theorem states that every finite dimensional Lie algebra is isomorphic to a matrix Lie algebra. For every finite dimensional matrix Lie algebra, there is a linear group (matrix Lie group) with this algebra as its Lie algebra. So every abstract finite dimensional Lie algebra is the Lie algebra of some (linear) Lie group.
3.1.2 SIM(2) Group Properties

The similitude group, \( \text{SIM}(2) = \mathbb{R}^2 \rtimes (\mathbb{R}^+ \times \text{SO}(2)) \), is the group of planar rotations (\( \text{SO}(2) \)), translations (\( \mathbb{R}^2 \)) and scaling (\( \mathbb{R}^+ \)). It can be parametrized by the group elements \( g = (x, a, \theta) \) where \( x = (x, y) \in \mathbb{R}^2 \) are the two spatial variables that in the case of scale orientation scores label the domain of the image \( f \), \( a \) is the scale and \( \theta \) is the orientation angle. Group elements will be denoted by \( g \) or the explicit notation \( (x, a, \theta) \).²

For \( g = (x, a, \theta) \), \( g' = (x', a', \theta') \in \text{SIM}(2) \) the group product \( g \cdot \text{SIM}(2) g' \) is given by³:

\[
g \cdot \text{SIM}(2) g' = (x, a, \theta) \cdot \text{SIM}(2) (x', a', \theta') = (x + aR \vec{x}', a \theta + \theta'),
\]

\[
g^{-1} = (-a^{-1}R_{\cdot \theta}x, a^{-1}, -\theta).
\]

The identity element is given by \( e_{\text{SIM}(2)} = (0, 0, 1, 0)^4 \). Note that the \( \text{SIM}(2) \) group is not commutative, i.e. in general, \( gg' \neq g'g \). This is due to the presence of the semi-direct product, denoted by the symbol "\( \rtimes \)", between \( \mathbb{R}^2 \) and \( (\mathbb{R}^+ \times \text{SO}(2)) \).

The Lie algebra of \( \mathfrak{g} \) of \( \text{SIM}(2) \) is spanned by \( \mathfrak{t}_e(\text{SIM}(2)) = \text{span}\{A_2, A_3, A_4, A_1\} \equiv \text{span}\{e_x, e_y, e_a, e_\theta\} \),⁴ where \( e_x, e_y, e_a, \) and \( e_\theta \) are the unit vectors in the \( x, y, a \) and \( \theta \) direction respectively. The choice of index in the notation \( A_i \) will become clear in the subsequent chapters. \( \mathfrak{t}_e(G) \) is equipped with a Lie bracket given by

\[
[A, B] = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}(t^{-2}(a(t)b(t)a(t)^{-1}b(t)^{-1} - e),
\]

where \( a, b : \mathbb{R} \to \text{SIM}(2) \) are any smooth curves in \( \text{SIM}(2) \) with \( a(0) = b(0) = e \) and \( a'(0) = A \) and \( b'(0) = B \). For example, to find the Lie brackets of \( \text{SIM}(2) \) we can use for \( A_2 : a(t) = (t, 0, 1, 0), A_3 : a(t) = (0, t, 1, 0), A_4 : a(t) = (0, 0, t + 1, 0) \) and for \( A_1 : a(t) = (0, 0, 1, t) \), yielding the following Lie brackets

\[
\]

Therefore, the structure constants \( c_{ij}^k \), satisfying \( [A_i, A_j] = \sum_{k=1}^{N} c_{ij}^k A_k \), are given by,

\[
\begin{pmatrix}
0 & A_3 & -A_2 & 0 \\
-A_3 & 0 & 0 & -A_2 \\
A_2 & 0 & 0 & -A_3 \\
0 & A_2 & A_3 & 0
\end{pmatrix}.
\]

where \( i \) enumerates vertically and \( j \) enumerates horizontally.

Next we express the Lie algebra \( \mathfrak{t}_e(\text{SIM}(2)) \) by the isomorphism of the group \( \text{SIM}(2) \) to matrix group \( \mathfrak{g} \mathfrak{m}(2) \) of \( 3 \times 3 \) matrices,

\[
\text{SIM}(2) \ni (x, a, \theta) \leftrightarrow \begin{pmatrix}
ar \theta \\
0 \\
1
\end{pmatrix}
\in \mathfrak{g} \mathfrak{m}(2), \quad \text{where} \quad R_\theta = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]

²Throughout this thesis these two alternative notations will be used interchangeably.
³For the sake of simplicity of notation, hereon we drop \( \cdot \) to denote group product of a group \( G \).
⁴Here on for the ease of notation, the identity element of any group is denoted by \( e \) and the group involved can be understood from the context.
⁵Note that the Lie algebra \( \mathfrak{t}_e(G) \) of a Lie group \( G \) is also the tangent space at the identity element \( e \) of \( G \).
Then the basis of the Lie algebra are given by,

\[
X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

where \(\{1, 2, 3, 4\} := \{\theta, x, y, a\}\). For a derivation of these basis elements see Appendix(SecA.2).

Consequently the commutators are given using (3.3),

\[
[X_2, X_3] = 0, [X_2, X_4] = -X_2, [X_2, X_1] = -X_3,
[X_3, X_4] = -X_3, [X_3, X_1] = X_2, [X_4, X_1] = 0.
\]

Note that, as expected the Lie brackets for the matrix representations match the commutators given in (3.8).

### 3.1.3 Group Representations

On orientation scores \(U \in L_2(SIM(2))\) we have two different representations, defined by

\[
(L_g \circ U)(h) = (U \circ L_g^{-1})(h) = U(g^{-1}h), \quad g, h \in SIM(2), \quad (3.11)
\]

\[
(R_g \circ U)(h) = (U \circ R_g^{-1})(h) = U(hg), \quad g, h \in SIM(2), \quad (3.12)
\]

where \(L_g\) and \(R_g\) are the left and right regular representations respectively\(^6\). \(L_g, R_g : SIM(2) \rightarrow SIM(2)\), are defined as,

left-multiplication : \(L_g(h) = gh\), right-multiplication : \(R_g(h) = hg\).

A function \(\Phi \in L_2(SIM(2))\) is called left invariant if it commutes with the left regular representation, i.e. \(\Phi \circ L_g = L_g \circ \Phi\), for all \(g \in SIM(2)\).

### 3.1.4 Left invariant Vector Fields

Now we present the concept of left-invariant vector fields in the context of \(G = SIM(2)\). A vector field (now considered as a differential operator\(^7\)) \(A\) on a group \(G\) is called left-invariant if it satisfies

\[
A_g \phi = A_c(\phi \circ L_g) = A_c(h \mapsto \phi(gh)),
\]

for all smooth functions \(\phi \in C^\infty(\Omega_g)\) where \(\Omega_g\) is an open set around \(g \in G\) and with the left multiplication \(L_g : G \rightarrow G\) as defined in the previous section. The linear space of left-invariant vector fields \(\mathcal{L}(G)\) equipped with the Lie product \([A, B] = AB - BA\) is isomorphic to \(T_e(G)\) by means of the isomorphism,

\[
T_e(G) \ni A \leftrightarrow A \in \mathcal{L}(G) \leftrightarrow A_g(\phi) = A(\phi \circ L_g) = A(h \mapsto \phi(gh)) \equiv (L_g)_*A(\phi)
\]

for all smooth \(\phi : G \ni \Omega_g \rightarrow \mathbb{R}\).

---

\(^6\)Note that, \(L_g, R_g : SIM(2) \rightarrow L_2(SIM(2))\)

\(^7\)Note that any tangent vector \(X \in T(G)\), where \(T(G)\) is the tangent space of lie group \(G\) can be considered as a differential operator acting on a function \(U : G \rightarrow \mathbb{R}\). So, for instance, if we are using \(X_\epsilon \in T_e(G)\) in the context of differential operators, all occurrences of \(\epsilon\) will be replaced by \(\partial_\epsilon\), which is the short-hand notation for \(\frac{\partial}{\epsilon\partial\epsilon}\).
We define an operator $dR : T_e(G) \rightarrow \mathcal{L}(G)$ as,
\[
(dR(A)\phi)(g) = \lim_{t \to 0} \frac{(R_{\text{exp}(tA)}\phi)(g) - \phi(g)}{t}, \quad A \in T_e(G), \phi \in \mathcal{L}_2(G), g \in G
\]  \tag{3.14}
and where $R$ and $\text{exp}$ are the right regular representation and the exponential map (defined previously) respectively. Using $dR$, we obtain the corresponding basis for left-invariant vector fields on $G$:
\[
\{A_1, A_2, A_3, A_4\} := \{dR(A_1), dR(A_2), dR(A_3), dR(A_4)\}
\]  \tag{3.15}
or explicitly in coordinates
\[
\{A_1, A_2, A_3, A_4\} = \{\partial_y, \partial_x, \partial_\theta, \partial_\beta\} = \{\partial_y, a(\cos \theta \partial_x + \sin \theta \partial_\phi), a(- \sin \theta \partial_x + \cos \theta \partial_\phi), a \partial_a\},
\]  \tag{3.16}
where recall that as differential operators,
\[
\{A_1, A_2, A_3, A_4\} = \{\partial_\beta, \partial_x, \partial_\theta, \partial_a\}.
\]

For example, we have
\[
A_3 = (dR(A_3)\phi)(g) = \lim_{t \to 0} \frac{(R_{\text{exp}(tA)}\phi)(g) - \phi(g)}{t} = \lim_{t \to 0} \frac{\phi(g \cdot_G \text{exp}(tA_3)) - \phi(g)}{t} = \lim_{t \to 0} \frac{\phi(b + te_\xi, a, \theta) - \phi(b, a, \theta)}{t} = \partial_\xi,
\]
where, $g = (b, a, \theta) \in \text{SIM}(2)$, $\cdot_G$ denotes the $\text{SIM}(2)$ group product and $e_\xi = \left(\begin{array}{c} -a \sin \theta \\ a \cos \theta \end{array}\right)$.

Left invariant vector fields constructed in such a way have a useful property that, if $X_\epsilon \in T_e(G)$ is such that $X_\epsilon = \sum_{j=1}^4 \epsilon_j A_j$, then for any $g \in G$, the corresponding tangent vector $X_g$, is given by $X_g = \sum_{j=1}^4 \epsilon_j A_j$.

To simplify the scale related left invariant differential operator, we introduce a new variable, $\tau = \log_a$, which leads to the following change in left invariant derivatives
\[
\{A_1, A_2, A_3, A_4\} = \{\partial_y, e^\tau (\cos \theta \partial_x + \sin \theta \partial_\phi), e^\tau (- \sin \theta \partial_x + \cos \theta \partial_\phi), \partial_\tau\}.
\]  \tag{3.17}
Also note that, this change of variable, leads to an alternative description of the $\text{SIM}(2)$ group,
\[
g = (x, \tau, \theta)
\]
\[
gg' = (x, \tau, \theta)(x', \tau', \theta') = (x + e^\tau R_\theta x', \tau + \tau', \theta + \theta'),
\]
\[
e = (0, 0, 0, 0),
\]
where $x = \left(\begin{array}{c} x \\ y \end{array}\right)$. Note that substituting $\tau = 0$, leads to $a = e^\tau = 1$ which explains the difference in identity element of the $\text{SIM}(2)$ group in the log description.

Recall that in the construction of the wavelet kernel, a log-scale based definition was given. The underlying motivation for that construction is the aforementioned variable change, $\tau = \log_a(a)$.

---

*Proposed in [12, Sec 4.8.1].

*For the sake of simplicity the base $e$ will be dropped in $\log_e$. 
Comparing the derivatives $\partial_y$ and $\partial_\eta$ we observe that $\partial_\eta$ is invariant under rotation, i.e. the interpretation of $\partial_\eta$ stays the same.

The set of differential operators $\{A_1, A_2, A_3, A_4\} = \{\partial_\theta, \partial_\xi, \partial_\eta, \partial_\tau\}$ is the appropriate set of differential operators to be used in orientation scores instead of the set $\{A_1, A_2, A_3, A_4\} = \{\partial_\theta, \partial_x, \partial_y, \partial_a\}$ (which would seem to be the logical choice because of the sampling axes), since it has the following advantages,

- Since all 4 differential operators $\{\partial_\theta, \partial_\xi, \partial_\eta, \partial_\tau\}$ are left-invariant, we can be sure that all $SIM(2)$-coordinate independent linear and nonlinear combinations of these operators are left invariant. This advantage will become clear in the subsequent chapters when we deal with evolutions on the $SIM(2)$ group.

- At each scale, the derivatives $\partial_\theta, \partial_\xi, \partial_\eta$ have a clear interpretation, since at each scale $\partial_\xi$ is always the spatial derivative tangent to the orientation $\theta$ and $\partial_\eta$ is always orthogonal to this orientation. Figure 3.1 illustrates this for $\delta_\eta$ versus $\delta_y$.

Note that when constructing higher order left-invariant derivatives, the order in which the derivatives are applied is important. This is due to the fact that unlike derivatives $\{\partial_x, \partial_y, \partial_a, \partial_\theta\}$ which commute, the left-invariant derivatives $\{\partial_\xi, \partial_\eta, \partial_\tau, \partial_\theta\}$ do not commute. The nonzero commutators are given by

$$
[\partial_\tau, \partial_\xi] = \partial_\xi, \quad [\partial_\tau, \partial_\eta] = \partial_\eta, \quad [\partial_\theta, \partial_\xi] = \partial_\eta, \quad [\partial_\theta, \partial_\eta] = -\partial_\xi,
$$

where recall that $[A_i, A_j] = A_i A_j - A_j A_i$.

For a more intuitive explanation of left-invariant differential operators in the case of $SE(2)$-group (easily visualized as it excludes scale) see Appendix(Sec. A.3).
3.1.5 Exponential Map

In this section we explicitly evaluate the exponential \((exp)\) map and the logarithm \((log)\) map. Recall the following definition from Section 3.1.1:

Let \(X \in T_e(SIM(2))\). Then, \(exp(X) = \gamma_c(1)\), where

\[
\gamma_c : \mathbb{R} \to SIM(2)
\]

is the unique one parameter subgroup of \(SIM(2)\) whose tangent vector at the identity is equal to \(X\). It follows from the chain rule that, \(exp(tX) = \gamma_c(t)\). This map \(\gamma_c\), can be constructed as the integral curve of the left invariant vector fields associated with \(X\).

An exponential curve is a curve \(\gamma_c : \mathbb{R} \to SIM(2)\) for which the components of the tangent vector expressed in the left invariant basis \(\{A_1, A_2, A_3, A_4\}\) are constant over the entire parametrization (if necessary the reparametrization of \(t\)), i.e.

\[
\frac{d}{dt} \gamma_c(t) = \sum_{i=1}^{4} c^i A_i \bigg|_{g=\gamma_c(t)}, \text{ for all } t \in \mathbb{R}.
\]

These lines are analogous to straight lines in \(\mathbb{R}^n\), which have a also have constant tangent vector relative to the cartesian basis \(\{e_x, e_y\}\).

An exponential curve is obtained by using the \(exp\) mapping of the Lie algebra elements, (3.19). That is, an exponential curve passing through the identity element \(e \in SIM(2)\) at \(t = 0\) can be written as

\[
\gamma_c(t) = \exp \left( \sum_{i=1}^{4} t^i A_i \bigg|_{g=e} \right) = \exp \left( \sum_{i=1}^{4} t^i A_i \right),
\]

and an exponential curve passing through \(g_0 \in SIM(2)\) can be obtained by left multiplication with \(g_0 = (x_0, y_0, e^{\tau_0}, \theta_0)\), i.e. \(g_0 \gamma_c(t)\).

Below we present two different methods of arriving at these exponential curves. The first method involves a matrix-algebra approach based on (3.1). In this method we calculate exponential curves by using the matrix Lie algebra elements in (3.9) and the matrix exponent defined in (3.1). Let \(m(g) \in \mathcal{M}(2)\) represent the matrix representation of \(g = (x, y, a, \theta) \in SIM(2)\). Then the exponential curves can be determined by solving

\[
g = g_0 \exp(t \sum_{i=1}^{4} c^i A_i)
\]

which is equivalent to solving

\[
m(g) = m(g_0) \exp \left( t \sum_{i=1}^{4} c^i A_i \right)
\]

or

\[
m(g) = m(g_0) \exp \left( t \sum_{i=1}^{4} c^i X_i \right).
\]

Recall that, this method works as \(exp\) map corresponds to the matrix exponential in case of matrix Lie groups.

Now we present a different method for evaluating exponential curves. This is a more elegant approach based on the method of characteristics for PDEs. We begin by proving the following result.
The Similitude Group

Theorem 3.1. \( e^{tdR(A)} = R_{e^{tA}}, \forall t > 0, \forall A \in T_c(SIM(2)) \) and \( dR \) is defined in (3.14).

Proof. We need to prove that \( e^{tdR(A)}U = R_{e^{tA}}U \), where \( U \in D(dR(A)) \), \( \forall t > 0 \). Note that \( e^{tdR(A)}U \) is a solution for \( \frac{dW}{dt} = dR(A)W \) with \( W(g, 0) = U \). So to prove this theorem, we need to show that, \( \frac{\partial(R_{e^{tA}}U)}{\partial t} = dR(A)(R_{e^{tA}}U) \), which follows from,

\[
\frac{\partial(R_{e^{tA}}U)}{\partial t} = \lim_{h \to 0} \frac{R_{e^{(t+h)A}}U - R_{e^{tA}}U}{h} = \lim_{h \to 0} \frac{R_{e^{hA}}R_{e^{tA}}U - R_{e^{tA}}U}{h} = \lim_{h \to 0} \left( \frac{R_{e^{hA}} - I}{h} \right) R_{e^{tA}}U = dR(A)(R_{e^{tA}}U). \tag{3.24}
\]

Note that the limit above is defined on the space \( H = L_2(SIM(2)) \). To justify the last step of (3.24) we need to show that, \( U \in D(dR(A)) \Rightarrow R_{e^{tA}}U \in D(dR(A)) \), which is proved in the next lemma.

Lemma 3.2. \( U \in D(dR(A)) \Rightarrow R_{e^{tA}}U \in D(dR(A)) \) and generalized derivatives exist.

Proof. We first note that, \( U \in D(dR(A)) \Rightarrow (dR(A)U) \in H \), where \( H = L_2(SIM(2)) \). Then, \( R_{e^{tA}}U \in D(dR(A)) \) follows as \( [dR(A)R_{e^{tA}}](U) = [R_{e^{tA}}dR(A)](U), \forall U \in D(dR(A)) \).

\[
[dR(A)R_{e^{tA}}](U)(g) = \lim_{h \to 0} \frac{R_{e^{hA}}R_{e^{tA}}(U(g)) - R_{e^{tA}}(U(g))}{h} = \lim_{h \to 0} \frac{R_{e^{tA}}R_{e^{hA}}(U(g)) - R_{e^{tA}}(U(g))}{h} = [dR(A)R_{e^{tA}}dR](U)(g)
\]

where we have used the fact that, \( R_{h}(U(g)) = U(gh) \) and \( e^{tA}e^{hA} = e^{(t+h)A} = e^{hA}e^{tA} \).

Corollary 3.3. \( \gamma_c(t) = g_0 \exp \left( t \sum_{i=1}^{4} e^i A_i \right) \) are the characteristics for the following PDE,

\[
\begin{cases}
\frac{dW}{dt} = - \sum_{i=1}^{4} e^i A_i W \\
W(g, 0) = U.
\end{cases} \tag{3.25}
\]

Proof. Let \( dR(A) = A = \sum_{i=1}^{4} e^i A_i \). Then the solution for (3.25) is, \( W(\cdot, t) = e^{-tdR(A)}U = R_{e^{-tA}}U \), where the second equality follows from Theorem 3.1. Thus we have,

\[
W(g, t) = U(ge^{-tA}) \quad \forall g \in SIM(2), \forall t > 0 \Rightarrow W(g_0 e^{tA}, t) = U(g_0) \quad \forall t > 0. \tag{3.26}
\]

Remark. The computation of the exponential curves in a fixed frame of reference follows from the chain rule,

\[
\frac{\partial w}{\partial t} = \sum_{i=1}^{4} f^i \frac{\partial w}{\partial \theta^i}. \tag{3.27}
\]

and that \( (dv, \dot{v}) = f^i \). Thus solving the ODE system \( \dot{\gamma}^i = f^i, i \in \{1, 2, 3, 4\} \), we arrive at the exponential curves.

\[^{10}D(A) \text{ denotes the domain of the operator } A.\]
Thereby the explicit formulation of the exponential curves \( g_0 \gamma_\gamma(t) = (x(t), y(t), \tau(t), \theta(t)) \) passing through \( g_0 = \{x_0, y_0, \epsilon^{\gamma}, \theta_0\} \) at \( t = 0 \) is,

\[
\begin{align*}
x(t) &= \frac{1}{c_1 + c_4} e^{\epsilon \gamma} c_1 \left(( - \sin[\theta_0] + e^{\epsilon \gamma} \sin[tc_1 + \theta_0] \right) c_2 + \left(- \cos[\theta_0] + e^{\epsilon \gamma} \cos[tc_1 + \theta_0] \right) c_3 + c_4 x_0 + c_4 (e^{\epsilon \gamma} (-\cos[\theta_0] + e^{\epsilon \gamma} \cos[tc_1 + \theta_0] \right) c_2 + e^{\epsilon \gamma} (\sin[\theta_0] - e^{\epsilon \gamma} \sin[tc_1 + \theta_0]) c_3 + c_4 x_0]) \\
y(t) &= \frac{1}{c_1 + c_4} e^{\epsilon \gamma} c_1 \left(( - \sin[\theta_0] - e^{\epsilon \gamma} \cos[tc_1 + \theta_0] \right) c_2 + \left(- \sin[\theta_0] + e^{\epsilon \gamma} \sin[tc_1 + \theta_0] \right) c_3 + c_4 y_0 + c_4 (e^{\epsilon \gamma} (-\sin[\theta_0] + e^{\epsilon \gamma} \sin[tc_1 + \theta_0] \right) c_2 + e^{\epsilon \gamma} (\cos[\theta_0] + e^{\epsilon \gamma} \cos[tc_1 + \theta_0]) c_3 + c_4 y_0]) \\
\tau(t) &= tc_4 + \tau_0 \\
\theta(t) &= tc_1 + \theta_0. (3.28)
\end{align*}
\]

Now we present an important result for the \( exp \) map in the \( SIM(2) \) group.

**Theorem 3.4.** The exponential map, \( exp : T_e(SIM(2)) \rightarrow SIM(2) \) is surjective, i.e. for all \( g \in SIM(2) \), there exists a \( X \in T_e(SIM(2)) \) such that, \( g = exp(X) \).

An elegant proof for this theorem is provided in Appendix (Sec. A.4).

From the above theorem \( exp \) map is surjective and thus we can define the \( log \) map, \( log = (exp)^{-1} : SIM(2) \rightarrow T_e(SIM(2)) \). To explicitly determine the \( log \) map, we solve for \( \{c_1, c_2, c_3, c_4\} \) from the equality, \( g = exp \left( t \sum_{i=1}^4 c^i A_i \right) \), where \( g \in SIM(2) \) and \( c^i \)'s are as defined earlier. This is achieved by substituting \( g_0 = c \), i.e. \( x_0 = y_0 = \theta_0 = \tau_0 = 0 \), in (3.28) yielding,

\[
\begin{align*}
c^1 &= \frac{\theta}{t} \\
c^2 &= \frac{y\theta - x\tau - e^{\epsilon \gamma}(y\theta - x\tau) \cos \theta + e^{\epsilon \gamma}(x\theta + y\tau) \sin \theta}{t(1 + e^{2\epsilon \gamma} - 2e^{\epsilon \gamma} \cos \theta)} \\
c^3 &= \frac{-x\theta - y\tau + e^{\epsilon \gamma}(x\theta + y\tau) \cos \theta + e^{\epsilon \gamma}(y\theta - x\tau) \sin \theta}{t(1 + e^{2\epsilon \gamma} - 2e^{\epsilon \gamma} \cos \theta)} \\
c^4 &= \frac{\tau}{t}, (3.29)
\end{align*}
\]

where recall that \( g = (x, y, e^{\epsilon \gamma}, \theta) \in SIM(2) \). The formulae in (3.29) can be written in a simplified form,

\[
\begin{align*}
c^1 &= \frac{\theta}{t} \\
c^2 &= \frac{y\theta - x\tau + (-\theta \eta + \tau \xi)}{t(1 + e^{2\epsilon \gamma} - 2e^{\epsilon \gamma} \cos \theta)} \\
c^3 &= \frac{-x\theta + y\eta + \theta \xi + \tau \eta}{t(1 + e^{2\epsilon \gamma} - 2e^{\epsilon \gamma} \cos \theta)} \\
c^4 &= \frac{\tau}{t}, (3.30)
\end{align*}
\]

where we have made use of the definition of \( \xi \) and \( \eta \),

\[\xi = e^{\epsilon \gamma}(x \cos \theta + y \sin \theta), \quad \eta = e^{\epsilon \gamma}(-x \sin \theta + y \cos \theta).\]

This explicit form of the \( log \) map will be used to approximate the solution for evolutions on \( SIM(2) \) in the coming chapters.

### 3.2 \( SIM(2)-\text{Convolution} \)

It will be shown in the next chapter, Section 4.2, that all linear left invariant operators on the space of scale-OS are a group convolution with an appropriate kernel. In this section we briefly
treat the mathematical properties of $SIM(2)$-convolution. The generalized group convolution in the case of the $SIM(2)$-group can be written as,

$$(Ψ *_{SIM(2)} U)(g) = \int_{SIM(2)} Ψ(h^{-1}g)U(h)dh,$$

where $U ∈ L_2(SIM(2))$ (in our case a scale orientation score of an image) and $Ψ ∈ L_2(SIM(2))$ is the $SIM(2)$ convolution kernel. Explicitly,

$$(Ψ *_{SIM(2)} U)(x,a,θ) = \int_{R^2} \int_0^{2\pi} Ψ \left( \frac{1}{a'} R_{θ'}^{-1}(x - x'), \frac{a}{a'}, θ - θ' \right) U(x',a',θ') \frac{1}{(a')^3} dx'da'dθ',$$

where the measure $dμ_{SIM(2)}(h) = dh = \frac{1}{(a')^3} dx'da'dθ' is the left-invariant Haar measure defined on $SIM(2)$ and $g = (x,a,θ), h = (x',a',θ') ∈ SIM(2)$.

Below we summarize a few properties of the $SIM(2)$-convolution. Most of these properties have a fairly straightforward proof. Assume that $Ψ, Ψ_1, Ψ_2, U, U_1, U_2 ∈ L_2(SIM(2)). Then,

- **Associativity:**

  $$(Ψ_1 *_{SIM(2)} Ψ_2) *_{SIM(2)} U = Ψ_1 *_{SIM(2)} (Ψ_2 *_{SIM(2)} U).$$

  This also holds for scalar multiplication, i.e.

  $$a(Ψ *_{SIM(2)} U) = ((aΨ) *_{SIM(2)} U) = (Ψ *_{SIM(2)} (aU)),$$

  for all $a ∈ ℂ$.

- **Distributivity:**

  $$(Ψ_1 + Ψ_2) *_{SIM(2)} U = Ψ_1 *_{SIM(2)} U + Ψ_1 *_{SIM(2)} U$$

  and

  $$Ψ *_{SIM(2)} (U_1 + U_2) = Ψ *_{SIM(2)} U_1 + Ψ *_{SIM(2)} U_2.$$  \hspace{1cm} (3.32)

- **Identity:** $δ *_{SIM(2)} U = U *_{SIM(2)} δ = U$, where $δ : SIM(2) → ℂ$ is defined by $δ(g) = δ(x,y,a,θ) = δ(x)δ(y)δ(a)δ(θ)$, $g = (x,y,a,θ) ∈ SIM(2)$.$^{11}$ Note that in our case of scale-OS we can reproduce back the data using the kernel, $δ : (x,y,a,θ) ↦ δ(x)δ(y)δ(θ - θ_0)δ(τ - τ_0)$, where $θ_0,τ_0$ belong to the set of discrete orientations and scales(log sampling), used in the construction of scale-OS, respectively.

- **Invariance:** A $SIM(2)$-convolution is left invariant on it’s left-input, i.e.

  $$L_g(Ψ *_{SIM(2)} U) = Ψ *_{SIM(2)} (L_gU), \text{ for all } g ∈ SIM(2).$$  \hspace{1cm} (3.33)

  See Def. 4.1 for the definition of left invariant operator. This follows from the discussion in the previous subsection. Note that (3.33) is similar to the translation invariance property of the usual convolution on $R^d$.

- **Differentiation Rule:** $D(Ψ *_{SIM(2)} U) = ((DΨ) *_{SIM(2)} U)$, where $D$ denotes a left invariant derivative operator (see Section 3.1.4). This property follows from the following argument

  $$D(Ψ *_{SIM(2)} U)(g) = D \int_{SIM(2)} (L_h ∘ Ψ)(g)U(h)dh \stackrel{3.32}{=} \int_{SIM(2)} (D ∘ L_h ∘ Ψ)(g)U(h)dh$$

  $$= \int_{SIM(2)} (L_h ∘ (D ∘ Ψ))(g)U(h)dh = ((DΨ) *_{SIM(2)} U)(g).$$  \hspace{1cm} (3.34)

  Note that this property does not hold for general differential operators.

$^{11}$Here the $δ$ defined on a one dimensional space are the usual dirac delta functions.
Figure 3.2: Schematic view of a \(SE(2)\)-convolution (left) and an \(R^3\)-convolution (right). Any convolution \(K \ast f\) can be envisioned by the kernel \(K\) ”moving” over the entire domain of \(f\) where the result of the convolution is found by taking inner products at all positions. The kernel is indicated as a box, and the 3 axes denote the orientation score domain and 3D image domain respectively. As indicated by the arrows, in both cases the kernel is moving over the \(x, y\)-plane. In the case of the \(SE(2)\)-convolution, there is an additional twist when moving in the orientation dimension.

Below we explicitly state the \(SE(2)\)-convolution and relate it to \(SIM(2)\)-convolution. This is important as it leads to an easier implementation of the \(SIM(2)\)-convolution. Let \(\Psi', U' \in L^2(SE(2))\). Explicitly,

\[
(\Psi' \ast_{SE(2)} U')(x, \theta) = \int_{\mathbb{R}^2} \Psi'(R_{\theta}^{-1}(x - x'), \theta - \theta')U(x', \theta')d\theta'dx',
\]

where \(g = (x, \theta) \in SE(2)\). Figure 3.2 schematically explains how an \(SE(2)\)-convolution can be envisioned.

Consider \((\Psi \ast_{SIM(2)} U)\), where \(\Phi, U \in L^2(SIM(2))\) and \(g = (x, a, \theta) \in SIM(2)\):

\[
(\Psi \ast_{SIM(2)} U)(g) = \int \frac{2\pi}{a} \int_{\mathbb{R}^+} \Psi(R_{\theta}^{-1}(x - x'), \frac{a}{a'}, \theta - \theta')U(x', a', \theta') \frac{1}{(a')^3}d\theta'da'dx' = \int_{\mathbb{R}^+} [\Psi_{\tau'} \ast_{SE(2)} U(\cdot, \tau', \cdot)](x, \theta)d\tau'
\]

where note that we have transformed the scale \(a\) as \(\tau = \log(a)\). For arbitrarily fixed values of \(\tau, \tau', \Psi_{\tau'} : SE(2) \to \mathbb{C}\) is defined as,

\[
\Psi_{\tau'}(x, \theta) = \frac{1}{(e^{\tau'})^2} \Psi \left(\frac{x}{e^{\tau'}}, \frac{\tau}{\tau'}, \theta\right).
\]

Note that as \(\Psi \in L^2(SIM(2))\), \(\Psi_{\tau'} \in L^2(SE(2))\) and hence the convolution is well defined.

We use this straightforward relation to implement \(SIM(2)\)-convolution in terms of \(SE(2)\)-convolution. In the Appendix (Sec A.5) we briefly describe the implementation of \(SE(2)\)-convolution. For a complete treatment of \(SE(2)\)-convolution see [21, Ch.3].

### 3.2.1 SIM(2)-convolution vs SIM(2)-Correlation

For the sake of completeness we present the concept of \(SIM(2)\)-correlation.
Figure 3.3: Example showing the difference between a kernel and its corresponding "mirrored" kernel. Left: an arbitrary $SE(2)$-kernel $\Psi$. Right: $\tilde{\Psi}(x, \theta) = \Psi(-R_{\theta}^{-1}x, -\theta)$, is a mirrored version of $\Psi$ with an additional twist over the orientation axis. Note that an $SE(2)$-convolution with the kernel on the left is the same as an $SE(2)$-correlation with the kernel on the right and vice versa.

Analogously to the $\mathbb{R}^2$-correlation, the $\text{SIM}(2)$-correlation is defined as

$$
(\Psi *_{\text{SIM}(2)} U)(g) = \int_{\text{SIM}(2)} \Psi(g^{-1}h)U(h)dh,
$$

(3.38)

where $\Psi, U \in L_2(\text{SIM}(2))$ and $\tilde{\Psi}(g) = \Psi(g^{-1})$ for all $g \in \text{SIM}(2)$. This shows that $\Psi *_{\text{SIM}(2)} U = \tilde{\Psi} *_{\text{SIM}(2)} U$. Explicitly this yeilds,

$$
(\tilde{\Psi} *_{\text{SIM}(2)} U)(x, a, \theta) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^+} \Psi \left( \frac{1}{a} R_{\theta}^{-1}(x' - x), \frac{a'}{a}, \theta' - \theta \right) U(x', a', \theta') \frac{1}{(a')^3} dx' da' d\theta',
$$

(3.39)

where $\Psi(x, a, \theta) = \Psi(-\frac{1}{a} R_{\theta}^{-1}x, \frac{1}{a}, -\theta)$. This structure of $\text{SIM}(2)$ group indicates that transforming an $\text{SIM}(2)$-convolution kernel into an $\text{SIM}(2)$-correlation kernel is not only accomplished by mirroring the $x, y, \theta$ axes but also involves a rotation as well as scaling, i.e. the correlation kernel would involve a twisting and scaling of the convolution kernel. Since visualizing this kernel is difficult for the $\text{SIM}(2)$ case due to the presence of four variables, $(x, y, a, \theta)$, we illustrate this difference between correlation and convolution graphically, using the a $SE(2)$ kernel. Let $\Psi \in L_2(SE(2))$, and the corresponding correlation kernel, $\Psi(x, \theta) = \Psi(-R_{\theta}^{-1}x, -\theta)$. Note that this can be graphically visualized as "twisting" of the kernel and not scaling, see Figure 3.3.

Note that , it is preferable to use $\text{SIM}(2)$-convolution instead of $\text{SIM}(2)$-correlation as $\text{SIM}(2)$-correlation does not fulfill the properties of associativity, identity and the differentiation rule.
Chapter 4

Image Enhancement via Left Invariant Operations on Scale-OS

We start by describing “legal” operators on scale-OS and then show that such legal linear operators are group convolutions with an appropriate kernel. We then give an approximation of the contour enhancement heat kernels for the linear left-invariant diffusion process on the $SIM(2)$ group. We then introduce adaptive diffusion on the $SIM(2)$ group in a rigorous manner using a differential-geometric approach. We conclude the chapter by applying $SE(2)$ based adaptive diffusion (CED-OS) to the scale-OS.

4.1 Legal Operators on the Scale-OS

Let $\psi$ be an admissible wavelet as described in Section 2.4 and let $G = SIM(2)$, then there exists a 1-to-1 correspondence between bounded operators $\Phi \in B(C_G^K)$ on orientation scores and bounded operators $\Upsilon \in B(L^2(\mathbb{R}^d))$:

$$T[f] = (W\psi)^*[\Phi(W\psi[f])], \ f \in L^2(\mathbb{R}^d),$$

which allows us to relate operations on orientation scores to operations on images in a robust manner. To get a schematic view of the operations, see Figure 4.1.

Recall from Corollary 2.11 that $C_G^K$ is the space of orientation scores as a closed linear subspace of $H_\psi$, which is a vector subspace of $L^2(G)$. For proper wavelets we have (approximative) $L^2$-norm preservation, so then we have $L^2(G) \cong H_\psi$. In this section we set $L^2(G) = H_\psi$, to avoid technical subtleties concerning approximations.

At this point we remark that if, $\Phi : L^2(G) \rightarrow L^2(G)$ is a bounded operator on $L^2(G)$, then the range of restriction of this operator to the subspace $C_G^K$ of orientation scores need not be contained in $C_G^K$, i.e. $\Phi(U_f)$ need not be the orientation score of an image. Recall the adjoint mapping of $W_\psi : L^2(\mathbb{R}^d) \rightarrow L^2(G)$, given by

$$(W_\psi)^*(V) = \int_G U_\phi \psi V(g) d\mu_G(g), \ V \in L^2(G).$$

The operator $P_\psi = W_\psi (W_\psi)^*$ is the orthogonal projection on the space of orientation scores $C_G^K$. This projection can be used to decompose the manipulated orientation score:

$$\Phi(U_f) = P_\psi (\Phi(U_f)) + (I - P_\psi) (\Phi(U_f)).$$
Notice that the orthogonal complement \((C^G_K)^\perp\), which equals \(\mathcal{R}(I - P_\psi)\), is exactly the null-space of \((W_\psi)^*\) as \(\mathcal{N}(W_\psi)^* = (\mathcal{R}(W_\psi))^\perp = (C^G_K)^\perp\) and so
\[
[(W_\psi)^* \circ \Phi \circ W_\psi][f] = [(W_\psi)^* \circ P_\psi \circ \Phi \circ W_\psi][f],
\]
(4.2)
for all \(f \in L^2(\mathbb{R}^d)\) and all \(\Phi \in \mathcal{B}(L^2(G))\).

**Definition 4.1.** An operator \(\Phi : L^2(G) \rightarrow L^2(G)\) is left invariant iff
\[
\Phi[\mathcal{L}_h f] = \mathcal{L}_h[\Phi f], \text{ for all } h \in G, \ f \in L^2(\mathbb{R}^d),
\]
(4.3)
where the left regular action \(\mathcal{L}_g\) of \(g \in G\) onto \(L^2(G)\) is given by
\[
\mathcal{L}_g \psi(h) = \psi(g^{-1}h) = \psi \left( \frac{1}{\alpha} R_{g}^{-1}(b' - b), \frac{a'}{\alpha}, \theta' - \theta \right),
\]
(4.4)
with \(g = (b, a, \theta) \in G, \ h = (b', a', \theta') \in G\).

**Theorem 4.2.** Let \(\Phi\) be a bounded operator on \(C^G_K\), \(\Phi : C^G_K \rightarrow L^2(G)\). Then the unique corresponding operator \(\Upsilon\) on \(L^2(\mathbb{R}^d)\), which is given by \(\Upsilon[f] = (W_\psi)^* \circ \Phi \circ W_\psi[f]\) is Euclidean invariant, i.e. \(U_g \Upsilon = \Upsilon U_g\) for all \(g \in G\) if and only if \(P_\psi \circ \Phi\) is left invariant, i.e. \(\mathcal{L}_g(P_\psi \circ \Phi) = (P_\psi \circ \Phi) \mathcal{L}_g\), for all \(g \in G\).

**Proof.** As,
\[
W_\psi[U_g[f]](h) = (U_h \psi, U_g f)_{L^2(\mathbb{R}^d)} = (U_g^{-1} h \psi, f)_{L^2(\mathbb{R}^d)} = \mathcal{L}_g[W_\psi[f]](h)
\]
we conclude that,
\[
W_\psi U_g = \mathcal{L}_g W_\psi, \text{ for all } g \in G.
\]
(4.5)
Moreover,
\[
(W_\psi U_g f, U)_{L^2(G)} = (\mathcal{L}_g W_\psi, U)_{L^2(G)} \Leftrightarrow (f, U_g^{-1} (W_\psi)^* U)_{L^2(G)} = (f, (W_\psi)^* \mathcal{L}_g^{-1} U)_{L^2(G)}
\]
for all \(U \in L^2(G), \ f \in L^2(\mathbb{R}^d), \ g \in G\) and therefore we have,
\[
U_g(W_\psi)^* = \mathcal{L}_g(W_\psi)^*, \text{ for all } g \in G.
\]
(4.6)
(Necessary condition) Assuming that \((P_\psi \circ \Phi)\) is left invariant it follows from (4.2), (4.5) and (4.6) that
\[
\Upsilon[U_g f] = (W_\psi)^* \circ \Phi \circ W_\psi \circ U_g[f]
\]
\[
= (W_\psi)^* \circ (P_\psi \circ \Phi) \circ W_\psi \circ U_g[f]
\]
\[
= (W_\psi)^* \circ (P_\psi \circ \Phi) \circ \mathcal{L}_g \circ W_\psi[f]
\]
\[
= (W_\psi)^* \circ \mathcal{L}_g \circ (P_\psi \circ \Phi) \circ W_\psi[f]
\]
\[
= U_g \circ (W_\psi)^* \circ \Phi \circ W_\psi[f] = U_g[\Upsilon[f]]
\]
(4.7)
for all \( f \in \mathbb{L}_2(\mathbb{R}^2) \) and \( g \in G \). Thus we have \( \Upsilon U_g = U_g \Upsilon \) for all \( g \in G \).

(Sufficient condition) Now suppose \( \Upsilon \) is Euclidean invariant. Then again by (4.5) and (4.6) we have that,

\[
(W_{\psi}^*)^* \circ \Phi \circ L_g \circ W_{\psi}[f] = (W_{\psi})^* \circ \Phi \circ L_g \circ W_{\psi}[f],
\]

for all \( f \in \mathbb{L}_2(\mathbb{R}^2) \) and \( g \in G \). Since the range of \( L_g |_{C^G_K} \) and the range of \( (P \circ \Phi) \) is contained in \( C^G_K \), we have \( (P \circ \Phi) \circ L_g \circ W_{\psi} = (P \circ \Phi) \circ W_{\psi} \). As the range of \( W_{\psi} \) equals \( C^G_K \), we have \( L_g \circ (P \circ \Phi) = (P \circ \Phi) \circ L_g \) for all \( g \in G \).

Practical Consequence: Euclidean invariance of \( \Upsilon \) is of great practical importance, since the result should not be essentially different if the original image is rotated or translated. In addition in our construction scaling the image also does not affect the outcome of the operation. Note that it is not a problem when the mapping \( \Phi : C^G_K \rightarrow \mathbb{L}_2(G) \) maps an orientation score to an element in \( \mathbb{L}_2(G) \backslash C^G_K \) as long as one is aware from (4.2) that \( P \circ \Phi : C^G_K \rightarrow C^G_K \) yields the same result.

### 4.2 Left Invariant Linear Bounded Operations are Group Convolutions

We start off by showing that all left invariant bounded linear operators on the space of orientation scores are group convolutions (defined appropriately). Note that this is possible as the group under consideration has a measure defined on itself (left/right invariant Haar measure).

**Theorem 4.3** (Dunford-Pettis). Let \( X \) be a measurable space, with measure \( \mu : X \rightarrow \mathbb{R}^+ \). Let \( 1 \leq p < \infty \). Let \( A \) be a bounded linear operator from \( \mathbb{L}_p(X) \) into \( \mathbb{L}_\infty(X) \), then there exists a \( K \in L_1(X \times X) \) such that

\[
\sup_x \left( \int_X |K(x,y)|^q d\mu(y) \right)^{\frac{1}{q}} = ||A||,
\]

with \( q > 0 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \) and for all \( f \in \mathbb{L}_p(X) \) we have that

\[
(Af)(x) = \int_X K(x,y)f(y)d\mu(y)
\]

for almost every \( x \in X \).

For a proof see [40]. In our case of the \( \text{SIM}(2) \) group this translates as follows: All bounded linear operators \( \Phi : \mathbb{L}_2(\text{SIM}(2)) \rightarrow \mathbb{L}_2(\text{SIM}(2)) \) are kernel operators, i.e. there exists a kernel \( K : \text{SIM}(2) \times \text{SIM}(2) \rightarrow \mathbb{C} \) such that,

\[
(\Phi(U))(g) = \int_{\text{SIM}(2)} K(g,h)U(h)dh, \ g, h \in \text{SIM}(2)
\]

and the kernel satisfies,

\[
\sup_g \left( \int_{\text{SIM}(2)} |K(g,h)|^2 dh \right)^{\frac{1}{2}} < \infty.
\]
Note that there is a subtlety involved which allows the usage of Dunford-Pettis theorem in our case. We assume, $\Phi : L_2(SIM(2)) \rightarrow L_2(SIM(2)) \cap L_\infty(SIM(2))$, such that, $\Phi : L_2(SIM(2)) \rightarrow L_2(SIM(2))$ and $\Phi : L_2(SIM(2)) \rightarrow L_\infty(SIM(2))$ is a bounded operator\(^1\).

If $\Phi$ is left invariant (as required), i.e. $\Phi \circ L_p \circ U = L_p \circ \Phi \circ U$ (recall Def. 4.1), using (4.8) we obtain,

$$
(\Phi \circ L_p \circ U)(g) = \int_{SIM(2)} K(g,h)L_p(h)dh
$$

$$
= \int_{SIM(2)} K(g,h)U(p^{-1}h)dh
$$

$$
= \int_{SIM(2)} K(g,ph)U(h)dh \quad \text{(substitute } h \leftarrow ph) \tag{4.9}
$$

and

$$
(L_p \circ \Phi \circ U)(g) = \int_{SIM(2)} K(p^{-1}g,h)U(h)dh. \tag{4.10}
$$

Note that in the arguments made above the choice of $p \in SIM(2)$ and $U \in L_2(SIM(2))$ is arbitrary. For the right-hand sides of (4.9) and (4.10) to be equal, $K(g,ph) = K(p^{-1}g,h)$ must hold for all $p,g,h \in SIM(2)$. Replacing $g$ by $pg$ we obtain $K(pg,ph) = K(g,h)$, and thus we see that the kernel $K$ should be invariant under the left-multiplication of its arguments by an arbitrary element of $SIM(2)$. So $K(g,h) = K(e,g^{-1}h)$, whence if we define a new function $\Psi : SIM(2) \rightarrow C$ such that, $\Psi(g) = K(e,g^{-1})$ for $g \in SIM(2)$, then using (4.8), we arrive at the $SIM(2)$ convolution

$$
(\Psi *_{SIM(2)} U)(g) = \int_{SIM(2)} \Psi(h^{-1}g)U(h)dh. \tag{4.11}
$$

(4.11) is a straightforward generalization of $\mathbb{R}^n$-convolution to any other Lie-group, by replacing $SIM(2)$ by any other group. For e.g. we obtain the $\mathbb{R}^n$-convolution if we use the translation group $\mathbb{R}^n$ in (4.11), i.e.

$$
(f *_{\mathbb{R}^n} g)(x) = \int_{\mathbb{R}^n} f(x - x')g(x')dx', \tag{4.12}
$$

where $f,g \in L_1(\mathbb{R}^d)$.

### 4.3 Basic Left Invariant Operations on Scale-OS

In image analysis, Euclidean invariant differential operators are used for corner/line/edge/blob detection. Often, these differential invariants are expressed in a local coordinate system (gauge coordinates). See [19] for construction of such detectors using gauge coordinates.

Below we present a few simple global left invariant operators used to enhance elongated structures.

\(^1\)This is a practical assumption as output of the operators acting on the orientation scores must be bounded in addition to being square integrable.
Introduction to Diffusions on SIM(2) group

- Normalization. See Figure (4.2).

\[ [\Phi(U_f)](b, a, \theta) = \begin{cases} 
U_f(b, a, \theta)/ \left( \int_{\mathbb{R}^2} |U_f(x, a, \theta)|^p dx \right)^{\frac{1}{p}}, & 1 < p < \infty \\
U_f(b, a, \theta)/ \max_{x \in \mathbb{R}^2} |U_f(x, a, \theta)|, & p = \infty.
\end{cases} 
\]

- Grey-value Transformations.

\[ \Phi(U_f) = (U_f - \min_g \{U_f(g)\})^p, \quad p > 0. \]

This operation can be used to enhance the strongly oriented structures in the score and reduce the noise/weakly oriented structures in the score, see Figure (4.3). Note that this operation does not correspond to a simple grey-value transformation on the original image as \((f)^p \neq (W_\psi)^*(W_\psi[f])^q\).

4.4 Introduction to Diffusions on SIM(2) group

In this section we apply the general theory of diffusions on Lie groups, [14], to the SIM(2) group and consider the following left-invariant second-order evolution equations,

\[ \begin{align*}
\partial_t W(g, t) &= Q^{D, a}(A_1, A_2, A_3, A_4)W(g, t), \\
W(\cdot, t = 0) &= W_\psi f(\cdot),
\end{align*} \tag{4.13} \]

where \(W : SIM(2) \times \mathbb{R}^+ \to \mathbb{C}\) and \(Q^{D, a}\) is the following quadratic form on \(\mathcal{L}(SIM(2))\),

\[ Q^{D, a}(A_1, A_2, A_3, A_4) = \sum_{i=1}^{4} \left( -a_i A_i + \sum_{j=1}^{4} D_{ij} A_i A_j \right), \quad a_i, D_{ij} \in \mathbb{R}, \quad D := [D_{ij}] \geq 0, \quad D^T = D. \tag{4.14} \]

When \(D_{ij} = D_{ij} \delta_{ij}, \quad i, j \in \{1, 2, 3, 4\}\), the quadratic form becomes,

\[ Q^{D, a}(A_1, A_2, A_3, A_4) = [-a_1 \partial_0 - a_2 \partial_k - a_3 \partial_\eta - a_4 \partial_r + D_{13}(\partial_\eta)^2 + D_{22}(\partial_k)^2 + D_{33}(\partial_\eta)^2 + D_{44}(\partial_r)^2]. \tag{4.15} \]

The first order part of (4.15) takes care of transport (convection) along the exponential curves\(^2\), deduced in Section 3.1.5. The second order part takes care of diffusion in the SIM(2) group. Note that the non-commutative nature of the SIM(2) group, recall (3.18), makes these evolution equations complicated. Also note that these evolution equations are left-invariant as they are constructed by linear combinations of left-invariant vector fields.

Hörmander, [26], gave necessary and sufficient conditions on the convection and diffusion parameters, respectively \(a = (a_1, a_2, a_3, a_4)\) and \(D = D_{ij}\), in order to get smooth Green’s functions of the left-invariant convection-diffusion equation (4.13) with generator (4.15). By these conditions the non-commutative nature of SIM(2), in certain cases, takes care of missing directions in the diffusion tensor. Below we formulate the Hörmander’s theorem.

A differential operator \(L\) defined on a manifold \(M\) of dimension \(n \in \mathbb{N}\), \(n < \infty\), is called hypo-elliptic if for all distributions \(f\) defined on an open subset of \(M\) such that \(Lf\) is \(C^\infty\) (smooth), \(f\) must also be \(C^\infty\). In his paper [26], Hörmander presented a sufficient and essentially necessary condition for an operator of the type \(L = c + X_0 + \sum_{i=1}^{r} \{X_i\} 2, \quad r \leq n\), where \(\{X_i\}\) are vector

\(^2\)Later in the chapter we prove that diffusion in our case take place only along exponential curves.
Figure 4.2: Image processing via Normalization of the SOS layers. Normalization of the orientation layers in the scale-OS reveals the most contrasting lines. Left: Original Image, Center: Normalization for $p < \infty$, Right: Normalization for $p = \infty$.

Figure 4.3: Image processing via Monotonic Grey-value transformation. Left: Noisy image, Right: Grey-value transformation for $p=3$. 
fields on \( M \), to be hypo-elliptic. This condition referred to as the Hörmander’s condition, is that among the set
\[
\{ X_{j1}, [X_{j1}, X_{j2}], [X_{j1}, [X_{j2}, X_{j3}]], \ldots, [X_{j1}, [X_{j2}, [X_{j3}, \ldots X_{jk}]][], j_i \in \{0, 1, \ldots, r\}\} \quad (4.16)
\]
there exist \( n \) elements which are linearly independent at any given point in \( M \). In case \( M \) is a Lie group and we restrict ourselves to left-invariant vector fields, then it is sufficient to check whether the vector fields span the tangent space at the unity element. So the necessary and sufficient conditions for smooth (resolvent) Green’s functions on \( SIM(2)\backslash \{e\} \) on the diffusion and convection parameters \((D, a)\) in the generator \((4.15)\) of \((4.13)\) for diagonal \(D\) are
\[
\{1, 2, 4\} \subset \{i \mid a_i \neq 0 \lor D_{ii} \neq 0\} \lor \{1, 3, 4\} \subset \{i \mid a_i \neq 0 \lor D_{ii} \neq 0\}. \quad (4.17)
\]
By results in Section 4.2, the solutions of \((4.13)\) are given by a \(SIM(2)\)-convolution with the corresponding Green’s function \(K_t^{D,a}\),
\[
W(g, t) = (K_t^{D,a} *_{SIM(2)} U)(g) = \int_{SIM(2)} K_t^{D,a}(h^{-1}g)U(h)d\mu_{SIM(2)}(h). \quad (4.18)
\]
In this study we are predominantly concerned with contour enhancement. So we consider a special case of \((4.13)\) where
\[
D_{ij} = D_{ii}\delta_{ij}, \ i,j \in \{1, 2, 3, 4\}, \ D_{33} = 0 \text{ and } a = 0, \quad (4.19)
\]
and arrive at Gaussian estimates of the Green’s function for this evolution equation. Note that this particular choice of parameters satisfies the Hörmander’s condition and therefore the Green’s function of this diffusion process will be smooth. The choice \(a = 0\), reflects that we do not allow any transport along the curves in the \(SIM(2)\) group while the choice of the diffusion tensor \(D\) is based on the fact that we wish to diffuse along the elongated structures and not orthogonal to it.

4.5 Approximate Contour Enhancement Kernels on Scale-OS

Compared to diffusion on \(\mathbb{R}^n\), where the Green’s function is simply a Gaussian, the Green’s function of diffusion on non-commutative groups is complicated. In [13, 17], the authors derive the exact Green’s function for \((4.13)\) with conditions similar to \((4.19)\) for the \(SE(2)\) case. Explicit and exact formulas for heat kernels for the \(SIM(2)\) case do not exist in the literature. However there exists a general theory, see [29, 35], which provides Gaussian estimates for Green’s function of left-invariant diffusions on Lie groups, generated by subcoercive operators. In this section we employ this general framework to our case of interest. We first carry out the method of contraction which serves as an essential pre-requisite for the Gaussian estimates and approximation kernels later in this chapter.

4.5.1 Approximation of \(SIM(2)\) by a Nilpotent Group via Contraction

We follow the general framework by ter Elst and Robinson [35], which involves semigroups on Lie groups generated by subcoercive operators. In their work we consider a particular case
by setting the Hilbert space, \( H = L^2(SIM(2)) \), the group \( G = SIM(2) \) and the right-regular representation \( \mathcal{R} \). Furthermore we consider the algebraic basis \( \{A_1 = \partial_\theta, A_2 = \partial_x, A_4 = \partial_x\} \) leading to the following filtration of the Lie algebra

\[
\mathfrak{g}_1 := span\{A_1, A_2, A_4\}, \\
\subset \mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1] = span\{A_1, A_2, A_3, A_4\} = \mathcal{L}(SIM(2)).
\]

(4.20)

Based on this filtration we assign the following weights to the generators:

\[
w_1 = w_2 = w_4 = 1 \text{ and } w_3 = 2.
\]

(4.21)

For e.g. \( w_2 = 1 \) since \( A_2 \) occurs first in \( \mathfrak{g}_1 \), while \( w(3) = 2 \) since \( A_3 \) occurs in \( \mathfrak{g}_2 \) and not \( \mathfrak{g}_1 \).

Now based on these weights we define the following dilations on the Lie algebra \( T_\epsilon(SIM(2)) \) (recall \( A_i = A_i|_\epsilon \)),

\[
\gamma_q \left( \sum_{i=1}^{4} c^i A_i \right) = \sum_{i=1}^{4} q^{w_i} c^i A_i, \quad \forall c^i \in \mathbb{R}
\]

\[
\tilde{\gamma}_q(x, y, \tau, \theta) = \left( \frac{x}{q^{w_x}}, \frac{y}{q^{w_y}}, \frac{\tau}{q^{w_{\tau}}}, e^{i \frac{\theta}{q^{w_{\theta}}}} \right)
\]

with the weights \( w_i \) defined in (4.21), and for \( 0 < q \leq 1 \) we define the Lie product \([A, B]_q = \gamma_q^{-1} \gamma_q(A) \gamma_q(B)\). Let \((SIM(2))\) be the simply connected Lie group generated by the Lie algebra \((T_\epsilon(SIM(2)), [\cdot, \cdot])\). The group products on these intermediate groups \((SIM(2))\) are given by,

\[
(x, y, \tau, \theta) \cdot (x', y', \tau', \theta') = (x + q^{w_x} \sin(q\theta)x' - q\sin(q\theta)y', y + q^{w_y} \sin(q\theta)y' + q\sin(q\theta)x', \tau + \tau', \theta + \theta').
\]

(4.22)

Note that the dilation on the Lie algebra coincides with the pushforward of the dilation on the group \( \gamma_q = (\tilde{\gamma}_q)_* \), and thus the left-invariant vector fields on \((SIM(2))\) are given by

\[
A_i^q|_g = (\tilde{\gamma}_q^{-1} \circ L_g \circ \tilde{\gamma}_q)_* A_i,
\]

\( \forall q \in (0, 1) \) which leads to,

\[
A_1^q|_g = q \left( \frac{1}{q} \partial_\theta \right) = \partial_\theta
\]

\[
A_2^q|_g = q \left[ e^{q\gamma} \left( \frac{\cos(q\theta)}{q} \partial_x + \frac{\sin(q\theta)}{q^2} \partial_y \right) \right] = e^{q\gamma} \left( \cos(q\theta) \partial_x + \frac{\sin(q\theta)}{q} \partial_y \right)
\]

(4.23)

\[
A_3^q|_g = q^2 \left[ e^{q\gamma} \left( -\frac{\sin(q\theta)}{q} \partial_x + \frac{\cos(q\theta)}{q^2} \partial_y \right) \right] = e^{q\gamma} \left( -q \sin(q\theta) \partial_x + \cos(q\theta) \partial_y \right)
\]

\[
A_4^q|_g = \partial_x.
\]

For example,

\[
A_3^q|_g = (\tilde{\gamma}_q^{-1} \circ L_g \circ \tilde{\gamma}_q)_* A_3 = (\tilde{\gamma}_q^{-1} \circ L_g)_* (\tilde{\gamma}_q^{-1})_* (A_3) = (\tilde{\gamma}_q^{-1} \circ L_g)_* (q^2 \partial_y)
\]

\[
= q^2 (\tilde{\gamma}_q^{-1})_* (\partial_y) = q^2 e^{q\gamma} (\tilde{\gamma}_q^{-1})_* (-\sin(q\theta) \partial_x + \cos(q\theta) \partial_y)
\]

\[
= q^2 e^{q\gamma} (-\sin(q\theta) \gamma_q^{-1}(\partial_x) + \cos(q\theta) \gamma_q^{-1}(\partial_y)) = q^2 e^{q\gamma} \left( \frac{-\sin(q\theta)}{q} \partial_x + \frac{\cos(q\theta)}{q^2} \partial_y \right).
\]

\(^3\)Recall that \((SIM(2)) = \exp(T_\epsilon(SIM(2)))\), Section 3.1.1.
Note that $[A_i, A_j]_q = \gamma_q^{-1}[\gamma_q(A_i), \gamma_q(A_j)] = \gamma_q^{-1}q^{w_i+w_j}[A_i, A_j] = \sum_{k=1}^{4} q^{w_i+w_j-w_k} c_{ij}^k A_k$ and therefore

$$[A_1, A_2]_q = A_3, \quad [A_1, A_3]_q = -q^2 A_2$$
$$[A_4, A_2]_q = q A_2, \quad [A_4, A_3]_q = q A_3.$$  \hfill (4.24)

Analogously to the case $q = 1$, $(SIM(2))_{q=1} = SIM(2)$ there exists an isomorphism of the Lie algebra at the unity element $T_c((SIM(2))_q)$ and the left-invariant vector fields on the group $L((SIM(2))_q)$:

$$A_i \leftrightarrow A_i^q \text{ and } A_j \leftrightarrow A_j^q \Rightarrow [A_i, A_j]_q \leftrightarrow [A_i^q, A_j^q]. \hfill (4.25)$$

It can be verified that the left invariant vector fields $A_i^q$ satisfy the same commutation relations as (4.24).

Consider the case $H \equiv \lim_{q \downarrow 0} (SIM(2))_q$. In this case the left-invariant vector fields are given by

$$A_0^q = \partial_t, \quad A_1^q = \partial_x + \theta \partial_y, \quad A_2^q = \partial_y, \quad A_3^q = \partial_r.$$  \hfill (4.26)

So, the homogeneous nilpotent contraction Lie group equals

$$H_3 = \lim_{q \downarrow 0} (SIM(2))_q \text{ and } SIM(2) = (SIM(2))_{q=1}/\{0\} \times \{0\} \times \{0\} \times 2\pi \mathbb{Z}, \hfill (4.27)$$

with the Lie algebra $L(H) = \text{span}\{\partial_t, \partial_x+\theta \partial_y, \partial_y, \partial_r\}$ and recall from Section 3.1.4, $L(SIM(2)) = \text{span}\{\partial_t, \partial_k, \partial_b, \partial_r\}$.

### 4.5.2 Gaussian Estimates for the Heat-kernels on $SIM(2)$

According to the general theory in [35], the heat-kernels $K_t^{q, D} : (SIM(2))_q \to \mathbb{R}^+$ (i.e. kernels for contour enhancement whose convolution yields diffusion on $(SIM(2))_q$) on the parametrized class of groups $(SIM(2))_q$, $q \in [0, 1]$ in between $SIM(2)$ and its nilpotent Heisenberg approximation $(SIM(2))_0$ satisfy the Gaussian estimates

$$|K_t^{q, D}(g)| \leq C t^{-\frac{3}{2}} \exp \left(-\frac{-b \|g\|_q^2}{4t}\right), \text{ with } C, b > 0 \text{ (constant), } g \in (SIM(2))_q, \hfill (4.28)$$

where the norm $\| \cdot \|_q : (SIM(2))_q \to \mathbb{R}^+$ is given by $\|g\|_q = \|\log((SIM(2))_q)(g)\|_q$. $\log((SIM(2))_q) : (SIM(2))_q \to T_c((SIM(2))_q)$, is the logarithmic mapping on $(SIM(2))_q$, which was computed explicitly for the case of $(SIM(2))_{q=1} = SIM(2)$ in Section 3.1.5, and where the weighted modulus, see [35, Prop 6.1], in our special case of interest is given by,

$$\left| \sum_{i=1}^{4} c_i^q A_i^q \right|_q = \sqrt{|c_1^q|^{2/w_1} + |c_2^q|^{2/w_2} + |c_3^q|^{2/w_3} + |c_4^q|^{2/w_4}}$$
$$= \sqrt{(c_1^q)^2 + (c_2^q)^2 + (c_3^q)^2 + (c_4^q)^2 + |c_1^q|^4}, \hfill (4.29)$$

where recall the weightings from (4.21).

**Remark.** The constants $b, C$ in (4.28) can be taken into account by the transformations $t \mapsto \frac{1}{t}$, $U \mapsto C_U U$ respectively.

**Lemma 4.4.** A sharp estimate for the front factor constant $C$ in (4.28) is given by

$$C = \frac{1}{4\pi D_{11} D_{22}} \frac{1}{\sqrt{D_{44}}}. \hfill (4.30)$$
Proof. According to general theory in [35] the choice of constant $C$ is uniform for all groups $(SIM(2))_{q}$, $q \in [0,1]$. Thus to determine $C$ we need to determine the front factor constant for the Green’s function $G$ of the following resolvent equation

$$((D_{11}(A_{1}^{0})^{2} + D_{22}(A_{2}^{0})^{2} + D_{44}(A_{4}^{0})^{2})) G(x,y,\tau,\theta) = +\delta_{e}$$

where $A_{i}^{0}$ denote the basis for $L(H)$ (recall $H = (SIM(2))_{q}[0]$) and $e$ is the identity of $H$. Since $\{A_{i}^{0}\}_{i=1,2}$ are independent of $\tau$ the Green’s function in our case is separable, i.e.

$$G(x,y,\tau,\theta) = G_{(SIM(2))_{q=0}}^{D_{11},D_{22}}(x,y,\theta) \cdot G_{\mathbb{R}}^{D_{44}}(\tau)$$

where $G_{(SIM(2))_{q=0}}^{D_{11},D_{22}}(x,y,\theta)$ and $G_{\mathbb{R}}^{D_{44}}(\tau)$ are Green’s function for linear diffusion on $SE(2)$ and $\mathbb{R}$ respectively. The Green’s function for the Heisenberg case of $SE(2)$ have been derived explicitly in [17] and the front factor is given by $\frac{1}{\sqrt{D_{11}D_{22}D_{44}}}$ and the Green’s function of diffusion on $\mathbb{R}$ is $G_{\mathbb{R}}^{D_{44}}(\tau) = \frac{|\tau|}{\sqrt{D_{44}}}$. Thus the choice of sharp estimate follows from multiplication of the front factor constants.

Using this theory we arrive at the Gaussian estimates for the case $SIM(2) = (SIM(2))_{q=1}$. Using the definition of $c_{i}$, $i \in \{1,2,3,4\}$ from (3.30) we arrive at,

$$|K_{t}^{q=1,D}(g)| \leq \frac{1}{4\pi t^{2}D_{11}D_{22}\sqrt{D_{44}}} \exp\left(-\frac{1}{4t}\left(\frac{\theta^{2}}{D_{11}} + \frac{(c_{2})^{2}}{D_{22}} + \frac{\tau^{2}}{D_{44}} + \frac{|c_{3}|}{\sqrt{D_{11}D_{22}D_{44}}}\right)\right) (4.31)$$

where,

$$c_{2} = \frac{(t_{\theta} - x\tau) + (-\theta\eta + \tau\xi)}{t(1 + e^{t\tau} - 2e^{t\eta}cos\theta)}, c_{3} = \frac{-(x\theta + y\tau) + (\theta\xi + \tau\eta)}{t(1 + e^{t\tau} - 2e^{t\eta}cos\theta)}.$$  

A problem with these estimates is that they might not be differentiable everywhere. This problem can be solved by applying the estimate

$$|a| + |b| \geq \sqrt{a^{2} + b^{2}} \geq \frac{1}{\sqrt{2}}(|a| + |b|),$$

which holds for all $a, b \in \mathbb{R}$, to the exponents of our Gaussian estimates. Thus we estimate the weighted modulus by the equivalent $(\forall q > 0)$ weighted modulus $|\cdot|_{q} : T_{c}(SIM(2))_{q} \to \mathbb{R}^{+}$ by $\left| \sum_{i=1}^{4} c_{i} A_{i}^{q} \right|_{q} = \sqrt{\sum (c_{i}^{q})^{2}}$, yielding the Gaussian estimate,

$$|K_{t}^{q=1,D}(g)| \leq \frac{1}{4\pi t^{2}D_{11}D_{22}\sqrt{D_{44}}} \exp\left(-\frac{1}{4t}\left(\frac{\theta^{2}}{D_{11}} + \frac{(c_{2})^{2}}{D_{22}} + \frac{\tau^{2}}{D_{44}} + \frac{|c_{3}|^{2}}{D_{11}D_{22}D_{44}}\right)\right). (4.32)$$

Figure 4.4 shows the typical structure of these enhancement kernels. In Figure 4.6 we provide the enhanced image for a variety of cases. The results clearly indicate that even with linear diffusion we achieve very good results due to the natural construction of the transform and operations on the $SIM(2)$ group. As seen in the figure, we are able to handle straight and curved lines for different scales. In addition this method can also enhance an image with single scale repeating structure.

Figure 4.5 clearly indicates the advantage of scale-OS constructed using the $SIM(2)$ group over the OS constructed over $SE(2)$ group, see [15, 17], in the handling of multi-scale images. Though there is definite improvement due to usage of $SIM(2)$ diffusion on $SE(2)$ OS as
compared to $SE(2)$ diffusion\(^4\) on $SE(2)$ OS, there are added artefacts such as unnecessary increase in luminescence. One obvious reason for this is that we convolve each $SIM(2)$ kernel with the entire $SE(2)$ OS thus leading to additional diffusion in multiple regions. Though this causes an increase in the brightness of thin lines, there is also increased brightness in the thick lines.

### 4.6 Nonlinear Left Invariant Diffusions on $SIM(2)$

From (4.13) we recall that the general left-invariant second order evolution equation under consideration in the context of scale-OS. In this section we consider nonlinear evolutions without convection given by,

$$
\begin{align*}
\partial_t U(g,t) &= Q^{D(U),a=0} U(g,t), \ g \in SIM(2), \ t > 0, \\
U(\cdot, t = 0) &= W_{\psi} f(\cdot), \ g \in SIM(2),
\end{align*}
$$

where the conductivity $D(U)$ depends on the local differential structure of $(g,t) \mapsto U(g,t)$. The advantage of this more elaborate nonlinear, adaptive diffusions on invertible scale-OS is that the domain of the scale-OS is the similitude group, which has a much richer structure than the domain of images $\mathbb{R}^2$, allowing us to deal with crossing curves (with different scales) in images where the commonly used nonlinear adaptive diffusion techniques usually fail, see Section 1.2.1.

We begin by choosing an appropriate metric on $SIM(2)$. We then use a Cartan connection to write (4.33) in the covariant derivative form and prove that diffusions locally take place only along exponential curves in $SIM(2)$ group even in the nonlinear adaptive case. We finally present a method to extract spatial curvature from the scale-OS. The ideas used in this section are analogous to nonlinear adaptive diffusion on the $SE(2)$ group by Duits and Franken [16, 18].

\(^4\)See [17] for the Gaussian estimates of the Green’s function of linear diffusion on the $SE(2)$ case.
Figure 4.5: Comparison of enhancement via Gaussian estimates for heat kernels for different cases. From left to right: original image; original image+noise; enhancement on $SE(2)$-OS via Gaussian estimates for $SE(2)$ heat kernels, see [17]; enhancement on $SE(2)$-OS via Gaussian estimates for $SIM(2)$ heat kernels; enhancement on scale-OS via Gaussian estimates for $SIM(2)$ heat kernels.
Parameters used: $N_\theta = 20, N_s = 5$; $SE(2)$ estimates: $D_{11} = 0.00015, D_{22} = 1, t = 15$; $SIM(2)$ estimates: $D_{11} = 0.05, D_{22} = 1, D_{44} = 0.01, t = 0.7$.

Figure 4.6: Plot of enhancement on a scale-OS via linear diffusion on $SIM(2)$ group using Gaussian estimated provided in this section. From left to right: Original Image; Original Image+Noise; Grey-Scale Transformation ($p = 1.6$); Contour enhancement. Row 1: Image with curved contours; Row 2: Image with repeating single scale structure; Row 3: Noisy microscopy image of bone tissue; Row 4: Noisy microscopy image of muscle cell.
4.6.1 Design of the metric on $SIM(2)$

In order to generalize the coherence enhancing diffusion schemes (see Section 1.2.1) on images to scale-OS we have to replace the left invariant vector fields $\{\partial_x, \partial_y\}$ on the additive group $(\mathbb{R}^2, +)$, by the left-invariant vector fields on $SIM(2)$ as in (4.13). In order to keep track of orthogonality and parallel transport in such diffusions we need an invariant first fundamental form $\mathcal{G}$ on $SIM(2)$, rather than the trivial, bi-invariant (i.e. left and right invariant), first fundamental form on $(\mathbb{R}^2, T(\mathbb{R}^2))$, where each tangent space $T_x(\mathbb{R}^2)$ is identified with $T_0(\mathbb{R}^2)$ by standard parallel transport on $\mathbb{R}^2$, i.e. $\mathcal{G}_{\mathbb{R}^2}(x, y) = x \cdot y = x^1 y^1 + x^2 y^2$.

Recall Theorem 4.2 which essentially states that operators $\Phi$ on scale-OS should be left-invariant (i.e. $L_y \circ \Phi = \Phi \circ L_y$) and not right-invariant in order to ensure that the effective operator $\Upsilon_\psi$ on the image is a Euclidean invariant operator. This suggests that the first fundamental form required for our diffusions on $SIM(2)$ should be left-invariant. The following theorem characterizes the formulation of a left-invariant first fundamental form w.r.t the $SIM(2)$ group.

**Theorem 4.5.** The only real valued left-invariant (symmetric, positive, semidefinite) first fundamental form $\mathcal{G} : SIM(2) \times T(SIM(2)) \times T(SIM(2)) \to \mathbb{C}$ on $SIM(2)$ are given by,

$$\mathcal{G} = \sum_{i=1}^{4} \sum_{j=1}^{4} g_{ij} \omega^i \otimes \omega^j, \quad g_{ij} \in \mathbb{R}. \quad (4.34)$$

where the dual basis $\{\omega^1, \omega^2, \omega^3, \omega^4\} \subset \mathcal{L}(SIM(2))^*$ of the dual space $(\mathcal{L}(SIM(2)))^*$ of the vector space $\mathcal{L}(SIM(2))$ of left-invariant vector fields spanned by

$$\{A_1, A_2, A_3, A_4\} = \{\partial_y, e^\tau (\cos \theta \partial_x + \sin \theta \partial_y), e^\tau (-\sin \theta \partial_x + \cos \theta \partial_y), \partial_x\}, \quad (4.35)$$

obtained by applying the operator $d\mathcal{R} : T_e(G) \to \mathcal{L}(G)$, defined as

$$(d\mathcal{R}(A))\phi(g) = \lim_{t \downarrow 0} \left(\mathcal{R}_{\exp(tA)}\phi)(g) - \phi(g) \right), \quad A \in T_e(G), \phi \in \mathbb{L}_2(G), g \in G. \quad (4.36)$$

to the standard basis in the Lie algebra

$$\{A_1, A_2, A_3, A_4\} = \{\partial_y, \partial_x, \partial_y, \partial_x\} \subset T_e(SIM(2)), \quad (4.37)$$

is given by

$$\{\omega^1, \omega^2, \omega^3, \omega^4\} = \{d\theta, \frac{1}{e^\tau} (\cos \theta dx + \sin \theta dy), \frac{1}{e^\tau} (-\sin \theta dx + \cos \theta dy), d\tau\}. \quad (4.38)$$

**Proof.** Recall that $d\mathcal{R}$ yields the fundamental isomorphism between the Lie algebra $T_e(SIM(2))$ and $\mathcal{L}(SIM(2))$. The dual basis (4.38) satisfy $\{A^*_1, A^*_j\} = \delta^*_j$. A first fundamental form $\mathcal{G} : SIM(2) \times T(SIM(2)) \times T(SIM(2)) \to \mathbb{C}$ on $SIM(2)$ by definition is

- left invariant if $\forall h, g \in SIM(2) \forall X, Y \in T(SIM(2)) : \mathcal{G}_h(X_h, Y_h) = \mathcal{G}_{gh}((L_g)_*X_h, (L_g)_*Y_h)$.
- right invariant if $\forall h, g \in SIM(2) \forall X, Y \in T(SIM(2)) : \mathcal{G}_h(X_h, Y_h) = \mathcal{G}_{gh}((R_g)_*X_h, (R_g)_*Y_h)$.
- inversion-invariant if $\forall h, g \in SIM(2) \forall X, Y \in T(SIM(2)) : \mathcal{G}_{r_e(g)}((r_e)_*X_g, (r_e)_*Y_g) = \mathcal{G}_g(X_g, Y_g)$.
- Ad-invariant if $\forall h, g \in SIM(2) \forall X, Y \in T(SIM(2)) : \mathcal{G}_{h_{gh}}((Ad(h)X_g, Ad(h)*Y_g) = \mathcal{G}_g(X_g, Y_g)$.
• inversion-invariant if \( \forall h, g \in SIM(2) \ \forall X, Y \in T(SIM(2)) : G_{r_h(g)}((r_h)_*X, (r_h)_*Y) = G_g((X, Y)) \).

The dual tangent space \( (T_g(SIM(2))^*) \), \( g \in SIM(2) \) is spanned by \( \{\omega^1|_g, \omega^2|_g, \omega^3|_g, \omega^4|_g\} \). Thus for all \( g \in SIM(2) \) there exist numbers \( g_{ij} \in \mathbb{R} \), \( i, j \in \{1, 2, 3, 4\} \) such that

\[
G_g = \sum_{i=1}^{4} \sum_{j=1}^{4} g_{ij}(g) \omega^i|_g \otimes \omega^j|_g.
\] (4.39)

\( G \) is left-invariant iff \( G_g(A_i|_g, A_j|_g) = G_e((L_{g^{-1}})_* A_i|_g, (L_{g^{-1}})_* A_j|_g) = G_e(A_i, A_j), \ \forall i, j \in \{1, 2, 3, 4\} \) \( \forall g \in SIM(2) \) which implies that \( \forall i, j \in \{1, 2, 3, 4\} \), \( g_{ij}(g) = g_{ij} \).

We consider the Maurer-Cartan form on \( SIM(2) \) (discussed in the remainder of the subsection) and impose the following left-invariant, first fundamental form \( G_\beta : SIM(2) \times T(SIM(2)) \times T(SIM(2)) \rightarrow C \) on \( SIM(2) \),

\[
G_\beta = \sum_{i=1}^{4} \sum_{j=1}^{4} g_{ij} \omega^i \otimes \omega^j = d\theta \otimes d\theta + \beta^2 d\xi \otimes d\xi + \beta^2 d\eta \otimes d\eta + \delta^2 d\tau \otimes d\tau.
\] (4.40)

Here the parameters \( \beta \) (physical dimensions equal \( 1/[\text{Length}] \)) and \( \delta \) (dimensionless) should be considered as a fundamental parameters which relate the notion of distances in the sets \( \{(x, \tau, \theta) : x \in \mathbb{R}^2\}, \{(x, \tau, \theta) : \tau \in \mathbb{R}\} \) and \( \{(x, \tau, \theta) : \theta \in [0, 2\pi]\} \).

In order to understand the meaning of the next two theorems we need some definitions from differential geometry. See [34] for more details on these concepts.

**Definition 4.6.** Let \( M \) be a smooth manifold and \( G \) be a Lie group. A principle fiber bundle \( P_G := (P, M, \pi, R) \) above a manifold \( M \) with structure group \( G \) is a tuple \( (P, M, \pi, R) \) such that \( P \) is a smooth manifold (called the total space of the principle bundle), \( \pi : P \rightarrow M \) is a smooth projection map with \( \pi(P) = M \) and \( \pi(u \cdot a) = \pi(u), \ \forall u \in P, \ a \in G, \ R \) a smooth right action \( R_g p = p \cdot g, \ p \in P, \ g \in G \). Finally it should satisfy the "local triviality" condition: For each \( p \in M \) there is a neighborhood \( U \) of \( p \) and a diffeomorphism \( t : \pi^{-1}(U) \rightarrow U \times G \) of the form \( t(u) = (\pi(u), \phi(u)) \) where \( \phi \) satisfies \( \phi(u \cdot a) = \phi(u)a \) where the latter product is in \( G \).

**Definition 4.7.** A principle fiber bundle \( P_G := (P, M, \pi, R) \) is commonly equipped with a Cartan-Ehresmann connection form \( \omega \). This by definition is a Lie algebra \( T_e(G) \)-valued 1-form \( \omega : P \times T(P) \rightarrow T_e(G) \) on \( P \) such that

\[
\omega(dR(A)) = A \text{ for all } A \in T_e(G)
\]

\[
\omega((R_h)_*A) = Ad(h^{-1})\omega(A) \text{ for all vector fields } A \text{ in } G.
\] (4.41)

It is also common practice to relate principle fiber bundles to vector bundles. Here one uses an external representation \( \rho : G \rightarrow F \) into a finite-dimensional vector space \( F \) of the structure group to put an appropriate vector space structure on the fibers \( \{\pi^{-1}(m)\} m \in M \) in the principle fiber bundles.

**Definition 4.8.** Let \( P \) be a principle fiber bundle with finite dimensional structure group \( G \). Let \( \rho : G \rightarrow F \) be a representation in a finite-dimensional vector space \( F \). Then the associated vector bundle is denoted by \( P \times_F F \) and equals the orbit space under the right action

\[
(P \times F) \times G \rightarrow P \times F \text{ given by } ((u, X), g) \mapsto (ug, \rho(g)X),
\]

for all \( g \in G \), \( X \in F \) and \( u \in P \).
Theorem 4.9. The Maurer-Cartan form \( w \) on \( SIM(2) \) can be formulated as
\[
\begin{align*}
\mathbf{w}_g(X_g) &= \sum_{i=1}^{4} (\omega^i|_{g}, X_g) A_i, \quad X_g \in T_g(SIM(2)),
\end{align*}
\]
where \( \{\omega^i\}_{i=1}^{4} \) is given by (4.38) and \( A_i = A_i|_{e} \); recall (4.35). It is a Cartan-Ehresmann connection form on the principle fiber bundle \( P = (SIM(2), e, SIM(2), \mathcal{L}(SIM(2))) \), where \( \pi(g) = e, R_g u = ug, u, g \in SIM(2) \). Let \( \text{Ad} \) denote the adjoint action of \( SIM(2) \) on its own Lie algebra \( T_e(SIM(2)) \), i.e. \( \text{Ad}(g) = (R_{g^{-1}}L_g)* \), i.e. the push-forward of conjugation. Then the adjoint representation of \( SIM(2) \) on the vector space \( \mathcal{L}(SIM(2)) \) of left-invariant vector-fields is given by
\[
\tilde{\text{Ad}}(g) = d\mathcal{R} \circ \text{Ad}(g) \circ \mathbf{w}.
\]
The adjoint representation gives rise to the associated vector bundle \( SIM(2) \times \tilde{\text{Ad}} \mathcal{L}(SIM(2)) \). The corresponding connecting form on this vector bundle is given by
\[
\mathbf{\tilde{w}} = A_2 \otimes \omega^3 \land \omega^1 + A_3 \otimes \omega^1 \land \omega^2 + A_2 \otimes \omega^4 \land \omega^2 + A_3 \otimes \omega^4 \land \omega^3.
\]
Then \( \mathbf{\tilde{w}} \) yields the following 4 \by\ 4 matrix-valued 1-form:
\[
\mathbf{\tilde{w}}^k_j(t) := -\mathbf{\tilde{w}}(\omega^k, \cdot, A_j), \quad k, j \in \{1, 2, 3, 4\}
\]
on the frame bundle, where the sections are moving frames. Let \( \{\mu_k\}_{k=1}^{4} \) denote the sections in the tangent bundle \( E := (SIM(2), T(SIM(2))) \) which coincides with the left-invariant vector fields \( \{A_k\}_{k=1}^{4} \). Then the matrix-valued 1-form (4.35) yields the Cartan connection \( D \) on the tangent bundle \( (SIM(2), T(SIM(2))) \) given by the covariant derivatives
\[
D_{X|_{\gamma(t)}}(\mu(\gamma(t))) := D(\mu(\gamma(t)))(X|_{\gamma(t)})
\]
\[
= \sum_{k=1}^{4} \hat{\gamma}^k(\gamma(t)) \mu_k(\gamma(t)) + \sum_{k=1}^{4} \hat{\gamma}^k(\gamma(t)) \sum_{k=1}^{4} \mathbf{\tilde{w}}^k_j(X|_{\gamma(t)}) \mu_j(\gamma(t))
\]
(4.46)
with \( \hat{a}^k = \gamma^k(t)(A_i|_{\gamma(t)} a^k) \), for all tangent vectors \( X|_{\gamma(t)} = \gamma^i(t)A_i|_{\gamma(t)} \) along a curve \( t \mapsto \gamma(t) \in SIM(2) \) and all sections \( \mu(\gamma(t)) = \sum_{k=1}^{4} a^k(\gamma(t)) \mu_k(\gamma(t)) \). The Christoffel symbols in (4.46) are constant \( \Gamma_{ik}^j = -c_{ik}^j \), with \( c_{ik}^j \) the structure constants of the Lie algebra \( T_e(SIM(2)) \).

Proof. As in Theorem 4.5, we set \( \{A_1, A_2, A_3, A_4\} = \{\partial_t, \partial_x, \partial_y, \partial_r\} \) as a basis for the Lie algebra \( T_e(SIM(2)) \) and \( \{A_1, A_2, A_3, A_4\} = \{dR(A_1), dR(A_2), dR(A_3), dR(A_4)\} = \{\partial_{\theta}, \partial_{\xi}, \partial_{\eta}, \partial_{\theta_r}\} \) as the basis for the space \( \mathcal{L}(SIM(2)) \) of left-invariant vector fields with corresponding dual basis \( \{\omega^1, \omega^2, \omega^3, \omega^4\} \subset (\mathcal{L}(SIM(2)))^* \). The Maurer-Cartan form \( \omega : (SIM(2), T(SIM(2))) \rightarrow T_e(SIM(2)) \) is defined as
\[
\mathbf{w}_g(Y_g) = (L_{g^{-1}})_*Y_g,
\]
where \( (L_{g^{-1}})_* \) denotes the push-forward of the inverse left multiplication \( h \mapsto L_{g^{-1}}h = g^{-1}h \) i.e. \( \mathbf{w}_g(Y_g) \phi = Y_g(\phi \circ L_{g^{-1}}) \) for all \( \phi : \Omega_e \rightarrow \mathbb{R} \) smooth and defined on some open local set \( \Omega_e \).
around the unity $e$. Recall that the left-invariant vector fields $\{A_i\}_{i=1}^4$ satisfy $A_i|_g = (L_g)_*A_i$ and therefore the dual elements (the corresponding co-vector fields) are obtained by the pull-back from $T_e(SIM(2))$, i.e. $\omega^i|_c = (L^{-1}_g)_*\omega^i|_e$ since $\langle (L^{-1}_g)_*\omega^i|_e, (L_g)_*A_j \rangle = \langle \omega^i|_e, A_j \rangle = \delta^i_j$.

Now for any $X_g \in T_g(SIM(2))$, direct computation yields (4.42),

$$w_g(X_g) = (L_{g^{-1}})_*X_g = \sum_{i=1}^4 (\omega^i|_e, (L_{g^{-1}})_*X_g)A_i$$

$$= \sum_{i=1}^4 ((L_g)_*\omega^i|_e, X_g)A_i = \sum_{i=1}^4 (\omega^i|, X_g)A_i.$$

Now we show that the Maurer-Cartan form indeed forms a Cartan-Ehresmann connection form, recall Def. 4.7, on the principle fiber bundle, $P = (SIM(2), e, SIM(2), \mathcal{L}(SIM(2)))$. Now recall from Theorem 4.5 that the left invariant vector fields are obtained from the operator $dR$, i.e. $A_i = dR(A_i)$. The Maurer-Cartan form does the reverse in the sense that it connects each tangent space $T_g(SIM(2))$ to $T_e(SIM(2))$. To this end note that

$$\lim_{h \to 0} \frac{\phi(g \cdot h A_i) - \phi(g)}{h} =: \left. (dR(A_i))_g \phi \right| (L_g)_*A_i \phi = A_i \phi \circ L_g \in \mathbb{R}$$

for all $g \in SIM(2)$ and all smooth $\phi : \Omega_g \to \mathbb{R}$. Therefore we have

$$\forall i \in \{1, 2, 3, 4\} : w \circ dR(A_i) = w(A_i) = A_i \Leftrightarrow w \circ dR = I.$$

The second requirement in Def. 4.7 follows from the following computation,

$$w_g((R_h)_*Y_g) = (L_{h^{-1}} \circ L_{g^{-1}})_*((R_h)_*Y_g) = (L_{h^{-1}})_* \circ (L_{g^{-1}})_* \circ (R_h)_*Y_g = Ad(h^{-1})w_gY_g.$$ 

To show that equality (4.43) holds we note that the left multiplication $L_g$ and the right multiplication $R_g$ commute and this leads to

$$(R_g^{-1}L_g)_* = (L_gR_g^{-1})_* = (L_g)_*(R_g^{-1})_* = (L_g)_*(R_g^{-1} L_g)_*(L_g^{-1})_*,$$

which implies that $\tilde{A}d(g) = dR \circ Ad(g) \circ w$. The adjoint representation $Ad : SIM(2) \to GL(T_e(SIM(2)))$ coincides with the derivative of the conjugate automorphism $h \mapsto \text{conj}(g)(h) = ghg^{-1}$ evaluated at $e$, i.e. $Ad(g) = D_e\text{conj}(g) = (R_g^{-1} L_g)_*$.

Remark. Here $GL(T_e(SIM(2)))$ is the collection of linear operators on the Lie algebra $T_e(SIM(2))$. Note that each linear operator $\overline{Q} \in GL(T_e(SIM(2)))$ on $T_e(SIM(2))$ is 1-to-1 related to bilinear form $Q$ on $(T_e(SIM(2)))^* \times T_e(SIM(2))$ by means of

$$\langle B, \overline{Q}A \rangle = Q(B, A), \text{ for all } B \in (T_e(SIM(2)))^*, A \in T_e(SIM(2)) \text{ and}$$

$$\overline{Q} = \sum_{i=1}^4 Q(\omega^i|_e, \cdot)A_i.$$ 

So a basis for $GL(T_e(SIM(2)))$ is given by $\{\omega^i|_e \otimes A_j\}_{i, j \in \{1, 2, 3, 4\}}$. For the simplicity of notation we omit the overline and write $\omega^i|_e \otimes A_j$ as it is clear from context whether we mean the bilinear form or the linear mapping.

Recall Def. 4.8 of an associated vector bundle and set

$$P = SIM(2), \quad M = e, \quad G = SIM(2), \quad F = \mathcal{L}(SIM(2)), \quad \rho = \tilde{A}d, \quad \pi(g) = e, \quad R_gu = ug.$$ 

(4.48)

where $\mathcal{L}(SIM(2))$ denotes the Lie algebra of left invariant vector fields on $SIM(2)$ and $\tilde{A}d$ the adjoint representation of $SIM(2)$ into $GL(\mathcal{L}(SIM(2)))$ given by

$$\tilde{A}d(g)X = (R^{-1}_g L_g)_*X, \quad X \in \mathcal{L}(SIM(2)), \quad g \in SIM(2).$$
A connection \( \omega \) on a principle fiber bundle \( P \) is 1-to-1 related to a connection \( \tilde{\omega} \) on the vector bundle \( P \times_{\rho} F \) by means of

\[
\omega = \sum_j A_j \otimes dx^j \leftrightarrow \tilde{\omega} = \sum_j \rho_\ast(A_j) \otimes dx^j,
\]

where \( \{dx^j\} \) are dual forms on \( F \), see [31] for more details on this bijection. In our case we have \( F = \mathcal{L}(SIM(2)) \) and the corresponding dual forms \( \{\omega^j\}_{j=1}^4 \). Note that we applied the convention mentioned in the remark. So in our case (4.48) the push-forward \( \rho_\ast \) of \( \rho = \text{Ad} \) equals

\[
(\text{Ad})_\ast = (dR \circ \text{Ad}_\ast)(A_j) = (dR \circ \text{ad} \circ \omega)(A_j).
\]

From \( \tilde{\omega} \) defined in (4.49) we can construct the 16-connection 1-form \( \{\tilde{\omega}^k_j(\cdot)\}_{k,j=1}^4 \) via (4.45) which together form a \( 4 \times 4 \) matrix-valued 1-form on the frame bundle where the sections are moving frames.

Let \( \{\mu_k\}_{k=1}^4 \) denote the sections in \( (SIM(2), T(SIM(2))) \) which coincide respectively with the left-invariant vector fields \( \{A_k\}_{k=1}^4 \). Then the Cartan connection \( D \) on the vector bundle \( SIM(2) \times_{\text{Ad}} \mathcal{L}(SIM(2)) \) is isomorphic to the tangent bundle \( SIM(2) \times T(SIM(2)) \) which equals

\[
D := d + \tilde{\omega} \text{ with } \tilde{\omega}(\mu_k)(\cdot) := \sum_{j=1}^4 \tilde{\omega}^j_k(\cdot)\mu_j.
\]

or more precisely the covariant derivatives are given by

\[
D_{X|\gamma(t)}(\mu(\gamma(t))) := (D\mu(\gamma(t)))(X|\gamma(t))
\]

\[
= \sum_{k=1}^4 \hat{a}^k \mu_k(\gamma(t)) + \sum_{k=1}^4 a^k(\gamma(t)) \sum_{j=1}^4 \tilde{\omega}^j_k(X|\gamma(t))\mu_j(\gamma(t))
\]

\[
= \sum_{k=1}^4 \hat{a}^k \mu_k(\gamma(t)) + \sum_{k=1}^4 \gamma^i(\gamma(t)) a^k(\gamma(t)) \Gamma^j_{ik} \mu_j(\gamma(t)),
\]

with \( \hat{a}^k = \dot{\gamma}^i(t)(A_i|\gamma(t))a^k(\gamma(t)) \), for all curves \( \gamma : \mathbb{R} \to SIM(2) \) and tangent vectors \( X|\gamma(t) = \sum_{i=1}^4 \dot{\gamma}^i(t)A_i|\gamma(t) \in T_{\gamma(t)}(SIM(2)) \) and all sections

\[
\mu(\gamma(t)) = \sum_{k=1}^4 a^k(\gamma(t))\mu_k(\gamma(t))
\]

and where the Christoffel symbols \( \Gamma^k_{ij} \) are constant

\[
\Gamma^k_{ij} = \tilde{\omega}^k_j(A_i) = -\tilde{\omega}(\omega^k, A_i, A_j) = -c^k_{ij} = c^k_{ji},
\]
As seen in the theorem above, we define the notion of covariant derivatives independent of the metric \( G \) on \( SIM(2) \), which is the underlying principle behind the Cartan connection. Although in principle these two entities need not be related, in Lemma 4.11 we show that the connection induced above is metric compatible.

**Definition 4.10.** Let \((M, G)\) be a Riemannian manifold (or pseudo-Riemannian manifold) where \( M \) and \( G \) denote the manifold and the metric defined on it respectively. Let \( \nabla \) denote a connection on \((M, G)\). Then \( \nabla \) is called metric compatible with respect to \( G \) if

\[
\nabla_Z G(X,Y) = G(\nabla_Z X, Y) + G(X, \nabla_Z Y),
\]

for all \( X, Y, Z \in T(M) \).

**Lemma 4.11.** The Cartan connection \( D \) on \((SIM(2), T(SIM(2)))\) is metric compatible with respect to \( G_{\beta, \delta} : SIM(2) \times T(SIM(2)) \times T(SIM(2)) \rightarrow C \) on \( SIM(2) \) defined in (4.40).

**Proof.** We first note that \( D_A G_{\beta, \delta}(A_j, A_k) = 0, \ i,j, k \in \{1, 2, 3, 4\} \) as the covariant derivative of a scalar field is the same as partial derivative. Here \( \{A_i\}_{i=1}^4 \) denote the basis of the left invariant vector fields \( L(SIM(2)) \). The following brief computation

\[
G_{\beta, \delta}(D_A A_j, A_k) + G_{\beta, \delta}(A_j, D_A A_k) = G_{\beta, \delta}(c^l_{ij} A_l, A_k) + G_{\beta, \delta}(A_j, c^m_n A_m) = -c^l_{ij} - c^m_n,
\]

where we have used the fact that \( \Gamma^k_{ij} = c^k_{ij} \), along with non-zero structure constants \( c^3_{12} = -c^3_{21} = c^3_{31} = -c^3_{43} = c^4_{24} = c^4_{24} = -c^4_{24} = 1 \), leads to the result. \( \square \)

The next theorem relates the previous results on Cartan connections and covariant derivatives to the nonlinear diffusion schemes on \( SIM(2) \).

**Theorem 4.12.** The covariant derivative of a co-vector field \( a \) on the manifold \((SIM(2), G_{\beta, \delta})\) is a \((0,2)\)-tensor field with components \( \nabla_j a_i = A_j a_i - \Gamma^i_{lj} a_l \), whereas the covariant derivative of a vector field \( v \) on \( SIM(2) \) is a \((1,1)\)-tensor field with components \( \nabla_j v^i = A_j v^i + \Gamma^i_{jk} v^k \), where we have made use of the notation \( \nabla_j := D_{A_j} \). The Christoffel symbols equal minus the structure constants of the Lie algebra \( L(SIM(2)) \), i.e. \( \Gamma^k_{ij} = -c^k_{ij} \). The Christoffel symbols are anti-symmetric as the underlying Cartan connection \( D \) has constant torsion. The left-invariant equations (4.33) can be rewritten in covariant derivatives

\[
\begin{align*}
\partial_t W(g, s) &= \sum_{i,j=1}^{4} A_i ((D_{ij}(W))(g, s)) A_j W(g, s) = \sum_{i,j=1}^{4} \nabla_i ((D_{ij}(W))(g, s)) \nabla_j W(g, s), \\
W(g, 0) &= W_0 f(g), \quad \text{for all } g \in SIM(2), \ s > 0.
\end{align*}
\]

(4.53)

Both convection and diffusion in the left-invariant evolution equations (4.33) take place along the exponential curves in \( SIM(2) \) which are covariantly constant curves with respect to the Cartan connection. These curves \( t \mapsto (x(t), y(t), \tau(t), \theta(t)) \) in \( \mathbb{R}^2 \times \mathbb{R}^2 \times SO(2) \) are given by,

\[
x(t) = \frac{1}{c_1 + c_4} \left[ e^{\kappa_2 c_1} \left( -\sin[\theta_0] + e^{\kappa_4} \sin[t c_1 + \theta_0] \right) c_2 + \left( -\cos[\theta_0] + e^{\kappa_4} \cos[t c_1 + \theta_0] \right) c_3 \right] + c_2 x_0 \\
+ c_4 \left[ e^{\kappa_2} \left( -\cos[\theta_0] + e^{\kappa_4} \cos[t c_1 + \theta_0] \right) c_2 + e^{\kappa_4} \sin[t c_1 + \theta_0] c_3 + c_4 x_0 \right]
\]

\[
y(t) = \frac{1}{c_1 + c_4} \left[ e^{\kappa_2 c_1} \left( \cos[\theta_0] - e^{\kappa_4} \cos[t c_1 + \theta_0] \right) c_2 + \left( -\sin[\theta_0] + e^{\kappa_4} \sin[t c_1 + \theta_0] \right) c_3 \right] + c_3 y_0 \\
+ c_4 \left[ e^{\kappa_2} \left( -\sin[\theta_0] + e^{\kappa_4} \sin[t c_1 + \theta_0] \right) c_2 + e^{\kappa_4} \left( -\cos[\theta_0] + e^{\kappa_4} \cos[t c_1 + \theta_0] \right) c_3 + c_4 y_0 \right]
\]

\[
\tau(t) = t c_3 + \tau_0
\]

\[
\theta(t) = t c_1 + \theta_0,
\]

(4.54)

where \( c_1^2 + c_4^2 \neq 0 \).
In this subsection we provide algorithms for curvature estimation at each point of the manifold. To achieve this we first present a few preliminaries.

Remark. Though the connection $\nabla$ is torsion-free, i.e. the torsion tensor $T(X,Y) = D_XY - D_YX - [X,Y]$ is constant, it follows from a simple computation

$$ T(A_i,A_j) = D_{A_i} A_j - D_{A_j} A_i - [A_i,A_j] = \sum_{k=1}^{4} (\Gamma^k_{ij} A_k - \Gamma^k_{kj} A_i - c^k_{ij} A_k) = -3 \sum_{k=1}^{4} c^k_{ij} A_k. $$

The covariant constant curves are by definition given by $D_\gamma \dot{\gamma} = 0$ on the tangent bundle $(SIM(2), T(SIM(2)))$

$$ D_\gamma \dot{\gamma} = D_{\gamma(t)} A_{\gamma(t)} = \dot{\gamma}^i A_i |_{\gamma(t)} = \ddot{\gamma}^i A_i |_{\gamma(t)} + \dot{\gamma}^i \dot{\gamma}^k \Gamma^i_{kj} A_j = \ddot{\gamma}^i A_i |_{\gamma(t)} = 0, $$

where we have made use of the Einstein’s summation convention and applied automatic summation over double indices and where $\Gamma^i_{ij} = -\Gamma^i_{ji} = c^k_{ij} = -c^k_{ji}$. Note that the tangent vectors to these autopo-parallel curves have constant coefficients with respect to $\{A_1, A_2, A_3, A_4\}$ as $\forall t > 0$, $\dot{\gamma}^i(t) = \langle \omega^i |_{\gamma(t)}, \dot{\gamma}(t) \rangle = \gamma(t), \dot{\gamma}(0) = c^i \in \mathbb{R}$, $i \in \{1,2,3,4\}$. For smooth $U : SIM(2) \to \mathbb{C}$ one has

$$ \frac{d}{dt} U(\gamma(t)) = \lim_{h \to 0} \frac{U(\gamma(t+h) - U(\gamma(t)))}{h} = (dR(\sum_{i=1}^{4} c^i A^i)U)(\gamma(t)) $$

$$ = \sum_{i=1}^{4} c^i (dR(A_i)U)(\gamma(t)) = \sum_{i=1}^{4} c^i A_i U |_{\gamma(t)}, $$

where $\gamma(t) = g_0 e^{\sum_{i=1}^{4} c^i A^i}$. Therefore these curves $\gamma(t)$ coincide with the exponential curves in $SIM(2)$, see (3.28). As mentioned earlier the connection $D$ is a Koszul connection which has the property that $\nabla_i (U A_j) \phi = \nabla_i (U \nabla_j \phi) = U \nabla_i \nabla_j \phi + (\nabla_i U) \nabla_j \phi$ for all $U \in C^1(SIM(2))$ and all smooth $\phi \in C^\infty(SIM(2))$. Set $U = D_{\gamma} (W(\cdot,s))$ and $\phi = W(\cdot,s)$ for all $s > 0$, use $D_{\gamma(t)} = D_{\gamma(t)}$ and $\Gamma^i_{ij} = -\gamma^k_{ij}$, take the sum over both indices $i, j$ and (4.53) follows.

4.6.2 Extraction of spatial curvature from scale-OS

Let $U : SIM(2) \to \mathbb{R}^+$ be a smooth function on $SIM(2)$. For instance this could be the absolute value $U = |W_\mu f| = \sqrt{(\Re(W_\mu f))^2 + (\Im(W_\mu f))^2}$ of an orientation score of an image. In this subsection we provide algorithms for curvature estimation at each $g_0 \in SIM(2)$ in the domain of $U$, by finding the exponential curves through $g_0$ that fits $U$ locally in an optimal way. To achieve this we first present a few preliminaries.

The inner product between two left-invariant vector fields is given by applying the first fundamental form (4.40) on $T(SIM(2)) \times T(SIM(2))$ and is denoted by $(\cdot, \cdot)_\nu$,

$$ (c^i A_i, c^j A_j)_\nu = \sum_{i,j=1}^{4} g_{ij} c_1^i c_1^j = c_1^i c_2^i + \beta^2 c_1^i c_2^j + \beta^2 c_1^j c_2^i + \delta^2 c_1^i c_2^j $$

Also called "auto-parallel" curves.
where we use the convention \( c_1^1 = c^\theta_k, c_2^1 = c^\xi_k, c_3^1 = c^\tau_k, c_4^1 = c^\nu_k, k = 1, 2 \). The norm of a left-invariant vector field \( c^iA_i \) is given by

\[
|c^iA_i|_\nu = \sqrt{(c^iA_i, c^jA_j)} = \sqrt{(c^\theta\theta)^2 + (c^\xi\xi)^2 + (c^\tau\tau)^2 + (c^\nu\nu)^2} = \|c\|_\nu,
\]

(4.57)

with \( c = (c^1, c^2, c^3, c^4) \in \mathbb{R}^4 \). Note that the norm \( \| \cdot \|_\nu : \mathcal{L}(SIM(2)) \to \mathbb{R}^+ \) is defined on the space \( \mathcal{L}(SIM(2)) \) of left-invariant vector fields on \( SIM(2) \), whereas the norm \( \| \cdot \|_\nu : \mathbb{R}^4 \to \mathbb{R}^+ \) is defined on \( \mathbb{R}^4 \). The gradient \( dU \) of \( U : SIM(2) \to \mathbb{R}^+ \) which is a co-vector field is given by

\[
dU = \frac{\partial U}{\partial \theta} \omega^1 + \frac{\partial U}{\partial \xi} \omega^2 + \frac{\partial U}{\partial \tau} \omega^3 + \frac{\partial U}{\partial \nu} \omega^4.
\]

The corresponding vector field is,

\[
G^{-1}_\beta \partial U = \frac{\partial U}{\partial \theta} \partial \theta + \beta^{-2} \frac{\partial U}{\partial \xi} \partial \xi + \beta^{-2} \frac{\partial U}{\partial \tau} \partial \tau + \beta^{-2} \frac{\partial U}{\partial \nu} \partial \nu,
\]

where \( G^{-1}_\beta \partial U : (T(SIM(2)))^* \to T(SIM(2)) \) is the inverse of the fundamental bijection between the tangent space and its dual and is defined as \( G^{-1}_\beta \omega^k = g^k_iA^i \), with \( g^{ij} g_{ij} = \delta_i^j \). The norm of a co-vector field (such as \( \partial \nu \)) is given by

\[
|a_i \omega^j|_\nu = g^{ij} a_i a_j = (a^\theta)^2 + \beta^{-2} (a_\xi)^2 + \beta^{-2} (a_\tau)^2 + \delta^{-2} (a_\nu)^2 = \|a\|_{\nu^{-1}}, \quad \text{with } a = (a^1, a^2, a^3, a^4).
\]

(4.58)

If we differentiate a smooth function \( U : SIM(2) \to \mathbb{R}^+ \) along an exponential curve \( \gamma(t) = g_0 \exp(t \sum c^iA_i) \) passing through \( g_0 \) we get (by application of chain rule)

\[
\frac{d}{dt} U(\gamma(t)) = \langle dU, \gamma'(t) \rangle = \frac{4}{\sum_{i=1}^4 c^iA_i} U(c^iA_i) U(\gamma(t)) = c^1 U_\theta(\gamma(t)) + c^2 U_\xi(\gamma(t)) + c^3 U_\tau(\gamma(t)) + c^4 U_\nu(\gamma(t)).
\]

(4.59)

Now we return to our goal of finding the optimal exponential curve at position \( g_0 \in SIM(2) \) given \( U : SIM(2) \to \mathbb{R}^+ \).

**Definition 4.13.** Consider the solution of the following minimization problem:

\[
c_* = \text{arg min}_{c \in c^iA_i} \left\{ \left| \frac{d}{dt} U(\gamma(t)) \right|_{t=0}^2 \right\}, \quad \gamma(t) = g_0 \exp(t \sum_{i=1}^4 c^iA_i) ; \quad \|c\|_\nu = 1.
\]

(4.60)

Then we call the covariantly constant curve \( t \mapsto g_0 \exp(t \sum_{i=1}^4 c^iA_i) \) the optimal exponential curve at \( g_0 \in SIM(2) \) given \( U : SIM(2) \to \mathbb{R}^+ \).

By means of (4.59) and the chain rule the energy in (4.60) can be rewritten as

\[
\left| \frac{d}{dt} U(\gamma(t)) \right|_{t=0}^2 = \|\nabla(\nabla U)^T(\gamma(0)) \cdot \gamma'(0)\|_{\nu^{-1}}^2
\]

\[
= \left\| \begin{bmatrix} \partial_\theta (\partial_\theta U) & \partial_\xi (\partial_\theta U) & \partial_\tau (\partial_\theta U) & \partial_\nu (\partial_\theta U) \\ \partial_\theta (\partial_\xi U) & \partial_\xi (\partial_\xi U) & \partial_\tau (\partial_\xi U) & \partial_\nu (\partial_\xi U) \\ \partial_\theta (\partial_\tau U) & \partial_\xi (\partial_\tau U) & \partial_\tau (\partial_\tau U) & \partial_\nu (\partial_\tau U) \\ \partial_\theta (\partial_\nu U) & \partial_\xi (\partial_\nu U) & \partial_\tau (\partial_\nu U) & \partial_\nu (\partial_\nu U) \end{bmatrix} \begin{bmatrix} c^1 \\ c^2 \\ c^3 \\ c^4 \end{bmatrix} \right\|_{g_0, \nu}^2
\]

(4.61)

where \( \nabla U := (\partial_\theta U, \partial_\xi U, \partial_\tau U, \partial_\nu U) \). Note that this noncovariant Hessian \( HU \) is not the same as the covariant Hessian form consisting of covariant derivatives of the Cartan connection.
provided in (4.12), which equals

\[ [\nabla, \nabla_j U] = [\nabla_i A_j U] = [A_i A_j U + \Gamma^i_{jk} A_k U] \]

\[
= \begin{pmatrix}
\partial_\theta (\partial_\theta U) & \partial_\theta (\partial_\eta U) & \partial_\theta (\partial_\tau U) & \partial_\theta (\partial_\tau U) \\
\partial_\eta (\partial_\theta U) & \partial_\eta (\partial_\eta U) & \partial_\eta (\partial_\tau U) & \partial_\eta (\partial_\tau U) \\
\partial_\tau (\partial_\theta U) & \partial_\tau (\partial_\eta U) & \partial_\tau (\partial_\tau U) & \partial_\tau (\partial_\tau U) \\
\partial_\tau (\partial_\eta U) & \partial_\tau (\partial_\eta U) & \partial_\tau (\partial_\tau U) & \partial_\tau (\partial_\tau U)
\end{pmatrix}.
\] (4.62)

For example, in the 2nd row and 1st column we have \( \nabla_1 \nabla_2 U = (\partial_\theta \partial_\theta \xi - c_1^2 \partial_\eta - c_1^4 \partial_\tau)U = \partial_\xi (\partial_\theta U) \). The minimization problem in (4.60) can now be rewritten as

\[
\arg\min_c \{ ||(HU)(g_0)\ c||^2_{\nu-1} \ ||c||_{\nu} = 1 \}.
\]

Set \( M_\nu := \text{diag}\{1, \beta^{-1}, \beta^{-1}, \delta^{-1}\} \in GL(4, \mathbb{R}) \) and \( H_\nu U := M_\nu H(U)M_\nu \). Then by the Euler Lagrange theory the gradient of \( ||(H(U)c)|| \) = \( (c, (H(U))^T M_\nu^2 (H(U))c) \) at the optimum \( c_\ast \) is linearly dependent on the gradient of the side condition, which can be written as \( (1 - ||c||^2) = 1 - (c, M_\nu^{-2} c) = 0 \), i.e.

\[
(HU(g_0))^T M_\nu^2 (HU(g_0)) c_\ast = \lambda M_\nu^{-2} c_\ast \iff (H(U))^T (H(U)) \tilde{c} = \lambda \tilde{c},
\]

for some Lagrange multiplier \( \lambda \in \mathbb{R} \), where \( \tilde{c} = M_\beta^{-1} c_\ast \). Thus we have shown that the minimization problem (4.53) requires eigensystem analysis of \( (H(U))^T H(U) \) rather than eigensystem analysis of the covariant Hessian given in (4.62). Making use of the discussion above we suggest the following method for curvature estimation.

Compute the curvature of the projection \( x(s(t)) = \mathbb{P}_x \left( g_0 \exp(t \sum_{i=1}^4 c_i A_i) \right) \) of the optimal exponential curve in \( SIM(2) \) on the ground plane from an eigenvector \( c_\ast = (c_\beta^\ast, c_\xi^\ast, c_\eta^\ast, c_\tau^\ast) \). This eigenvector of \( (\tilde{H}_\nu U)^T (\tilde{H}_\nu U) \), with a \( 4 \times 4 \) Hessian

\[
\tilde{H}_\nu U = \begin{pmatrix}
\beta^2 \partial_\theta \partial_\theta |U| & \beta \partial_\xi \partial_\theta |U| & \beta \partial_\eta \partial_\theta |U| & \beta \partial_\tau \partial_\theta |U| \\
\beta \partial_\xi \partial_\theta |U| & \partial_\xi \partial_\xi |U| & \partial_\xi \partial_\eta |U| & \partial_\xi \partial_\tau |U| \\
\beta \partial_\eta \partial_\theta |U| & \partial_\eta \partial_\xi |U| & \partial_\eta \partial_\eta |U| & \partial_\eta \partial_\tau |U| \\
\beta \partial_\tau \partial_\theta |U| & \partial_\tau \partial_\xi |U| & \partial_\tau \partial_\eta |U| & \partial_\tau \partial_\tau |U|
\end{pmatrix},
\] (4.63)

corresponds to the smallest eigenvalue. The curvature estimation is then given by the expression \( \kappa_{est} = ||\tilde{x}(s)|| \text{sgn}(\tilde{x}(s) \cdot \mathbf{e}_\eta) \).

### 4.6.3 Adaptive diffusion on Scale-OS

In order to obtain adaptive diffusion on scale-OS we consider the following nonlinear left-invariant evolution equation on \( SIM(2) \):

\[
\begin{cases}
\partial_t U(g,t) = \\
(\beta \partial_\theta \partial_\theta \partial_\eta \partial_\tau) \begin{pmatrix}
(D_{11}(U))(g,t) & 0 & 0 & 0 \\
0 & (D_{22}(U))(g,t) & 0 & 0 \\
0 & 0 & (D_{33}(U))(g,t) & 0 \\
0 & 0 & 0 & (D_{44}(U))(g,t)
\end{pmatrix} \begin{pmatrix}
\beta \partial_\theta \\
\partial_\xi \\
\partial_\eta \\
\frac{\beta}{4} \partial_\tau
\end{pmatrix} U(g,t),
\end{cases}
\]

for all \( g \in SIM(2) \), \( t > 0 \),

\( U(g,t = 0) = \mathcal{W}_\psi[f](g) \) for all \( g \in SIM(2) \),

(4.64)
where the positive functions $D_{kk} : L(SIM(2) \times \mathbb{R}^+) \cap C^2(SIM(2) \times \mathbb{R}^+) \to C^1(SIM(2) \times \mathbb{R}^+)$, $k \in \{1, 2, 3, 4\}$ are given by

$$(g, t) \mapsto (D_{kk}(U))(g, t) \geq 0, \; U \in L(SIM(2) \times \mathbb{R}^+)$$

These functions $D_{kk}$, $k \in \{1, 2, 3, 4\}$ should be chosen dependent on the local Hessian $HU(\cdot, t)$ of $U(\cdot, t)$ such that, at strong orientations, $D_{33}$ should be small so that we have anisotropic diffusion in the spatial plane along the preferred direction $\partial_\xi$, while at weak orientations $D_{22}$ and $D_{33}$ should be relatively comparable for isotropic diffusion.

In the previous subsection we discussed a method to obtain curvature estimates in scale-OS. This was done by finding the best exponential curve fit to the absolute value of $W_{\psi} f$, of the scale-OS $W_{\psi} f : SIM(2) \to \mathbb{C}$. We can include curvature in the scheme (4.64) by replacing $\partial_\xi$ by $\partial_\xi + \kappa \partial_\eta$.

**Remark.** We note that $\{\partial_\theta, \partial_\xi + \kappa \partial_\eta, \partial_\eta, \partial_\zeta\}$ are, in contrast to $\{\partial_\theta, \frac{1}{2} \partial_\xi, \frac{1}{2} \partial_\eta, \frac{1}{2} \partial_\zeta\}$, not orthonormal with respect to the $(\cdot, \cdot)$ inner product (4.40). Therefore to make use of this different set of coordinates, gauge coordinates aligned with the optimally fitting exponential curve will need to be introduced. A similar approach is followed in the case of adaptive diffusion on $SE(2)$, see [18, Sec. 4.1].

### 4.7 Adaptive $SE(2)$ Diffusion on Scale-OS

In this section we apply nonlinear adaptive diffusions on the 2D Euclidean motion group $SE(2)$, called coherence enhancing diffusion on orientation scores (CED-OS), to each scale of our scale-OS. We can apply CED-OS in our case as at each fixed scale the scale-OS is a function on $SE(2)$ group defined of $0 < a^- < a^+ < \infty$ from Section 2.5.

Recall that for an image $f \in L^2(\mathbb{R}^2)$, the corresponding scale-OS $W_{\psi}(f) \in \mathbb{C}^{SIM(2)} \subset L^2(SIM(2))$. For any $a \in (0, \infty)$, $W_{\psi}(f)(\cdot, a, \cdot) \in L_2(SE(2))$. Let $\Phi : \mathbb{L}_2(SE(2)) \to L_2(SIM(2))$ denote nonlinear adaptive diffusion (CED-OS) on the $SE(2)$ group. In this section we propose the operator $\Lambda$ on scale-OS defined as

$$\Lambda[(\mathbb{L}_2)(\cdot))(g)] = \sum_{i=1}^{m} \Phi[\mathbb{L}_2(\cdot, a_i, \cdot)]]$$

where $g \in \mathbb{R}^2 \times [0, 2\pi) \times [a^-, a^+]$ and $\{a_i\}_{i=1}^{m}$ is the discretization of $[a^-, a^+]$. Recall the definition of $0 < a^- < a^+ < \infty$ from Section 2.5.

### 4.7.1 CED-OS Implementation

Curvature estimation of a spatial curve using $SE(2)$-OS is similar to the scheme followed in Section 4.6.2 wherein we make use of the optimal exponential curve fit at each point. We

---

*The absolute value $|W_{\psi} f|$ is phase invariant. This is important for local feature estimation. For e.g. the curvature estimate at the border of a thick line should be the similar to the estimate on top of the line.*

*Due to constraints of time, adaptive diffusions on the $SIM(2)$ group have not been implemented.*
make use of the same notation as used in Section 4.6.2, the only difference being that in the $SE(2)$ group setting there is no scaling group involved. Two methods suggested for curvature estimation are as follows:

- Compute the curvature of the projection $x(s(t)) = \mathbb{P}_2(g_0 \exp(t \sum_{i=1}^3 c_i A_i))$ of the optimal exponential curve in $SE(2)$ on the ground plane from an eigenvector $e_\ast = (e_\ast^0, e_\ast^1, e_\ast^2)$. This eigenvector of $(\tilde{H}_\beta[U])^T (\tilde{H}_\beta[U])$, with a $3 \times 3$ Hessian
  \[ H_\beta[U] = \begin{pmatrix} \beta^2 \partial_\theta \partial_\theta |U| & \beta \partial_\theta \partial_s |U| & \beta \partial_s \partial_s |U| \\ \partial_\theta \partial_s |U| & \partial_s \partial_s |U| & \partial_s \partial_s |U| \\ \partial_\theta \partial_s |U| & \partial_s \partial_s |U| & \partial_s \partial_s |U| \end{pmatrix}, \]  
  (4.66)
corresponds to the smallest eigenvalue. The curvature estimation is now given,
  \[ \kappa_{est} = ||\tilde{x}(s)|| \text{sgn}(\tilde{x}(s) \cdot e_\eta) = \frac{c^0 \text{sgn}(c^0_\beta)}{\sqrt{(c^0_\beta)^2 + (c^1_\beta)^2}} \]
  Note that unlike the $SIM(2)$ case curvature is a constant in this case.

- In this method we only the choice of optimal exponential curve to horizontal exponential curves. Horizontal curves are curves in the $(SE(2), \omega^2)$ Sub-Riemannian manifold. For a brief explanation of horizontal curves see App(Sec. B.2). We wish to diffuse along horizontal curves because typically the mass of a $SE(2)$-OS is concentrated around a horizontal curve [16] and therefore this is a good and fast curvature estimation method.

Compute the curvature of the projection $x(s(t)) = \mathbb{P}_2(g_0 \exp(t \sum_{i=1}^3 c_i A_i))$ of the optimal exponential curve in $SE(2)$ on the ground plane from the eigenvector $e_\ast = (e_\ast^0, e_\ast^1)$. This eigenvector of $(\tilde{H}_\beta^\text{hor}[U])^T (\tilde{H}_\beta^\text{hor}[U])$, with a $3 \times 2$ horizontal Hessian
  \[ H_\beta^\text{hor}[U] = \begin{pmatrix} \beta^2 \partial_\theta \partial_\theta |U| & \beta \partial_\theta \partial_s |U| \\ \partial_\theta \partial_s |U| & \partial_s \partial_s |U| \end{pmatrix}, \]  
  (4.67)
corresponds to the smallest eigenvalue. The curvature estimation is now given,
  \[ \kappa_{est}^\text{hor} = ||\tilde{x}(s)|| \text{sgn}(\tilde{x}(s) \cdot e_\eta) = \frac{c^0_\beta}{c^1_\beta}. \]
  For numerical experiments on these curvature estimates on orientation scores of noisy images, see [16].

For experiments in this study we enforce horizontality and restrict ourselves to the second method.

We are interested in the nonlinear diffusion equation given by
  \[ \begin{aligned}
      \partial_t W(g, t) &= \nabla \cdot (D(W)(g, t) \nabla W(g, t)), \ g \in SE(2), \ t \geq 0, \\
      W(g, 0) &= U(g)
  \end{aligned} \]  
  (4.68)
where $U : SE(2) \to \mathbb{C}$. The diffusion tensor is given by an adaptive mixture of diffusion along the exponential curve and $\beta$-isotropic diffusion
  \[ D(W)(g) = (1 - D_a) \frac{\mu^2}{||c||_\beta} cc^T + D_a \text{diag}\{1, 1, \beta_0^2\}, \ 0 < D_a < 1. \]
Figure 4.7: Finite difference scheme of (4.68), where second-order B-spline interpolation, [38], is employed for sampling on the grid of the moving frame \( \{ \mathbf{e}_\theta, \mathbf{e}_\xi, \mathbf{e}_\eta \} \). where we have to substitute \( D_a = D_a(s(W(\cdot, t))) \) and \( \approx c_*(W(\cdot, t))(g). s(W(\cdot, t)) \) is a measure of orientation confidence and is determined by taking the Laplacian in the plane orthogonal to the line which is calculated at \( g \in \text{SE}(2) \) by
\[
s(g) = -\Delta |U|(g) = -((e_1(g))^T \tilde{H}_\beta |U| e_1(g) + (e_2(g))^T \tilde{H}_\beta |U| e_2(g)),
\]
where \( \tilde{H}_\beta |U| \) is defined in (4.66) and \( e_1 \) and \( e_2 \) are vectors orthonormal to the tangent vector \( c. \) Below we briefly explain \( \beta \)-isotropic diffusion.

There is no inherent notion of isotropy in \( \text{SIM}(2) \). We can however define an artificial notion of \( \beta \)-isotropic diffusion, where the diffusion tensor is defined as
\[
\mathbf{D}_a^{\beta o} = \text{diag}\{1, 1, \beta^2\}.
\]

The diffusion equation is \( \beta \)-isotropic in the sense that \( \| \partial_\xi \|_\beta = \| \partial_\eta \|_\beta = \beta^2 \| \partial_\theta \|_\beta = \beta \) and so with respect to the metric on \( \text{SE}(2) \) (same as the \( \text{SIM}(2) \) case (4.40) but without the \( d\tau \otimes d\tau \) term) we apply the same amount of diffusion in all directions. The \( \beta \)-isotropic diffusion equation can be written as
\[
\partial_t W = ((\partial_\xi^2 + \partial_\eta^2) + \mu^2 \partial_\theta^2) W = ((\partial_\xi^2 + \partial_\eta^2) + \mu^2 \partial_\theta^2) W,
\]
where \( W : \text{SE}(2) \to \mathbb{C} \).

\( D_a \) in CED-OS is given by
\[
D_a(W(\cdot, t)) = \zeta \left( \frac{s(W(\cdot, t))}{\max_{g \in \text{SE}(2)} s(W(g, t))} \right)
\]
where \( \zeta \) is defined as
\[
\zeta = \begin{cases} 
1 & s \leq 0 \\
1 - \exp(-(\frac{c}{\tau})^n) & n > 0 \text{ and } s > 0 \\
\exp(-(\frac{c}{\tau})^n) & n < 0 \text{ and } s > 0,
\end{cases}
\]
with \( n \in \mathbb{Z} \setminus \{0\} \). The explicit inclusion of adaptive curvature is achieved via a Gauge frame interpretation of anisotropic diffusion in \( \text{SE}(2) \). For more details on the motivation for the choice of \( D_a \) and \( \zeta \) and construction of Gauge frames see [21, Chapter 6].
Figure 4.8: Illustration of scale invariance of left-invariant operators on scale-OS. From left to right: input image; enhancement via linear diffusion on input image sampled on 200 × 200 grid; enhancement via linear diffusion on input image sampled on 300 × 300 grid.

Figure 4.7 illustrates the numerical schemes used for the discretization of (4.68). These schemes make use of second-order B-spline interpolation, [38], for sampling on the grid of moving frames \( \{e_\theta, e_\xi = \cos \theta e_x + \sin \theta e_y, e_\eta = -\sin \theta e_x + \cos \theta e_y \} \). For the stability analysis of these numerical schemes see [16].

4.7.2 Advantage of scale adaptive diffusion

Figure 4.8 illustrates the idea of scale invariance. This is a useful mathematical property and intuitively it means that large particles in an image diffuse more than small particles. However in most medical imaging applications this is not desirable as the interest usually lies in a particular interval of scales. Figure 4.9 illustrates this idea. The advantage of scale based application of CED-OS lies in the adaptability of diffusion parameters per scale, for e.g. by adapting the end time of diffusion per scale. This is a clear advantage of CED on scale-OS in comparison to CED on \( SE(2) \)-OS.

4.7.3 Results

In this section present the results of applying CED-OS at each layer of the scale-OS. The parameters used in the construction of scale-OS are the same for all the images: \( k = 3 \) (order of B-spline for construction), \( N_s = 4 \) (number of scales) and \( N_\theta = 32 \) (number of orientations). \( k = 3, N_\theta = 32 \) is the parameter used in the construction of \( SE(2) \)-OS.

Figure 4.10 and Figure 4.11 illustrate the main idea of this method. Figure 4.10 shows the result of enhancement via applying CED-OS to each scale of the scale-OS and Figure 4.11 shows the enhancement achieved at each scale. Parameters used for CED-OS: \( \tau = 0.1, \beta = 0.058, endt = \{0.5, 0.5, 0.5, 0.5\} \).

In Figure 4.12 we compare enhancement via Gaussian estimates for linear diffusion on scale-OS (recall Section 4.5.2) with nonlinear diffusion via CED-OS per scale on scale-OS for a test-image with crossing lines. Parameters used for the linear diffusion scheme: \( D_{11} = 0.05, D_{22} = 1, D_{44} = 0.01, endt = 0.7 \). Parameters used for CED-OS on scale-OS: \( \tau = 0.1, endt = \{1, 1, 1, 1\}, \beta = 0.058 \).

In Figure 4.13 we compare enhancement via Gaussian estimates for linear diffusion on scale-OS (recall Section 4.5.2) with nonlinear diffusion via CED-OS per scale on scale-OS for a Retinal
Figure 4.9: Illustration of disadvantage of scale-invariance of diffusion (CED-OS on scale-OS) on retinal fundus image. Left: Fundus image; right: Enhancement via CED-OS per scale. Scale-invariant diffusion causes additional diffusion of the optic disc in the retina which is undesirable when the goal is blood-vessel extraction.

fundus image. Parameters used for the linear diffusion scheme: $D_{11} = 0.05$, $D_{22} = 1$, $D_{44} = 0.01$, $endt = 0.7$. Parameters used for CED-OS on scale-OS: $\tau = 0.1$, $endt = \{0.5, 0.5, 1.2, 1.5\}$, $\beta = 0.058$. Figure 4.12 and Figure 4.13 clearly indicate that enhancement via nonlinear diffusion give better results than linear diffusion.

In Figure 4.14 we perform CED-OS on both the $SE(2)$-OS and the scale-OS. Parameters chosen for CED-OS on $SE(2)$-OS: $\tau = 0.1$, $endt = 3$, $\beta = 0.058$. Parameters used for CED-OS on scale-OS: $\tau = 0.1$, $endt = \{0.5, 0.5, 0.5, 0.5\} \beta = 0.058$. As seen in the results, for an image with same scale structures, CED-OS on scale-OS and CED-OS on $SE(2)$-OS give very similar results. Note that our schemes handle crossings in an image in a much better way than standard adaptive diffusion techniques suggested in [30] and [41]. For a comparison (in the context of crossing curves in a medical image) of CED technique suggested in [41] and CED-OS see [16].

Figures 4.14, 4.15, 4.16, 4.17 and 4.18 compare enhancement via CED-OS on $SE(2)$-OS and scale-OS of different images. For all these images the parameters for CED-OS are: $\tau = 0.1$, and $\beta = 0.058$.

Figure 4.14 illustrates that for an image with uniform scale structure CED-OS on $SE(2)$-OS ($endt = 10$) and scale-OS ($endt = \{0.5, 0.5, 0.5, 0.5\}$) have similar results.

Figures 4.15, 4.16, 4.17, 4.18 clearly indicate that CED-OS on scale-OS leads to better enhancement when multiple scale structures are present in the image. These experiments are performed on different medical images.
Figure 4.10: CED-OS applied to the scale-OS of a noisy test-image with concentric circles. Left: Noisy input image; Right: Result of enhancement at each scale on scale-OS.

Figure 4.11: Enhancement per scale of the test image in Figure 4.10. As clearly visible most of the noise is concentrated at the lower scales and this suggests the need for scale based enhancement operators.
Figure 4.12: Comparison between behaviour of linear diffusion on scale-OS and CED-OS per layer on scale-OS for a test-image. From left to right: original image; original image + noise; enhancement via Gaussian estimates for heat kernels of linear diffusion on $SIM(2)$ group (see Section 4.5.2); enhancement via CED-OS per scale layer of the scale-OS. As expected the nonlinear adaptive diffusion scheme gives much better results than the linear scheme.

Figure 4.13: Comparison between behaviour of linear diffusion on scale-OS and CED-OS per layer on scale-OS for a Retinal fundus image. From left to right: input image; enhancement via Gaussian estimates for heat kernels of linear diffusion on $SIM(2)$ group (see Section 4.5.2); enhancement via CED-OS per scale layer of the scale-OS.
Figure 4.14: CED-OS on $SE$-OS, [16], and scale-OS. Top- left: original image; right: original image + noise. Bottom- left: CED-OS on $SE(2)$-OS; right: CED-OS at each scale in scale-OS and then adding the layers.
Figure 4.15: CED-OS on $SE$-OS and scale-OS for a multi-scale test-image. Top to bottom: input image; CEDOS on $SE(2)$-OS; CED-OS on scale-OS
Figure 4.16: CED-OS on $SE$-OS and scale-OS for a collagen tissue image. Top to bottom: input image; CEDOS on $SE(2)$-OS; CED-OS on scale-OS
Figure 4.17: CED-OS on $SE$-OS and scale-OS for a X-ray catheter image. Top to bottom: input image; CEDOS on $SE(2)$-OS; CED-OS on scale-OS
Figure 4.18: CED-OS on $SE$-OS and scale-OS for a bone-microscopy image. Top to bottom: input image; CEDOS on $SE(2)$-OS; CED-OS on scale-OS
Chapter 5

Summary and Future Research

5.1 Evolutions and Construction of Wavelet Transform on the Similitude Group (Summary)

The enhancement of structures in noisy image data is relevant for many (bio)medical imaging applications. Commonly used image processing methods cannot appropriately handle elongated structures that cross each other. In this thesis we propose to extend the orientation score framework (successful in handling crossing structures) to a scale orientation score framework by introducing the notion of scaling. This is needed because in many imaging applications elongated structures with different scales (diameters) appear.

The construction of scale orientation score (scale-OS) is introduced in Chapter 2. We begin by reviewing the construction of unitary maps on functional Hilbert spaces and arrive at the celebrated result for wavelet transform in a group theoretic setting by Grossman, Morlet and Paul. Then a more generalized version of the wavelet transform is presented which avoids the requirement of irreducibility of group representations. From this generalization it followed that our wavelet transformation is a unitary mapping between the space of images with finite support in the Fourier domain and a functional Hilbert space. We then explicitly quantify the stability of the inverse wavelet transform and then present criterion for the choice of admissible wavelets that allow stable invertible wavelet transformation. Finally a suitable wavelet for the purpose of our applications is proposed.

In Chapter 3 differential-geometric structure of the $SIM(2)$ group is introduced which is essential for designing left-invariant enhancement-contextual operators on the scale-OS. We present the idea of left-invariance in the context of the $SIM(2)$ group where we deal with left-invariant vector fields and derivatives. This is followed by an explicit derivation of the exponential and logarithmic curves for $SIM(2)$ group. We conclude by presenting $SIM(2)$-convolutions which are essential for implementation of linear operators on scale-OS.

Chapter 4 deals with operators on scale-OS. The chapter begins by showing that operators on scale-OS should be left invariant so that the net operator on the corresponding image is Euclidean and scaling invariant, i.e. translation, rotation and scaling of the image does not affect the result of left-invariant operations on the corresponding scale-OS. Using the Dunford-Pettis theorem it is shown that all linear left-invariant operators on scale-OS are essentially group convolutions (defined appropriately). We then introduce the generic left-invariant convection-diffusion equation on the $SIM(2)$ group and using the results by Hörmander provide conditions under which this equation has a smooth Green’s function. Sharp Gaussian estimates for the
Green’s function of linear left-invariant diffusion on $SIM(2)$ group generated by subcoercive operators are then introduced. This is achieved by approximating the $SIM(2)$ group by a nilpotent group via contraction. Results of enhancement via linear diffusions are then presented. By choosing an appropriate metric on $SIM(2)$ and employing the Cartan connection we present the nonlinear convection-diffusion equation in a covariant form and show that diffusion takes place only along the exponential curves in $SIM(2)$. Following this, a method for extraction of spatial curvature in this abstract setting of the convection-diffusion equation is proposed. Finally by applying nonlinear adaptive diffusion defined on the $SE(2)$ group we enhance the scale-OS per scale and present experiments. These experiments show that CED application on scale-OS is preferable over CED on $SE(2)$-OS. The typical advantages are

- Adaptability of diffusion can be scale dependant. This is often required in medical imaging applications, wherein a particular scale is more important than others.
- In the $SE(2)$-OS framework if $|Re(U(x, \theta))|$ (or $|Im(U(x, \theta))|$) is large, where the real part corresponds to a line (and imaginary part to an edge), we do not know which scale is responsible for the high response. This is not a problem in the scale-OS framework.

5.2 Future Research

In this section we discuss potentially interesting topics for future research related to the work described in this thesis.

We have presented sharp Gaussian estimates for Green’s function of linear left-invariant diffusions on $SIM(2)$. It is an interesting mathematical problem to explicitly derive these Green’s functions both for the Heisenberg group $H = (SIM(2))_0$ and the $SIM(2)$ group.

Implementation of curvature estimation in the $SIM(2)$ setting and the resulting diffusion equations proposed in this study would be an interesting numerical problem.

Due to noncommutative structure of the Lie algebra of $SIM(2)$, applying diffusion along an elongated structure also leads to some diffusion in the direction orthogonal to the elongated structure. This leads to undesirable blurring which is especially noticeable in the case of linear left-invariant diffusions via Gaussian estimates. The introduction of a “thinning” mechanism accomplished by adaptive morphological operations on scale-OS via left-invariant Hamilton-Jacobi equations is a promising approach to obtain sharper results.

This thesis focusses on contour enhancement (via diffusions on scale-OS) in images. Contour completion by considering convection based evolutions on scale-OS would be a challenging topic in the further development of scale-OS based image processing.
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Joseph Addison famously wrote, “There is not a more pleasing exercise of the mind than gratitude. It is accompanied with such an inward satisfaction that the duty is sufficiently rewarded by the performance”.

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It seems that my academic sojourn is far from over. At this moment I cannot help but recall the immortal quote by Dory in Finding Nemo, “Just keep swimming”.

Appendix A

A.1 Results from Chapter 2

Theorem A.1 (Haar’s theorem[see [25] for proof]). Let $G$ be a locally compact topological group. There is, up to a positive multiplicative constant, a unique countably additive, nontrivial measure $\mu$ on the Borel subsets of $G$ satisfying the following properties:

1. $\mu(gE) = \mu(E)$ for any $g$ in $G$ and Borel set $E$ (left-translation-invariance).
2. $\mu(K)$ is finite for every compact set $K$.
3. Every Borel set $E$ is outer regular:
   $$\mu(E) = \inf \{\mu(U) : E \subseteq U, \text{Open}\}$$
4. Every Borel set $E$ is inner regular:
   $$\mu(E) = \sup \{\mu(K) : K \subseteq E, \text{Kcompact}\}.$$  

Such a measure on $G$ is called a left Haar measure.

It can also be proved that there exists a unique (up to multiplication by a positive constant) right-translation-invariant Borel measure $\nu$ satisfying the above regularity conditions and being finite on compact sets, but it need not coincide with the left-translation-invariant measure $\mu$. The left and right Haar measures are the same only for so-called unimodular groups.

Expression for left invariant Haar measure for $SIM(2)$ and $SE(2)$ groups

- Similitude group has the structure of a semi-direct product, $SIM(2) = \mathbb{R}^2 \rtimes (SO(2) \times \mathbb{R}^+)$, where $\mathbb{R}^2$ is the group of translations, $SO(2)$ that of rotations and $\mathbb{R}^+$ of dilations/scaling.

For this particular case we compute the left invariant Haar measure. If $(b_0, \theta_0, a_0)$ is a fixed element of the group and $(b, \theta, a)$ arbitrary then,

$$(b', \theta', a') = (b_0, \theta_0, a_0)(b, \theta, a) = (b_0 + a_0R_\theta b, \theta_0 + \theta, a_0a).$$

Noting that $det[R_{\theta_0}] = 1$ we arrive at,

$$db' = a_0^2db, \ d\theta' = d\theta, \ da' = a_0da.$$  

Thus the measure $d\mu(b, e^{i\theta}, a) = \frac{1}{a_0}dbd\theta da$, is the left invariant Haar measure for the $SIM(2)$ group.
• Euclidean Motion group, $SE(2) = \mathbb{R}^2 \times SO(2)$, where $\mathbb{R}^2$ is the group of translations, $SO(2)$ that of planar rotations. The expression for the left invariant Haar measure can be arrived at as above. $d\mu(b, \theta) = dbd\theta$

**Evaluation of (2.15):**

From Theorem 2.4, we know that,

$$(W_{\psi}f, W_{\psi}f)_{L^2(G)} = C_{\psi}(f,f)_H.$$ 

This implies $W_{\psi}SW_{\psi} = C_{\psi}I$, where $I : H \rightarrow H$ is the identity operator, and therefore $W_{\psi}^{-1} = \frac{1}{C_{\psi}}(W_{\psi})^*$. Now we arrive at (2.15). By the definition of adjoint of an operator,

$$(\phi, W_{\psi}f)_{L^2(G,d\mu_G)} = (W_{\psi}^*(\phi), f)_{L^2(\mathbb{R}^2)}, \text{ where } \phi \in C^G_{K_{\psi}}.$$ (A.1)

Note that we have used $L^2(G, d\mu_G)$ norm for $C^G_{K_{\psi}}$ because $C^G_{K_{\psi}}$ is a closed subspace of $L^2(G, d\mu_G)$. Here we drop the usage of $d\mu_G$ from the notation of Hilbert space. Also note that we have used $L^2(\mathbb{R}^2)$ as the domain of our functions, which is the space to which all images belong. (A.1) can be written as,

$$\int_G \overline{\phi}(g)(W_{\psi}f)(g)d\mu_G(g) = \int_{\mathbb{R}^2} \overline{(W_{\psi}f)(x)}f(x)dx.$$ (A.2)

The LHS can be written as,

$$\int_G \overline{\phi}(g)(W_{\psi}f)(g)d\mu_G(g) = \int_G \overline{\phi}(g) \left[ \int_{\mathbb{R}^2} (U_g\phi)(x)f(x)dx \right] d\mu_G(g)$$

$$= \int_{\mathbb{R}^2} \left[ \int_G \phi(g)(U_g\phi)(x)d\mu_G(g) \right] f(x)dx.$$ (A.3)

The last equality follows from Fubini’s theorem. Combining (A.2) and (A.3) we have the required result.

**Proof of Lemma 2.6:** Let $f \in (V_{\psi})^\perp$. Thus

$$\forall b \in \mathbb{R}^d \forall t \in T(U_b,t\psi,f)_{L^2(\mathbb{R}^d)} = 0.$$ (A.4)

We know that,

$$(U_b,t\psi,f)_{L^2(\mathbb{R}^d)} = (FR_tT\psi,Ff)_{L^2(\mathbb{R}^d)} = (F^{-1}[FR_t\psi\Omega Ff])(b).$$

Thus (A.4) implies

$$\forall b \in \mathbb{R}^d \forall t \in T \left( (b,t) \rightarrow F^{-1}[FR_t\psi\Omega Ff])(b) \right) = 0$$

$$\Rightarrow \left( (\omega, t) \rightarrow FR_t\psi(\omega)Ff(\omega) \right) = 0 \ a.e. \ on \ \mathbb{R}^d \times T$$

$$\Rightarrow \left( (\omega, t) \rightarrow |FR_t\psi(\omega)Ff(\omega)|^2 \right) = 0 \ a.e. \ on \ \mathbb{R}^d \times T.$$ Therefore,

$$M_{\psi}(\omega)(Ff)(\omega) = (2\pi)^{d/2} \int_T \left| FR_t\psi(\omega)Ff(\omega) \right|^2 d\mu_T(t) = 0 \ a.e. \ on \ \mathbb{R}^d.$$ Since $M_{\psi} \neq 0 \ a.e. \ as \ \psi$ is an admissible wavelet, this implies that $|Ff|^2 = 0$ which in turn implies that $f = 0.$
A.2 Derivation for basis of Lie Algebra

The four basic sets of operations of dilation, rotation and the two translations, each constitute one-parameter subgroups of $SIM(2)$. More precisely, these subgroups are generated by group elements of the type

$$(x, a, \theta) = (0^T, e^t, 0), \ t \in \mathbb{R},$$
or

$$(x, a, \theta) = (0^T, 1, t), \ t \in [0, 2\pi),$$
or

$$(x, a, \theta) = (e^t, 1, 0), \ t \in \mathbb{R}, \ i = 1, 2,$$

where,

$$0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

constitute one-parameter subgroups. A general element of $SIM(2)$ can be written as a product of elements of these subgroups:

$$(x, a, \theta) = (e_1 b_1, 1, 0)(e_2 b_2, 1, 0)(0^T, 1, \theta)(0^T, a, 0), \ \text{where} \ b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$ 

Generically, writing elements in any one of these subgroups as $g(t)$ and computing the derivative at the identity $\frac{d}{dt}g(t) \bigg|_{t=0}$, we obtain the four $3 \times 3$ matrices,

$$X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ X_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\{1, 2, 3, 4\} := \{\theta, x, y, a\}$.

A.3 Left-Invariant Differential Operators on $SE(2)$- An intuitive explanation

In Figure A.1(a), the left side depicts a curve $\gamma : \mathbb{R} \rightarrow SE(2)$ passing through the identity element $e$, such that the tangent vector $X_e = c^x e_x + c^y e_y + c^\theta e_\theta \in T_e(SE(2))$. When the curve $\gamma$ is left multiplied by $g = (x, \theta) \in SE(2)$, it yields a new curve $g\gamma$, shown on the right side. At position $g$, the curve $g\gamma$ has a tangent vector $X_g \in T_g(SE(2))$ which corresponds to $X_e \in T_e(SE(2))$. This operation of transporting vectors from $T_e(SE(2))$ to $T_g(SE(2))$ is called the push-forward operation and is denoted by $X_g = (L_g)_* X_e$. Intuitively the push-forward operator allows the movement of tangent vectors to tangent spaces at other group elements. Using the notion of push-forward operator the concept of left-invariant vector field is introduced as a tangent vector field $SE(2) \rightarrow T_g(SE(2))$, that fulfils the property $X_g = (L_g)_* X_e$, for all $g \in SE(2)$ and a fixed $X_e \in T_e(SE(2))$.

The Lie Algebra $T_e(SE(2))$ is spanned by the basis $e_x, e_y, e_\theta$ as shown on the left side of Figure A.1(a). Though one could use the same basis for $T_g(SE(2))$ for all $g \in SE(2)$, it is more natural to introduce a new set of basis vectors constructed by push-forward of $e_x, e_y, e_\theta$,

$$\{e_x(g), e_y(g), e_\theta(g)\} = \{(L_g)_* e_x, (L_g)_* e_y, (L_g)_* e_\theta\} = \{\cos \theta e_x + \sin \theta e_y, -\sin \theta e_x + \cos \theta e_y, e_\theta\}.$$
Figure A.1: Illustration of the concept of left-invariance, from two different perspectives: (a) considered as tangent vectors to the curves, i.e. $X_g = c^i e_i(g) + c^\theta e_\theta(g)$, for all $g \in SE(2)$, (b) considered as differential operators on a locally defined smooth function $U$, i.e. $X_g = c^i \partial_i(g) + c^\theta \partial_\theta(g)$, for all $g \in SE(2)$. The push forward $L_g^* : T_e(SE(2)) \rightarrow T_g(SE(2))$ connects the tangent space at the unity element $T_e(SE(2))$ to all tangent spaces $T_g(SE(2))$ in a "left-invariant" way.

In Figure A.1(b) we utilize the alternate description of a tangent vectors $(X_e, X_g)$ as a differential operators, acting on a function $U : SE(2) \rightarrow \mathbb{R}$, i.e. we replace all occurrences of $e_i$ by $\partial_i$. $X_g$ can be viewed as an operator that calculated the derivative of $U$ at $g$ in the direction specified by $X_g$,

$$X_g(U) = (c^i \partial_i U + c^\theta \partial_\theta U) = (c^i (\cos \theta \partial_x + \sin \theta \partial_y) + c^\theta (-\sin \theta \partial_x + \cos \theta \partial_y) + c^\eta \partial_\eta)U,$$

yielding a scalar on the $\mathbb{R}$-axis on the bottom of the figure. Alternatively we could also arrive at the same result by first translating and rotating $U$ over $g$, i.e. $L_g(U) = U \circ L_g$, such that the original neighbourhood $\omega_g$ is shifted back to $\omega_e$ giving,

$$X_g(U) = X_e(U \circ L_g) = (c^i \partial_i + c^\theta \partial_\theta + c^\eta \partial_\eta)(U \circ L_g).$$

### A.4 Exponential map on SIM(2) is surjective

In this section we give a proof for the surjectivity of the $\exp$ map on $SIM(2)$ group, cf. Theorem 3.4. This proof is based on the results given in [3, Chap.4].

**Theorem A.2.** The exponential map, $\exp : T_e(SIM(2)) \rightarrow SIM(2)$ is surjective, i.e. for all $g \in SIM(2)$, there exists a $X \in T_e(SIM(2))$ such that, $g = \exp(X)$.
Proof. Any general element $X \in T_S(SIM(2))$ has the form,

$$X(\tilde{\alpha}, \tilde{\beta}) = \sum_{i=1}^{4} c^i X_i = \begin{pmatrix} c^4 & -c^1 & c^2 \\ c^1 & c^4 & c^3 \\ 0 & 0 & 0 \end{pmatrix},$$

where $c^i \in \mathbb{R}$ for all $i$ and $\tilde{\alpha} = \begin{pmatrix} c^4 \\ c^1 \end{pmatrix}$ and $\tilde{\beta} = \begin{pmatrix} c^2 \\ c^4 \end{pmatrix}$.

Corresponding to a 2-vector $\vec{v}$ consider, a 2 × 2 matrix $s(\vec{v})$,

$$s(\vec{v}) = \begin{pmatrix} v^1 & -v^2 \\ v^2 & v^1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}.$$

For any two vectors $\vec{v}, \vec{w}$ we note that, $s(\vec{v})s(\vec{w}) = s(\vec{w})s(\vec{v})$ and $s(\vec{v})\vec{w} = s(\vec{w})\vec{v}$.

Using $s$, $X(\tilde{\alpha}, \tilde{\beta})$ can be rewritten as,

$$X(\tilde{\alpha}, \tilde{\beta}) = \begin{pmatrix} s(\tilde{\alpha}) & \tilde{\beta} \\ 0^T & 1 \end{pmatrix}.$$

Taking the exponential of the matrix $X(\tilde{\alpha}, \tilde{\beta})$ yields $g \in SIM(2)$

$$g = \exp(X(\tilde{\alpha}, \tilde{\beta})) = \begin{pmatrix} e^{c^4} R_{\theta} & F(s(\tilde{\alpha})) \tilde{\beta} \\ 0^T & 1 \end{pmatrix},$$

where $0 \leq \theta < 2\pi$ and $F(s(\tilde{\alpha})) \tilde{\beta}$ is a $2 \times 2$ matrix defined as the sum of the following infinite series,

$$F(s(\tilde{\alpha})) = I + \frac{s(\tilde{\alpha})}{2!} + \frac{|s(\tilde{\alpha})|^2}{3!} + \cdots$$

Define, \( \sinh : \mathbb{R} \to \mathbb{R} \) as \( \sinh(u) = \frac{\sinh(u)}{u} \), \( \sinh(0) = 1 \). This function is a positive, infinitely differentiable function as is, $F(u) = e^{u/2} \sinh(u/2)$. Expanding $F(u)$ using Taylor series we get,

$$F(u) = e^{u/2} \sinh(u/2) = 1 + \frac{u}{2!} + \frac{u^2}{3!} + \cdots.$$ For $u \neq 0$, we can also write,

$$F(u) = e^{u} \sinh(u/2) = \frac{e^u - 1}{u}.$$ We now define the $2 \times 2$ matrix valued function,

$$e^{A/2} \sinh(A/2) = F(A) = I + \frac{A}{2!} + \frac{A^2}{3!} + \cdots, \quad F(N) = I,$$

for any $2 \times 2$ real matrix $A$ and where $N$ and $I$ are the $2 \times 2$ null and identity matrices respectively. Further, if $\det(A) \neq 0$ then,

$$F(A) = e^{A/2} \sinh(A/2) = A^{-1} [e^A - I].$$ Further if $\det(e^A - I) \neq 0$ we may also write,

$$(F(A))^{-1} = \frac{e^{-A/2}}{\sinh(A/2)} = [e^A - I]A.$$ Hence for $|\tilde{\alpha}| \neq 0$ (recall $\tilde{\alpha} = \begin{pmatrix} c^4 \\ c^1 \end{pmatrix}$) and $\det(s(\tilde{\alpha})) = |\tilde{\alpha}|$,

$$F(s(\tilde{\alpha})) = s(\tilde{\alpha})^{-1} [e^{s(\tilde{\alpha})} - I] = \frac{1}{(c^1)^2 + (c^4)^2} \begin{pmatrix} c^4 & -c^1 & c^2 & c^4 \\ -c^1 & c^4 & c^3 & -c^4 \\ e^4 \cos(c^1) - 1 & -e^4 \sin(c^1) & e^4 \sin(c^1) & e^4 \cos(c^1) - 1 \end{pmatrix}.$$
and

\[
[F(g(\bar{\alpha}))]^{-1} = \frac{1}{2(\cosh(c^4) - \cos(c^1))} \begin{pmatrix}
\cos(c^1) - e^{-c^4} & \sin(c^1) \\
-\sin(c^1) & \cos(c^1) - e^-c^4
\end{pmatrix} \begin{pmatrix}
c^4 & -c^1 \\
c^1 & c^4
\end{pmatrix}
\]

Thus, (A.5) can be written as

\[
g = \exp(X(\bar{\alpha}, \bar{\beta})) = \left( e^{c^4} R_\theta e^{a(\pi)/2 \sinh(\frac{N_o}{2})} \right),
\]

and thus writing \(g = (b, a, \theta)\) in matrix form and comparing, we get the relations

\[
c^4 = \log(a), \quad \bar{\beta} = \left( \frac{c^2}{c^3} \right) = [F(g(\bar{\alpha}))]^{-1} b,
\]

between group parameters and the lie algebra. This shows that any group element \((b, a, \theta)\) can be written as the exponential of some element \(X(\bar{\alpha}, \bar{\beta})\) in the lie algebra. \(\square\)

### A.5 Implementation of G-convolution, \(G = \mathbb{R}^2 \ltimes SO(2)\)

Recall that the \(G\)-convolution is given by (3.35). There are two reasonable options for implementing this expression, either by means of discretization or by expressing everything in terms of steerable components. Below we briefly discuss the discretization of the orientation score followed by a brief treatment of both these approaches.

In order to discretize \((SE(2))\) orientation scores equidistant samples in the \(SE(2)\) group using the standard \((x, y, \theta)\) coordinates are taken. In the spatial dimensions, the sampling distance is set equal to the sampling distance of the corresponding image. Assume that the image is square with \(N_x \times N_y\) pixels. In the orientation dimension, the number of samples taken is denoted by \(N_o\), and therefore the sampling distance is given by \(s_o = 2\pi/N_o\).

#### A.5.1 Implementation of \(SE(2)\)-convolution via Discretization

The most straightforward implementation of \(SE(2)\)-convolution is obtained by replacing the integrals in (3.35) by finite sums. We index the samples of the \(SE(2)\) kernel, \(\Psi(x, y, l)\), with \(x, y \in \{-K_o^H, \ldots, K_o^H\}\), \(K_o^H = \lfloor K_o/2 \rfloor\) and \(l \in \{-K_o^H, \ldots, K_o^H\}\). Here it has been assumed that the sampled \(\Psi\) has odd dimensions and the central value \(\Psi(0, 0, 0)\) is represented by the middle sample. The samples of the orientation scores are indexed by \(U(x, y, l)\) with \(x, y \in 0, 1, \ldots, N_x - 1\) and \(l \in 0, 1, \ldots, N_o - 1\). We arrive at,

\[
R(x, y, l) = \sum_{l'=l-K_o^H}^{l+K_o^H} \sum_{x'=x-K_o^H}^{x+K_o^H} \sum_{y'=y-K_o^H}^{y+K_o^H} \mathcal{R}_{l', s_o}[\Psi](x-x', y-y', l-l') U_{ext}(x', y', l'),
\]

where \(\mathcal{R}_{l', s_o}[\Psi]\) denotes the rotated \(SE(2)\) kernel and \(U_{ext}\) denotes the sampled orientation score \(U\) on the extended domain, i.e. \(U_{ext}\) is defined on \(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\), taking into account the appropriate boundary conditions. For the orientation dimensions, the boundary condition is periodic. The choice of spatial boundary conditions is a more delicate issue since any boundary condition would create undesired discontinuities on the spatial boundary leading to boundary artefacts. In the implementation used during this study, a zero-padding boundary condition is employed.
The rotated kernel $R_{\theta}[\Psi]$ can be obtained in various ways. If an analytical formula for $\Psi$ is known, the rotated kernel is obtained by sampling it for $N_0$ required rotations. Note that while this method is quite accurate, it takes a large amount of memory. Alternatively, $\Psi$ is sampled for a single orientation, and its rotated versions are obtained using interpolation. In the implementation used for this study, B-spline interpolation proposed in [38] is employed. 

A.5.2 Implementation of $SE(2)$-convolution via Steerability

Steerable filters, which find widespread use in image analysis, were introduced by Freeman et al. [22]. For a complete treatment along with implementation details of the steerable $SE(2)$-convolution, see [21, Chap. 3].

The idea is that steerable convolution kernels $\psi \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$, which by definition can written as

$$\psi(x) = \sum_{m=-N}^{N} \psi_m(x) = \sum_{m=-N}^{N} f_m(r)e^{-im\phi}, \quad N \in \mathbb{N},$$

for some $f_m \in C^1(\mathbb{R}, \mathbb{C})$, are easily rotated (without interpolation artefacts on a rectangular pixel grid) by means of

$$R_{\theta}\psi(x) = \sum_{m=-N}^{N} \psi_m(r)e^{im\theta}e^{-im\phi},$$

with the additional advantage (regarding computation time, in case multiple orientations are needed) that

$$(R_{\theta}\psi * f)(x) = \sum_{m=-N}^{N} e^{im\theta}(\psi * f)(x), \quad x \in \mathbb{R}^2, \quad \theta \in [0, 2\pi).$$

Though it may seem that any convolution kernel within $L_2(\mathbb{R}^2)$ is steerable after expansion in the complete orthonormal base of eigenfunctions of the Harmonic oscillator, note that, in in this case we have a finite expansion ($N < \infty$).

The input equals a function $U : G \rightarrow \mathbb{C}$ (for example an orientation score $U_f$ of an image $f$) and a convolution kernel $K : G \rightarrow \mathbb{C}$. The output equals $W = K * G U$, where the $G$-convolution is given by (3.35) and is related to the $G$-correlation as $W = \tilde{K} * G U$, and is explicitly in terms of $G$-correlation is given by

$$W(b, \theta) = (K * G U)(b, \theta) = (\tilde{K} * G U)(b, \theta)$$

$$= \int_{\mathbb{R}^2} \int_{0}^{2\pi} \tilde{K}(R_{\theta}^{-1}(b' - b), \theta' - \theta)U(b', \theta')d\theta'd\theta'. \quad (A.9)$$

Expand $\tilde{K}$ in steerable components (with respect to both angle and space):

$$\tilde{K}(b, \theta) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \tilde{K}_{mn}(b)e^{im\theta}, \quad b = (r \cos \phi, r \sin \phi)$$

such that $\tilde{K}_{mn}(R_{\theta}^{-1}b) = e^{im\theta}\tilde{K}_{mn}(b)$ and

$$\tilde{K}_{mn}(b) = \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} K(R_{\theta}^{-1}b, e^{i\theta})e^{im\theta}e^{in\theta'}d\theta d\theta'. \quad (A.11)$$

1The implementations of $SE(2)$-convolution employed throughout this study have been taken from the work done by Erik Franken [21, Chap. 3] and Markus van Almsick [39].
Similarly expanding $U$ into steerable components (with respect to angle only we obtain,

$$W(b, e^{i\theta}) = \int_{\mathbb{R}^2} \int_0^{2\pi} \hat{K}(b' - b, e^{i\theta'}) db' d\theta'$$

$$= \int_{\mathbb{R}^2} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \sum_{\hat{n} \in \mathbb{Z}} \hat{K}_{mn}(b' - b) U_{\hat{n}}(b') e^{i(n+m)\theta} e^{-i(m+\hat{n})\theta}$$

$$= \int_{\mathbb{R}^2} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \sum_{\hat{n} \in \mathbb{Z}} \hat{K}_{mn}(b' - b) U_{\hat{n}}(b') e^{i(n+m)\theta} \left( \int_0^{2\pi} e^{-i(m+\hat{n})\theta'} d\theta' \right) db'$$

$$= 2\pi \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} e^{i(n+m)\theta} (\hat{K}_{mn} * U_{-m})(b),$$

where $(\hat{K}_{mn} * U_{-m})(b)$ are just ordinary 2D-correlations in $\mathbb{R}^2$, i.e.

$$(\hat{K}_{mn} * U_{-m})(b) = \int_{\mathbb{R}^2} \hat{K}_{mn}(b' - b) U_{-m}(b') db'.$$

So in practice when the series is truncated at say $|m| \leq M_{\text{max}}$, a $G$-convolution can be implemented by means of $(1 + 2 * N_{\text{max}})(1 + 2 * M_{\text{max}})$ 2D-convolutions. The evaluation of $W$ is sped up by using a fast Fourier Transform instead of a discrete Fourier Transform.
Appendix B

B.1 Regularized Derivatives and Local Features

In this section we present a brief synopsis of the ill-posed nature of numerical differentiation of an image (in the sense of Hadamard). This question has been dealt with in detail by various authors, such as in the context of edge detection, see [37]. We then make this operation well posed by adding regularization.

In machine vision, as well as in most numerical problems, the data is noisy. Sensor noise arises at least in part from quantum fluctuations in the number of absorbed photons per sensor and unit time. This represents a fundamental limitation for real-time imagery when integration time and size of the sensors are limited by the need of high temporal and spatial resolution. It is therefore important, that the results of numerical operations performed on the data are not too sensitive to noise. Even a small amount of noise may disrupt differentiation. For instance, consider a function \( f(x) \) and \( \hat{f}(x) = f(x) + \epsilon \sin(\omega x) \). \( f(x) \) may be close to \( \hat{f}(x) \) according to standard norms (\( L^2, L^\infty, \cdots \)), provided \( \epsilon \) is sufficiently small. On the other hand, \( f'(x) \) may be quite different from \( \hat{f}'(x) \) if \( \omega \) is large enough.

The mathematical term well-posed problem stems from a definition given by Jacques Hadamard, [24]. Differentiation of the function \( f \) is a typically ill-posed problem, since it can be seen that the solution of the inverse problem

\[
g(x) = Af(x)
\]

where \( Af \) is the integral operator

\[
\int_{-\infty}^{x} f(\bar{x})d\bar{x} = \int_{-\infty}^{+\infty} h(x-\bar{x})f(\bar{x})d\bar{x},
\]

where \( h \) is the step function. It is well known that inverse linear problems in which \( g \) and \( f \) belong to Hilbert spaces are ill-posed, [36]. Various rigorous methods have been proposed for transforming ill-posed problems into well-posed problems, the chief of which are regularization techniques, see [36, Chap II] for more details. Gaussian derivatives, [19], is an often used technique to regularize derivatives of images.

Recall that the general diffusion equation for orientation scores using left-invariant differential operators is

\[
\begin{align*}
\partial_t u &= \left( \begin{array}{c} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \tau} \end{array} \right)^T \left( \begin{array}{cccc} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{array} \right) \left( \begin{array}{c} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \tau} \end{array} \right) u = Q(A)
\end{align*}
\]  

(B.1)
with \( u(x, \tau, \theta; 0) = U_f(x, \tau, \theta) \) where \( U(x, \tau, \theta) \) is the scale-OS of an image \( f \). The solution of (B.1) can be written as, \( u(\cdot, \cdot, \cdot; t) = e^{tA}U \). Our aim is to find local features using well posed left-invariant derivative operators. We therefore wish to operationalize \( D(e^{tA}U) \) where \( D \) is any derivative constructed from \( \{ \partial_\theta, \partial_\xi, \partial_\eta, \partial_x, \partial_y \} \) and \( e^{tA} \) is the left-invariant diffusion operator described above. Note that due to the noncommutative nature of left-invariant derivatives, the order of regularization operator and differential operator matters. We provide two implementation of the regularized derivatives. In the first one we consider the use of diffusion with a diagonal diffusion tensor as regularizer. Subsequently we consider the case of isotropic diffusion over the spatial derivatives as regularizer.

### B.1.1 General Regularized Derivatives in Scale-OS

We wish to use anisotropic diffusion as regularizer to calculate left-invariant derivatives in \( SIM(2) \),

\[
\begin{align*}
\partial_t u(g, t) &= (D_{11}\partial_\theta^2 + D_{22}\partial_\xi^2 + D_{44}\partial_\eta^2) u(g, t), \quad g \in SIM(2), \quad t \geq 0, \\
u(g, t = 0) &= U(g).
\end{align*}
\]  

(B.2)

Note that for proper regularization in all directions we require \( D_{11} \neq 0, \ D_{22} \neq 0 \) and \( D_{44} \neq 0 \). However, \( D_{43} \neq 0 \) is not a requirement because non-zero diffusion in \( \theta \) and \( \xi \) directions induces smoothing in \( \eta \) direction due to the commutator relationships, recall (3.18). Also \( D_{22} \neq 0 \) and \( D_{33} = 0 \) implies diffusion tangent to the contours and not orthogonal to it. Recall that an approximate analytical expression was derived for the Green’s function of (B.2) in Section 4.5,

\[
|K^{\eta=-1, D}_{\xi}(g)| \leq \frac{1}{4\pi t^2 D_{11} D_{22} D_{44}} \exp \left( -\frac{1}{4t} \left( \frac{\theta^2}{D_{11}} + \frac{(c^2)^2}{D_{22}} + \frac{\tau^2}{D_{44}} + \frac{|c^3|}{\sqrt{D_{11} D_{22} D_{44}}} \right) \right)
\]  

(B.3)

where,

\[
c^2 = \frac{(y\theta - x\tau) + (-\theta\eta + \tau\xi)}{t(1 + e^{2\tau} - 2e^{\tau}\cos\theta)}, \quad c^3 = \frac{-(x\theta + y\tau) + (\theta\xi + \tau\eta)}{t(1 + e^{2\tau} - 2e^{\tau}\cos\theta)}.
\]

Note that this approximation renders a non-differentiable approximation for the Green’s function and is therefore not usable for the purpose of regularized derivatives. A smooth approximation of (B.3) was provided in Section 4.5,

\[
|K^{\eta=-1, D}_{\xi}(g)| \leq \frac{1}{4\pi t^2 D_{11} D_{22} D_{44}} \exp \left( -\frac{1}{4t} \left( \frac{\theta^2}{D_{11}} + \frac{(c^2)^2}{D_{22}} + \frac{\tau^2}{D_{44}} + \frac{|c^3|^2}{D_{11} D_{22} D_{44}} \right) \right)
\]  

(B.4)

To arrive at regularized derivatives we sample the required derivatives of this approximate Green’s function and then apply a \( SIM(2) \)-convolution to the scale-OS. Since the formulae for such kernels is rather cumbersome below we provide a different approach to regularized derivatives.

### B.1.2 Gaussian Derivatives in Scale-OS

The implementation in the previous section is rather complicated. However, the implementation becomes much simpler if we restrict to the case of isotropic diffusion \( (D_{22} = D_{33}) \) with no diffusion over scaling \( (D_{44} = 0) \), i.e.

\[
\partial_t u = (D_{11}\partial_\theta^2 + D_{22}(\partial_\xi^2 + \partial_\eta^2)) u = (D_{11}\partial_\theta^2 + D_{22}e^{2\tau}(\partial_\xi^2 + \partial_\eta^2)) u.
\]  

(B.5)
Since at a fixed scale (B.5) can be described using commutative $\partial_x$, $\partial_y$ and $\partial\theta$ derivatives, this equation simplifies to the diffusion equation in $\mathbb{R}^3$, except the $2\pi$ periodicity of the $\theta$ dimension. Therefore, the Green’s function is approximated by the Gaussian kernel

$$
G_{t, t_0}(x, y, \theta) = \frac{1}{8\sqrt{\pi^3 t_0}} e^{-\frac{x^2+y^2}{4t_0} - \frac{\theta^2}{4t}},
$$

(B.6)

where $t_0 = tD_{11}$ and $t_s = t(e^{2\tau} D_{22})$. Though in this case we can use standard separable implementations of Gaussian derivatives we have to take into account the non-commutative nature of the derivatives. A $(i, j, k)$th order isotropic Gaussian derivative implementation for a $3D$ image $f$ satisfies the following relation,

$$
\partial_x^i \partial_y^j \partial_z^k e^{i(\partial_x^2 + \partial_y^2 + \partial_z^2)} f = \partial_x^i e^{i\partial_x^2} \left( \partial_y^j e^{i\partial_y^2} \left( \partial_z^k e^{i\partial_z^2} f \right) \right),
$$

(B.7)

where the equality implies separability along the three dimensions. We want to make use of similar implementations to construct Gaussian derivatives in the orientation scores, meaning we have to ensure that the same permutation of differential operators and regularization operators is allowed. By noting that

$$
\partial_x^{\xi} \partial_y^{\eta} \partial_z^{\theta} e^{i(\partial_x^2 + \partial_y^2 + \partial_z^2)} = \partial_x^{\xi} \partial_y^{i\eta} \partial_z^{i\theta},
$$

(B.8)

$$
\partial_x^{\xi} \partial_y^{\eta} \partial_z^{\theta} e^{i(\partial_x^2 + \partial_y^2 + \partial_z^2)} \neq \partial_x^{\xi} \partial_y^{\theta} \partial_z^{\eta},
$$

we conclude that we always should ensure a certain order of the derivative operators, i.e. one should first calculate the derivative $\partial_x$ and then the commuting spatial derivatives $\{\partial_y, \partial_\theta\}$, which are calculated from the Cartesian derivatives $\{\partial_x, \partial_y\}$ using $\partial_x = e^\tau (\cos \theta \partial_x + \sin \theta \partial_y)$ and $\partial_y = e^{-\tau} (-\sin \theta \partial_x + \cos \theta \partial_y)$. The commutator relations $[\partial_x, \partial_y] = \partial_\theta$ and $[\partial_y, \partial_\theta] = -\partial_x$ (see Section (3.1.4)) allow to rewrite the derivatives in this order. For e.g. the derivative $\partial_\theta \partial_y$ can be calculated directly with Gaussian derivatives but the derivative $\partial_\theta \partial_\theta$ must be constructed with Gaussian derivatives $\partial_\theta \partial_x + \partial_\theta$. It is very important to note that this implementation for Gaussian derivatives works $\tau$ is a constant. Thus at every scale in our OS we have a different implementation of the regularized derivatives.

### B.2 Horizontal Curves

**Definition B.1.** Choose $L > 0$. A curve $\gamma = (x, y, \theta) : [0, L] \to SE(2)$, with $x, y \in C^1([0, L])$ and $\tau \in C^0([0, L])$ is called horizontal iff

$$
(A^3, \dot{\gamma}(s)) = 0 \Leftrightarrow \begin{pmatrix}
\frac{1}{e^\tau} \sin \theta dx + \frac{1}{e^\tau} \cos \theta dy \\
\gamma(s)
\end{pmatrix} = 0,
$$

(B.9)

where $\gamma(s) \in T_g(SE(2))$.

A vector $X_g \in T_g(SE(2))$ is called horizontal if it is the tangent vector to some horizontal curve through $g \in SE(2)$. A vector field $\mathcal{H}$ is called horizontal if $X_g$ is horizontal for all $g \in SE(2)$. (B.9) implies that a smooth curve $s \mapsto \gamma(s)$ is horizontal iff

$$
\dot{\gamma}(s) \in \text{span}\{e_x | \gamma(s), e_\xi | \gamma(s)\}, \text{ for all } s > 0.
$$

(B.10)

Let $\mathcal{H}$ be a horizontal vector field (horizontal section of the tangent bundle). Then $\forall U \in \mathcal{H}$ we have $(A^3, U) = 0$. Thus in $\mathcal{H}$ we have that $-\frac{1}{e^\tau} \sin \theta dx + \frac{1}{e^\tau} \cos \theta dy = 0$, i.e. in $\mathcal{H}$, $\tan \theta = \frac{dy}{dx}$. This provides us with an alternate description of a horizontal curve.

**Definition B.2.** A curve $s \mapsto \gamma(s) = (x(s), y(s), \theta(s))$ is called horizontal iff $\theta(s) = \arg(x(s) + iy(s))$. Then $\gamma$ is called the lifted curve in $SE(2)$ of the projected curve $s \mapsto x(s) = (x(s), y(s))$ in $\mathbb{R}^2$. 

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*Horizontal Curves* 

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A smooth horizontal curve $\gamma = (x, \theta)$ in $SE(2)$ given by $s \mapsto (x(s), y(s), \theta(s))$ can be parametrized by the arc length $s > 0$ of it’s projection $x = P_{\mathbb{R}^2} \gamma$ on the spatial plane. Using the spatial arc length parametrization and (B.2), along a horizontal curve $\gamma = (x, \theta): \mathbb{R}^2 \to SE(2)$ one has

$$\gamma = (x, \theta) \text{ is horizontal } \Rightarrow \kappa(s) = \dot{\theta}(s)$$ \hspace{1cm} (B.11)$$

where $\kappa(s) = \pm \|\dot{x}(s)\|_{\mathbb{R}^2}$, with $\| \cdot \|_{\mathbb{R}^2}$ the euclidean norm on $\mathbb{R}^2$, is the curvature of the curve $s \mapsto x(s) = P_{\mathbb{R}^2} \gamma(s)$.

By (B.10) and Def. B.2 it follows that the horizontal part $\mathcal{H}_0 \subset T_g(SE(2))$ of each tangent space $T_g(SE(2))$ is spanned by $\mathcal{H}_0 = \{\partial_\theta|_g, \partial_\xi|_g\}$ and the space of horizontal left-invariant vector fields is spanned by $\{A_1, A_2\} = \{\partial_\theta, \partial_\xi\}$. However it is important to note that a horizontal curve itself $s \mapsto \gamma(s) \in SIM(2)$, can have components in all directions $\{e_\theta, e_\xi, e_\eta\}$ in contrast to $\dot{\gamma}(s) \in T_{\gamma(s)}(SIM(2))$. 
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