Hyperbolicity in nonautonomous velocity fields with applications to point vortex systems

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Abstract

A theoretical study is presented on the dynamics of two-dimensional nonautonomous velocity fields, with an application to point vortex flows. Flows induced by mutually advected points vortices with negligible viscosity are examined, and transport barriers and chaotic regions are determined.

An arbitrary number of point vortices of equal strength placed on a circle, with equal distances between neighbouring vortices, induce a stationary velocity field in a frame that rotates with the system. In such a setup, there are two hyperbolic critical points for every vortex. The corresponding separatrices are independent of the vortex strength and scale invariant. The hyperbolic critical points have an analogue in time dependent flows: hyperbolic trajectories, the associated invariant manifolds of which form the transport barriers. A path of a hyperbolic critical point and a hyperbolic trajectory coincide in stationary flows, and in the case of small time dependence, they remain close together. The existence and uniqueness of a hyperbolic trajectory close to the path of a hyperbolic critical point are shown, provided that the velocity of the path and the second derivative of the velocity field are sufficiently small.

A numerical approach reveals that the number of hyperbolic trajectories in three vortex systems depends on the configuration of the vortices. There are six hyperbolic trajectories if the vortices are placed on the vertices of an equilateral triangle. If the deviation from this setup, and therefore the time dependence in the flow, is sufficiently large, the number of hyperbolic trajectories decreases to five. One hyperbolic point vanishes near the centre of rotation, where the interaction of the manifolds, i.e., the transport, primarily takes place. For larger deviations, the number is four, which is retained when two vortices are so close together that they rotate around each other. The behaviour of the invariant manifolds is more complicated than what can be explained by the established theory on the interaction of the manifolds of two hyperbolic points. A similar disappearance is observed for two of the six hyperbolic critical points.
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1. Introduction

The work described in this report was conducted at the Eindhoven University of Technology, from March 2006 till June 2007. Object of study was the dynamics of two-dimensional vortex systems with localised vorticity distributions, so-called point vortices. More specifically, we examined the flows induced by mutually advected points vortices at high Reynolds number and attempted to determine transport barriers and chaotic regions. This was done both analytically and numerically.

In section 1.1, it is first explained why this work is focussed on two-dimensional flows, and we make clear why two-dimensional turbulent flows can be modelled by a system of a finite number of point vortices; next, the dynamical systems approach to analysing these flows is substantiated in section 1.2. Finally, the remainder of the report is outlined in section 1.3.

1.1. Two-dimensional point vortex flows

Many large scale geophysical fluid flows are approximately incompressible and two-dimensional. Although there may be important vertical variations in the horizontal flow, vertical velocities are generally much smaller than horizontal velocities. This means that in three-dimensional large scale flows, two-dimensional, incompressible models can accurately describe the velocity field at fixed levels or on quasi-horizontal surfaces [1].

In geophysical flows, inertial and viscous forces are often small compared to the Coriolis force. This means that these flows are governed by a balance of the Coriolis force and a horizontal pressure gradient. These are the so-called geostrophic flows, which obey the Taylor-Proudman theorem. This theorem states that geostrophically balanced flows are independent of the axial (vertical) coordinate, making them effectively two-dimensional [2].

A third reason that fluid flows in nature are often practically two-dimensional, is that they have a density stratification. Temperature gradients in the atmosphere and salinity gradients in the ocean make that geophysical flows are restricted to two dimensions, because gravity confines fluid elements to a layer of fluid of constant density.

Two-dimensional turbulence in fluids with negligible viscosity is generally dominated by small scale, coherent vortices that emerge from an initial state, as shown by experiments and numerical calculations. These coherent structures contain most
of the vorticity and energy of the system [3]. The interaction between these structures steepens gradients in the vorticity distribution. The dynamics of these two-dimensional fluids can be described by a finite system of point vortices, in the limit where the typical size of these patches of approximately constant vorticity is much smaller than the distances between their centres of gravity.

The notion of a point vortex gives a description of the flow on a certain level, which has no universal range (although they appear in superfluid $^4$He [4]). The mathematical convenience, combined with the fact they form an accurate model of the flows away from the vortices, justifies a point vortex model for flows with small patches of vorticity.

Point vortices mutually interact through the velocity field that they create and have no self-induced motion: they are advected by the flow induced by other vortices. They form a Hamiltonian system, which yields a first constant on the motion. In the infinite plane, the linear and angular momenta are also conserved.

Systems of three vortices are integrable, meaning that their behaviour is regular (often even quasi-periodic). However, the particle trajectories in the induced velocity field can exhibit chaotic behaviour. In the case of four or more vortices, the vortex motion can become chaotic. This is the reason that in this work systems of three point vortices have been chosen to model two-dimensional turbulent flows.

1.2. Dynamical systems approach

Fluid particle trajectories have long been studied by integrating the ordinary differential equation defined by the velocity field [5]. In the case that the field is linear in the spatial variables, different trajectories only differ by their initial conditions. If the velocity field depends nonlinearly on the spatial variables, then completely different particle trajectories are found for different initial conditions, generally. Transport properties and partial barriers can then be obtained by integrating many different trajectories with different initial conditions. The number of trajectories needed to acquire the desired information may be impractically large.

Nevertheless, with today’s computational possibilities, finding a large number of trajectories of a nonlinear velocity field might be feasible numerically. In the computational fluid dynamics community, there are accurate algorithms for solving the Navier-Stokes equations. However, these solutions are only a prerequisite for analysing mixing and transport in chaotic systems. It can still be very difficult to derive transport properties, given an enormous number of trajectories, as illustrated in figure 1.1. Many trajectories displayed together show complex tangles, which are hard to interpret.
A different approach to analysing the motion of particles was initiated by Poincaré. Instead of finding a trajectory associated with an initial condition, he sought a relation between all possible trajectories. This led to an approach using geometrical structures that organise the flow in different regions, which contain different types of trajectories. The mathematical framework for this method is provided by dynamical systems theory, which can be employed to study Lagrangian transport and mixing in geophysical flows.

The application of dynamical systems theory to analysing flows starts with the input of the numerical solutions of the velocity field, or with experimental data sets, e.g., obtained by oceanographic measurements. In both cases the field is given as a data file, meaning that the field is only known for discrete times and spatial coordinates. The dynamical systems approach is useful for analytical velocity fields, but also for fields described by numerically or experimentally obtained data. For the dynamical systems results do not depend on an analytical form of the system under study; they only require the presence of certain geometrical features.

If the velocity field is unknown analytically, the domains of chaotic and regular advection in the flow can also be visualised by means of Poincaré sections. This method consists of placing passive particles in the flow, and examining the distribution of the particles, when the system reaches the same state (e.g., the same configuration of vortices). However, this method requires that the system is periodic or at least quasi-periodic, so that it is not applicable to systems with general time dependence. The dynamical systems approach is not limited by these requirements, and can be applied to any time dependent flow.

The concepts from dynamical systems theory are applied to analyse the structure of the velocity field, which is generated by a set of point-vortices, by studying the dy-
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The dynamics of the hyperbolic trajectories and their associated invariant manifolds. These trajectories and manifolds are the generalisations of the hyperbolic critical points (saddle points) and separatrices in time independent flows to aperiodic, time dependent systems [7]. The invariant manifolds act as transport barriers in unsteady flow fields.

1.3. Report outline

The remainder of this report is organised as follows. In chapter 2, we give a general overview of the theory, which is at the basis of this work. It covers theory on incompressible two-dimensional flows, as well as some background on critical points. Theory on point vortices is discussed in a separate part, chapter 3. It contains standard theory of point vortex systems and point vortex configurations in a rotating frame, both time independent and time dependent.

Chapter 4 is the most mathematically oriented one. It is discussed how the hyperbolic trajectories and the associated manifolds follow from their analogues in stationary flows. Furthermore, the existence and uniqueness of a hyperbolic trajectory close to a certain curve in space is presented, with restrictions on the velocity field and the curve. The numerical methods that have been used and the results of the simulations can be found in chapters 5 and 6, respectively. The focus in these simulations is on time dependent three vortex systems, where the hyperbolic hyperbolic trajectory and the invariant manifolds are visualised for different vortex configurations. Chapter 7 is devoted to the conclusions and the recommendations.
2. General theory

A general overview of the theory on two-dimensional velocity fields is presented, without any restriction to point vortices. First, we start in section 2.1 with the theory of incompressible flows, where motion can be described by a stream function. Next, the equations expressing conservation of mass and momentum are discussed in section 2.2. The conserved quantities in an incompressible flow are the topic of section 2.3.

After that, we shift to the topic of critical points. The characteristics of the flow near these points can be obtained by an eigenvalue analysis of the linearised velocity field around such a critical point. Different types of linearisations and the possible types of critical points are discussed in section 2.4.

Finally, section 2.5 deals with the number and nature of critical points in a given flow. The number of critical points and singularities of a continuous vector field, enclosed by a closed curve, is related to the Poincaré index of that curve. This imposes certain restrictions on the division of the critical points into different types.

2.1. Two-dimensional incompressible flows

In a two-dimensional, incompressible flow the motion can be described in terms of a scalar-valued stream function \( \Psi \), which is defined by

\[
\begin{align*}
\dot{x} & = \frac{\partial \Psi}{\partial y}, \\
\dot{y} & = -\frac{\partial \Psi}{\partial x},
\end{align*}
\]

(2.1)

where the superimposed dot denotes a derivative with respect to time.\(^1\) \( \dot{x} \) and \( \dot{y} \) are the Cartesian \( x \)- and \( y \)-component of the flow velocity \( \mathbf{v} = (u, v) = (\dot{x}, \dot{y}) \), respectively [2]. Equations (2.1) form a Hamiltonian system, regardless of the form of \( \Psi \) [8]. The system is said to have one degree of freedom is the flow is steady, i.e., \( \Psi = \Psi(x, y) \), and two if the flow is unsteady, \( \Psi = \Psi(x, y, t) \). In the case of a time-periodic flow, the number of degrees of freedom is referred to as one and a half [9]. Hamiltonian systems with one degree of freedom are integrable, which means that they cannot be chaotic. Systems with one and a half degree of freedom or more may be non-integrable and can therefore be chaotic.\(^2\)

\(^1\)In order to avoid confusion, it is used solely with functions that have time as the only argument.
\(^2\)Non-integrability is a necessary condition for chaos, but not sufficient [9].
The vorticity $\omega$, which is defined as
\[ \omega = \nabla \times v, \]
has $x$- and $y$-component equal to zero, because the flow is two-dimensional: $v$ has no $z$-component and $\partial_z v = 0$. Consequently, the only nonzero component of $\omega$ is
\[ \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial y^2}. \] (2.2)

Thus, the stream function satisfies the Poisson equation with the vorticity as the source term. The solution of this equation can be written in terms of the Green’s function $G$ for the Laplacian operator. The Green’s function is the solution of the Poisson equations, which has a Dirac delta function as source: 3
\[ \nabla^2 G(x) + \delta(x) = 0. \]

In two dimensions, the solution reads \[ G(x) = -\frac{1}{2\pi} \log |x|. \]
The stream function in (2.2) can then be retrieved from
\[ \Psi(x, y) = \int G(x - \xi) \omega_z(\xi) d\xi. \] (2.3)

### 2.2. Conservation equations

The differential form of the law of mass conservation reads
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, v) = 0, \] (2.4)
where $\rho$ is the density of the fluid and $v$ is a two- or three-dimensional velocity field [2]. Equation (2.4) is also called the continuity equation. The second term can be written as
\[ \nabla \cdot (\rho \, v) = v \cdot \nabla \rho + \rho \nabla \cdot v. \]
Then (2.4) can be rewritten using the definition of the material derivative,
\[ \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + v \cdot \nabla, \]
\[ \text{Sometimes the notation } \delta^2 \text{ is adopted for the Dirac Delta function in two dimension: } \delta^2(x) = \delta(x)\delta(y). \]
which describes the rate of change of a quantity when travelling with the flow. Then (2.4) becomes
\[ \frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot v = 0. \] (2.5)

A fluid is called incompressible, if its density does not change with pressure, which is approximately true for liquids. In most cases the density changes in the flow are small. Therefore, the first term of (2.5) is often neglected, so that it reduces to
\[ \nabla \cdot v = 0. \] (2.6)

Henceforth, we assume that the flow under study is incompressible, so that (2.6) holds. The momentum balance is provided by the Navier-Stokes equation, which reads
\[ \frac{Dv}{Dt} = \frac{1}{\rho} \nabla p + g + \nu \nabla^2 v, \] (2.7)

where \( p \) is the thermodynamic pressure, \( g \) is the gravitational acceleration and \( \nu \) is the kinematic viscosity [2]. When viscous effects are neglected, the Navier-Stokes-equation reduces to the Euler equation
\[ \frac{Dv}{Dt} = \frac{1}{\rho} \nabla p + g. \] (2.8)

The point vortex motion, which is considered in this work, satisfies the Euler equation. Taking the curl of this equation yields an equation for the vorticity \( \omega \):
\[ \frac{D\omega}{Dt} - \omega \cdot \nabla v = 0, \]
as the curl of a gradient is zero (and the density is constant) and \( g \) is constant. This formulation is for a three-dimensional velocity field \( v \). In two dimensions, \( \omega \) has only a \( z \)-component, whereas \( v \) has no \( z \)-component. This gives that \( \omega \cdot \nabla v = 0 \) and that
\[ \frac{D\omega_z}{Dt} = 0. \]

A point vortex has a vorticity with a singular distribution, where all vorticity is concentrated in a single point. Therefore, this distribution is a solution of the above equation which holds everywhere, except at the vortex.

2.3. Conserved quantities

The two-dimensional inviscid Euler equations have a number of invariants. If the velocity field decays sufficiently rapidly at infinity, and external body forces are absent, the total vorticity \( \omega_T \) and the total kinetic energy per unit mass \( E \) are conserved:
\[ \omega_T(t) = \int \omega(x, t) \, dx, \]
\[ E(t) = \frac{1}{2} \int |v|^2 \, dx = \frac{1}{2} \int \omega \Psi \, dx. \]

The last expression follows from integration by parts, with appropriate boundary conditions [11]. In the case of viscous flows, the kinetic energy is no longer conserved, as it is transformed into internal energy.

A fundamental scalar quantity associated with the vorticity is the circulation \( \Gamma \), which is defined as the line integral of the velocity field around a closed curve:

\[
\Gamma = \oint_C \mathbf{v} \cdot d\ell = \iint_A (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, dA = \iint_A \omega \, dA.
\]

Here, \( d\ell \) is an infinitesimal line element tangent to the closed curve \( C \). Using Stokes’s integral theorem [12], this line integral is rewritten as a surface integral over the area \( A \) of \( C \). The vector \( \mathbf{n} \) is the normal vector on an element \( dA \). Consequently, the circulation equals the flux of vorticity \( \omega \) through \( A \).

A principal theorem on the circulation is Kelvin’s circulation theorem, which states that in an ideal, incompressible fluid, acted on by conservative forces, the circulation around a closed material curve moving with the fluid, is constant [13]. This means that vorticity cannot be created, unless the conditions are violated, e.g., by viscous boundary layers or by stratification. If the circulation is zero for all closed curves, then the flow is irrotational [11].

The Euler equations are invariant with respect to spatial translations and rotations, which means that the linear momentum \( \mathbf{P} \) and angular momentum \( \mathbf{L} \) (both per unit mass) of the fluid are conserved:

\[
\mathbf{P} = \frac{1}{2} \int (x \times \omega) \, dx,
\]

\[
\mathbf{L} = \frac{1}{3} \int x \times (x \times \omega) \, dx = -\frac{1}{2} \int |x|^2 \omega \, dx.
\]

A quantity that is constant in two, but not in three dimensions, is the enstrophy \( W \):

\[
W = \frac{1}{2} \int |\omega|^2 \, dx.
\]

Therefore, in two dimensions, the energy as well as the enstrophy are conserved, whereas in three dimensions, the enstrophy can increase by vortex tube stretching [11]. This conservation in two dimensions is the cause of prohibiting an energy cascade to smaller scales.
2.4. Linearisation of the velocity field

The local properties of the flow near a point or a trajectory, can be obtained by linearising the velocity field around it. Consider the following velocity field:

\[ \dot{x} = v(x(t), t). \] (2.9)

This equation states that the Lagrangian velocity \( \dot{x} \) of a fluid parcel is equal to the Eulerian flow velocity \( v \) at the position \( x \) of the parcel. This equation also describes the advection of a passive tracer in the flow, which is assumed to have no affect on the dynamics of the fluid. A passive tracer is massless and has infinitesimally small size, so that inertial effects can be discarded [14].

Suppose \( \hat{x} \) is a trajectory of a fluid parcel, so it is a solution of (2.9). In order to check the local properties of \( \hat{x} \), we examine the velocity field linearised around it, with deviation \( \xi \):

\[ \dot{\hat{x}} + \dot{\xi} = v(\hat{x}(t), t) + \nabla v(\hat{x}(t), t) \xi, \]

such that, since \( \dot{\hat{x}} = v(\hat{x}(t), t) \),

\[ \dot{\xi} = \nabla v(\hat{x}(t), t) \xi. \] (2.10)

Here, \( \nabla v \) is the velocity gradient tensor or the Jacobian matrix. Since we are considering the linearised behaviour around a fluid particle trajectory, it is called Lagrangian linearisation. Linearisation around a specified point in space is referred to as Eulerian linearisation. Equation (2.10) can be used to find the local stability properties of a critical point, which is defined as a point \( x_0 \) in space, such that at a fixed time \( t_0 \):\(^4\)

\[ v(x_0, t_0) = 0. \]

Note that generally the orbits traced out by critical points are no particle trajectories;\(^5\) these paths do not even necessarily behave like particle trajectories. For example, they can be eliminated by a coordinate transformation [15], and they do not have to move with the characteristic velocity of the fluid. Moreover, critical points can bifurcate, which makes their velocity approach infinity, so that the existence in time of a critical point is not guaranteed. Critical points are interesting, because the flow characteristics often change there [1]. Along any line through a critical point, the flow on opposite sides of the point is in opposite directions in general.

The characteristic properties of a critical point are determined by the eigenvalues of the Jacobian matrix of the velocity field in (2.10), evaluated at the critical point at that instant of time [5, 1].

---

\(^4\)Often a critical point of a function is defined by the property that the gradient of the function vanishes in that point. In that case \( x_0 \) is a critical point of the stream function.

\(^5\)It follows directly from the velocity field equation \( \dot{x} = v(x(t), t) \) that this is true unless \( \dot{x}_0 = 0 \).
2. General theory

If both eigenvalues $\lambda_i$ of the $2 \times 2$ matrix $\nabla \mathbf{v}(\mathbf{x}_0)$ are real, and of opposite signs, then the critical point is a hyperbolic point. If the eigenvalues are complex (and each other’s complex conjugate), then the critical point is an elliptic point. Except for the case that both eigenvalues are zero, these are the only possible cases in incompressible flows. This can be seen from the characteristic equation of $\nabla \mathbf{v}$:

$$(\partial_x u - \lambda)(\partial_y v - \lambda) - \partial_x v \partial_y u = 0,$$

so $\lambda^2 - \lambda(\partial_x u + \partial_y v) = -\partial_x u \partial_y v + \partial_x v \partial_y u \equiv c^2$, where $\partial_x$ and $\partial_y$ denote partial derivatives with respect to $x$ and $y$, respectively. Hence, $\lambda = \pm c$ only if the flow is incompressible. The critical point is hyperbolic if $c^2 > 0$, so if $\partial_x v \partial_y u > \partial_x u \partial_y v$, and elliptic otherwise.\(^6\) This excludes other possibilities, such as sources ($\lambda_1 > \lambda_2 > 0$), sinks ($\lambda_1 < \lambda_2 < 0$), and spiral points ($\lambda_{1,2} = \alpha \pm i\beta$).

If, in addition to being incompressible, the flow is locally irrotational, so that $\omega_z = \partial_x v - \partial_y u = 0$, then the eigenvalues satisfy

$$\lambda^2 = (\partial_x u)^2 + (\partial_x v)^2 > 0,$$

so they are real and of opposite sign. The flow induced by a point vortex is irrotational around the vortices, so that the critical points in these flows are hyperbolic in nature.

2.5. Index of a velocity field

In a vector field a number can be associated with a topological characteristic of the field. This is particularly interesting near critical points, i.e., hyperbolic and elliptic

\(^6\)This requirement on an elliptic point corresponds to Weiss’s criterion of a vortex in a two-dimensional incompressible flow: $x$ and $y$ with $\det(\nabla \mathbf{v}(x, y)) > 0$ [16].
2.5. Index of a velocity field

The number of critical points flow enclosed by a certain curve is related to the so-called Poincaré index of that curve [17, 18].

Consider the following vector field \((\dot{x}, \dot{y}) = (P(x, y, t), Q(x, y, t))\). Suppose we have a closed curve \(\chi\) in the plane. At each point on \(\chi\), we examine the vector field. When going around the closed path \(\chi\), the field vector rotates around the intersection point with \(\chi\), or it does not. Thus the vector makes a whole number revolutions. This number of rotations of the field vectors around the intersection points with \(\chi\), is called the Poincaré index \(I_\chi\) of the curve \(\chi\), or the winding number [19, 20]. Here \(I_\chi > 0\), if the rotation is in the same sense as the rotation taken to follow the path \(\chi\); \(I_\chi < 0\), in the case of opposite sense of rotation. It is important to point out that the index is independent of the chosen direction of the path. The index of a curve \(\chi\) can be expressed as the line integral

\[
I_\chi = \frac{1}{2\pi} \oint_\chi d \left( \frac{\arctan \frac{Q}{P}}{P} \right) = \frac{1}{2\pi} \oint_\chi \left( \frac{P \frac{\partial Q}{\partial x} - Q \frac{\partial P}{\partial x}}{P^2 + Q^2} \right) dx + \left( P \frac{\partial Q}{\partial y} - Q \frac{\partial P}{\partial y} \right) dy.
\] (2.11)

The fundamental theorem on indices states the following. Suppose that \(D\) is the area spanned by the closed path \(\chi\). If the continuous vector field \(V\) is nowhere zero on \(D\), and the field has a defined direction everywhere on \(D\), then \(I_\chi = 0\) [19]. This means that if \(I_\chi \neq 0\), then \(V\) has at least one critical point in \(D\). This theorem by Poincaré is a first indication that the index can give information on the vector field inside the path.

An important property of the index of a curve is the following: if a closed curve \(\chi\) or the vector field is continuously deformed, the index of \(\chi\) does not change, provided that it does not pass through a critical point of the vector field during this deformation. The critical points in this theory are points, where the direction of the vector is undefined, so that either the vector field at that point vanishes, or the field is not defined (a singularity) [20].

A significant consequence of this property on the connection between the indices of critical points and the Poincaré index \(I_\chi\) is the Poincaré index theorem, which states the following [19]. Suppose that \(V\) is a continuous vector field. Let \(D\) be the area spanned by a closed curve \(C\). If the field vector is defined everywhere on \(C\), then

\[
I_\chi = \sum_{i=1}^{n} I(P_i),
\]

where \(P_i, i = 1, \ldots, n\) are the critical points of \(V\) inside \(D\).\(^7\) Thus the total index can

\(^7\)Notice the similarities with the residue theorem in complex function theory.
be obtained, by considering the critical points of the vector field. As the index of a curve is invariant under continuous deformations, this imposes restrictions on the number and type of critical points. This is employed in chapter 3.

Figure 2.2-i: A hyperbolic critical point has Poincaré index (winding number) of $-1$, as the vector rotates once in a direction opposite of that of the path taken around the point.

Figure 2.2-ii: A vector on a contour around an elliptic point rotates once in the direction in which the contour is taken, so the point has index 1.

It follows that the index of a path around an elliptic point is $I_x = 1$, and if the path is around a hyperbolic point, then $I_x = -1$. This is illustrated in figures 2.2-i and 2.2-ii. This can be verified analytically by computing the contour integral (2.11), see appendix A.2. This analysis can also be done for complex functions instead of vector fields, so that the relation with Cauchy’s residue theorem becomes more apparent.

The theory can be applied to time dependent flows at every instant of time. However, it is difficult to make the connection between the critical points at different times. In chapter 3, the theory is applied to configurations of point vortices in a rotating frame, where the field far away is circular, meaning that the total index equals 1. Flows are considered with different angular frequencies of the frame, which leaves the index of the system invariant. In order to maintain the total Poincaré index of the flow, the emergence or disappearance of a critical point is bound to occur simultaneously with that of a point (or more) with an index of opposite sign.
3. Point vortex theory

The theory and governing equations concerning two-dimensional systems of point vortices form the topic of this chapter, where the concepts that were introduced in chapter 2 are applied. We start in section 3.1 with the equations describing the induced vortex motion and the corresponding stream function. Also, an energy consideration and the Hamiltonian with other constants of motion are presented. Furthermore, the transport of passively transported tracers in the flow is discussed. Passive tracers are particles that advected by the flow, without affecting it. Next, the location and number of the critical points form the topic of section 3.2.

As a system of vortices of equal sign rotates around its centre of vorticity, it is convenient to examine the motion of the vortices and the tracers in a frame, that rotates with the average angular velocity of the vortices. Therefore, the equations valid in a corotating frame are discussed in section 3.3. This approach is also taken in the numerical simulations. Subsequently, configurations of vortices, which are in equilibrium in the corotating frame are considered, supposing that they are stable. The motive is that these steady flows are often analytically solvable, so that the location of the vortices and the location and number of hyperbolic critical points can be computed exactly. These flows then form the basis for weakly time dependent flows.

The first consideration of a flow in a configuration in equilibrium in a revolving frame, is that of three vortices of equal strength on a line, in section 3.4. The line setup is unstable and a flip occurs, meaning that the middle vortex changes places with one of the other two, until the line configuration is approached again. The only stable setup of three vortices permitting a time independent solution in a revolving frame, is when the three vortices are equally far apart. This equilateral triangle configuration, as discussed in section 3.5, forms the point of departure for time dependent three vortex flows.

Similar to this triangle configuration, the setup of an arbitrary number of vortices placed on a circle, with equal distances between neighbouring vortices, has an equilibrium solution. This is the subject of section 3.6. Assuming that this configuration is stable, the associated angular frequency can be expressed in terms of the number of vortices, the vortex strength, and the radius of the circle. As the flow is independent of time, the separatrices can be found from the critical points of the velocity field. It is shown that the shape of the separatrices is independent of the vortex strength, and that it is scale invariant. In section 3.7 it is shown that the number of critical
points can be calculated, given the number of vortices. Employing the Poincaré index theorem from section 2.5, the number of hyperbolic critical points can be obtained.

Finally, we shift to a time dependent flow. In the case of three vortices, the number of critical points at any instant of time can be computed numerically, as is shown in section 3.8. The number found for the equilateral triangle proves to be a maximum, and the number of critical points decreases with the deviation of from the equilateral triangle configuration. This provides interesting information for the numerical simulations of time dependent three vortex flows in chapter 6.

It applies to entire chapter that the well-known part of the theory and governing equations are dealt with only briefly; more extensive derivations can be retrieved in appendix B.

### 3.1. Vortex motion

We consider the limiting case of localised vorticity distributions in two dimensions and formulate the equations on point vortex dynamics. It is assumed that the vorticity \( \omega_z = \omega \) of a vortex with strength \( \kappa \) is concentrated in a single point \( x_0 \), so

\[
\omega(x) = \kappa \delta(x - x_0),
\]

where \( \delta \) is the two-dimensional Dirac delta function, which makes the vorticity infinite at the point vortex and zero elsewhere. This is not a perfect representation of finite patches of vorticity, yet it proves to be mathematically very convenient and highly accurate for modelling flows, away from the patches. The stream function for this point vortex of strength \( \kappa \) at position \( x_0 \) can be found from (2.3):

\[
\Psi(x, y, t) = \int G(x - \xi) \omega(\xi) \, d\xi
= -\frac{1}{2\pi} \int \log |x - \xi| \kappa \delta(x_0 - \xi) \, d\xi
= -\frac{\kappa}{2\pi} \log |x - x_0|.
\]

As the stream function is linearly related to the vorticity, the stream function for a set of \( N \) of point vortices of strength \( \kappa_j \) located at \( x_j \) is obtained by linear superposition:

\[
\Psi(x, y, t) = -\frac{1}{2\pi} \sum_{j=1}^{N} \kappa_j \log |x - x_j|
= -\frac{1}{4\pi} \sum_{i=1}^{N} \kappa_i \log \left( (x - x_i)^2 + (y - y_i)^2 \right).
\]  \hspace{1cm} (3.1)
This stream function has singularities at the location of the point vortices, so $\Psi$ describes the motion in the entire flow domain, except for these points. This imposes no restraints, as we are merely interested in the flow properties away from the vortices.

If we view the motion in polar coordinates $(r, \varphi)$ instead of Cartesian coordinates, such that $\mathbf{v} = (v_r, v_\varphi)$, we find for a vortex of strength $\kappa$ at $(x_i, y_i) = (0, 0)$:

$$\hat{\Psi}(r, \varphi) = -\frac{\kappa}{2\pi} \log r,$$

and, consequently,

$$v_r = \frac{1}{r} \frac{\partial \hat{\Psi}}{\partial \varphi} = 0,$$

$$v_\varphi = -\frac{\partial \hat{\Psi}}{\partial r} = \frac{\kappa}{2\pi r}.$$

Hence, the velocity field induced by a point vortex has solely an azimuthal component. A point vortex has no self-induced motion, so that its motion is completely determined by other vortices. As a consequence, two vortices remain at a fixed distance and rotate around a common centre. Note that the circular velocity field, induced by a vortex of positive strength, has a counterclockwise direction.

It proves to be convenient to express the equations in complex variables. The stream function (3.1) in complex coordinates $z = x + iy$ ($i \equiv \sqrt{-1}$ being the complex unit) for the flow generated by a system of $N$ point vortices, located at positions $z_j = x_j + iy_j$, is

$$\Psi(z, z^*, t) = -\frac{1}{4\pi} \sum_{j=1}^{N} \kappa_j \log |z - z_j|^2.$$  \hspace{1cm} (3.2)

Next, we discuss the energy of the point vortex system [21, 11]. Integration of the kinetic energy per unit mass $\frac{1}{2} v_\varphi^2$ of a single vortex over an area $A$ enclosed by a circle of radius $R$ and a smaller circle of radius $\varepsilon$:

$$T = \frac{1}{2} \int_A v_\varphi^2 \, dA$$

$$= \frac{1}{2} \int_\varepsilon^R \left( \frac{\kappa}{2\pi r} \right)^2 2\pi r \, dr$$

$$= \frac{\kappa^2}{4\pi} \left( \log R - \log \varepsilon \right).$$

It follows that in the limit $\varepsilon \to 0$, $T \to \infty$, so the kinetic energy of a single point vortex is infinite. For general vorticity distribution, the quantity

$$W = \frac{1}{2} \int_A \omega \Psi \, dA$$  \hspace{1cm} (3.3)
represents the part of the kinetic energy associated with the way the vorticity is continuously being redistributed [21]. It measures the kinetic energy associated with the relative motion of the vortices. It turns out that this $W$ is the Hamiltonian $H$ of this system, which is a constant of the motion because of the assumption that no energy is lost by dissipation. The fact that interacting point vortices form a Hamiltonian system, implies that vortices cannot merge. For a system of point vortices the Hamiltonian is then found by substitution of $\Psi$ from (3.1) into (3.3):

$$H = -\frac{1}{8\pi} \sum_{j=1}^{N} \sum_{i=1, j \neq i}^{N} \kappa_i \kappa_j \log\left( (x_i - x_j)^2 + (y_i - y_j)^2 \right),$$  

(3.4)

Here, the energy of the vortices themselves has been omitted, which is the term with $j = i$ in the first sum of (3.4). In the case of finite patches of vorticity, this is no longer true. In that case, there is also energy associated with the interaction of particles within a patch. The fact that the Hamiltonian is a constant of the motion implies the absence of explicit time dependence:

$$\frac{\partial H}{\partial t} = 0.$$

The Hamiltonian is invariant under translation and rotation of the system, as a result of which two linear momenta ($P_x$ and $P_y$) and the angular momentum $L^2$ are three other invariants of the motion:

$$P_x = \sum_{i=1}^{N} \kappa_i x_i,$$

$$P_y = \sum_{i=1}^{N} \kappa_i y_i,$$

$$L^2 = \frac{1}{2} \sum_{i=1}^{N} \kappa_i (x_i^2 + y_i^2).$$

As we view the system in the infinite plane, it is invariant under translations, so that the origin can be shifted such that $P_x = P_y = 0$ without loss of generality. This is equivalent to adding a constant to the numerical value of $P_x$ and $P_y$, which does not affect the conservation in the flow [11]. The consequence is that $L^2$ is equal to the average squared distance between the vortices [22]:

$$L^2 = \frac{1}{2N} \sum_{j=1}^{N} \sum_{m=1}^{N} |z_j - z_m|^2$$

1. This can be seen by writing out the double sum: $\frac{1}{2N} \sum_{j}^{N} \sum_{m}^{N} |z_j - z_m|^2 = \frac{1}{2N} \sum_{j}^{N} \sum_{m}^{N} |z_j|^2 + |z_m|^2 - 2 \sum_{m}^{N} \sum_{j}^{N} \Re(z_j z_m) = L^2$, as the last two terms in the double sum are zero.
3.1. Vortex motion

\[
= \frac{1}{N} \sum_{j=1}^{N} \sum_{m=1}^{N} |z_j - z_m|^2.
\]

This quantity defines a spatial scale of the motion. By rescaling the coordinates \( z \rightarrow z/L \), it can be made equal to 1. The vortex strength \( \kappa \) can be scaled out by rescaling the time variable \( t \rightarrow (\kappa/L^2)t \). Then different flows are distinguished only by the Hamiltonian \( H \), which is related to the vortex configuration:

\[
\Lambda \equiv e^{-4\pi H} = \prod_{j=1}^{N} \prod_{m=1}^{N} |z_j - z_m|,
\]

(3.5)

where \( k = L = 1 \) has been taken. This is employed in the numerical simulations of the three vortex systems. In that case (3.5) reduces to

\[
\Lambda = e^{-4\pi H} = |z_1 - z_2||z_1 - z_3||z_2 - z_3|.
\]

(3.6)

This means that \( H \) is related to the product of the side lengths of the triangle spanned by the vortices. Since \( H \) is a conserved quantity, this product is constant. The range of \( \Lambda \) is \( \Lambda = 0 \), where \( H = \infty \) and two vortices coalesce, to \( \Lambda = 1 \), where \( H = 0 \) and the vortices lie on the vertices of an equilateral triangle.

From the four invariants three integrals in involution can be constructed, which means that they have the special property that they are in involution with respect to the Poisson bracket. For functions \( f \) and \( g \) this Poisson bracket is defined as

\[
\{f, g\} = \sum_{i=1}^{N} \frac{1}{\kappa_i} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right).
\]

(3.7)

The three integrals are \( H, L^2 \) and \( P_x^2 + P_y^2 \), which are mutually involutive for all \( \kappa_i \), meaning that for all three combinations of these three invariants the Poisson bracket vanishes. A Hamiltonian system with \( N \) degrees of freedom is integrable, if it has \( N \) integrals in involution [23]. Therefore, the system of three point vortices is integrable for any vortex strengths and positions.

In the special case that the sum of the vortex strengths is zero, \( H, L^2, P_x \) and \( P_y \) are mutually in involution, so that the four point vortex system is integrable. The fundamental Poisson brackets are

\[
\{z_i, z_j\} = 0,
\]

\[
\{z_i, z_j^*\} = \frac{-2i}{\kappa_i} \delta_{ij},
\]
with $\delta_{ij}$ the Kronecker delta: $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. The equations of motion of the point vortices can be written as

$$\dot{z}_\alpha = \{z_\alpha, H\},$$

or, written out per component,

$$\dot{x}_\alpha = \left\{ x_\alpha, H \right\} = \sum_j \frac{1}{\kappa_j} \left( \frac{\partial x_\alpha}{\partial x_j} \frac{\partial H}{\partial y_j} - \frac{\partial x_\alpha}{\partial y_j} \frac{\partial H}{\partial x_j} \right),$$

$$\dot{y}_\alpha = \left\{ y_\alpha, H \right\} = \sum_j \frac{1}{\kappa_j} \left( \frac{\partial y_\alpha}{\partial x_j} \frac{\partial H}{\partial y_j} - \frac{\partial y_\alpha}{\partial y_j} \frac{\partial H}{\partial x_j} \right),$$

which are Hamilton’s canonical equations. Substitution of the Hamiltonian (3.4) gives

$$\dot{x}_k = -\frac{1}{2\pi} \sum_{j=1}^N \kappa_j \left( \frac{y_k - y_j}{(x_k - x_j)^2 + (y_k - y_j)^2} \right),$$

$$\dot{y}_k = \frac{1}{2\pi} \sum_{j=1}^N \kappa_j \left( \frac{x_k - x_j}{(x_k - x_j)^2 + (y_k - y_j)^2} \right).$$

The differential equations (3.8) can be captured in one equation in the complex positions $z_k = x_k + iy_k$ and its complex conjugate $z_k^* = x_k - iy_k$:

$$\dot{z}_k^* = \frac{1}{2\pi i} \sum_{j=1}^N \kappa_j \left( \frac{1}{z_k^* - z_j} \right).$$

The motion of a passive tracer, which can be considered as a vortex with zero strength, is governed by (2.1), where the stream function $\Psi$ was found in (3.1). This stream function $\Psi$ equals the nonautonomous Hamiltonian $H_p$ of the dynamics of the passive particles (subscript p), so that (2.1) are Hamilton’s canonical equations. This Hamiltonian $H_p$ depends explicitly on time through the positions $(x_j, y_j)$ of the vortices. This is the reason that the passive particles may exhibit chaotic behaviour, even if the system (3.8) is regular (integrable) [3].
Substituting the streaming function $\Psi$ of (3.1) into (2.1), gives an equation for the velocity field $(\dot{x}, \dot{y})$ of the passive particles:

$$
\dot{x} = -\frac{1}{2\pi} \sum_{j=1}^{N} \kappa_j \frac{y - y_j}{(x - x_j)^2 + (y - y_j)^2},
$$

$$
\dot{y} = \frac{1}{2\pi} \sum_{j=1}^{N} \kappa_j \frac{x - x_j}{(x - x_j)^2 + (y - y_j)^2},
$$

(3.10)

Analogous to the derivation of (3.9), which describes the vortex motion in complex coordinates, the system (3.10) can be assembled in one equation:

$$
\dot{z}^* = \frac{1}{2\pi i} \sum_{j=1}^{N} \kappa_j \frac{1}{z - z_j},
$$

(3.11)

which follows from the stream function $\Psi$ in (3.2).

### 3.2. Critical points

The remaining sections of this chapter focus on flows induced by vortices, which are time independent or on flows at a fixed time. In the latter case, we are considering the so-called frozen time velocity field. In both cases, important properties of the flow are contained in the critical points, which are the locations where the velocity field is zero. They can be obtained by solving the coordinates for which the right-hand side of (3.11) vanishes. Although the critical points may be time dependent, the paths that they travel are no particle trajectories, unless the system is time independent. Moreover, critical points can disappear or emerge, so that their existence in time is not guaranteed.

The nature of these critical points is hyperbolic, as is shown in section 2.5. The location of the points as a function of the vortex strengths and positions in three vortex systems is discussed in appendix B.3.

The critical points in a flow induced by a system of $N$ point vortices are found by solving the roots of the polynomial $P$, which follows from multiplying (3.11) with $\prod_{j=1}^{N} (z - z_j)$, after diving out the factor $\kappa/(2\pi i)$ [24]:

$$
P(z) = \kappa_1 (z - z_2)(z - z_3) \ldots (z - z_N) +
\kappa_2 (z - z_1)(z - z_3) \ldots (z - z_N) + \ldots +
\kappa_N (z - z_1)(z - z_2) \ldots (z - z_{N-1}).
$$

(3.12)

This means that the critical points of a system of $N$ point vortices are the roots of a polynomial of degree $N - 1$. Therefore, the critical points can be found analytically.
for at most five point vortices. The maximum number of critical points for any configuration is \( N - 1 \). This maximum may not be attained, e.g., if roots coincide. Expanding the powers in \((3.12)\) gives

\[
P(z) = (\kappa_1 + \kappa_2 + \ldots + \kappa_N) z^{N-1} - (\kappa_1 z_2 + \kappa_1 z_3 + \ldots + \kappa_N z_1 + \kappa_N z_3 + \ldots + \kappa_2 z_N + \ldots + \kappa_N z_1 + \kappa_N z_2 + \ldots + \kappa_N z_{N-1}) z^{N-2} + O(z^{N-3})
\]

Therefore, if the sum of the vortex strengths is zero, there are at most \( N - 2 \) roots of \( P \), meaning at most \( N - 2 \) critical points.

The number of critical points is frame dependent, which means that they can be transformed away or created by switching to another frame. In the numerical simulations, we shift to a rotating frame, such that the vortex motion is quasi-stationary. In that frame, the critical points found above are no longer zeros of the velocity field. However, the transformation to the revolving frame introduces other and more zeros. If the appropriate angular frequency of the frame is chosen, the number of critical points can be larger than the number of vortices. Section 3.3 covers a more detailed discussion on the number of critical points in a corotating frame.

The midpoint, or centroid or centre of mass, \( z_M \) of positions \( z_\alpha \) is defined by

\[
z_M = \frac{1}{N} \sum_{\alpha=1}^{N} z_\alpha.
\]

This is a constant of the motion, as discussed in section 3.1. The centre of vorticity \( z_V \) is given by

\[
z_V = \frac{\sum_{\alpha=1}^{N} \kappa_\alpha z_\alpha}{\sum_{\alpha=1}^{N} \kappa_\alpha}.
\]

The sum of vectors from the midpoint to the critical points equals the vector from the centre of vorticity to the midpoint [24]. Note that \( z_V \) is a constant of the motion, because of the conservation of the linear momenta from section 2.3.

In the remaining sections of this chapter we focus on vortices of equal strength. In the simulations \( \kappa_\alpha = 1 \) is taken for every \( \alpha \) such that \( z_V \) equals \( z_M \). The centre of vorticity is then identical to the centre of rotation, which is chosen to be the origin.

### 3.3. Corotating frame

It is convenient to view the advection of the passive tracers in a frame that rotates with the average angular frequency of the point vortices, such that the motion of the
vortices is quasi-stationary. Often this motion in the corotating frame is quasi-periodic around an equilibrium point.

However, this transformation has major physical consequences. Namely, the Hamiltonian in the corotating frame is no longer an invariant quantity. Furthermore, the critical points in this frame are no longer hyperbolic by definition. This is because the eigenvalues of the velocity gradient tensor can become complex, such that elliptic critical points are possible. These points are called ghost vortices, because they are a consequence of the transformation to a corotating frame, which means that they are not recognised as vortices in the stationary frame.

A shift to a frame that rotates with frequency $\Omega$, amounts to adding an extra background vorticity $\omega'$ to the system of $\omega' = 2\Omega$. This can be seen by considering the flux of this extra vorticity through a circle of radius $R$:

$$\int \int \omega' \, dA = \pi R^2 \omega'. \tag{3.13}$$

As shown in section 2.3, this flux can also be written as

$$\oint v \cdot d\ell = \int_0^{2\pi} v_{\phi} R \, d\phi,$$

where $v_{\phi}$ is equal to $v_{\phi} = \Omega R$ for a solid body rotation, such that the last integral yields $2\pi R^2 \Omega$. This in combination with (3.13) shows that $\omega' = 2\Omega$.

In the fixed frame the equations of motion of the point vortices and the passive tracers can be derived from the Hamiltonian, which reads in complex coordinates:

$$H = -\frac{1}{8\pi} \sum_{j=1}^{N} \sum_{m=1}^{N} \kappa_j \kappa_m \log |z_m - z_j|^2.$$

Applying the canonical equations (3.1), the equations governing the motion of the vortices at position $z_m$ were found to be as in (3.9), and advection of passive particles at complex position $z$ satisfies (3.11). The transformation to a frame rotating with frequency $\Omega$ gives for the trajectories of the vortices

$$\tilde{z}_m = z_m e^{-i\Omega t}, \quad m = 1, \ldots, N, \tag{3.14}$$

where the tilde indicates a coordinate in the corotating frame. Differentiation of (3.14) with respect to $t$ yields

$$\dot{\tilde{z}}_m = (\dot{z}_m - i\Omega z_m) e^{-i\Omega t}. \tag{3.15}$$
After taking the complex conjugate of (3.15), substituting (3.9) and applying (3.14), the equation describing the motion of the vortices in the corotating frame becomes

\[
\dot{\tilde{z}}^* = \dot{z}_m^* e^{i\Omega t} + i\Omega z_m^* e^{i\Omega t} = \frac{1}{2\pi i} \sum_{j=1}^{N} \frac{\kappa_j}{\tilde{z}_m - \tilde{z}_j} + i\Omega \tilde{z}_m^*. 
\]  
(3.16)

A similar approach for the coordinates \(\tilde{z} = z e^{-i\Omega t}\) of the passive tracers in the rotating frame gives

\[
\dot{\tilde{z}}^* = \frac{1}{2\pi i} \sum_{m=1}^{N} \kappa_m \frac{1}{z - \tilde{z}_m} + i\Omega \tilde{z}^*. 
\]  
(3.17)

The stream function \(\tilde{\Psi}(\tilde{z}, \tilde{z}^*, t)\) in the moving frame has, besides a logarithmic term as in (3.2), an additional term:

\[
\tilde{\Psi}(\tilde{z}, \tilde{z}^*, t) = -\frac{1}{4\pi} \sum_{m=1}^{N} \kappa_m \log |\tilde{z} - \tilde{z}_m|^2 + \frac{1}{2} \Omega \tilde{z} \tilde{z}^*. 
\]  
(3.18)

A consequence of the shift to the corotating frame is that the instantaneous critical points are no longer necessarily hyperbolic in nature. This can be seen from equation (3.17) in \(x\) - and \(y\)-coordinates:

\[
\dot{x} = -\frac{1}{2\pi} \sum_{j=1}^{N} \kappa_j \frac{y - y_j}{(x - x_j)^2 + (y - y_j)^2} + \Omega y, \\
\dot{y} = \frac{1}{2\pi} \sum_{j=1}^{N} \kappa_j \frac{x - x_j}{(x - x_j)^2 + (y - y_j)^2} - \Omega x. 
\]  
(3.19)

For the vorticity \(\omega_z\) we now have

\[\omega_z = \partial_x v - \partial_y u = -2\Omega,\]

which is consistent with the remark on the background rotation at the beginning of this section.\(^2\) In the corotating frame the matrix \(\nabla v\) for any \(x\) then has the form

\[
\nabla v(x) = \begin{pmatrix} A & B + \Omega \\ B - \Omega & -A \end{pmatrix},
\]

\(^2\)Note that \(\omega_z = -2\Omega\) holds in the rotating frame, so that the added background rotation in the nonrotating frame is +2\(\Omega\).
for real constants $A$ and $B$. The eigenvalues $\lambda_{\pm}$ of this matrix are

$$
\lambda_{\pm} = \sqrt{A^2 + B^2 - \Omega^2},
$$

which becomes imaginary if $\Omega^2 > A^2 + B^2$. In that case, the critical points become elliptic, i.e., ghost vortices.

In the following sections flows are considered, which are stationary in a rotating frame, such that the theory on the Poincaré index from section 2.5 can be applied for all time. The critical points in this theory are points where the velocity field vanishes, or where the field is not defined, meaning a singularity [20]. A velocity field induced by point vortices is continuous, except at the vortices. As the vortex is approached, the azimuthal velocity goes to infinity, with opposite directions on either side of the vortex. Therefore, the field is undefined, just as in the case of a critical point of the vector field. This makes that in this context, an elliptic critical point and a vortex are topologically equivalent, even though the flow characteristics can be completely different, such as the decay of the induced velocity as a function of the distance to the vortex [20]. This is consistent with the fact that the index is independent of the radius of the path taken in the computation. In conclusion, a vortex is a critical point of the flow, having an index of 1.

Henceforth, we are considering systems of three point vortices of equal, positive strengths. Then for the contour a circle can be taken, which is large enough such that the velocity field is tangent to this curve along the entire path. This gives an index of 1, because sufficiently far away the field induced by three vortices equals that of a single vortex (with triple strength), i.e., an elliptic point.

Note that this index equals 1, regardless of the presence of background rotation, i.e., regardless of whether we view the system in a rotating frame or not. In the nonrotating frame, the field far away is oriented counterclockwise for vortices of positive strength. When the background rotation is turned on, this orientation reverses immediately, because of the rigid body rotation, which introduces large azimuthal velocities in the opposite direction far away. Nevertheless, the index remains 1, as it is independent of the direction of the path taken in calculation the contour integral.

### 3.4. Three vortices on a line

A configuration of three vortices system that allows for an equilibrium situation in the corotating frame, is one where vortices of equal strength $\kappa$ are initially positioned on a line, with equal distances between neighbouring vortices: $z_{k,0} : (r,0), (0,0), (0,r)$. 

For $z_{10} = r$, we substitute $z_k(t) = z_{k0} e^{i\Omega t}$ into (3.11), which gives\(^3\) for $z_1$

$$z_1^* = -i\Omega r e^{i\Omega t} = \frac{\kappa}{2\pi i} \left( \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} \right)$$

$$= \frac{\kappa}{2\pi i} \left( \frac{1}{r e^{i\Omega t}} + \frac{1}{2r e^{i\Omega t}} \right)$$

$$= \frac{3\kappa}{4\pi i r} e^{-i\Omega t}.$$

The angular frequency $\Omega$ is then found to be

$$\Omega = \frac{3\kappa}{4\pi r^2}.$$

As $\Omega \propto \kappa$, the vortex strength does not affect the solution of (3.11), or alter the lines of constant $\tilde{\Psi}$, which is equation (3.18) with $\kappa_m = \kappa$. So $\kappa = 1$ can be assumed

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\(^3\)The angular frequency can be derived from $z_1$, as the same result follows for $z_3$ from symmetry, and there is no induced velocity at the middle vortex, making it the centre of rotation.
3.5. Equilateral triangle

without loss of generality. If the size of the configuration is modified by a factor $\alpha$, so $r \to \alpha r$, we find for the location of the critical points:

$$\dot{\tilde{z}}^* = \frac{1}{2\pi i} \left( \frac{1}{\tilde{z} - \alpha r} + \frac{1}{\tilde{z}} + \frac{1}{\tilde{z} + \alpha r} \right) + i \frac{3\tilde{z}^*}{4\pi \alpha^2 r^2} = 0.$$  

Multiplication with $\alpha$ gives

$$\frac{1}{2\pi i} \left( \frac{1}{\alpha \tilde{z} - r} + \frac{1}{\alpha \tilde{z}} + \frac{1}{\alpha \tilde{z} + r} \right) + i \frac{3\tilde{z}^*}{4\pi r^2} = 0,$$

meaning that the location of the critical points scale with $\alpha$. It follows by a similar argument that the lines of constant $\tilde{\Psi}$, which include the separatrices, scale with $\alpha$ as well. This has been worked out for the configuration of an arbitrary number of vortices on a circle, see section 3.6. We can conclude that the figure of the critical points and the associated separatrices is independent of $\kappa$ and scale invariant.

The separatrices for this configuration are shown in figure 3.1. These are the lines of constant $\tilde{\Psi}$, equation (3.18), through the zeros. This approach yields the separatrices, as there are no lines of constant $\tilde{\Psi}$ through the elliptic critical points and the lines crossing the hyperbolic points are the separatrices by definition. The four hyperbolic and two elliptic critical points satisfy the Poincaré index theorem, since the associated indices of the hyperbolic points ($-4$) and the elliptic points ($+2$), combined with $+3$ from the vortices, gives a total index of $-4 + 2 + 3 = 1$. Six flow domains can be distinguished: three around the vortices, two around the ghost vortices, and one path between the separatrices.

The configuration discussed above is unstable for motions perpendicular to the line. After being close to the equilibrium for a certain amount of time, the centre vortex and one of the outer vortices quickly swap places in a circular motion. The stationary line configuration is then again approached, until another flip occurs. This process has been observed in the numerical simulations.

3.5. Equilateral triangle

In systems of three point vortices, there is only one stable equilibrium, in which the vortices are at rest in a frame rotating with the correct angular frequency. The vortices are placed on the vertices of an equilateral triangle, so that the three distances between the vortices are equal. In the nonrotating frame there is only one critical point for all time, because of symmetry. The transformation to a corotating frame introduces extra critical points, both of the elliptic and the hyperbolic type.

---

4This is a degenerate hyperbolic critical point, with an index of $-2$, see figure 3.9.
In the equilateral triangle configuration the vortex stationary positions in the revolving frame are taken to be $(0, 1), (-\frac{1}{2}\sqrt{3}, -\frac{1}{2})$ and $(\frac{1}{2}\sqrt{3}, -\frac{1}{2})$, so that the radius $r$ of the circle is 1. As is shown in section 3.6.1, the angular frequency $\Omega$ of this system is $\kappa/(2\pi)$, where $\kappa$ can be taken equal to 1 without loss of generality. The zeros of (3.17) are tracked numerically, and the method of finding the separatrices is the same as described before.

The setup with associated critical points and separatrices is shown in figure 3.2. There are six hyperbolic points, three of which are at equal distance close to the origin and three are on the lines through the origin and the vortices. The other four critical points are of the elliptic type: three are enclosed by the separatrices associated with the inner three hyperbolic points and one is in the origin. These elliptic ghost vortices, which have an opposite sense of rotation as compared to the vortices, ensure that the Poincaré index theorem is satisfied: the total index is +1 (because of the azimuthal field outside the outer separatrices) and the vortices yield +3 and the hyperbolic points give a total index of $-6$: $-6 + 4 + 3 = 1$. The three outer hyperbolic points are connected by a set of separatrices; the inner three points are connected by a different set of separatrices.

These transport barriers define eight flow domains. The separatrices belonging to the three outer hyperbolic points mark out three areas around the vortices. The separatrices associated with the inner three hyperbolic points enclose a region around the origin, and define three domains surrounding the ghost vortices. Finally, there is a very narrow path between the two independent set of separatrices. This latter region is the only one which contains no critical point.

The orbits traced out by the critical points are trajectories, and we can use these trajectories and the associated separatrices as a guideline for what to expect from the weakly time dependent flows. Time dependence in the flow is introduced by perturbing the equilateral triangle configuration. Under small perturbations, the flow characteristics and transport barriers are only slightly disturbed. With the idea that the trajectories we are looking for stay close to the orbits of the critical points, the analysis in this section facilitates the localisation of the trajectories and the flow barriers. However, if the time dependence is substantial, the separation of the paths of the critical points and the corresponding trajectories complicates the quest for these trajectories enormously. Then we have to resort to other methods, which are described in chapter 5.
Figure 3.2: The separatrices in the stationary velocity field of the equilateral triangle of vortices at rest in the corotating frame with frequency $\Omega = \kappa/(2\pi)$ (unit radius), which has a counterclockwise sense. The vortices are indicated by the circles and the ghost vortices by the dots; their direction of rotation (which is opposite for both types) is shown by the curved arrows. There are six hyperbolic points (intersections of the separatrices) and four elliptic ghost vortices, which ensure that the Poincaré index theorem is satisfied. The arrows near the hyperbolic points and outside the separatrices display the forward time evolution of particles. There are three hyperbolic points are connected by the dashed separatrices; the solid lines are the separatrices of the inner three points. These transport barriers define eight flow domains: three around the vortices enclosed by the dashed lines, four around the elliptic points encompassed with the solid lines (one of which is around the origin), and the very narrow path between the solid and dashed lines.
3.6. Vortices on a circle

In this section we consider a system $N$ point vortices located on a circle, with equal distances between the neighbouring vortices. It is shown that such a configuration exhibits a stationary rotation. In a frame where the vortices are at rest, numerous interesting flow characteristics can be examined, e.g., the angular frequency (section 3.6.1), scale invariance (section 3.6.2), the number of critical points, and the location of transport barriers (section 3.7).

Just as the equilateral triangle setup serves as a starting point for the analysis of time dependent three vortex systems, the motion of $N$ vortices on a circle is the basis for general $N$ vortex motion. An elaboration on this motion and the accompanying equations can be found in appendix B.5.

3.6.1. Angular frequency

It can be shown that a configuration of $N$ point vortices of equal strength $\kappa$ initially located on a circle with radius $r$ and with equal distances between two neighbouring vortices, rotates on this circle\footnote{It is assumed that this configuration is stable. However, it can be shown that for $N = 7$ the system is marginally stable, and for $N > 7$ it is unstable [25]. This has been verified numerically.} with angular frequency $\Omega$ given by

$$\Omega = \frac{(N - 1)\kappa}{4\pi r^2}.$$  

In order to show that the vortices rotate on the circle, we need to show that the circular motion is a solution of the differential equations.

Without loss of generality, we assume that the vortices $p_i$ initially have coordinates $\mathbf{x}_{k_0} = (x_{k_0}, y_{k_0}) = (r \cos \varphi_k, r \sin \varphi_k)$, with $\varphi_{k_0} = 2\pi \frac{k - 1}{N}$. This means that vortex $p_1$ lies on the $x$-axis with $\mathbf{x}_1 = (r, 0)$. In complex notation, we have $z_{k_0} = re^{i\varphi_{k_0}}$. The motion of the vortices then needs to obey

$$z_k(t) = z_{k_0} e^{i\Omega t}, \quad k = 1, \ldots, N.$$  

(3.20)

It is verified whether the trajectories (3.20) satisfy

$$\dot{z}_k^* = \frac{\kappa}{2\pi i} \sum_{j=1}^{N} 1 \frac{1}{z_k - z_j}.$$  

(3.21)

This is done for vortex $p_1$; it then follows from rotational symmetry (permutation of the indices) that (3.20) holds for all vortices. Substitution into the right hand side of
(3.21) yields
\[ \dot{z}_1^* = \frac{\kappa}{2\pi i} \sum_{j=2}^{N} \frac{1}{z_{10} e^{i\Omega t} - z_j e^{i\Omega t} - r e^{i\Omega t} e^{2\pi i(j-1)/N}} \]
\[ = \frac{\kappa}{2\pi i} e^{-i\Omega t} \sum_{j=2}^{N} \frac{1}{1 - e^{2\pi i(j-1)/N}}. \]

It is shown in appendix B.5 that
\[ \sum_{j=2}^{N} \frac{1}{1 - e^{2\pi i(j-1)/N}} = \frac{N - 1}{2}, \]
(3.22)
such that (3.20) and (3.21) demand that
\[ \dot{z}_1^* = -i\Omega r e^{-i\Omega t} = \frac{\kappa e^{-i\Omega t}}{4\pi i r} (N - 1). \]

We then find for \( \Omega \):
\[ \Omega = \frac{(N - 1)\kappa}{4\pi r^2}. \]
(3.23)

The fact that the angular frequency (dimension: \( s^{-1} \)) is proportional to the strength of the vortices is logical, given the dimension of the strength (m\(^2\) s\(^{-1} \)) and the fact that \( \kappa \) contains the only time dependence in (3.23).

Note that, as the azimuthal velocity induced by a vortex of strength \( \kappa \) at distance \( r \) is \( \kappa/(2\pi r) \), shifting to a frame that rotates with this frequency is equivalent to placing a vortex in the origin of strength \( -(N - 1)\kappa/2 \).

3.6.2. Scale invariance

As \( \Omega \propto \kappa \), the location of the critical points is independent of \( \kappa \). This follows directly from (3.17), because the factor \( \kappa \) can divided out. For the same reason, the stream function \( \tilde{\Psi} \) also scales with \( \kappa \), which means that the lines of constant \( \tilde{\Psi} \), which include the separatrices, are invariant under change of \( \kappa \).

The only remaining parameter is the radius \( r \) of the circle. It affects the location of the critical points and the shape of the separatrices in the following way. The critical points are the solutions \( \tilde{z} \) of
\[ \frac{\kappa}{2\pi i} \sum_{m=1}^{N} \frac{1}{\tilde{z} - \tilde{z}_m} + i\Omega \tilde{z}^* = 0. \]
(3.24)
If we alter the radius by a factor $\alpha$, so we transform $r \rightarrow \alpha r$, then, consequently, $\Omega \rightarrow \alpha^{-2}\Omega$ and $\hat{z}_m \rightarrow \alpha \hat{z}_m$. So the new roots satisfy

$$\frac{\kappa}{2\pi i} \sum_{m=1}^{N} \frac{1}{\hat{z} - \alpha \hat{z}_m} + i \frac{\Omega}{\alpha^2} \hat{z}^* = 0.$$ 

Multiplication with $\alpha$ then yields

$$\frac{\kappa}{2\pi i} \sum_{m=1}^{N} \frac{1}{\alpha \hat{z} - \hat{z}_m} + i \frac{\Omega}{\alpha} \hat{z}^* = 0.$$ (3.25)

A comparison of (3.8) and (3.25) shows that the roots transform via $\hat{z} \rightarrow \alpha \hat{z}$.

The shape of the separatrices is not affected by this transformation, meaning that they scale with the constant factor $\alpha$. This can be seen from considering $\tilde{\Psi}$ in (3.18):

$$\tilde{\Psi}(\hat{z}, \hat{z}^*, t) = -\frac{\kappa}{4\pi} \sum_{m=1}^{N} \log |\hat{z} - \hat{z}_m|^2 + \frac{\Omega}{2} \hat{z} \hat{z}^*.$$ (3.26)

For given hyperbolic point $\tilde{h}$, the associated separatrices are given by the points $\tau$ that satisfy

$$\tilde{\Psi}(\tau, \tau^*, t) = \tilde{\Psi}(\tilde{h}, \tilde{h}^*, t).$$

If the transformation $r \rightarrow \alpha r$ is performed, so that $\Omega \rightarrow \alpha^{-2}\Omega$ and $\hat{z}_m \rightarrow \alpha \hat{z}_m$ and $\tilde{h} \rightarrow \alpha \tilde{h}$, then the separatrices in the transformed system are the points $\chi$, satisfying

$$\tilde{\Psi}(\chi, \chi^*, t) = \tilde{\Psi}(\alpha \tilde{h}, \alpha \tilde{h}^*, t)$$

$$= -\frac{\kappa}{4\pi} \sum_{m=1}^{N} \log |\alpha \tilde{h} - \alpha \tilde{z}_m|^2 + \frac{\Omega}{2\alpha^2} \alpha \tilde{h} \alpha \tilde{h}^*$$

$$= -\frac{\kappa}{4\pi} \sum_{m=1}^{N} \left( \log |\tilde{h} - \tilde{z}_m|^2 + \log \alpha^2 \right) + \frac{1}{2} \Omega \tilde{h} \tilde{h}^*$$

$$= \tilde{\Psi}(\tilde{h}, \tilde{h}^*, t) - \frac{Nk \log \alpha}{2\pi}.$$ 

These points are the points in the original situation scaled by a factor $\alpha$:

$$\tilde{\Psi}(\alpha \tau, \alpha \tau^*, t) = -\frac{\kappa}{4\pi} \sum_{m=1}^{N} \log |\alpha \tau - \alpha \tilde{z}_m|^2 + \frac{\Omega}{2\alpha^2} \alpha \tau \alpha \tau^*$$

$$= -\frac{\kappa}{4\pi} \sum_{m=1}^{N} \left( \log |\tau - \tilde{z}_m|^2 + \log \alpha^2 \right) + \frac{1}{2} \Omega \tau \tau^*$$

$$= \tilde{\Psi}(\tau, \tau^*, t) - \frac{Nk \log \alpha}{2\pi}$$

$$= \tilde{\Psi}(\tilde{h}, \tilde{h}^*, t) - \frac{Nk \log \alpha}{2\pi}.$$
3.7. Critical points in circle configurations

From \( \tilde{\Psi}(\alpha \tau, \alpha \tau^*, t) = \tilde{\Psi}(\chi, \chi^*, t) \) it follows that the separatrices scale with the radius of the circle.

In conclusion, the shape of the separatrices is independent of \( \kappa \) and scale invariant. Thus the values \( \kappa = 1 \) and \( r = 1 \) can be taken henceforth without loss of generality, and there is no free parameter in the system.\(^6\)

3.7. Critical points in circle configurations

![Diagram of separatrices and vortices](image)

**Figure 3.3:** The separatrices in the stationary velocity field of the square configuration of vortices in the corotating frame with frequency \( \Omega = \frac{3 \kappa}{4 \pi} \) (\( r = 1 \)). The direction of rotation of the vortices (circles) and the ghost vortices (dots) is opposite for both types and is shown by the curved arrows. There are eight hyperbolic points and five elliptic ghost vortices, which ensure that the Poincaré index theorem is satisfied. The arrows near the hyperbolic points and outside the separatrices display the forward time evolution of particles. There are four hyperbolic points that are connected by the dashed separatrices; the solid lines are the separatrices belonging to the inner four points. These transport barriers define ten flow domains: four around the vortices enclosed by the dashed lines, five around the elliptic points encompassed with the solid lines, and the very narrow path between the solid and dashed lines. This figure is consistent with the results found in [26].

\(^6\) The angular frequency is not free, because it needs to satisfy (3.23) in order to have an equilibrium situation in the corotating frame.
3. Point vortex theory

Figure 3.4: The separatrices in the stationary velocity field of the pentagon of vortices (circles) in the corotating frame with frequency $\Omega = \kappa/\pi$ (unit radius). The intersection of the separatrices show that there are ten hyperbolic critical points, and the separatrices enclose six elliptic critical points (dots, five between the dashed lines and one in the origin). There are five critical elliptic points (the vortices), which have an index of $+1$, which complies with the Poincaré index theorem: $5 + 6 - 10 = 1$.

The configurations of an arbitrary number of vortices is further examined in this section. Using the independence of the vortex strength and the scale invariance, as shown in section 3.6, the number and nature of the critical points can be subjected to an investigation for general circular distribution of the vortices. In section 3.7.1 it is shown, that given the number of vortices, the number and location of the critical points can be retrieved. The distribution of these points between the elliptic and hyperbolic type follows by means of the Poincaré index theorem, see section 3.7.2. The limit case of an infinite number of vortices is discussed in section 3.7.3.

In order to gain an insight into the flow induced by $N$ vortices on a circle, the hyperbolic and elliptic critical points and the associated separatrices of the circle configurations of 4, 5, 6, 8, and 16 vortices, are shown in figures 3.3, 3.4, 3.5, 3.6, and 3.7, respectively.

In the setup of the square in figure 3.3, there are eight hyperbolic points and five elliptic ghost vortices, satisfying the Poincaré index theorem. The total index is $+1$ (because of the azimuthal field far away): the vortices yield $+4$ and the hyperbolic points give a total index of $-8$; the additional index of $+5$ is provided by the ghost
3.7. Critical points in circle configurations

The separatrices in the stationary velocity field of the hexagon of vortices (circles) in the corotating frame with frequency \( \Omega = \frac{5\kappa}{4\pi} \) \((r = 1)\). The intersection of the separatrices show that there are twelve hyperbolic critical points, and the separatrices enclose seven elliptic critical points (dots, six between the dashed lines and one in the origin). The Poincaré index theorem is satisfied, as there are six critical elliptic points (the vortices), which have an index of \(+1\) : \(6 + 7 - 12 = 1\).

Vortices. The four outer and four inner hyperbolic points, which are connected by two different sets of separatrices, define ten flow domains.

3.7.1. Number of critical points

**Theorem 3.1.** In a rotation frame where a circle configuration of \(N\) point vortices is in equilibrium, there are \(3N + 1\) critical points. There is one critical point on each line through the origin and a vortex and each angle bisector of these lines contains two critical points.

**Proof.** In order to find the critical points we need to solve

\[
\frac{\kappa}{2\pi i} \sum_{m=1}^{N} \frac{1}{z - z_m} + i\Omega z^* = 0, \quad (3.27)
\]

where the coordinates are in the corotating frame. It has been shown that the location of the critical point is independent of \(\kappa\) and scale invariant. Therefore, we take \(\kappa = 1\).
3. Point vortex theory

Figure 3.6: The separatrices in the stationary velocity field of the octagon of vortices (circles) in the corotating frame with frequency $\Omega = 7\kappa/(4\pi)$ ($r = 1$). The intersection of the separatrices show that there are sixteen hyperbolic critical points, and the separatrices enclose nine elliptic critical points (dots, eight between the dashed lines and one in the origin). The Poincaré index theorem is satisfied, because there are eight critical elliptic points (the vortices), which have an index of $+1 : 8 + 9 - 16 = 1$.

and $r = 1$ such that $\Omega = (N - 1)/(4\pi)$. This gives for (3.27):

$$\sum_{m=1}^{N} \frac{1}{z - z_m} - \frac{1}{2} (N - 1) z^* = 0.$$  \hfill (3.28)

Now consider the complex equation $z^N = 1$. The roots are the locations $z_j = e^{2\pi i (j-1)/N}, j = 1, \ldots, N$ of the vortices. It then follows that

$$\prod_{j=1}^{N} (z - e^{2\pi i (j-1)/N}) = z^N - 1.$$

Both polynomials are equal, because they have the same roots, the same leading coefficient (1) and the same degree. For readability, we write $z_j$ instead of $e^{2\pi i (j-1)/N}$ henceforth. Equation (3.28) is multiplied with $\prod_{j=1}^{N}(z - z_j)$ to yield

$$\prod_{j=1}^{N}(z - z_j) \sum_{m=1}^{N} \frac{1}{z - z_m} - \frac{1}{2} (N - 1) z^* \prod_{j=1}^{N}(z - z_j) = 0.$$  \hfill (3.29)
3.7. Critical points in circle configurations

Figure 3.7: The separatrices in the stationary velocity field of the hexadecagon configuration of vortices (circles) in the corotating frame with frequency \( \Omega = 15\kappa/(4\pi) \) (unit radius). The intersection of the separatrices show that there are 32 hyperbolic critical points and the separatrices enclose 17 elliptic critical points (dots, 16 between the dashed lines and one in the origin). The Poincaré index theorem is satisfied, because there are 16 critical elliptic points (the vortices), which have an index of +1: 16 + 17 − 32 = 1.

Furthermore, note that

\[
\prod_{j=1}^{N}(z - z_j) \sum_{k=1}^{N} \frac{1}{z - z_k} = \frac{d}{dz} \left( \prod_{j=1}^{N}(z - z_j) \right) = Nz^{N-1}.
\]

This gives for the critical points, equation (3.29),

\[
Nz^{N-1} - \frac{N-1}{2} z^*(z^N - 1) = 0.
\]

(3.30)

Figures 3.2 up to and including 3.7 have led to the conjecture that all critical points lie on a few lines through the origin. This suggests the use of polar coordinates \( r \) and \( \varphi \), such that equation (3.30) can be considered for constant \( \varphi \). It then becomes an equation in \( r \), instead of an equation in \( z \) and \( z^* \). Substitution of \( z = re^{i\varphi} \) and \( z^* = re^{-i\varphi} \) into (3.30) gives

\[
Nr^{N-1}e^{(N-1)i\varphi} - \frac{N-1}{2} re^{-i\varphi}(r^Ne^{Ni\varphi} - 1) = 0,
\]

\[\text{This identity follows from the fact that } \prod (z - z_j) \sum (z - z_k)^{-1} \text{ contains } N \text{ terms of } N-1 \text{ factors } z - z_m, \text{ where in any of the } N \text{ terms one factor } z - z_m \text{ is missing. This is the polynomial } P \text{ in (3.12) with } \kappa_j = 1.\]
so that \( r = 0 \) or
\[
Nr^{N-2}e^{(N-1)i\varphi} - \frac{N-1}{2} r^N e^{(N-1)i\varphi} + \frac{N-1}{2} e^{-i\varphi} = 0.
\]
After rearrangements we find \( r = 0 \) or
\[
r^N - \frac{2N}{N-1} r^{N-2} - e^{-i\varphi} = 0.
\] (3.31)
This equation has real solutions, only if \( N\varphi = k\pi, \ k \in \mathbb{N} \), so if \( \varphi = k\pi/N, k \in \mathbb{N} \). The vortices are positioned at angles \( \varphi = 2\pi k/N \), so the critical points lie on the lines through the vortices and the origin and on the lines exactly in between, i.e., the angle bisectors. Consequently, the equations to be solved are
\[
p_-(r) = r^N - \frac{2N}{N-1} r^{N-2} - 1 = 0, \quad \text{for } \varphi = \frac{2k\pi}{N}, \ k = 1, \ldots, N,
\] (3.32)
which hold on lines that pass through the vortices, and
\[
p_+(r) = r^N - \frac{2N}{N-1} r^{N-2} + 1 = 0, \quad \text{for } \varphi = \frac{(2k-1)\pi}{N}, \ k = 1, \ldots, N,
\] (3.33)
which is valid for the angle bisectors of the lines connecting the vortices with the origin. Both polynomials in \( r \) are shown in figure 3.8 for \( N = 4 \). For the number of

![Figure 3.8: The polynomials \( p_- \) and \( p_+ \) for \( N = 4 \). \( p_- \) has only one root for \( r > 0 \), whereas \( p_+ \) has two.](image)

zeros of \( p_+ \) and \( p_- \), we examine the derivatives:
\[
p'_+(r) = p'_-(r) = Nr^{N-1} - \frac{2N(N-2)}{N-1} r^{N-3},
\]
which is zero if \( r = 0 \) (a solution already found), and if
\[
r = r_0 = \sqrt{\frac{2(N-2)}{N-1}},
\]
3.7. Critical points in circle configurations

which is a minimum, as can be easily checked. At this root, \( p_- \) and \( p_+ \) have the values:

\[
p_{\pm}(r_0) = \pm 1 - \frac{2}{N-2} \left( \frac{2N-4}{N-1} \right)^{N/2}.
\]

\( p_+(r_0) \) is negative for \( N \geq 3 \) and \( p_-(r_0) < -1 \) for \( N \geq 3 \). So \( p_+ \) has one negative minimum and \( p_+(0) > 0 \), so \( p_+ \) has two zeros, one smaller and one larger than \( r_0 \). \( p_- \) has a minimum smaller than its starting value \( p_-(0) = -1 \), so \( p_- \) has only one zero, which is larger than \( r_0 \).

In conclusion, the lines through the vortices and the origin contain one zero and the angle bisectors of these lines have two critical points. As we can see from (3.32) and (3.33), both sets of solutions are valid for \( N \) values of \( \varphi \). This gives a total of \( 3N \) zeros, and combined with trivial root \( z = 0 \), we find a total of \( 3N + 1 \) critical points in a system of \( N \) point vortices placed on a circle.

\[\square\]

### 3.7.2. Nature of critical points

Next, we want to know how this number of critical points is divided into hyperbolic and elliptic points. To this end, the Poincaré index theorem can be applied. However, it is not guaranteed that each critical point is a regular version of one of these two types. This means that essentially more 'exotic', degenerate variants are possible, as explained in figure 3.9.

![Figure 3.9: A hyperbolic critical point with index \(-2\), which is also called a monkey-saddle (left), and a point with index \(+2\), also referred to as a dipole (right). However, the latter point is not critical, as the field vector is uniquely determined.](image)

Besides hyperbolic points with two stable and two unstable branches, there can also be saddle points with three or more stable and unstable arms. Such a special hyperbolic point has a different index. For instance, the so-called monkey-saddle with six bifurcations has an index of \(-2\). It is shown in appendix B.3, that this monkey-saddle lies in the centre of the equilateral triangle configuration in the nonrotating...
Points with a Poincaré index more than +1 are also possible. The dipole from figure 3.9 has an index of +2, however, it is not a critical point, as the field vector is uniquely determined. It follows that elliptic critical points have an index of +1, even if they are degenerate, because these points still have closed curves of constant value of the stream function around them (which deform from circular to square for higher order degeneracies).

We show here that in flows induced by point vortices in the corotating frame, only nondegenerate hyperbolic critical points can exist. This means that every hyperbolic critical point has index $-1$. The method of the proof is explained in figure 3.10.

Figure 3.10: A circle of arbitrarily small radius $\varepsilon$ around a hyperbolic point $z_h$. If there are at most four points where the value of $\tilde{\Psi}$ is equal to that of $z_h$, then hyperbolic points with at most two stable and two unstable branches can exist, so that critical points with in total six branches and a Poincaré index of $-2$ (and higher order degeneracies) are excluded.

A hyperbolic point and the associated separatrices are characterised by the fact that they have an equal value of the stream function $\tilde{\Psi}$. The number of lines departing from the critical point $z_h$ and having the same value of $\tilde{\Psi}$ (say, $c_h$) can be found as follows.

We consider a small circle of radius $\varepsilon$ around the critical point. The stream function is then examined at positions on the circle around $z_h$, i.e., $z = z_h + \varepsilon e^{i\varphi}$, $\varphi \in [0, 2\pi)$ being the polar angle. The tilde, indicating a coordinate in the corotating frame, has been omitted for readability. The values of $\varphi$ are then computed for which $z_h + \varepsilon e^{i\varphi}$ has a $\tilde{\Psi}$-value of $c_h$. Note that this cannot be done for an arbitrary value of $\varepsilon$. Hence, it is treated as a perturbation parameter, meaning that initially, terms of first order in $\varepsilon$ are matched and terms of higher order are neglected.

---

Note that in the case of only one critical point, its index needs to be $-2$ in order to keep the total index of the field equal to 1.
3.7. Critical points in circle configurations

We find for the value \( c_h \) at a hyperbolic point \( z_h \) in the corotating frame:

\[
\tilde{\Psi}(z_h, z^*_h, t) = -\frac{1}{4\pi} \sum_{j=1}^{N} \log |z_h - z_j|^2 + \frac{N - 1}{8\pi} z_h z^*_h \equiv c_h, \tag{3.34}
\]

where \( \kappa = 1 \) and \( r = 1 \) have been taken. We then search for the values of \( \varphi \in [0, 2\pi) \), for which

\[
\tilde{\Psi}(z_h + \varepsilon e^{i\varphi}, z^*_h + \varepsilon e^{-i\varphi}, t) = c_h.
\]

Substitution gives

\[
\tilde{\Psi}(z_h + \varepsilon e^{i\varphi}, z^*_h + \varepsilon e^{-i\varphi}, t) = -\frac{1}{4\pi} \sum_{j=1}^{N} \log |z_h + \varepsilon e^{i\varphi} - z_j|^2 + \frac{N - 1}{8\pi} (z_h + \varepsilon e^{i\varphi})(z^*_h + \varepsilon e^{-i\varphi}) = c_h.
\]

The logarithm is expanded for small \( \varepsilon \):

\[
\log((x + \varepsilon y)(x^* + \varepsilon y^*)) = \log((x x^*) + \left( \frac{y}{x} + \frac{y^*}{x^*} \right) \varepsilon - \left( \frac{y^2}{2x^2} + \frac{(y^*)^2}{2(x^*)^2} \right) \varepsilon^2 + O(\varepsilon^3),
\]

which gives

\[
-\frac{1}{4\pi} \sum_{j=1}^{N} \log |z_h - z_j|^2 + \frac{N - 1}{8\pi} z_h z^*_h + \varepsilon \left(-\frac{1}{4\pi} \sum_{j=1}^{N} \left( \frac{e^{i\varphi}}{z_h - z_j} + \frac{e^{-i\varphi}}{z^*_h - z_j} \right) + \frac{N - 1}{8\pi} (e^{i\varphi}z^*_h + e^{-i\varphi}z_h) \right) + \varepsilon^2 \left(\frac{1}{4\pi} \sum_{j=1}^{N} \left( \frac{e^{2i\varphi}}{2(z_h - z_j)^2} + \frac{e^{-2i\varphi}}{2(z^*_h - z_j)^2} \right) + \frac{N - 1}{8\pi} \right) + O(\varepsilon^3) = c_h. \tag{3.35}
\]

Matching the terms of \( O(1) \) in (3.35) gives equation (3.34), which can be subtracted to yield the following balance for the terms of \( O(\varepsilon) \):

\[
\sum_{j=1}^{N} \left( \frac{e^{i\varphi}}{z_h - z_j} + \frac{e^{-i\varphi}}{z^*_h - z_j} \right) - \frac{N - 1}{2} (e^{i\varphi}z^*_h + e^{-i\varphi}z_h) = 0.
\]

This equation is the sum of \( e^{i\varphi} \) times equation (3.28) and \( e^{-i\varphi} \) times the complex conjugate of (3.28), so that it is satisfied for any \( \varphi \). Consequently, we must examine the second order terms in \( \varepsilon \) in (3.35):

\[
\frac{1}{4\pi} \sum_{j=1}^{N} \left( \frac{e^{2i\varphi}}{2(z_h - z_j)^2} + \frac{e^{-2i\varphi}}{2(z^*_h - z_j)^2} \right) + \frac{N - 1}{8\pi} = 0,
\]
which can be written as\(^9\)
\[
e^{2i\varphi} \Theta + e^{-2i\varphi} \Theta^* + N - 1 = 0,
\]
with
\[
\Theta = \sum_{j=1}^{N} \frac{1}{(z_h - z_j)^2}.
\]

We now have one real-valued equation (3.36), which can be rewritten as
\[
2 \cos(2\varphi) \text{Re}(\Theta) - 2 \sin(2\varphi) \text{Im}(\Theta) + N - 1 = 0,
\]
where \(\text{Re}(\Theta)\) and \(\text{Im}(\Theta)\) denote the real and imaginary part of \(\Theta\), respectively. An equation of the form \(\cos(2\varphi) = c\) or \(\sin(2\varphi) = c\), \(c \in \mathbb{R}\) has at most four solutions \(\varphi \in [0, 2\pi)\). This also holds for the sum of these functions, as given by (3.37).

This concludes the proof that there are at most four bifurcations of a critical point, with the same value of the stream function in the corotating frame. So degenerate hyperbolic points are excluded, and the Poincaré index of a hyperbolic critical point is \(-1\). Note that this analysis is not restricted to circle configurations of vortices, and that it is valid for any distribution of vortices at any instant of time.

A verification for the equilateral triangle configuration in the nonrotating frame shows the following. Note that this is done for a fixed time \(t^*\), as the vortex positions are time dependent. Take \(z_1 = 1, z_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{3}i, z_3 = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i\) so that for the only critical point at \(z = 0\), we find
\[
\Psi(z = 0, t^*) = -\frac{1}{4\pi}(|z_1|^2 + |z_2|^2 + |z_3|^2) = -\frac{3}{4\pi} = c_h.
\]
For coordinate \(z = \varepsilon e^{i\varphi}\), we find that the \(\mathcal{O}(\varepsilon)\) and \(\mathcal{O}(\varepsilon^2)\) balances are satisfied automatically. For the terms of \(\mathcal{O}(\varepsilon^3)\) we find
\[
\sum_{j=1}^{3} \frac{e^{3i\varphi}}{3z_j^3} + \frac{e^{-3i\varphi}}{3(\zeta_j^*)^3} = 0,
\]
yielding \(\cos(3\varphi) = 0\), so that \(\varphi = \pi/6 + k\pi/3, k = 1, \ldots, 6\). This is the monkey-saddle as shown in figure 3.9.

Returning to the \(3N + 1\) of critical points of the circle configurations of \(N\) vortices, suppose that in a configuration of \(N\) point vortices, \(N_{cp}\) critical points are found. These critical points are either hyperbolic or elliptic in nature, so that
\[
N_{hyp} + N_{ell} = N_{cp},
\]
\(^9\)This follows from the fact that for complex number \(w = a + bi\) it holds that \((w^*)^2 = a^2 - b^2 - 2abi = (a^2 - b^2 + 2abi)^* = (w^2)^*\).
where \( N_{\text{hyp}} \) and \( N_{\text{ell}} \) represent the number of hyperbolic and elliptic critical points, respectively. It has been explained that the total index in the corotating frame equals +1 for a system of vortices of equal sign. As the hyperbolic critical points have index \(-1\), and the elliptic critical points and the vortices have index +1, we find
\[
-N_{\text{hyp}} + N_{\text{ell}} + N = 1. \tag{3.39}
\]
The combination of (3.38) and (3.39) gives
\[
N_{\text{hyp}} = \frac{1}{2} (N_{\text{cp}} + N - 1),
\]
\[
N_{\text{ell}} = \frac{1}{2} (N_{\text{cp}} - N + 1). \tag{3.40}
\]
From (3.40) we can derive that
\[
N_{\text{hyp}} = \frac{1}{2} (3N + 1 + N - 1) = 2N,
\]
\[
N_{\text{ell}} = \frac{1}{2} (3N + 1 - N + 1) = N + 1. \tag{3.41}
\]
Therefore, in a circle configuration of \( N \) point vortices which is in equilibrium in the corotating frame there are \( 2N \) hyperbolic and \( N + 1 \) elliptic critical points. Furthermore, every vortex and every elliptic point has a closed flow domain around it, and every configuration of vortices on a circle has a close path in between the sets of separatrices. This gives a total of \( 2N + 2 \) flow domains for a setup of \( N \) vortices on a circle.

Although the theory discussed above applies to systems of an arbitrary number of vortices, there is a striking difference between on the one hand, the setups of three and four vortices, and on the other hand, the configurations of five and more. In the case of three and four vortices, figures 3.2 and 3.3, the vortices are enclosed by the separatrices corresponding to the outer hyperbolic points, whereas the separatrices associated with the inner hyperbolic points enclose the ghost vortices. The narrow path in between has a clockwise orientation. For five and more vortices, the situation is exactly opposite: the lines of constant value of the stream function of the inner hyperbolic points surround only the vortices, and the elliptic points are encompassed with the separatrices of the outer hyperbolic points. The path in between is now taken in a counterclockwise direction.

### 3.7.3. Limit case

Note that since \( p_+(r) > p_-(r) \) for all \( r, \ r_- > r_+ \), where \( p_-(r_-) = 0 \) and \( r_+ \) is the largest root of \( p_+ \). In the limit \( N \rightarrow \infty \), the terms \( \pm 1 \) become irrelevant in the balance of terms for \( r > 1 \) and
\[
r^N - \frac{2N}{N - 1} r^{N-2} = r^{N-2} \left( r^2 - \frac{2N}{N - 1} \right) = 0,
\]
such that for large $N$ we have $r_\pm \to \sqrt{2}$. For the smallest zero $r_{m+}$ of $p_+$ we can write

$$r^{N-2} \left( r^2 - \frac{2N}{N-1} \right) + 1 = 0,$$

which gives $r_{m+} \to 1$ for $N \to \infty$. For a large number of vortices the separatrices converge to two sets of two lines that intersect each other infinitely many times. In

Figure 3.11: The separatrices in the corotating frame in the limit $N \to \infty$ and $\kappa \to 0$ such that their product $N\kappa$ is constant. The transport barriers are formed by a ring of radius 1 (bold) of continuous vorticity and a ring of radius $\sqrt{2}$. The outer ring contains a continuous distribution of ‘ghost vorticity’ of opposite sign as compared to the inner ring. The elliptic point in the origin has vanished, because $\kappa \to 0$.

the limit $N \to \infty$, the transport barriers are formed by two rings. One has radius 1, and contains the vortices and the hyperbolic points associated with the smallest root of $p_+$; the other ring is of radius $\sqrt{2}$, and it comprises the hyperbolic points, associated with the lines connecting origin with the vortices (the root of $r_-$), and the elliptic points on the bisectors of these lines (the largest root of $r_+$).

In the limit $N \to \infty$, the total strength becomes infinite, which would make the system rotate infinitely fast, as $\Omega \propto N$. If $N \to \infty$ and $\kappa \to 0$ such that $N\kappa$ converges to a constant value, the inner ring has a continuous vorticity distribution, and the fluid moves in a counterclockwise direction between the rings and in a clockwise direction in the inner and outer area, see figure 3.11. As the number of ghost vortices in the outer ring increases and their strength decreases accordingly, the outer rings then has a continuous distribution of ‘ghost vorticity’ of opposite sign as compared to the inner ring. The elliptic point in the origin has vanished because $\kappa \to 0$.  

3.8. Time dependent three vortex systems

It is shown in section 3.5, that a system of three point vortices in the equilateral triangle configuration gives rise to ten critical points in the corotating frame. The numerical simulations focus on time dependent three vortex systems, where the time dependence is enforced by perturbing the steady equilateral triangle flow. In this section, it is examined what the number and type of critical points is in a time dependent three vortex system in a corotating frame. It is important to realise that this number may vary in time for any flow, as critical points can bifurcate.

This theory is significantly different from the theory of section 3.2, where the critical points in a nonrotating frame are discussed. Then for \( N \) vortices at any time at most \( N - 1 \) critical points are possible, which are hyperbolic by definition.

Recall the equation in complex coordinates describing the location of the critical points in a frame rotating with angular frequency \( \Omega \) and \( \kappa = 1 \), which holds at any time \( t \) for the vortex positions \( z_m \) at that time:

\[
\frac{1}{2\pi i} \left( \frac{1}{z - z_1} + \frac{1}{z - z_2} + \frac{1}{z - z_3} \right) + i\Omega z^* = 0. \tag{3.42}
\]

Multiplication of (3.42) with \( 2\pi i \prod_{j=1}^{3}(z - z_j) \) gives

\[
(z - z_1)(z - z_2) + (z - z_1)(z - z_3) + (z - z_2)(z - z_3) - 2\pi \Omega z^* \prod_{j=1}^{3}(z - z_j) = 0,
\]

such that, using the conservation of linear momenta, \( z_1 + z_2 + z_3 = 0 \),

\[
3z^2 + z_1z_2 + z_1z_3 + z_2z_3 - 2\pi \Omega z^* \prod_{j=1}^{3}(z - z_j) = 0.
\]

Employing once more that the centre of vorticity is in the origin, we find the following equation for the critical points of the velocity field:

\[
-z^*\Omega' z^3 + 3z^2 - \rho \Omega' z^* z + \rho + \tau \Omega' z^* = 0, \tag{3.43}
\]

where \( \Omega' = 2\pi \Omega \), \( \rho = z_1z_2 + z_1z_3 + z_2z_3 \), and \( \tau = z_1z_2z_3 \). Note that the time dependence of the flow in (3.43) is contained completely in \( \rho \) and \( \tau \), which are therefore no invariants of the motion. Otherwise, the equation would be the same for all time, and the number and location of critical points would be equal during the flow, which would exclude disappearance and emergence of critical points.

Switching back to real coordinates \( x \) and \( y \) by substitution of \( z = x + iy \) and \( z^* = x - iy \), we find the following coupled set of fourth order polynomials, by demanding
that both the real and the imaginary part of (3.43) are zero:

\[
\begin{align*}
     y\Omega'\tau_2 + \rho_1 + 3x^2 - 3y^2 + \Omega'\tau_1 x - \Omega'(x^2 + y^2)(\rho_1 + x^2 - y^2) &= 0, \\
     \rho_2 + \Omega'\tau_2 x - \Omega'\rho_2(x^2 + y^2) - \Omega'\tau_1 y - 2xy(-3 + \Omega'(x^2 + y^2)) &= 0,
\end{align*}
\]

(3.44)

where \(\tau_1 = \text{Re}(\tau), \tau_2 = \text{Im}(\tau), \rho_1 = \text{Re}(\rho), \rho_2 = \text{Im}(\rho)\). A quick check of the validity of this set for the equilateral triangle configuration from section 3.5, which has \(\Omega' = 1, \rho = 0\) and \(\tau = -i\), yields the ten real roots \((x, y)\) as found before.

A second verification is the case that \(\Omega' = 0\). Then only two real roots can be obtained, because then the velocity field is viewed in the stationary frame, in which generally there are two critical points, as shown in section 3.2. The last example is the line configuration, with \(\Omega' = 3/2\) and \(\rho = -1\) and \(\tau = 0\), yielding six real roots. It is shown in section 3.4 that this number is correct.

For any value of \(\Omega', \rho, \text{ and } \tau\) the set (3.44) gives at most ten zeros. We know that this number \(N_{cp}\) is divided by the number of elliptic critical points \(N_{ell}\) and hyperbolic critical points \(N_{hyp}\): \(N_{cp} = N_{ell} + N_{hyp}\). Furthermore, the Poincaré index theorem prescribes that, since there are three vortices with index +1 and the total index equals +1: \(-N_{hyp} + N_{ell} + 3 = 1\), such that

\[
\begin{align*}
     N_{hyp} &= N_{ell} + 2, \\
     N_{cp} &= 2(N_{ell} + 1),
\end{align*}
\]

meaning that the number of critical points is even. So in three vortex systems, the distribution of critical points between the elliptic and hyperbolic type at any instant of time, is restricted to \((N_{hyp}, N_{ell}) = (6, 4), (5, 3), (4, 2), (3, 1) \text{ or } (2, 0)\). The latter possibility is the case in the stationary frame, \(\Omega = 0\). For any \(\Omega \neq 0\), there are at least four real zeros, which means at least three hyperbolic and one elliptic critical points.
4. Theory on hyperbolic trajectories

In this chapter, the theory on hyperbolic trajectories is discussed. A hyperbolic fluid particle trajectory is the generalisation to time dependent flows of a hyperbolic critical point in steady flows [27], which was treated in section 2.4. The definition of a hyperbolic trajectory is built up in a number of steps in section 4.1. The starting point is a stationary flow with corresponding stable and unstable subspaces in section 4.1.1. In time dependent flows natural generalisations of hyperbolic points and associated subspaces are possible. However, the theory becomes more complex, so that we start with introducing some needed concepts in section 4.1.2.

The Lyapunov exponent, which describes particle separation in time, is a first step toward the time dependent generalisation of the hyperbolic critical point. However, Lyapunov exponents only describe particle separation in one temporal direction. Therefore, it is necessary to consider the equivalent of the Lyapunov exponent in the opposite temporal limit. The consideration of the two limits \( t \to \pm \infty \) can be combined in a single construction, called the exponential dichotomy [1], which is introduced in section 4.1.3. The associated invariant manifolds are the time dependent generalisations of the separatrices of the hyperbolic critical point in the stationary case, see section 4.1.4. Section 4.1.5 then contains a brief explanation of lobes dynamics, which describe the transport of fluid via intersecting manifolds.

The starting point of the theory on hyperbolic trajectories is a hyperbolic path in space, for instance the path of a hyperbolic critical point. A localisation around this path is made in section 4.2, after which a differential equation is derived for the distance between the path and the sought trajectory. It is then shown that an associated integral equation selects special trajectories from the solutions of the differential equations, which have a limitation on their growth in time. Under certain conditions on the velocity field and the hyperbolic path, this integral equation has a unique solution close to the path, as is proved in section 4.2.2. The physical interpretation of this theory is presented in section 4.2.3.

In order to clarify this part of the theory, it is applied to an example flow in appendix E. A simple set of differential equations is chosen, so that the trajectories and associated characteristics can be computed analytically. The theory is also applied to the point vortex flow of an equilateral triangle configuration. In the latter case, the parameters of the exponential dichotomy of the critical points in the corotating frame can be calculated explicitly. The results can be found in appendix C.3.
4.1. Definition of a hyperbolic trajectory

The concept of a hyperbolic trajectory is constructed in a number of steps. The starting point is a stationary flow, for which in the case of a linear velocity field, there are stable and unstable subspaces that divide the flow into different regions.

4.1.1. Stationary flow

We first analyse the time independent situation, where the theory is illustrated by an example.

Example 1. Consider the (linearised) velocity field of a flow around a hyperbolic critical point:

$$\begin{align*}
\dot{x} &= \lambda x, \\
\dot{y} &= -\lambda y,
\end{align*}$$

for some positive constant $\lambda$, with solution

$$\begin{align*}
x(t) &= x(0) e^{\lambda t}, \\
y(t) &= y(0) e^{-\lambda t}.
\end{align*}$$

The hyperbolic critical point $x_{cp}$ is given by $x_{cp} = 0$, so $x(0) = y(0) = 0$. This is a hyperbolic point in the classical sense, since the trajectories of (4.1) obey $x(t)y(t) = c$ for all $t$, $c$ being a constant, so that the trajectories form hyperbolas. The velocity field is depicted in figure 4.1. The lines are trajectories where the arrows indicate the forward time evolution. A trajectory that starts on either axis, remains on it. This means that the axes are material curves, which are defined as curves in the fluid composed of points that move with the flow. As such a curve is a trajectory, no other trajectory can ever cross it. Thus an axis divides the velocity field in two flow domains. Trajectories on the $x$-axis have the property that their distance from the critical point grows exponentially in time; trajectories on the $y$-axis approach the critical point at an exponential rate. Therefore, the $x$-axis is referred to as the unstable subspace $E^u$ and the $y$-axis as the stable subspace $E^s$. Note that the critical point cannot be reached in finite time, otherwise it would be the intersection of multiple trajectories.

In the example, the velocity field is linear in $x$, so that the subspaces divide the flow into four domains. A similar pair of material curves exists if the velocity field is nonlinear, and then the subspaces are tangent to them. This special pair of material curves are called the stable and unstable manifold of the hyperbolic point. The stable manifold consist of those points through which the corresponding trajectories are asymptotic to the saddle point as $t \rightarrow \infty$. Analogously, the unstable manifold is composed of the points that approach the hyperbolic point for $t \rightarrow -\infty$. These
4.1. Definition of a hyperbolic trajectory

Curves are referred to as material curves, to emphasise that they consist of points that move with the fluid. The term invariant is used to express that the manifolds are particles trajectories, meaning that points on them cannot leave them. The time independent invariant manifolds are called the separatrices in fluid mechanics.

4.1.2. Lyapunov exponents

In time dependent flows, natural generalisations of hyperbolic points and associated subspaces are possible, with a pair of exponentially growing and decaying solutions. However, the theory is much more complex and we start with introducing some needed concepts.

Even in simple velocity fields, passive particles can display chaotic trajectories in two-dimensional time dependent flows. Close particles separate very quickly from each other and tracer patches spread rapidly to fill the entire chaotic region. This phenomenon is called chaotic advection [9, 28]. A way to quantify this chaotic nondiffusive mixing is to compute the exponential rate of particle separation. The associated exponent is called the Lyapunov exponent. Lyapunov exponents were first introduced by Lyapunov [29] in order to study the stability of nonstationary solutions of ordinary differential equations [30].
In a general velocity field the trajectory of a passive tracer is found from solving (2.9). The Lagrangian chaos is measured by the exponential divergence of trajectories in the neighbourhood in time, which shows the sensitive dependence on the initial conditions. This exponential divergence of a nearby Lagrangian orbit is characterised by the maximum Lyapunov exponent. Performing the Lagrangian linearisation around $x \rightarrow x + \xi$ for a trajectory $x$ gives (2.10), which describes the growth of infinitesimal perturbations of a trajectory. The Lyapunov exponent $\lambda_L$ is defined as

$$
\lambda_L = \lim_{|t-t_0| \to \infty \atop \xi(t_0) \to 0} \lambda(t) = \lim_{|t-t_0| \to \infty \atop \xi(t_0) \to 0} \frac{1}{t-t_0} \log \frac{|\xi(t)|}{|\xi(t_0)|}.
$$

$\lambda$ is the finite time Lyapunov exponent (FTLE). The limit $\xi(t_0) \to 0$ indicates that the separation of two particles is considered, which are initially infinitesimally close together. For the flow around the hyperbolic point of the set (4.1) we find that the Lyapunov exponents are $\lambda$ for the trajectory given by the $x$-axis and $-\lambda$ for the trajectory given by the $y$-axis. This is consistent with the following physical interpretation.

Consider a very small circle centred at the starting point of the trajectory. This circle deforms under the linearised flow about the trajectory into an ellipse. The Lyapunov exponents are the average logarithmic expansion rates of the principal axes of this ellipse [5].

The concept of the Lyapunov exponent can be used in the time dependent generalisation of the hyperbolic critical point. The limit $t \to \infty$ replaces the analysis of the eigenvalue of the Jacobian matrix, and a set of asymptotically exponential solutions is identified.

However, Lyapunov exponents have no unique exponentially growing counterpart to the decaying direction, because they describe particle separation in one temporal direction. Therefore, it is necessary to consider the equivalent of the Lyapunov exponent in both limits $t \to \pm \infty$. The consideration of the two limits can be combined in one construction called exponential dichotomy [1]. In section 4.1.3 we define the notion of hyperbolicity of a trajectory or a path by introducing this concept.

### 4.1.3. Exponential dichotomy

Consider this nonautonomous velocity field:

$$
\dot{x}(t) = v(x(t), t).
$$

(4.2)
The dot \( \dot{\gamma} \) denotes differentiation with respect to time. A trajectory \( \gamma \) of (4.2) is said to be hyperbolic if the linearised equation of (4.2) around \( \gamma \),

\[
\dot{\xi} = \nabla v(\gamma(t), t)\xi,
\]

has an exponential dichotomy, which is defined as follows. Consider these linear differential equations [7, 32].

\[
\dot{\xi} = A(t)\xi, \quad \xi \in \mathbb{R}^2,
\]

with \( A \in \mathbb{R}^{2 \times 2} \) a continuous function of time. Suppose \( X \) is the fundamental solution matrix of (4.3), so for any initial condition \( \xi_0 \), \( \xi(t) = X(t)\xi_0 \) is the solution passing through \( \xi_0 \) at \( t = 0 \) and \( X(0) = I \), \( I \) being the identity matrix on \( \mathbb{R}^{2 \times 2} \). From this general solution \( \xi \), the following identity follows:

\[\dot{X}(t) = A(t)X(t).\]

The columns of the fundamental solution matrix consist of linearly independent solutions\(^1\) of the linear system.

**Lemma 4.1.** The fundamental solution matrix \( X(t) \) is invertible for all \( t \).

**Proof.** Suppose \( x_1 \) and \( x_2 \) are the linearly independent solutions of (4.3). We have that \( X(0) = I \) which means \( x_1(0) \) and \( x_2(0) \) are linearly independent and that \( X(0) \) is invertible. We need to show that \( x_1(t) \) and \( x_2(t) \) are linearly independent for all \( t \), so that \( X(t) \) is invertible for all \( t \).

If there is a \( t \) such that \( x_1(t) \) and \( x_2(t) \) are linearly dependent, i.e., \( x_2(t) = c x_1(t) \) for some \( c \in \mathbb{R} \), then it follows from the uniqueness of the solutions that \( x_2(\tau) = c x_1(\tau) \) for all \( \tau \). This contradicts the invertibility of \( X(0) \), so that we conclude that \( X(t) \) is invertible for all \( t \). \( \square \)

Equation (4.3) is said to have an exponential dichotomy if there is a projection \( P \in \mathbb{R}^{2 \times 2} \), i.e., a map \( P \) that satisfies \( P^2 = P \), and positive constants \( k, \ell, \alpha \) and \( \beta \), such that

\[
\| X(t)PX^{-1}(s) \|_2 \leq k e^{-\alpha(t-s)}, \quad s \leq t \leq \infty \tag{4.4-i}
\]

\[
\| X(t)QX^{-1}(s) \|_2 \leq \ell e^{-\beta(s-t)}, \quad s \geq t \geq -\infty, \tag{4.4-ii}
\]

where \( Q = I - P \) and \( \| \cdot \|_2 \) denotes a matrix norm:\(^2\)

\[
\|A\|_2 \equiv \sup \left\{ \frac{|Aw|_2^2}{|w|_2^2} \mid w \in \mathbb{R}^2 \wedge w \neq 0 \right\} = \sqrt{\lambda_{\max}(A^*A)}, \tag{4.5}
\]

\(^1\)The solutions \( x_1 = (x_{11}, x_{12}) \) and \( x_2 = (x_{21}, x_{22}) \) on \( \mathbb{R} \) are called linearly independent if there are no constants \( c_1 \neq 0 \) and \( c_2 \neq 0 \) such that \( c_1 x_1 + c_2 x_2 = 0 \) on \( \mathbb{R} \).

\(^2\)Note that \( Q \) is also a projection, because \( Q^2 = (I - P)^2 = (I - P)(I - P) = I^2 - IP - PI + P^2 = I - P - P + P = I - P = Q \).
where $\lambda_{\text{max}}(A^*A)$ denotes the largest eigenvalue of $A^*A$ and $A^*$ is the conjugate transpose of $A$ [31]. This means that $\|A\|_2$ is the largest singular value of $A$, a concept which is discussed more extensively in appendix D.

Note that from this definition of $\|A\|_2$, it holds for arbitrary vector $w$ that

$$|A\|_2 \leq \|\|A\|_2\|_2.$$ 

This is used frequently in estimates hereafter.

If the matrix $A$ in (4.3) is constant, then an exponential dichotomy is equivalent to the property that none of the eigenvalues of the matrix has zero real parts. This would be the case for a steady velocity field linearised around a critical point [15].

For two-dimensional, incompressible flows the projection operators $P$ and $Q$ must each identify exactly one linearly independent solution; generally, it holds that the sum of their ranks equals the dimension of the full system [1].

The meaning of an exponential dichotomy can be recognised by rewriting (4.4-i) and (4.4-ii) in an equivalent form:

$$\|X(t)P\xi\|_2 \leq k' e^{-\alpha(t-s)} \|X(s)P\xi\|_2, \quad s \leq t \leq \infty,$$

$$\|X(t)Q\xi\|_2 \leq \ell' e^{-\beta(s-t)} \|X(s)Q\xi\|_2, \quad s \geq t \geq -\infty,$$

$$\|X(t)PX^{-1}(t)\|_2 \leq m', \quad \text{for all } t,$$

where $k'$, $\ell'$ and $m'$ are positive constants and $\xi$ is an arbitrary constant vector. Now if $P$ has rank $k$, the equation (4.6-i) says that there is a $k$-dimensional subspace $E^s$ of initial conditions of trajectories of the linearised equations (4.1.3), with the property that these trajectories decay to the hyperbolic trajectory as $t \to \infty$. Equation (4.6-ii) says that there is a $(n-k)$-dimensional subspace $E^u$ of initial conditions of trajectories of the linearised equations (4.1.3), for which the trajectories approach the hyperbolic trajectory exponentially as $t \to -\infty$. $P$ projects onto the stable subspace $E^s$; $Q$ onto the unstable subspace $E^u$. Inequality (4.6-iii) means that the angle between these two subspaces remains bounded away from zero for all time [32, 30]. Generally, this means that the lines indicating the stable and unstable direction cannot coincide.

This can be expressed more geometrically in the so-called extended phase space. This approach yields an autonomous set of equations, at the cost of a higher dimension of the new phase space $E$. This is done by appending the time variable $t$ to the phase space \{x $\in$ $\mathbb{R}^2$\} [33, 34]:

$$E \equiv \{(x, t) \in \mathbb{R}^2 \times \mathbb{R}\}.$$ 

$^3$This angle is measured at a time slice in the ($x, y, t$)-space, so that it is the angle between two intersections of the subspaces with a plane of constant $t$. 
4.1. Definition of a hyperbolic trajectory

The 'new' vector field is defined by adding the trivial evolution of $t$

$$\begin{align*}
\dot{x} &= v(x, t), \\
\dot{t} &= 1.
\end{align*}$$

The hyperbolic trajectory $\gamma$ in $\mathcal{E}$ is denoted by

$$\Gamma(t) = (\gamma(t), t),$$

which has become a curve in the three-dimensional extended phase space. A time slice of $\mathcal{E}$ at time $\tau$ is defined by

$$\Sigma_\tau \equiv \{(x, t) \in \mathcal{E} \mid t = \tau\}.$$ 

Expressed in the extended phase space, the hyperbolic trajectory $\Gamma$ intersects $\Sigma_\tau$ in

The extended phase space $\mathcal{E}$ with time on the vertical axis. $\Sigma_\tau$ is the time slice at $t = \tau$. $E^u(\tau)$ and $E^s(\tau)$ are the unstable and stable subspaces of the linearised system (in this case one-dimensional), corresponding to initial conditions for which the trajectories approach $\Gamma(t)$ (thick line) exponentially in time for $t \to \infty$ and $t \to -\infty$, respectively [34].

Figure 4.2: The extended phase space $\mathcal{E}$ with time on the vertical axis. $\Sigma_\tau$ is the time slice at $t = \tau$. $E^u(\tau)$ and $E^s(\tau)$ are the unstable and stable subspaces of the linearised system (in this case one-dimensional), corresponding to initial conditions for which the trajectories approach $\Gamma(t)$ (thick line) exponentially in time for $t \to \infty$ and $t \to -\infty$, respectively [34].

the phase space in the unique point $\gamma(\tau)$. For arbitrary vector $\xi$ in $\Sigma_\tau$, $P\xi$ is a vector in $E^s(\tau)$. It is evolved forward in time by the matrix $X$ a time step $t$ and a time step $s$, with $t > s$. Equation (4.6-i) then states that the distance to the hyperbolic trajectory $\gamma$, which is measured at a time slices $\Sigma_t$ and $\Sigma_s$, decreases exponentially. A similar statement holds for equation (4.6-ii), where a vector in $\Sigma_\tau$ projected onto $E^u(\tau)$, is used as an initial condition of a trajectory that approaches $\gamma$ exponentially.
in backward time.

The parameters $\alpha$ and $\beta$ (dimension: $\text{time}^{-1}$) in (4.4-i) and (4.4-ii) measure the 'strength' of hyperbolicity of a trajectory $\gamma$: the larger $\alpha$ and $\beta$, the faster the trajectories are attracted to it. This characteristic becomes important in later analysis. Another significant feature of $\alpha$ and $\beta$ is that they are upper bounds of the time dependent eigenvalues of $\nabla v$. This is used in the interpretation of a theorem on the existence and uniqueness of a hyperbolic trajectory in section 4.2.2.

The meaning of the operators can be illustrated as follows [1]. Suppose $Y$ is the fundamental solution matrix of the linearised system $\dot{y} = Ay$. The solution passing through $y_0 = y(0) = (y_{01}, y_{02})$ at $t = 0$, can then for all $t$ be written as

$$y(t) = Y(t)y_0.$$  \hfill (4.7)

so that it holds that $Y(0) = I$. The columns of $Y$ consist of the two linearly independent solutions, $Y = (y_1, y_2)$, one of which can be isolated from (4.7) by setting $y_{01} = 0$ or $y_{02} = 0$. Take the latter option for example. Mathematically, this implies that we define a projection $P$, with

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The projection $P$ and its counterpart $I - P$ map an arbitrary initial vector $y_0$ onto the corresponding initial condition for one of the two linearly independent solutions:

$$Py_0 = \begin{pmatrix} 0 \\ y_{02} \end{pmatrix}, \quad (I - P)y_0 = \begin{pmatrix} y_{01} \\ 0 \end{pmatrix}. $$

The solutions are time dependent and therefore, the operators cannot be applied directly to $y = Yy_0$ at an arbitrary $t$ with the same effect. The solution must be inverted in order to obtain the corresponding initial condition, and the operator can be applied to this initial condition:

$$PY^{-1}(t)y(t) = Py_0 = \begin{pmatrix} 0 \\ y_{02} \end{pmatrix}$$

$$\quad (I - P)Y^{-1}(t)y(t) = (I - P)y_0 = \begin{pmatrix} y_{01} \\ 0 \end{pmatrix}. $$

The two linearly independent solutions at time $t$ can be retrieved by applying the operator $Y$:

$$Y(t)PY^{-1}(t)y(t) = y_{02}y_2(t),$$

$$Y(t)(I - P)Y^{-1}(t)y(t) = y_{01}y_1(t).$$
4.1. Definition of a hyperbolic trajectory

In this way, a general trajectory \( y(t) = Y(t)y_0 \) for all \( t \) can be separated into its linearly independent components by combining the fundamental solution matrix \( Y \) and the operator \( P \).

In order to gain an insight into the concept of an exponential dichotomy, we apply it to the velocity field of example 1, equations (4.1). Even though it is a linear velocity field without explicit time dependence, it serves its purpose well to illuminate the theory.

**Example 1 (continued).** We find for the Jacobian

\[
A(t) = \nabla v(x_{cp}(t)) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix},
\]

which gives for the fundamental solution matrix \( X \) and its inverse \( X^{-1} \)

\[
X(t) = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix}, \quad X^{-1}(t) = \begin{pmatrix} e^{-\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}.
\]

The matrices \( P \) and \( Q \) project onto the stable subspace \( x = 0 \) and unstable subspace \( y = 0 \), respectively, which yields:

\[
P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

such that

\[
X(t)PX^{-1}(s) = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\lambda s} & 0 \\ 0 & e^{\lambda s} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & e^{\lambda(s-t)} \end{pmatrix},
\]

\[
X(t)QX^{-1}(s) = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-\lambda s} & 0 \\ 0 & e^{\lambda s} \end{pmatrix} = \begin{pmatrix} e^{\lambda(t-s)} & 0 \\ 0 & 0 \end{pmatrix}.
\]

Therefore, the conditions (4.4-i) and (4.4-ii) are satisfied with \( k = \ell = 1 \) and \( \alpha = \beta = \lambda \). This means that the path of the critical point \( x = 0 \) for all \( t \), is a hyperbolic trajectory in this example, due to the absence of time dependence in the velocity field. Note that, since \( \partial_x v \) is independent of \( x \), any trajectory is hyperbolic. This is a consequence of the linearity in \( x \) of the velocity field.

4.1.4. Invariant manifolds

The exponential dichotomy guarantees the existence of two sets of initial conditions of asymptotically exponential trajectories of the linearised velocity field, one approaching the hyperbolic trajectory for \( t \to \infty \) and one for \( t \to -\infty \). Furthermore, it provides a projection \( P \) that identifies these two trajectories.
In the linear, stationary field (4.1) the subspace $E^s$ and $E^u$ consist of sets of decaying and growing solutions, respectively. A generalisation of these subspaces is possible for hyperbolic trajectories. The sets of growing and decaying solutions are again material curves, which are straight lines at each fixed time. However, the orientation of the subspaces depends on time in general. These are the linearised representations of trajectories that approach the hyperbolic trajectory exponentially.

The stable manifold $W^s$ and unstable manifold $W^u$ of a hyperbolic trajectory $\gamma$ can now be defined as follows. The stable invariant manifold $W^s(t)$ of a hyperbolic trajectory $\gamma$ at time $t$ consists of all the points through which at time $t$ the trajectories of the nonlinear velocity field pass, that approach $\gamma$ for $t \to \infty$. The unstable invariant manifold $W^u(t)$ of $\gamma$ at time $t$ consists of the points through which the trajectories of the nonlinear velocity field pass that approach $\gamma$ for $t \to -\infty$. So the manifolds can be considered as initial conditions of trajectories that approach $\gamma$ in forward or backward time. This is illustrated in figure 4.3.

![Figure 4.3: The stable $W^s$ and unstable $W^u$ manifolds, and tangent to them the corresponding stable $E^s$ and unstable $E^u$ subspaces at time $t$. Both sets intersect in the hyperbolic point $\gamma(t)$. Particles on $W^s$ ($W^u$) approach $\gamma$ for $t \to \infty$ ($t \to -\infty$), where the arrows indicate the forward time evolution. The entire system has become time dependent so that this is a snapshot at time $t$.](image)

This theory gives rise the to following method of finding the stable and unstable invariant manifolds of a trajectory $\Gamma$ at time $\tau$ [34]. Given a segment of the unstable manifold $W^u(t_1)$ through the hyperbolic trajectory on the time slice $\Sigma_{t_1}$, the unstable manifold $W^u(\tau)$ on the time slice $\Sigma_\tau$ can be obtained for $\tau > t_1$ by evolving the trajectories starting on the segment forward in time to $t = \tau$. In the same manner, given a segment of the stable manifold $W^s(t_2)$ through the hyperbolic trajectory on
the time slice $\Sigma_{t_2}$, the stable manifold $W^s(\tau)$ on the time slice $\Sigma_\tau$ can be obtained for $\tau < t_2$ by evolving the trajectories starting on the segment backward in time to $t = \tau$. This procedure is shown in figure 4.4. It is explained in chapter 5 how this is executed numerically.

Figure 4.4: The extended phase space $\mathcal{E}$ with the stable and unstable manifolds, which follow from evolving a small segment manifolds through the hyperbolic trajectory in time [34]. This approach is taken in the numerical simulations. The forward evolution of a segment of $W^u(t_1)$ of $\Gamma$ (thick piece) from the time slice $\Sigma_{t_1}$ to the time slice $\Sigma_\tau$ yields the unstable manifold of $W^u(\tau)$ of $\Gamma$ on the time slice $\Sigma_\tau$. Similarly, the backward evolution of a small segment of $W^s(t_2)$ of $\Gamma$ (thick piece) from the time slice $\Sigma_{t_2}$ to the time slice $\Sigma_\tau$ yields the unstable manifold $W^s(\tau)$ of $\Gamma$ on the time slice $\Sigma_\tau$.

4.1.5. Lobe dynamics

In time independent flows the stable and unstable manifolds corresponding to different hyperbolic trajectories coincide. This is the case in the vortex configurations on a circle in the corotating frame, as discussed in chapter 3. If this setup is perturbed in a way that it becomes time dependent, this symmetry is broken and the manifolds intersect at more locations than at the hyperbolic points.

This is a fundamental difference with respect to steady flows, and gives rise to moving regions of fluid, bounded by pieces of stable and unstable manifolds, the so-called lobes. Since the manifolds are material lines, fluid can not cross them by purely
advective processes and thus they are transport barriers. Motion of the lobes is thus the mechanism responsible for Lagrangian transport between different regions [35]. A background on lobe dynamics can be found in [36, 37].

If the flow is time independent, the manifolds intersect only in the hyperbolic trajectory. The intersection of the stable and unstable manifold associated with one hyperbolic trajectory is called a homoclinic point, and intersections of the stable and unstable manifolds of different hyperbolic trajectories are called heteroclinic points. If the flow becomes time dependent, the stable and unstable manifolds of the hyperbolic points intersect in more points, and one intersection implies infinitely many [9]. This is explained in figures 4.5 and 4.6. As the flow is orientation preserving, the area $A$ is mapped to $A'$, which in turn is mapped to $A''$. If $A$ was mapped to $B$, it would not preserve orientation. This mapping implies a periodicity in the flow, so that these figures of the homoclinic and heteroclinic tangles are Poincaré sections [38].

The transport mechanism can be explained as follows. Hamiltonian systems are area conserving (appendix A.1): a particle on the stable manifold approaches the hyperbolic point asymptotically slowly, so that a homoclinic or heteroclinic point on the unstable manifold must travel increasing distances normal to the stable manifold [9]. The aspect ratio of the lobes increases when the hyperbolic point is approached.

![Figure 4.5: The intersections of the stable and unstable manifolds of one hyperbolic point $P$ are called homoclinic points. These intersections are a consequence of the time dependence in the flow and transport takes place via the spanned areas, the lobes. The area $A$ is mapped to $A'$, which in turn is mapped to $A''$. A map to $B$ would not preserve orientation.](image-url)
Figure 4.6: The intersections of the stable and unstable manifolds of distinct hyperbolic points \( P \) and \( Q \), which are a consequence of the time dependence in the flow, are called heteroclinic points. The area \( A \) is mapped to \( A' \), which in turn is mapped to \( A'' \). A map to \( B \) would not preserve orientation.

4.2. Localisation around hyperbolic path

We return to the hyperbolic trajectories and focus on the characteristics. The starting point is a hyperbolic path in space. We assume that we have a given a continuous curve \( x_{cp}(t), t \in \mathbb{R} \), which is hyperbolic in a sense described below. This curve is not necessarily a solution of \((4.2)\). In the derivation by Mancho et al. [5], an arbitrary hyperbolic path is taken. Ju et al. [7] assume that this path is a curve of critical points (this explains the subscript \( cp \)). The reason for this is that in situations where the velocity field varies slowly in time, the curve of the frozen time critical points may stay close to a trajectory of the vector field. However, this assumption on the curve is not necessary in the following derivation.

Suppose we can prove that the linear equation

\[
\dot{\xi} = \nabla v(x_{cp}(t), t)\xi
\]

has an exponential dichotomy with corresponding projections \( P \) and \( Q \) and fundamental solution matrix \( X \). This means that we assume that the curve \( x_{cp} \) is hyperbolic. The distance from a trajectory \( x \) to \( x_{cp} \) is given by \( z = x - x_{cp} \). It is governed by
the following equation:

\[ \dot{z}(t) = A(t)z(t) + f(z(t), t), \quad (4.10) \]

with

\[ A(t) = \nabla v(x_{cp}(t), t), \tag{4.11} \]

\[ f(z(t), t) = v(z(t) + x_{cp}(t), t) - \nabla v(x_{cp}(t), t)z(t) - x_{cp}(t). \tag{4.12} \]

### 4.2.1. Integral formulation

Next, we switch from the implicit differential equation for the distance \( z \) between the path and the trajectory we are looking for, to an implicit integral equation. The solutions of this equation are solutions of the differential equation and, furthermore, they have an additional constraint on their growth in time. Consequently, the converse does not hold, i.e., solutions of the differential equation are not necessarily solutions of the integral equation.

**Theorem 4.2.** Let \( z \) be a function of \( t \in \mathbb{R} \) with values in \( \mathbb{R}^2 \). The following statements are equivalent:

1. \( z \) is a solution of the differential equation (4.10) that satisfies

\[
|QPX^{-1}(t)z(t)| \to 0 \quad \text{for} \quad t \to -\infty, \tag{4.13a}
\]

\[
|PX^{-1}(t)z(t)| \to 0 \quad \text{for} \quad t \to \infty, \tag{4.13b}
\]

where \( P \) and \( Q \) are the projections associated with the exponential dichotomy of the linear equation (4.9), which has fundamental solution matrix \( X \).

2. \( z \) is a solution of the integral equation

\[
z(t) = X(t) \int_{-\infty}^{t} PX^{-1}(s)f(z(s), s)ds - X(t) \int_{t}^{\infty} PX^{-1}(s)f(z(s), s)ds, \quad (4.14)
\]

where \( f \) is given in (4.12).

Remark: if it holds that

\[
k \, e^{\alpha t}|z(t)| \to 0 \quad \text{for} \quad t \to -\infty,
\]

\[
\ell \, e^{-\beta t}|z(t)| \to 0 \quad \text{for} \quad t \to \infty,
\]

then the conditions (4.13) are satisfied. This follows from the exponential dichotomy of (4.9), equation (4.4-i) with \( t = 0 \) and \( s = t \). Namely,

\[
k \, e^{\alpha t}|z(t)| \geq \|PX^{-1}(t)\|_2|z(t)| \geq |PX^{-1}(t)z(t)| \to 0,
\]
and likewise for $t \to -\infty$. This means that the conditions (4.13) are weaker than those of an exponential dichotomy. This remark can also be substantiated using the alternative definition of the exponential dichotomy, equation (4.6-i).

The proof of theorem 4.2 follows the continuation of example 1.

**Example 1 (continued).** Consider example 1 again, for which we formulate the differential and integral equation. We find for $f$:

\[
 f(z(t), t) = v(z(t) + x_{cp}(t), t) - \nabla v(x_{cp}(t), t) z(t) - \dot{x}_{cp}(t)
 = \begin{pmatrix}
 z_1 + x_{cp,1} \\
 -z_2 - x_{cp,2}
\end{pmatrix} - \begin{pmatrix}
 1 & 0 \\
 0 & -1
\end{pmatrix} \begin{pmatrix}
 z_1 \\
 z_2
\end{pmatrix} - \begin{pmatrix}
 0 \\
 0
\end{pmatrix}
 = \begin{pmatrix}
 x_{cp,1} \\
 -x_{cp,2}
\end{pmatrix} - \begin{pmatrix}
 0 \\
 0
\end{pmatrix}.
\]

Consequently, the integral equation (4.14), which has $f$ as a factor in both integrals, yields $z = 0$ as the only solution. Furthermore, the differential equation (4.10) reduces to $\dot{z} = Az$, with $A$ given in (4.8), which has general solutions

\[
 z_1 = z_1(0) e^{\lambda t}, \\
 z_2 = z_2(0) e^{-\lambda t}.
\]

The superimposed conditions (4.13) on these solutions read

\[
 QX^{-1}(t)z(t) = \begin{pmatrix}
 0 & 0 \\
 0 & 1
\end{pmatrix} \begin{pmatrix}
 e^{-\lambda t} & 0 \\
 0 & e^{\lambda t}
\end{pmatrix} \begin{pmatrix}
 z_1 \\
 z_2
\end{pmatrix} = \begin{pmatrix}
 0 \\
 e^{\lambda t} z_2
\end{pmatrix} \to \begin{pmatrix}
 0 \\
 0
\end{pmatrix}, ~ t \to \infty,
\]

\[
 PX^{-1}(t)z(t) = \begin{pmatrix}
 1 & 0 \\
 0 & 0
\end{pmatrix} \begin{pmatrix}
 e^{-\lambda t} & 0 \\
 0 & e^{\lambda t}
\end{pmatrix} \begin{pmatrix}
 z_1 \\
 z_2
\end{pmatrix} = \begin{pmatrix}
 e^{-\lambda t} z_1 \\
 0
\end{pmatrix} \to \begin{pmatrix}
 0 \\
 0
\end{pmatrix}, ~ t \to -\infty.
\]

Since the general solution $z$ is given by (4.15), these conditions can only be satisfied if $z_1(0) = z_2(0) = 0$, i.e., $z = 0$ for all $t$. This shows that the conditions (4.13) select from the solutions of the differential equation (4.10) the one that has time growth smaller than $e^{\pm \lambda t}$. This selective procedure is contained in the integral equation (4.14), which gives the solution $z = 0$ immediately.

**Proof.** We first prove that a solution of (4.14) is a solution of (4.10) that satisfies (4.13); we then show the converse.
The fact that a solution of the integral equation (4.14) is a solution of the differential equation (4.10), can be seen from direct substitution on both sides:

\[
\dot{z}(t) = A(t)z(t) + f(z(t), t)
\]

\[
= A(t)X(t) \left( \int_{-\infty}^{t} PX^{-1}(s)f(z(s), s)ds - \int_{t}^{\infty} QX^{-1}(s)f(z(s), s)ds \right) + f(z(t), t).
\]

(4.16)

Taking the derivative with respect to \( t \) of (4.14) yields:

\[
\dot{z}(t) = \dot{X}(t) \int_{-\infty}^{t} PX^{-1}(s)f(z(s), s)ds + X(t)PX^{-1}(t)f(z(t), t) - \dot{X}(t) \int_{t}^{\infty} QX^{-1}(s)f(z(s), s)ds + X(t)QX^{-1}(t)f(z(t), t)
\]

\[
= \dot{X}(t) \left( \int_{-\infty}^{t} PX^{-1}(s)f(z(s), s)ds - \int_{t}^{\infty} QX^{-1}(s)f(z(s), s)ds \right) + \left[ X(t)(P + Q)X^{-1}(t)f(z(t), t) \right]_{P+Q=I}
\]

\[
= \dot{X}(t) \left( \int_{-\infty}^{t} PX^{-1}(s)f(z(s), s)ds - \int_{t}^{\infty} QX^{-1}(s)f(z(s), s)ds \right) + f(z(t), t)
\]

(4.17)

Equating (4.16) and (4.17) then requires

\[
\dot{X}(t) = A(t)X(t),
\]

which is true, since \( X \) is the fundamental solution matrix of (4.9). This concludes the proof that a solution of (4.14) is a solution of (4.10).

In order to complete the first part of the proof, it remains to show that this solution obeys (4.13). We multiply (4.14) with \( QX^{-1}(t) \) and use that \( PQ = 0 \) and \( Q^2 = Q \):

\[
QX^{-1}z(t) = - \int_{t}^{\infty} QX^{-1}(s)f(z(s), s)ds.
\]

(4.18)

The right hand side of (4.18) goes to zero, because \( QX^{-1}f \) is an integrable function and the integration boundaries become equal in the limit \( t \to \infty \). Therefore, the left hand side of (4.18) goes to zero as well, so (4.13a) is satisfied.

On the other hand, multiplication of (4.14) with \( PX^{-1}(t) \) yields, using that \( P^2 = P \) and \( PQ = 0 \):

\[
PX^{-1}z(t) = \int_{-\infty}^{t} PX^{-1}(s)f(z(s), s)ds.
\]

(4.19)
An analogous argument for (4.19) in the limit \( t \to -\infty \) shows that (4.13b) is satisfied as well. This completes the proof that a solution of the integral equation (4.14) is a solution of the differential equation (4.10), satisfying (4.13).

In order to check whether a solution of the differential equation (4.10), obeying (4.13), is a solution of (4.14), suppose that (4.10) holds. Now use that \( \dot{X} = AX \) and the following, \( I \) begin the \( 2 \times 2 \) identity matrix:

\[
0 = \frac{d}{dt} = \frac{d}{dt}(X^{-1}(t)X(t)) = \frac{d}{dt}(X^{-1}(t))X(t) + X^{-1}(t)\frac{d}{dt}X(t)
= \frac{d}{dt}(X^{-1}(t))X(t) + X^{-1}(t)A(t)X(t),
\]
such that, by left multiplication with \( X(t) \) and right multiplication with \( X^{-1}(t) \),

\[
A(t) = -X(t)\frac{d}{dt}(X^{-1}(t)).
\] (4.20)

Equation (4.20) for \( A(t) \) is substituted into (4.10) to yield

\[
\dot{z}(t) = f(z(t), t) - X(t)\frac{d}{dt}(X^{-1}(t))z(t),
\]
so that

\[
X^{-1}(t)\dot{z}(t) = X^{-1}(t)f(z(t), t) - \frac{d}{dt}(X^{-1}(t))z(t),
\]
which can be factored as

\[
\frac{d}{dt}(X^{-1}z(t)) = X^{-1}(t)f(z(t), t).
\] (4.21)

Integration and subsequent left multiplication with \( X(t) \) then yields

\[
z(t) = X(t)\int_0^t X^{-1}(s)f(z(s), s)\,ds + X(t)z(0).
\] (4.22)

This result can be separated into two parts, using \( P + Q = I \):

\[
z(t) = X(t)\int_0^t X^{-1}(s)f(z(s), s)\,ds + X(t)z(0)
= X(t)\int_0^t (P + Q)X^{-1}(s)f(z(s), s)\,ds + X(t)z(0)
= X(t)\left(\int_0^t PX^{-1}(s)f(z(s), s)\,ds - \int_t^0 QX^{-1}(s)f(z(s), s)\,ds + z(0)\right).
\] (4.23)

\(^4\)Note that \( \frac{d}{dt}(X^{-1}(t)) \) is written deliberately without the dot-notation, to avoid confusion with \((\dot{X}(t))^{-1}\). The simple one-dimensional example \( X(t) = t \) shows that these are not the same: \((\dot{X}(t))^{-1} = 1 \) and \( \frac{d}{dt}(X^{-1}(t)) = -\frac{1}{t^2} \).
4. Theory on hyperbolic trajectories

Multiplication of (4.23) with $QX^{-1}(t)$ gives, since $QP = 0$ and $Q^2 = Q$:

$$QX^{-1}(t)z(t) = -\int_t^0 QX^{-1}(s)f(z(s), s)\, ds + Qz(0). \quad (4.24)$$

If we let $t \to \infty$, then the left hand side of (4.24) goes to zero if because of (4.13a). Then the right hand side of (4.24) also vanishes for $t \to \infty$:

$$Qz(0) = \int_0^\infty QX^{-1}(s)f(z(s), s)\, ds \quad (4.25)$$

Analogously, we can multiply (4.23) with $PX^{-1}(t)$, which leads to, since $P^2 = P$ and $PQ = 0$:

$$PX^{-1}(t)z(t) = \int_0^t PX^{-1}(s)f(z(s), s)\, ds + Pz(0). \quad (4.26)$$

Again, if $z$ obeys (4.13b), then the left hand side of (4.26) goes to zero for $t \to -\infty$. This then gives, since the right hand side also goes to zero,

$$Pz(0) = -\int_0^{-\infty} PX^{-1}(s)f(z(s), s)\, ds \quad (4.27)$$

With (4.25) and (4.27) we can rewrite (4.23):

$$z(t) = X(t) \left( \int_0^t PX^{-1}(s)f(z(s), s)\, ds - \int_t^0 QX^{-1}(s)f(z(s), s)\, ds + z(0) \right)$$

$$= X(t) \left( \int_0^t PX^{-1}(s)f(z(s), s)\, ds - \int_t^0 QX^{-1}(s)f(z(s), s)\, ds + (P + Q)z(0) \right)$$

$$= X(t) \left( \int_0^t PX^{-1}(s)f(z(s), s)\, ds - \int_t^0 QX^{-1}(s)f(z(s), s)\, ds \right.$$  

$$- \int_0^{-\infty} PX^{-1}(s)f(z(s), s)\, ds + \int_{-\infty}^0 QX^{-1}(s)f(z(s), s)\, ds \Big)$$

$$= X(t) \left( \int_{-\infty}^t PX^{-1}(s)f(z(s), s)\, ds - \int_t^\infty QX^{-1}(s)f(z(s), s)\, ds \right). \quad (4.28)$$

Note that the choice of $t = 0$ in (4.22) is arbitrary: any constant $c$ suffices, so in the derivation all zeros in the integration boundaries and in $z(0)$ can be replaced by $c$, which would not change the result.

In conclusion, a solution of (4.10) satisfying (4.13) is a solution of the (4.14). Since the converse has also been shown, the proof of theorem 4.2 is complete. ■
4.2. Localisation around hyperbolic path

4.2.2. Unique hyperbolic trajectory

Theorem 4.2 claims that the integral equation for \( z \) is more selective than the differential equation, in the sense that it yields trajectories in a frame moving with a hyperbolic path that satisfy an additional condition: (4.13). The integral equations provide these special trajectories immediately (assuming one can solve this implicit set of equations), whereas they are selected from the solutions of the differential equation by the conditions (4.13) [39]. As is shown later from a similar derivation in a transformed system, these limits are connected to what can be considered as growth in time which is weaker than exponential (also referred to as sub-exponential growth [15, 5]). This additional characteristic proves to be special when it concerns hyperbolic trajectories, because these trajectories are generally characterised by exponential growth in time.

The next theorem 4.3 gives sufficient conditions for the existence of a unique solution of the integral equation (4.14). Theorem 4.5 then asserts that this unique solution \( z \) is actually a hyperbolic trajectory, where the hyperbolicity is inherited from the hyperbolicity of the path \( x_{cp} \). It is a modification of a theorem presented by Ju et al. [7]. In that theorem, \( \|f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2)} < \infty \) and \( \|\nabla f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R}^2 \times \mathbb{R}^2)} \eta < 1 \), where \( \eta \) is defined by the constants in (4.4-i) and (4.4-ii) associated with the exponential dichotomy of the hyperbolic path \( x_{cp} \):

\[
\eta \equiv \frac{k}{\alpha} + \frac{\ell}{\beta},
\]

(4.29)

are shown to be sufficient conditions for the existence of a unique solution of the integral equation (4.14). The norms \( \|f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2)} \) and \( \|\nabla f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R}^2 \times \mathbb{R}^2)} \) are defined by

\[
\|f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2)} = \sup_{t \in \mathbb{R}} \sup_{\zeta \in \mathbb{R}^2} |f(\zeta, t)|,
\]

and

\[
\|\nabla f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2 \times \mathbb{R}^2)} = \sup_{t \in \mathbb{R}} \sup_{\zeta \in \mathbb{R}^2} \|\nabla f(\zeta, t)\|_2
\]

\[
= \sup_{\zeta \in \mathbb{R}^2} \sup_{t \in \mathbb{R}} \left\{ \frac{|\nabla f(\zeta, t) \cdot w|_{\mathbb{R}^2}}{|w|_{\mathbb{R}^2}} \mid w \in \mathbb{R}^2 \land w \neq 0 \right\}.
\]

where the norm \( \| \cdot \|_2 \) is as defined in (4.5) and \( z \) is a solution of the integral equation (4.14).

In our modified theorem, we want to prove the existence and uniqueness of a hyperbolic trajectory in a small tube around the hyperbolic path \( x_{cp} \) in the \((x, y, t)\)-space. If \( x_{cp} \) is the curve of instantaneous critical points, which is not necessary,
we have the following. If there is no time dependence, we know that the hyperbolic trajectory \( x \) coincides with the curve of instantaneous critical points \( x_{cp} \), because then \( x_{cp} \) is a trajectory. From equation (4.12) it follows that if \( z = 0 \), then \( x = x_{cp} \) and
\[
 f(0,t) = v(x_{cp}(t),t) - \dot{x}_{cp}(t) = -\dot{x}_{cp}(t).
\]
According to the differential equation (4.10), \( z = 0 \) is a solution only if \( f(z,t) = 0 \), meaning that the position of the critical point is independent of time. The conjecture is that if the time dependence is weak, then there still exists a unique hyperbolic trajectory, as expressed by theorem 4.3. The conditions on the existence of a unique solution are expressed in terms of the temporal dependence of the hyperbolic path and the spatial dependence of the velocity field.

**Theorem 4.3.** Suppose that
\[
 \|\dot{x}_{cp}\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} \|\nabla(\nabla v)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R};\mathbb{R}^2 \times 2 \times 2)} < \frac{1}{4\eta^2},
\]
where \( \eta \), as defined in (4.29), follows from the exponential dichotomy of the path \( x_{cp} \). Then in the space \( BC[\mathbb{R}, D] \), the space of all bounded and continuous functions defined on \( \mathbb{R} \) with values in \( D \equiv \{ u \in \mathbb{R}^2 | |u|_{\mathbb{R}^2} \leq \delta \} \), the integral equation (4.14) has a unique solution \( z \), where
\[
 \delta^2 = \frac{\|\dot{x}_{cp}\|_{L^\infty(\mathbb{R};\mathbb{R}^2)}}{\|\nabla(\nabla v)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R};\mathbb{R}^2 \times 2 \times 2)}}.
\]
Remark: if we can find a \( \delta \) such that the norm on \( \nabla(\nabla v) \) is restricted to \( D \) instead of \( \mathbb{R}^2 \), i.e.,
\[
 \|\dot{x}_{cp}\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} < \frac{\delta}{2\eta},
\]
\[
 \|\nabla(\nabla v)\|_{L^\infty(D \times \mathbb{R};\mathbb{R}^2 \times 2 \times 2)} < \frac{1}{2\eta}\delta,
\]
then ensuring the condition
\[
 \|\dot{x}_{cp}\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} \|\nabla(\nabla v)\|_{L^\infty(D \times \mathbb{R};\mathbb{R}^2 \times 2 \times 2)} < \frac{1}{4\eta^2}
\]
is sufficient for the existence of a unique solution in \( BC[\mathbb{R}, D] \). Generally, this is difficult, because with the restriction to \( D \), \( \delta \) is implicitly defined in (4.31).

This \( \delta \) defines the radius of the tube around the hyperbolic path \( x_{cp} \), which contains a unique hyperbolic trajectory if the condition (4.30) is satisfied.

---

5The notation \( \nabla(\nabla v) \in L^\infty(\mathbb{R}^2 \times \mathbb{R};\mathbb{R}^2 \times 2 \times 2) \) is adopted in order to avoid confusion with \( \nabla^2 v \in L^\infty(\mathbb{R};\mathbb{R}^2) \). Therefore, \( \nabla(\nabla v) \) is a \( 2 \times 2 \times 2 \)-tensor with elements \( u_{xx}, u_{xy} = u_{yx}, u_{yy}, v_{xx}, v_{xy} = v_{yx}, v_{yy} \), where the subscripts \( x \) and \( y \) denote differentiation with respect to \( x \) and \( y \).
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Proof. The condition (4.30) and the definition (4.31) of $\delta$ can combined into

$$
\|\dot{x}_{cp}\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} < \frac{\delta^2}{4\eta^2},
$$

$$
\|\nabla(\nabla v)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2)} < \frac{1}{4\eta^2\delta^2},
$$

so that

$$
\|\dot{x}_{cp}\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} < \frac{\delta}{2\eta}, \quad (4.32-i)
$$

$$
\|\nabla(\nabla v)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2)} < \frac{1}{2\eta\delta}. \quad (4.32-ii)
$$

The space $BC[\mathbb{R}, D]$ is a subset of the Banach space\(^6\) $BC[\mathbb{R}, \mathbb{R}^n]$, which is the space of all bounded and continuous functions defined on $\mathbb{R}$ with values in $\mathbb{R}^n$, with a norm defined as $\|\zeta\|_\infty = \sup_{t \in \mathbb{R}} |\zeta(t)|$ for $\zeta \in BC[\mathbb{R}, D]$. Define the map $T$ by

$$
Tz(t) = X(t) \int_{-\infty}^{t} PX^{-1}(s)f(z(s), s) \, ds - X(t) \int_{t}^{\infty} QX^{-1}(s)f(z(s), s) \, ds, \quad (4.33)
$$

such that a fixed point of $T$, i.e., $z$ with $Tz = z$, is a solution of (4.14). First we show that $T$ is well defined in $BC[\mathbb{R}, D]$ in lemma 4.4, so that if $|z| < \delta$, then also $|Tz| < \delta$. After that, we show that $T$ is a contraction mapping on the metric space $(BC[\mathbb{R}, D], \| \cdot \|)$.\(^7\) This is a metric space, because $BC[\mathbb{R}, D]$ is a subset of the Banach space $BC[\mathbb{R}, \mathbb{R}^n]$, which is a metric space. The Banach fixed point theorem (also known as the contraction mapping theorem or principle) states that every contraction mapping on a nonempty complete metric space has a unique fixed point [40]. Therefore, it suffices to show that $T$ is a contraction mapping on $(BC[\mathbb{R}, D], \| \cdot \|)$, so that the fixed point of $T$ gives us the unique solution of (4.14).

We introduce the following notation:

$$(f \circ z)(t) \equiv f(z(t), t),$$

\(^6\)A Banach space is a complete vector space with a norm $\| \cdot \|$. The space of all continuous functions $f : [a, b] \to K$ defined on a closed interval $[a, b]$ becomes a Banach space if the norm of $K$ is defined as the norm of such a function: $\|f\| = \sup_{x \in [a, b]} |f(x)|$. This example can be generalised to the space $C(X)$ of all continuous functions $X \to K$, where $X$ is a compact space, or to the space of all bounded continuous functions $X \to K$, where $X$ is any topological space.

\(^7\)A map $M$ is a contraction mapping (or contraction) on a metric space $(V, \| \cdot \|)$ if $M$ is a function from $V$ to itself, with the property that there is some real number $0 < k < 1$ such that, for all $x$ and $y$ in $V$, $\|Mx - My\| \leq k\|x - y\|$. The smallest value of $k$ is called the Lipschitz constant of $M$. 

Before proceeding with (4.35), we first estimate \( \|f \circ z\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} \) from \( \|f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R};\mathbb{R}^2)} \):

\[
\|f \circ z\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} \equiv \sup_{t \in \mathbb{R}} |f(z(t), t)| \leq \sup_{z \in \mathbb{R}^2} |f(z, t)| \equiv \|f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R};\mathbb{R}^2)}.
\]

**Lemma 4.4.** Under the assumptions of theorem 4.3, the map \( T \) as defined in (4.33) is a map from \( BC[\mathbb{R}, D] \) to itself.

**Proof.** The fact that \( Tz \) defines a continuous function for any continuous function \( z \) follows from the continuity of \( f \) and \( X \); the fact that \( Tz \) is bounded can be shown as follows. Estimate \( |Tz(t)| \) for arbitrary \( t \), using the triangle inequality combined with the exponential dichotomy:

\[
|Tz(t)|_{\mathbb{R}^2} = \left| \int_{-\infty}^{t} X(t)PX^{-1}(s)f(z(s), s) \, ds - \int_{t}^{\infty} X(t)QX^{-1}(s)f(z(s), s) \, ds \right|_{\mathbb{R}^2}
\]

\[
\leq \left| \int_{-\infty}^{t} X(t)PX^{-1}(s)f(z(s), s) \, ds \right|_{\mathbb{R}^2} + \left| \int_{t}^{\infty} X(t)QX^{-1}(s)f(z(s), s) \, ds \right|_{\mathbb{R}^2}
\]

\[
\leq \int_{-\infty}^{t} \|X(t)PX^{-1}(s)f(z(s), s)\|_{\mathbb{R}^2} \, ds + \int_{t}^{\infty} \|X(t)QX^{-1}(s)f(z(s), s)\|_{\mathbb{R}^2} \, ds
\]

\[
\leq \|f \circ z\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} \int_{-\infty}^{t} \|X(t)PX^{-1}(s)\|_{2} \, ds + \|f \circ z\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} \int_{t}^{\infty} \|X(t)QX^{-1}(s)\|_{2} \, ds
\]

\[
\leq \|f \circ z\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} \left( \int_{-\infty}^{t} k e^{-\alpha(t-s)} \, ds + \int_{t}^{\infty} \ell e^{-\beta(s-t)} \, ds \right)
\]

\[
= \|f \circ z\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} \lim_{R \to \infty} \left( \frac{k}{\alpha} e^{-\alpha(t-s)} \bigg|_{s=-R}^{s=t} - \frac{\ell}{\beta} e^{-\beta(s-t)} \bigg|_{s=t}^{s=R} \right)
\]

\[
= \|f \circ z\|_{L^\infty(\mathbb{R};\mathbb{R}^2)}, \tag{4.34}
\]

where the definition of the exponential dichotomy, (4.4-i) and (4.4-ii), has been used. We now have the inequality \( |Tz(t)| \leq \|f \circ z\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} \eta \) for arbitrary \( t \), such that the supremum over \( t \) yields

\[
\|Tz\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} \leq \|f \circ z\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} \eta. \tag{4.35}
\]

Before proceeding with (4.35), we first estimate \( \|\nabla f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R};\mathbb{R}^2 \times \mathbb{R}^2)} \). Note that for any \( \zeta \in BC[\mathbb{R}, D] \) and arbitrary \( t \)

\[
\|\nabla f(\zeta, t)\|_{2} = \|\nabla (\zeta + x_{cp}(t), t) - \nabla (x_{cp}(t), t)\|_{2}
\]
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\[ \int_0^1 \left\| \nabla (\nabla v(\tau \zeta + x_{cp}, t)) \cdot \zeta \right\|_2 \, d\tau \]

\[ \leq \int_0^1 \left\| \nabla (\nabla v(\tau \zeta + x_{cp}, t)) \right\|_2 \, d\tau \]

\[ \leq \int_0^1 \left\| \nabla (\nabla v) \|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})} \right\|_2 \, d\tau \]

\[ \leq \| \nabla (\nabla v) \|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})} \delta. \]

Since this holds for arbitrary \( t \), we can take the supremum over \( t \) and \( \zeta \) to find

\[ \| \nabla f \|_{L^\infty(D \times \mathbb{R}; \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})} = \sup_{\zeta \in \mathbb{D}} \| \nabla f(\zeta, t) \|_2 \]

\[ \leq \| \nabla (\nabla v) \|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})} \delta. \] (4.36)

This estimate can be employed in the next estimate, in which we have for \( z \in \mathbb{D} \) and arbitrary \( t \)

\[ \| \nabla f(\vartheta z(t), t) \|_2 \leq \| \nabla f \|_{L^\infty(D \times \mathbb{R}; \mathbb{R}^2 \times \mathbb{R})}. \] (4.37)

Now note that for all \( z \in BC[\mathbb{R}, D] \) and \( \vartheta \in [0, 1] \), we have \( \vartheta z \in BC[\mathbb{R}, D] \), such that

\[ \| \nabla f(\vartheta z(t), t) \|_2 \leq \| \nabla f \|_{L^\infty(D \times \mathbb{R}; \mathbb{R}^2 \times \mathbb{R})}. \]

This can be used in (4.37), so that

\[ \| f(z(t), t) - f(0, t) \|_{\mathbb{R}^2} \leq \left\| \int_0^1 \nabla f(\vartheta z(t), t) \cdot z(t) \, d\vartheta \right\|_{\mathbb{R}^2} \]

\[ \leq \left\| \int_0^1 \nabla f(\vartheta z(t), t) \|_2 | z(t) \|_{\mathbb{R}^2} \, d\vartheta \right\|_{\mathbb{R}^2}. \] (4.38)

After rearranging the latter inequality, taking the supremum over \( t \), and using (4.36), we have:

\[ \| f \circ z \|_{L^\infty(\mathbb{R}, \mathbb{R}^2)} \leq \| f(0, \cdot) \|_{L^\infty(\mathbb{R}, \mathbb{R}^2)} + \| \nabla f \|_{L^\infty(D \times \mathbb{R}; \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})} \delta \]

\[ \leq \| f(0, \cdot) \|_{L^\infty(\mathbb{R}, \mathbb{R}^2)} + \| \nabla (\nabla v) \|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})} \delta^2. \] (4.38)
If we use the assumptions (4.32-i) and (4.32-ii)

\[ \| f(0, \cdot) \|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} = \| \dot{x}_{\text{cp}} \|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} < \frac{\delta}{2\eta}, \]

\[ \| \nabla(\nabla v) \|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2 \times 2 \times 2 \times 2)} < \frac{1}{2\eta \delta}, \]

then (4.35) in combination with (4.38) gives

\[ \| Tz \|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} \leq \| f \circ z \|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} \eta \]

\[ \leq \left( \| f(0, \cdot) \|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} + \| \nabla(\nabla v) \|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2 \times 2 \times 2)} \delta^2 \right) \eta \]

\[ \leq \left( \frac{\delta}{2\eta} + \frac{1}{2\eta \delta} \right) \eta \]

\[ \leq \frac{\delta}{2} + \frac{\delta}{2} \]

\[ = \delta. \]

These properties of \( Tz \) show that \( T \) is well defined in \( BC[\mathbb{R}, D] \).

Next, we show that \( T \) is a contraction mapping on the metric space \( (BC[\mathbb{R}, D], || \cdot ||) \). We have for solutions \( z_1 \) and \( z_2 \) of (4.33) that

\[ Tz_1(t) - Tz_2(t) = X(t) \int_{-\infty}^{t} PX^{-1}(s) \left( f(z_1(s), s) - f(z_2(s), s) \right) ds \]

\[ - X(t) \int_{t}^{\infty} QX^{-1}(s) \left( f(z_1(s), s) - f(z_2(s), s) \right) ds, \]

so that, successively using the triangle inequality and the exponential dichotomy, we find for \( |Tz_1(t) - Tz_2(t)|_{\mathbb{R}^2} \):

\[ |Tz_1(t) - Tz_2(t)|_{\mathbb{R}^2} \]

\[ = \left| X(t) \int_{-\infty}^{t} PX^{-1}(s) \left( f(z_1(s), s) - f(z_2(s), s) \right) ds \right|_{\mathbb{R}^2} - \]

\[ \left| X(t) \int_{t}^{\infty} QX^{-1}(s) \left( f(z_1(s), s) - f(z_2(s), s) \right) ds \right|_{\mathbb{R}^2} \]

\[ \leq \left| X(t) \int_{-\infty}^{t} PX^{-1}(s) \left( f(z_1(s), s) - f(z_2(s), s) \right) ds \right|_{\mathbb{R}^2} + \]

\[ \left| X(t) \int_{t}^{\infty} QX^{-1}(s) \left( f(z_1(s), s) - f(z_2(s), s) \right) ds \right|_{\mathbb{R}^2} \]

\[ \leq \int_{-\infty}^{t} \left| X(t)PX^{-1}(s) \left( f(z_1(s), s) - f(z_2(s), s) \right) \right|_{\mathbb{R}^2} ds + \]

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\[
\int_{t}^{\infty} \left| X(t)QX^{-1}(s) \left( f(z_1(s), s) - f(z_2(s), s) \right) \right|_{\mathbb{R}^2} ds \\
\leq \int_{-\infty}^{t} \| X(t)PX^{-1}(s) \|_{2} \left| (f \circ z_1)(s) - (f \circ z_2)(s) \right|_{\mathbb{R}^2} ds + \\
\int_{t}^{\infty} \| X(t)QX^{-1}(s) \|_{2} \left| (f \circ z_1)(s) - (f \circ z_2)(s) \right|_{\mathbb{R}^2} ds \\
\leq \int_{t}^{t} k e^{-\alpha(t-s)} \left| (f \circ z_1)(s) - (f \circ z_2)(s) \right|_{\mathbb{R}^2} ds + \\
\int_{t}^{t} \ell e^{-\beta(s-t)} \left| (f \circ z_1)(s) - (f \circ z_2)(s) \right|_{\mathbb{R}^2} ds \\
\leq \| \nabla f \|_{L^{\infty}(D \times \mathbb{R} ; \mathbb{R}^{2} \times 2)} \| z_1 - z_2 \|_{L^{\infty}(\mathbb{R} ; \mathbb{R}^{2})} \left( \int_{-\infty}^{t} k e^{-\alpha(t-s)} ds + \int_{t}^{\infty} \ell e^{-\beta(s-t)} ds \right) \\
= \| \nabla f \|_{L^{\infty}(D \times \mathbb{R} ; \mathbb{R}^{2} \times 2)} \| z_1 - z_2 \|_{L^{\infty}(\mathbb{R} ; \mathbb{R}^{2})} \eta. \tag{4.39}
\]

The inequality \((\dagger)\) can be clarified as follows.

\[
\left| (f \circ z_1)(s) - (f \circ z_2)(s) \right|_{\mathbb{R}^2} \\
\leq \left| \int_{0}^{1} \frac{\partial}{\partial \tau} \left( f(\tau z_1(s) + (1 - \tau)z_2(s)) \right) d\tau \right|_{\mathbb{R}^2} \\
= \left| \int_{0}^{1} \left( \nabla f \left( \tau z_1(s) + (1 - \tau)z_2(s) \right) \right) \cdot \left( z_1(s) - z_2(s) \right) d\tau \right|_{\mathbb{R}^2} \\
\leq \left| \int_{0}^{1} \left( \nabla f \left( \tau z_1(s) + (1 - \tau)z_2(s) \right) \right) \cdot \left( z_1(s) - z_2(s) \right) \right|_{\mathbb{R}^2} d\tau \\
\leq \int_{0}^{1} \left\| \nabla f \left( \tau z_1(s) + (1 - \tau)z_2(s) \right) \right\|_{2} \left| z_1(s) - z_2(s) \right|_{\mathbb{R}^2} d\tau. \tag{4.40}
\]

For \(\tau \in [0, 1]\) and \(z_1, z_2 \in BC[\mathbb{R}, D]\), we have that

\[
|\tau z_1 + (1 - \tau)z_2| \leq |\tau z_1| + |(1 - \tau)z_2| \\
\leq \tau |z_1| + (1 - \tau)|z_2| \\
\leq \tau \delta + (1 - \tau)\delta \\
= \delta.
\]
So \( \tau z_1 + (1 - \tau)z_2 \in BC[\mathbb{R}, D] \). This gives for (4.40):
\[
|f \circ z_1(s) - f \circ z_2(s)|_{\mathbb{R}^2} \leq \int_0^1 \|\nabla f\|_{L^\infty(D \times \mathbb{R}^2 \times \mathbb{R}^2)} \|z_1(s) - z_2(s)\|_{\mathbb{R}^2} \, d\tau
\]
\[
= \|\nabla f\|_{L^\infty(D \times \mathbb{R}^2 \times \mathbb{R}^2)} \|z_1(s) - z_2(s)\|_{\mathbb{R}^2}
\]
(4.41)

We then take the supremum over \( t \) of \( |Tz_1(t) - Tz_2(t)|_{\mathbb{R}^2} \), so that (4.39) gives an estimate for \( \|Tz_1 - Tz_2\|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} \):
\[
\|Tz_1 - Tz_2\|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} \leq \|\nabla f\|_{L^\infty(D \times \mathbb{R}^2 \times \mathbb{R}^2)} \|z_1 - z_2\|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} \eta.
\]
(4.42)

The strict inequality now follows from (4.42) in combination with (4.36) and the assumption on \( \|\nabla(v)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2)} \):
\[
\|Tz_1 - Tz_2\|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} \leq \|\nabla f\|_{L^\infty(D \times \mathbb{R}^2 \times \mathbb{R}^2)} \|z_1 - z_2\|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} \eta
\]
\[
\leq \frac{1}{2\eta\delta} \delta \|z_1 - z_2\|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} \eta
\]
\[
= \frac{1}{2} \|z_1 - z_2\|_{L^\infty(\mathbb{R}; \mathbb{R}^2)}
\]

Therefore, \( T \) is a contraction mapping. This completes the proof of theorem 4.3.

It should be mentioned from the last estimate that if \( T \) is considered to be a mapping from \( BC[\mathbb{R}, \mathbb{R}^n] \) to itself (as is done by Ju et al. [7]), it suffices that
\[
\|\nabla(v)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2)} < \frac{1}{\eta\delta}.
\]

However, we need the bound \((2\eta\delta)^{-1}\) for \( T \) to be a mapping from \( BC[\mathbb{R}, D] \) to itself, i.e., \( \|Tz\|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} < \delta \) for \( \|z\|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} < \delta \).

Notice that, as expected intuitively, if the velocity field is stationary, i.e., \( \hat{x}_{\text{cp}}(t) = 0 \) for all \( t \), then \( \delta \) can be taken arbitrarily small such that the condition \( \|\nabla(v)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2 \times \mathbb{R}^2)} < (2\eta\delta)^{-1} \) is satisfied automatically. This is the case that the hyperbolic path \( x_{\text{cp}} \) is the hyperbolic trajectory, as discussed before.

We have shown that the integral equation for \( z \) has as unique solution under the given conditions (4.30). The next theorem states that this solution is a hyperbolic trajectory of the corresponding differential equation (4.10).

**Theorem 4.5.** The unique solution of the integral equation (4.14) is a hyperbolic trajectory of (4.10).
4.2. Localisation around hyperbolic path

Proof. For the proof we need the following important result on the stability of an exponential dichotomy, which is presented in [41].

Lemma 4.6. Suppose that the system of equations
\[
\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R}
\] (4.43)
has an exponential dichotomy with associated projection \(P\) and positive constants \(k, \ell, \alpha\) and \(\beta\) as in (4.4-i) and (4.4-ii), which yield \(\eta\) as defined in (4.29). Suppose further that
\[
\sup_{t \in \mathbb{R}}|B(t)|\eta < 1, \quad (4.44)
\]
Then perturbed system of (4.43),
\[
\dot{y}(t) = (A(t) + B(t))y(t), \quad t \in \mathbb{R},
\] (4.45)
also admits an exponential dichotomy with positive constants \(k' = \ell' = m\) and \(\alpha' = \beta' = \mu > 0\). Moreover, the projection \(R\) associated with this dichotomy is similar\(^8\) to \(P\) and
\[
\|Y(t)RY^{-1}(t) - X(t)PX^{-1}(t)\| \leq (k + \ell)m, \quad t \in \mathbb{R},
\]
where \(X\) and \(Y\) are the fundamental solution matrices of (4.43) and (4.45), respectively.

Proof. See [41]. \(\square\)

If \(z\) is the unique solution of (4.14) and therefore, the unique solution of (4.10) satisfying (4.13), it is a hyperbolic trajectory if the linearised equation of (4.10),
\[
\dot{\xi} = \nabla_z \left( A(t)z(t) + f(z(t), t) \right) \xi \\
= \left( \nabla v(x_{cp}(t), t) + \nabla f(z(t), t) \right) \xi,
\] (4.46)
has an exponential dichotomy. The subscript of the gradient denotes a derivative with respect to \(z\). The equation
\[
\dot{\xi} = \nabla v(x_{cp}(t), t) \xi
\]
has an exponential dichotomy because \(x_{cp}\) is a hyperbolic path. The matrix \(\nabla f\) in (4.46) can be identified as the perturbation matrix \(B\) in (4.45). The restriction (4.44) is satisfied, because
\[
\frac{\eta\|\nabla f\|_{L^\infty(D \times \mathbb{R}; \mathbb{R}^{2 \times 2})}}{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^{2 \times 2} \times 2)} \leq \frac{\eta\|\nabla v\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^{2 \times 2} \times 2)}}{L^\infty(D \times \mathbb{R}; \mathbb{R}^{2 \times 2})}
\]
\[
\leq \eta \frac{1}{2\eta \delta} \delta
\]
\[
= \frac{1}{2},
\]
\(^8\)Two matrices \(A\) and \(B\) are said to be similar if there exists a square nonsingular matrix \(S\) such that \(B = S^{-1}AS\).
where (4.36) and (4.32-ii) have been used. Then theorem 4.6 can then be applied, so that equation (4.10) also has an exponential dichotomy and that the found trajectory $z$ is hyperbolic. This shows that the unique solution of (4.14) is a hyperbolic trajectory.

The combination of the result of theorem 4.3 that the integral equation (4.14) has a unique bounded and continuous solution, and the result of theorem 4.5 that this solution is a hyperbolic trajectory of (4.10), shows that the integral equation yields a unique hyperbolic trajectory.

Theorems 4.3 and 4.5 can be used to find a hyperbolic trajectory of the original velocity field (4.2) by transforming the solution of (4.14) back to the stationary frame, using the hyperbolic path $x_{cp}$.

**Theorem 4.7.** Suppose $z$ is the unique solution of (4.14) for a given hyperbolic path $x_{cp}$. Then there exists a unique hyperbolic trajectory $x = z + x_{cp}$ of (4.2).

**Proof.** Suppose $z$ is the unique solution of (4.14) and therefore of (4.10). Then, $x = z + x_{cp}$ is a hyperbolic trajectory of (4.2) if the linearised equation

$$\dot{\xi} = \nabla v(x(t), t)\xi,$$

has an exponential dichotomy. This follows from the linearised equation (4.46), where

$$f(z(t), t) = v(z(t) + x_{cp}(t), t) - \nabla v(x_{cp}(t), t)z(t) - x_{cp}(t),$$

such that (4.46) becomes:

$$\dot{\xi} = \left(\nabla v(x_{cp}(t), t) + \nabla f(z(t), t)\right)\xi$$

$$= \left(\nabla v(x_{cp}(t), t) + \nabla v(z(t) + x_{cp}(t), t) - \nabla v(x_{cp}(t), t)\right)\xi$$

$$= \left(\nabla v(x_{cp}(t), t) + \nabla v(x(t), t) - \nabla v(x_{cp}(t), t)\right)\xi$$

$$= \left(\nabla v(x(t), t)\right)\xi.$$

This equation has an exponential dichotomy, as argued in lemma 4.6, so $x$ is a hyperbolic trajectory of the original problem. The uniqueness of $x$ follows from the fact that $z$ is unique for a given path $x_{cp}$.  

Note that it is not assumed that there are hyperbolic trajectories in the vector field; hyperbolicity arises from the hyperbolic instantaneous critical points. The bound in (4.44) is expressed solely in terms of the constants from the exponential dichotomy of the hyperbolic path $x_{cp}$.

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\(^9\)In the theorem by Ju et al. [7], one of the requirements is $\|\nabla f\|_{L^\infty(R^2 \times R^2; R^2)} \eta < 1$, such that the bound in (4.44) is sharp.
4.2. Localisation around hyperbolic path

4.2.3. Physical interpretation

Theorems 4.3 and 4.5 show that the existence and uniqueness of a hyperbolic trajectory can be expressed in terms of the velocity of the hyperbolic path, and the second derivative of the velocity field. In this section two physical interpretations of these theorems are proposed.

First, rewrite the inequality (4.30) as follows, where $\eta$ has been redefined by $2\eta$ in order to omit the factor 4:

$$\eta \| \dot{x}_{cp} \| < \frac{1}{\eta \| \nabla (\nabla v) \|} ,$$

(4.47)

where the exact norms have been left out for readability. Inequality (4.47) defines two length scales. On the one hand, $\eta \| \dot{x}_{cp} \|$ is the distance travelled by the hyperbolic path in the time $\eta$, where the scale $\eta$ measures the strength of hyperbolicity of the path $x_{cp}$. It is the typical time in which particles are attracted to the hyperbolic trajectory. If $\eta \| \dot{x}_{cp} \|$ is too large, then the hyperbolic path is too fast for the particles to follow, making a hyperbolic trajectory impossible.

On the other hand, the gradient length $\ell_g$ defines a different length scale:

$$\ell_g = \frac{\| \nabla v \|}{\| \nabla (\nabla v) \|} .$$

This $\ell_g$ defines a length scale over which $\nabla v$ typically changes. In addition, $\nabla v$ generates four inverse time scales generally, given by the matrix entries. However, the characteristics of the flow are invariant under rotations and translations, so that they are implied in the eigenvalues of $\nabla v$. These two eigenvalues are equal for incompressible flows (apart from a sign) and measured by $\eta^{-1}$, as explained in section 4.1.3. Therefore, the gradient length $\ell_g$ can be estimated by the right-hand side of (4.47).

If $\ell_g$ is small, then $\nabla v$ varies rapidly in space, so that the nonlinearities in the velocity field, which can destroy the hyperbolicity over a large range, are strong. If $\ell_g < \eta \| \dot{x}_{cp} \|$, then the spatial variations are then too large for the hyperbolicity of the path to be felt by a trajectory.

An alternate interpretation is found by defining two time scales. A Lagrangian time scale $t_{Lag}$ is provided by $\eta$, as it describes the time of separation of particle trajectories. A Eulerian time scale is given by $t_{Eul}^{-1} = \| \nabla v \|$. Inequality (4.30) can then be written as (again omitting the factor 4)

$$\| \dot{x}_{cp} \| \| \nabla \| t_{Eul}^{-1} \| < \frac{1}{t_{Lag}^2} .$$

(4.48)

The left-hand side of (4.48) describes the change in $t_{Eul}$ when travelling with the path $x_{cp}$. The combination $\dot{x}_{cp} \nabla$ is an inverse time scale, which is typical for the particle...
separation near the path, so can be estimated by $\eta^{-1}$. With these estimates, the inequality (4.48) reduces to

$$t_{\text{Eul}} > t_{\text{Lag}}.$$ 

This means that the velocity of the field as a whole is small compared to the typical velocity of a trajectory. This is according to expectation, because the existence of hyperbolic trajectories is expected for weakly time dependent flows, where the deviation from the stationary situation is small. Weakly time dependent flows are characterised by a large Eulerian time scale. In the limit of stationary flows, this time scale is infinitely large, so that the requirements are met automatically.
5. Numerical methods

Here, it is explained how the numerical results have been obtained. The computations of the manifolds have been done by coupling two codes: one that calculates the velocity field induced by point vortices, and one that computes the trajectories of passively advected particles given this field. This is explained in more detail in section 5.1. In order to find the invariant manifolds, initial pieces need to be found. Furthermore, backward time integration is performed for finding the stable manifold. It is shown in section 5.2 how this is done. In section 5.3 it is shown how the three vortex systems are simulated in such a way, that a comparison can be made with results found in literature. The calculations of the invariant manifolds were made on a 2.5 GHz computer with RISC processor and 1.2 GB memory, using variables with double precision.

The separatrices in the time independent flows in the circle configurations from chapter 3, have been calculated with Mathematica 5.2 on a 500 MHz computer, by use of an implicit plot of the stream function $\Psi$. First, the locations of the hyperbolic critical points have been found numerically by equating the velocity field to zero. The separatrices then follow from implicitly determining the lines were $\Psi$ has the same value as at the hyperbolic critical points. Using the same programme, the instantaneous locations of the critical points in the three vortex systems have been obtained, by solving the set (3.44) numerically.

5.1. Used techniques

The road to visualising the stable and unstable manifold in the point vortex systems is twofold. Despite the fact that the point vortex motion might be an integrable system (meaning that the motion is not chaotic, which is the case for three vortices or fewer), the vortex trajectories cannot be calculated analytically, as the vortex positions are unknown in general. Only in special configurations, the vortex motion is a steady rotation around the centre of vorticity, as described in chapter 3.

For the vortex positions as a function of time a code has been used, that was written in Java by Kuvshinov [3]. It is a symplectic code, which means that the constants of motion (see chapter 3) are kept constant extremely accurately. This characteristic is important, because growth of errors can easily lead to completely wrong vortex trajectories, especially when the number of vortices is larger than three and consequently, the vortex motion can be chaotic. The output of this code comprises the
vortex positions at discrete times.

This output has been used as the input of a modified contour kinematics code, which was written in Fortran by Meleshko et al. [42]. It uses Runge-Kutta time integration of variable order [43]. The time step is adjusted if needed, for instance because of a strongly nonlinear velocity field. The code was written to visualise material lines of a flow, for which the velocity field is known analytically. Therefore, we have adapted the code, such that the velocity field of the point vortex system at an arbitrary time can be found by applying linear interpolation of the positions of the point vortices at different times from the symplectic code.

Initially, a circle of passive particles is placed in the flow, and is stretched and folded by the velocity field. If at time $t = t'$ particles are too close together, they are deleted, and particles are added, if the distance between neighbouring particles becomes too large. If the angle between neighbouring linear segments is smaller than 120°, an extra marker is added to retain a smooth line. For accuracy purposes, the added tracers are placed midway between two particles in the initial circle at $t = 0$ and advected forward in time to $t = t'$. A check is preformed by computing the surface area spanned by this contour, which needs to remain constant in time, as is shown in appendix A.1. This is done by using Stokes’s theorem for surface area $A$:

$$A = \iint_S \, dx \, dy = \frac{1}{2} \oint_C (-y \, dx + x \, dy) \approx \frac{1}{2} \sum_{i=1}^k x_i y_{i+1} - y_i x_{i+1},$$

where the sum is taken over the $k$ tracer particles at positions $(x_i, y_i)$ which form the contour $C$ of the domain $S$.

### 5.2. Computation of invariant manifolds

The first manifolds in chapter 6 are drawn at a numerical time $t = t^*$, so that the manifolds are developed in such a way that the hyperbolic points can be clearly distinguished as the unique intersection of a stable and an unstable manifold. At this time $t = t^*$, a manifold typically consists of a few tens of particles and the computing time is at most ten seconds. The next snapshots of the manifolds are then made at $t = 2t^*$ and $t = 3t^*$ (or $t = 1.6t^*$ and $t = 2t^*$ if the development is very rapid), where the number of particles of manifold has increased to hundreds of particles, because of the stretching and folding. The computing time is then typically tens of seconds, up to at most a minute.

As explained in section 4.1.4, the invariant manifolds at a time $t = \tau$ can found by evolving a piece of the manifolds around the hyperbolic point forward and backward in time to $t = \tau$. A piece of the unstable manifold around the hyperbolic point is
found by placing a circular material curve in the flow and by evolving it backward in time. If the circle is placed correctly (which may take a few attempts), the particles are attracted to a hyperbolic trajectory at an exponential rate, and shrinks to a small piece at $t = t_1$. This means that the required integration time is small.

![Diagram](image)

Figure 5.1: The forward evolution of a segment of $W^u(t_1)$ of $\Gamma$ (thick piece) at $t = t_1$ to $t = \tau$ yields the unstable manifold of $W^u(\tau)$ of $\Gamma$ on the time slice $\Sigma_\tau$. Similarly, the backward evolution of a small segment of $W^s(t_2)$ of $\Gamma$ (thick piece) at $t = t_2$ to $t = \tau$ yields the unstable manifold $W^u(\tau)$ of $\Gamma$ on the time slice $\Sigma_\tau$.

Next, in order to find the unstable manifold at $t = \tau$, a small circular material curve is placed around this piece at $t = t_1$, and then evolved forward in time to $t = \tau$, as explained in figure 5.1. Similarly, the stable manifold at $t = \tau$ is then found from backward integration of a blob around a piece of the stable manifold from $t = t_2 = \tau + t_1$ to $t = \tau$. Then the integration time is the same for both manifolds, so that their developments can be compared. This piece of the stable manifold is obtained in an analogous manner to the unstable piece, by first evolving a blob of particles forward in time, which shrinks to a small piece at $t = t_2$. In chapter 6, we take $t_1 = 0$ and compute the manifolds for $\tau = t^*, 1.6t^*, 2t^*$, and $t = 3t^*$, as explained above.

A verification that the hyperbolic trajectory has been found, is provided by the characteristic that the material curve stretches exponentially in one direction, and it contracts in the direction perpendicular to that at an exponential rate. So the small circle becomes a line, which remains a closed loop, as the surface area is conserved.
The backward integration is executed as follows. It follows from equation (3.17) for the tracer motion in the revolving frame, that the vortex motion in the opposite temporal direction is given by (3.17) with $t \to -t$:

$$\hat{\mathbf{z}}^*(t) = -\hat{\mathbf{z}}^*(t) = \frac{-1}{2\pi i} \sum_{m=1}^{N} \kappa_m \mathbf{\bar{z}}(t) - \mathbf{\bar{z}}_m(t) - i\Omega \hat{\mathbf{z}}^*(t).$$

Therefore, to find the backward in time motion of the vortices, it suffices to change the signs of the vortex strengths: $\kappa_j \to -\kappa_j$ [11] and the angular frequency: $\Omega \to -\Omega$.

5.3. Three vortex motion

In the three vortex systems in chapter 6, all vortices are of equal strength and have a counterclockwise orientation. The variables have been scaled as described in section 3.1. This means that $L^2 = 1$, where

$$L^2 = \frac{1}{3} \left( |z_1 - z_2|^2 + |z_1 - z_3|^2 + |z_2 - z_3|^2 \right).$$

So the coordinates $z$ have been rescaled by $z \to z/L$. The vortex strength $\kappa$ has been scaled out, by rescaling the time variable $t \to (\kappa/L^2)t$. Different flows are distinguished only by the Hamiltonian $H$, which is related to the vortex configuration,

$$\Lambda = e^{-4\pi H} = |z_1 - z_2| |z_1 - z_3| |z_2 - z_3|,$$

where $k = L = 1$ has been taken. This $\Lambda$ is the variable describing the configuration and is constant in a flow, as $H$ is an invariant of the motion. $\Lambda = 1$ corresponds to the equilateral triangle configuration (with minimum energy) and all other setups have a smaller value of $\Lambda$ (larger energy). The critical value where the flip occurs, as discussed in section 3.4, is $\Lambda_c = 1/\sqrt{2} \approx 0.71$ [22]. The starting configuration is an isosceles triangle with two sides of length $a$ and one of length $b$. Given a choice of $\Lambda$, the values of $a$ and $b$ are calculated from

$$a^2b = \Lambda, \quad \frac{1}{3} (2a^2 + b^2) = 1.$$

Putting one vortex on the positive vertical (imaginary) axis, the isosceles triangle is constructed by taking the following initial positions $z_i$ of the vortices:

$$z_{1,2} = \pm \frac{b}{2} - \frac{1}{3} \sqrt{a^2 - \frac{b^2}{4}} i, \quad z_3 = \frac{2}{3} \sqrt{a^2 - \frac{b^2}{4}} i.$$

In the equilateral triangle, the value of $\Omega$ is $(2\pi r)^{-1}$, with $r$ the radius of the circle, i.e., the distance form the vortices to the origin. For the shift to a corotating frame the appropriate angular frequency $\Omega$ must be estimated, as $r$ is not constant. A Matlab programme written by R. Schuitvlot was used to compute the average value of $\Omega$ of the point vortices, measured over the length of the table given by the first code.
6. Numerical simulations

As described in section 5.3, the three vortex system has been scaled such that $L^2 = 1$ and the vortex strength is equal to unity. The only remaining parameter is $\Lambda = |z_1 - z_2||z_1 - z_3||z_2 - z_3|$. The starting configuration is the equilateral triangle with $\Lambda = 1$ and six hyperbolic critical points. This configuration is in equilibrium in the corotating frame, so that the critical points form six hyperbolic trajectories. The value of $\Lambda$ is then gradually decreased, and the number of hyperbolic trajectories is examined. The values of time where the manifolds are drawn ($t = t^*$), are chosen in such a way that the trajectories of the hyperbolic points and the development of the manifolds can be followed. It is important to point out that the figures of the manifolds are drawn at a fixed time (time slice in the extended phase space), so that the lines represent no particle trajectories, but initial conditions of trajectories that approach the hyperbolic trajectories in forward or backward time.

6.1. Transition from six to five hyperbolic points

![Figure 6.1: The stable (black) and unstable (red) manifolds associated with the six hyperbolic points in the corotating frame in the time dependent velocity field for $\Lambda = 1 - 9.6 \cdot 10^{-7}$ (left). The separatrices from the steady case are disturbed only slightly near the hyperbolic points, where transport can take place. The curved arrows indicate the orientation of the vortices, which is the same for all simulations.](image)
We start with a value of $\Lambda$ that deviates only slightly from the stationary flow with $\Lambda = 1$. The stable and unstable manifolds for $\Lambda = 1 - 9.6 \cdot 10^{-7}$ are shown in figure 6.1. This configuration has very small time dependence in the corotating frame, such that, as in the case of the equilateral triangle, there are six hyperbolic trajectories. The separatrices from the steady case are disturbed only a little near the hyperbolic points, where transport of particles can take place, and the flow domains are largely intact. This figure agrees with the results found in [22].

![Figure 6.2: Two hyperbolic points approach each other with an elliptic point in between (left). When the unstable and stable branches of both hyperbolic points intersect, the elliptic point is enclosed. Viewed from far away, this configuration is effectively one hyperbolic point (centre). The three points have merged into one hyperbolic point (right).](image)

Before dealing with smaller values of $\Lambda$, we first discuss some results on the critical points. For given initial configuration of vortices (the isosceles triangle), the number and location of the hyperbolic and elliptic critical points have been computed numerically by solving the roots of the coupled set of equations (3.44), where the same angular frequency is chosen as in the calculation of the manifolds. In figure 6.3 it is shown that for decreasing value of $\Lambda$ (starting at 1), two of the three hyperbolic points close to the centre and the elliptic in the origin start to align, and for $\Lambda \approx 0.989$ they merge to form just one hyperbolic point. Note that this is permitted topologically, as the Poincaré index is not changed by the elimination of one hyperbolic and one elliptic point. This merging process is explained in more detail in figure 6.2.

It is important to realise that this transition does not imply anything for a possible transition of the number of hyperbolic trajectories. Hyperbolic critical points are not coupled one-to-one to hyperbolic trajectories and they can provide misleading information for possible hyperbolic trajectories [5]; this analysis is merely meant to illustrate how the number of critical points can change for different values of $\Lambda$. 
6.1. Transition from six to five hyperbolic points

Figure 6.3: The hyperbolic (stars) and elliptic critical points (dots) in the initial isosceles triangle configuration for various values of \( \Lambda \). The vortices are shown by the circles. Top left: \( \Lambda = 1 \), with six hyperbolic and four elliptic critical points. Two of the three hyperbolic points close to the centre and the elliptic in the origin start to align and interact for \( \Lambda \approx 0.995 \) (top right). Bottom left: \( \Lambda = 0.99 \), bottom right: \( \Lambda = 0.988 \) where the two hyperbolic and the elliptic point have merged into one hyperbolic critical point, so that there are five hyperbolic and three elliptic critical points left.
6. Numerical simulations

Figure 6.4-i: $\Lambda = 0.999$, $t = t^*$.  

Figure 6.4-ii: $\Lambda = 0.999$, $t = 2t^*$.  

Figure 6.4-iii: $\Lambda = 0.999$, $t = 3t^*$.  

Figure 6.4-iv: A location of hyperbolic trajectory is indicated by a square to discern it from other intersections of the manifolds.

Figure 6.4: The stable (black) and unstable (red) manifolds associated with the six hyperbolic points in the corotating frame in the time dependent velocity field with $\Lambda = 0.999$ at three different times. The three inner hyperbolic points are approximately at the same positions as in the steady flow, i.e., at equal distance from the origin. The interaction of the manifolds occurs primarily around the origin, whereas the other flow domains are hardly perturbed.
6.1. Transition from six to five hyperbolic points

Furthermore, a topological argument as shown in section 3.8, demands that the transition occurs via the simultaneous elimination of a hyperbolic and an elliptic point. Note that the number of critical points depends on the value of the background frequency $\Omega$, so that the transition value is not uniquely determined.

The stable and unstable manifolds of the configurations with $\Lambda = 0.999$ are shown in figure 6.4. There are still six hyperbolic trajectories, the inner three of which are approximately at the same positions as in the steady equilateral triangle configuration. The locations of these trajectories for later times are found by following them from the early developments of the manifolds at $t = t^*$, where the hyperbolic points are the unique intersections of a stable and an unstable manifolds. For $\Lambda = 0.99$ in figure 6.5, there are also six hyperbolic trajectories, but the central equilateral triangle is disrupted, and two hyperbolic trajectories approach each other. This proximity of other hyperbolic points causes that not all manifolds in the centre are equally developed. The unstable and stable manifolds of the two interacting hyperbolic points in the centre start to intersect very quickly after the initial configuration of the isosceles triangle.

Note that the numerous intersections of the stable and unstable manifolds (primarily in the centre) do not behave according to the theory on lobe dynamics (see section 4.1.5). By this we mean that the typical creation of the lobes via the intersection of the stable and unstable of two distinct hyperbolic points is not observed: there are stable manifolds intersecting two different unstable manifolds and vice versa, in contrast to the picture as sketched by the homoclinic and heteroclinic tangles as described in section 4.1.5. The reason is that the figures are made at a fixed time slice, without any periodic time dependence between them.

For $\Lambda = 0.95$, a hyperbolic trajectory in the centre has vanished, so that only five remain, see figure 6.6. Note that the two sets of manifolds in the centre are well developed, suggesting that the 'total hyperbolicity' of the inner hyperbolic points is conserved. Mathematically, this means that the strengths of the remaining hyperbolic trajectories, as expressed by the parameters $\alpha$ and $\beta$ in the exponential dichotomy of equations (4.4-i) and (4.4-ii), is larger than those of the individual hyperbolic points from before the merging process, and also larger than those of the other three hyperbolic points.

So a transition from six to five occurs for both the hyperbolic trajectories and the hyperbolic critical points. As noted before, it is very difficult (if not impossible) to determine the transition values of $\Lambda$. The hyperbolic trajectory might disappear for a larger value of $\Lambda$ than the critical point does. In that case, theorem 4.3 possibly offers an explanation: the velocity field has become so time dependent that $\dot{x}_{cp}$ has become too large, or the nonlinearities in the field (measured by $\nabla(\nabla v)$) have increased too much because of the proximity of two vortices. Then the requirements for the existence of a hyperbolic trajectory close to the critical point are no longer satisfied.
Figure 6.5: The stable (black) and unstable (red) manifolds associated with the six hyperbolic points in the corotating frame in the time dependent velocity field with $\Lambda = 0.99$ at three different times. There are six hyperbolic trajectories, but the central hyperbolic trajectories are no longer positioned on the vertices of an equilateral triangle and two hyperbolic trajectories approach each other. The unstable and stable manifolds of the two distinct hyperbolic points in the centre start to intersect very quickly after the release of the isosceles triangle configuration. Some parts of the manifolds of the inner points nearly coincide.
6.1. Transition from six to five hyperbolic points

Figure 6.6-i: $\Lambda = 0.95$, $t = t^\star$.

Figure 6.6-ii: $\Lambda = 0.95$, $t = 1.6t^\star$.

Figure 6.6-iii: $\Lambda = 0.95$, $t = 2t^\star$.

Figure 6.6: The stable (black) and unstable (red) manifolds associated with the hyperbolic points in the corotating frame in the time dependent velocity field with $\Lambda = 0.95$ at three different times. A hyperbolic trajectory in the centre has vanished so that only five remain, and the two sets of manifolds in the centre are more developed than the three in the centre for larger values of $\Lambda$ and more than the outer three points. This means that the parameters $\alpha$ and $\beta$ in the exponential dichotomy are larger.
6.2. Transition from five to four hyperbolic points

For \( \Lambda \lesssim 0.9 \) a similar transition in the critical points occurs to that of section 6.1. In figure 6.7 it is shown that for decreasing value of \( \Lambda \) (starting at 0.8), two hyperbolic critical points with an elliptic critical point in between start to align, and for \( \Lambda \approx 0.75 \) they merge to form just one hyperbolic point. Again the total Poincaré index is not changed by the elimination of one hyperbolic and one elliptic point. This transition value is larger than \( \Lambda_c \approx 0.71 \), where flip occurs and the vortices pass through the line configuration. Recall that this does not mean anything for a possible transition from five to four hyperbolic trajectories.

A similar transition occurs in the number of hyperbolic trajectories, as shown in figures 6.8 and 6.9. For \( \Lambda = 0.80 \) there are five hyperbolic trajectories. The two transitions in the number of critical points and the number of hyperbolic trajectories need not occur at the same value of \( \Lambda \), because the existence of critical points depends on the frame that has been chosen, i.e., it depends on the angular frequency of the corotating frame.

The two sets of manifolds of the hyperbolic trajectories between the vortices are well developed, in contrast with the outer three. The transport barriers around the elliptic regions (the vortices) are visualised after short integration times. For \( \Lambda = 0.72 \), there are only four hyperbolic trajectories remaining. The manifolds around the origin intersect multiple times, but again the typical behaviour of the homoclinic and heteroclinic tangles forming lobes is not observed. One stable and one unstable manifold seem to approach a circular orbit around the vortices.

For both values of \( \Lambda \) the interaction of two of the three vortices becomes apparent, as the critical flip value \( \Lambda_c \approx 0.71 \) is approached. This holds particularly for \( \Lambda = 0.72 \) in figure 6.9-ii, where the two-plus-one state is almost complete, so that the two vortices hardly feel the presence of the third. The manifolds of the pair form a lemniscate (eight-shape). For smaller values of \( \Lambda \) (where the flip occurs), this pair can be considered as one vortex with double strength, which forms another lemniscate with the third vortex.
6.2. Transition from five to four hyperbolic points

Figure 6.7-i: $\Lambda = 0.8$, $t = 0$. Figure 6.7-ii: $\Lambda = 0.75$, $t = 0$.

Figure 6.7-iii: $\Lambda = 0.742$, $t = 0$. Figure 6.7-iv: $\Lambda = 0.741$, $t = 0$.

Figure 6.7: The hyperbolic (stars) and elliptic (dots) critical points for various values of $\Lambda$. The vortices are shown by the circles. Top left: $\Lambda = 0.80$, with five hyperbolic and three elliptic critical points. The two hyperbolic points on the bottom left and bottom right start to interact with the elliptic point in between for $\Lambda = 0.75$ (top right). Bottom left: $\Lambda = 0.742$, bottom right: $\Lambda = 0.741$ where the two hyperbolic and the elliptic point have merged into one hyperbolic critical point, so that there are four hyperbolic and two elliptic critical points left. Note that this transition value is larger than $\Lambda_c \approx 0.71$, although it depends on the angular frequency of the system.
6. Numerical simulations

Figure 6.8: The stable (black) and unstable (red) manifolds in the corotating frame for $\Lambda = 0.80$. The vortices are shown by the circles and there are five hyperbolic trajectories. The two sets of manifolds of the hyperbolic trajectories between the vortices are well developed, in contrast with the outer three. Figure 6.8-ii shows that the two-plus-one state is approached, where the manifolds of a dipole start to become visible.

Figure 6.9: The stable (black) and unstable (red) manifolds for $\Lambda = 0.72$. There are only four hyperbolic trajectories remaining. The manifolds around the origin intersect multiple times, but again the typical behaviour of the heteroclinic tangle is not observed. One stable and one unstable manifold seem to approach a limit cycle around the vortices. In figure 6.9-ii, a lemniscate of the manifolds of two vortices is apparent, excluding the third vortex.
7. Conclusions and recommendations

7.1. Conclusions

The dynamics of two-dimensional nonautonomous velocity fields have been studied, with an application to point vortex flows. More specifically, the flows induced by mutually advected points vortices with negligible viscosity have been examined and the transport barriers and chaotic regions in three vortex systems have been determined, using concepts from dynamical systems theory.

The starting point for time dependent flows is formed by the stationary velocity fields in equilibrium configurations. Point vortices of equal strength, placed on a circle with equal distances between neighbouring vortices, are in equilibrium in a frame that rotates with the system. The angular frequency of such a system is proportional to the strength of the vortices and inversely proportional to the square of the radius of the circle.

In a nonrotating frame, only critical points of the hyperbolic type are possible, however, a shift to a rotating frame introduces the possibility of elliptic critical points. The Poincaré index theorem, which relates the characteristics of a vector field at large distance to the number and types of critical points in the field, can be applied to these configurations. A setup of \( n \) vortices on a circle (which is stable for \( n \leq 7 \)) has been shown to give rise to \( 2n \) hyperbolic and \( n+1 \) elliptic critical points. This means that degenerate hyperbolic points are excluded.

The location of the critical points in circle configurations has a certain symmetry. On every line through a vortex and the origin, there is one hyperbolic critical point; every angle bisector of these lines contains one hyperbolic and one elliptic critical point. The exact locations of the points can be found by solving the roots of a polynomial of a degree equal to the number of vortices. The hyperbolic critical points, which are on the lines through the vortices and the origin, are connected by one set of separatrices, i.e., lines which have a constant value of the stream function; the other hyperbolic points, which are on the angle bisectors, are connected by a different set of separatrices. These separatrices define \( 2n+2 \) flow domains: \( n \) around the vortices, \( n+1 \) around the elliptic points and one path in between the sets of separatrices.

A critical point, which is stationary in the rotating frame, is not critical in the nonrotating frame, however, the orbit it traces out in the nonrotating frame is a
trajectory of the velocity field. This orbit of a hyperbolic critical point and the hyperbolic trajectory coincide in time independent flows, and in the case of small time dependence, they remain close together. The hyperbolic trajectory is defined by having the property that the linearised velocity field around it admits an exponential dichotomy. This concept is characterised by a time scale $\eta$, which is a typical time of attraction and repulsion of trajectories with respect to the hyperbolic trajectory in forward and backward time. There exists a special class of hyperbolic trajectories, which are limited by a weaker than exponential growth in time. Given a hyperbolic path in space, for which a hyperbolic critical point can be taken, the existence and uniqueness of such a special hyperbolic trajectory close to the hyperbolic path have been shown. The requirements are that the velocity of this path and the second derivative of the velocity field are sufficiently small.

These requirements amount to a separation of two length scales. The distance travelled by the hyperbolic path in the time $\eta$, must be smaller than a gradient length, which expresses a typical scale of the nonlinearities in the velocity field. If the distance travelled by the path is too large, then it is too fast for the particles to follow, making a hyperbolic trajectory impossible. On the other hand, if the gradient length is too large, then the spatial variations are then too large for the hyperbolicity of the path to be felt by a trajectory.

An alternate interpretation is that the requirements are equivalent to a separation of a Lagrangian and a Eulerian time scale. The time $\eta$ is a typical Lagrangian time, as it describes separation of trajectories. It must be larger than the Eulerian time scale, so that the velocity of the field as a whole is small compared to the typical velocity of a trajectory. The Eulerian time scale is then provided by inverse of the derivative of the velocity field. Then for the existence of a hyperbolic trajectory, the Eulerian time must be sufficiently large, meaning that the spatial variations in the velocity field must be sufficiently small.

In three point vortex systems, the hyperbolic trajectories and the associated invariant manifolds, which are the analogue of the separatrices in time dependent flows, have been examined numerically as a function of the vortex configuration. The equilibrium triangle has six hyperbolic critical points, giving rise to six hyperbolic trajectories. Three are far apart outside the triangle, and three are close to the centre of rotation. The latter region is where the intersections of the stable and unstable manifolds of different hyperbolic trajectories start for small deviations of the triangle. The manifolds of the inner three start to intersect multiple times quickly after the release of the initial positions.

The number of hyperbolic trajectories decreases from six to five, when two of the three close to the centre of rotation interact, until one remains after the transition. The next transition from five to four hyperbolic trajectories occurs, when two of the outer three hyperbolic trajectories turn into one. This transition occurs at a deviation from the stationary setup, where two vortices are almost so close together that the
7.2. Suggestions for further research

This work has provided many points for further investigation, and has raised numerous new questions, which can be tackled both analytically and numerically. This applies to the theory on point vortices as well as to the theory on hyperbolic trajectories.

It has been shown in chapters 4 and 6 that the relation between hyperbolic critical points and hyperbolic trajectories is complicated. The critical points are at the basis of the analysis of the hyperbolic trajectories and corresponding invariant manifolds, so that a thorough understanding of this relation is invaluable. A remarkable resemblance has been observed in the transitions of the numbers of hyperbolic trajectories and hyperbolic critical points for increasing deviation from the equilateral triangle configuration in three vortex systems. Furthermore, sufficient conditions on the velocity field and the path of a hyperbolic critical point for the existence and uniqueness of a hyperbolic trajectory have been presented in chapter 4. However, little is known about the necessary conditions in order to make the connection one-to-one.

Concerning point vortices, the critical points of the velocity field can be calculated at any instant of time, as shown in section 3.8. The number of critical points has been
computed for the initial positions of the vortices that was used in numerical simulation. However, it is not guaranteed that this number is maintained in the time of the evolution. Therefore, it is interesting to determine how the number of hyperbolic critical points, which exist continuously in time, depends on the configuration of the vortices (given by the parameter $\Lambda$), in order to make the connection with the number of hyperbolic trajectories.

In periodically time dependent flows, the transport mechanism for interactions of stable and unstable manifolds is provided by lobe dynamics, which is treated briefly in section 4.1.5. In the numerical simulations of the three point vortex systems, many figures of entangled stable and unstable manifolds have been drawn, but an analysis of the corresponding transport is yet to be made. This is particularly interesting, because the observed interactions are not in accordance with the known theory of the homoclinic and heteroclinic tangles, which are typical for one and two time dependent hyperbolic trajectories, respectively. In order to apply the theory of lobe dynamics to the three vortex systems, the figures need to be made at times corresponding to the period in the flow. These figures from different periods can then be compared, so that the transport in the system could be analysed.

For three vortex systems, it is difficult to determine the exact transition values of the parameter $\Lambda$, where the number of hyperbolic critical points and trajectories changes. Therefore, although the results found are consistent with the theory on the existence and uniqueness of hyperbolic trajectories, it has not really been tested. This means that no ranges of $\Lambda$ have been observed, where a hyperbolic trajectory has disappeared, even though its hyperbolic critical point still existed. This could then be explained by theorem 4.3, by stating that the velocity of the hyperbolic path or the second derivative of the velocity field (i.e., the nonlinearities) is too large. Perhaps such a phenomenon could be observed in systems of more than three vortices. The starting configuration could be four vortices with a small deviation from a square, such that the vortex motion is quasi-periodic.

The analytical work with the dynamical systems approach has focussed on the theory of the special hyperbolic trajectories. In a number of papers by Haller et al. [45, 46], the notion of a uniformly hyperbolic trajectory is discussed. A quick study has led to the conjecture that both names describe the same special trajectory. An investigation of the connection between these concepts might be worthwhile. Furthermore, there may be a close relation between the invariant manifolds as examined in this work, and the Lagrangian coherent structures, which are mentioned in these papers.
References


A. General theory

A.1. Volume conservation

An incompressible flow preserves volume, so in two dimensions the surface area of a domain spanned by a closed material curve is constant [1]. This is used as a verification in the contour kinematics code as described in chapter 5, where the surface area enclosed by the blob of tracer particles is calculated every time step. A derivation of this statement is given here.

Suppose $D_{t_0}$ denotes a domain in an three-dimensional space at time $t_0$, and $D_t$ denote the time evolution of $D_{t_0}$ under the transformation that maps points along fluid trajectories. Let $V(t)$ denote the volume of $D_t$, which has boundary $S_t$. The rate of change in volume equals the velocity flux over the boundary. Then, using Gauss’s divergence theorem [12],

$$\frac{dV}{dt}\bigg|_{t=t_0} = \int\int_{S_{t_0}} \mathbf{v} \cdot \mathbf{n} \, dS = \int\int\int_{D_{t_0}} \nabla \cdot \mathbf{v} \, d\mathbf{x},$$  \hspace{1cm} (A.1)

with $\nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$ the divergence of the three-dimensional velocity field. Equation (A.1) is known as Liouville’s theorem [9]. If the divergence is constant everywhere, $\nabla \cdot \mathbf{v} = C$ for constant $C$, then, since $t_0$ is arbitrary, equation (A.1) yields

$$\dot{V}(t) = C \iint_{D_t} d\mathbf{x} = CV(t)$$

with solution

$$V(t) = V(0)e^{Ct}.$$  

If the flow is incompressible, i.e., $C = 0$, then it follows that

$$V(t) = V(0)$$

and therefore, incompressible flows preserve volume.
A.2. Index of a velocity field

Consider the following vector field \((P, Q)\):

\[
\begin{align*}
\dot{x} &= P(x, y), \\
\dot{y} &= Q(x, y).
\end{align*}
\]

The index of a curve \(\chi\) can be expressed as the line integral

\[
I_\chi = \frac{1}{2\pi} \oint_\chi \left( \arctan \frac{Q}{P} \right) d\gamma = \frac{1}{2\pi} \oint_\chi \left( \frac{P \frac{\partial Q}{\partial x} - Q \frac{\partial P}{\partial x}}{P^2 + Q^2} \right) dx + \left( \frac{P \frac{\partial Q}{\partial y} - Q \frac{\partial P}{\partial y}}{P^2 + Q^2} \right) dy.
\]  
\tag{A.2}
\]

For a field with \(P(x, y) = \lambda x, Q(x, y) = -\lambda y\), which is typical for the flow around a hyperbolic point, we find, with \(\chi\) the unit circle taken counterclockwise:

\[
I_\chi = \frac{1}{2\pi} \oint_\chi \frac{\lambda^2 y dx - \lambda^2 x dy}{\lambda^2 (x^2 + y^2)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{-\sin^2 \varphi d\varphi - \cos^2 \varphi d\varphi}{\cos^2 \varphi + \sin^2 \varphi} = -1.
\]

For a velocity field with \(P(x, y) = \mu y, Q(x, y) = -\mu x\), describing the flow around an elliptic point, the \((A.2)\) gives for the same \(\chi\)

\[
I_\chi = \frac{1}{2\pi} \oint_\chi \frac{-\mu^2 y dx + \mu^2 x dy}{\mu^2 (y^2 + x^2)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin^2 \varphi d\varphi + \cos^2 \varphi d\varphi}{\sin^2 \varphi + \cos^2 \varphi} = 1.
\]
B. Point vortex theory

B.1. Point vortex motion

The Hamiltonian \( H \) of a system of point vortices is

\[
H = \frac{1}{2} \int_A \omega \Psi(x, y, t) \, dA \\
= \frac{1}{2} \int_A \sum_{i=1}^{N} \kappa_i \delta(x - x_i) \delta(y - y_i) \Psi(x, y, t) \, dA \\
= \frac{1}{2} \sum_{i=1}^{N} \kappa_i \Psi(x_i, y_i, t) \\
= \frac{1}{8\pi} \sum_{j=1}^{N} \sum_{i=1}^{N} \kappa_i \kappa_j \log \left( \left( x_i - x_j \right)^2 + \left( y_i - y_j \right)^2 \right). \tag{B.1}
\]

The fact that it is a constant of the motion, can also be recognised from the absence of explicit time dependence in the Hamiltonian. Namely, the derivative of \( H \) with respect to \( t \) gives

\[
\frac{dH}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt} + \frac{\partial H}{\partial t} = 0.
\]

Using that in a Hamiltonian system \( \frac{\partial H}{\partial x} = -\frac{dy}{dt} \) and \( \frac{\partial H}{\partial y} = \frac{dx}{dt} \), we find that

\[
\frac{\partial H}{\partial t} = 0.
\]

The equations of motion of the point vortices can be written as

\[
\dot{z}_\alpha = \{z_\alpha, \hat{H}\},
\]
or, written out per component,
\[
\dot{x}_\alpha = \{x_\alpha, H\} = \sum_j \frac{1}{\kappa_j} \left( \frac{\partial x_\alpha}{\partial x_j} \frac{\partial H}{\partial y_j} - \frac{\partial x_\alpha}{\partial y_j} \frac{\partial H}{\partial x_j} \right)
\]
\[
\dot{y}_\alpha = \{y_\alpha, H\} = \sum_j \frac{1}{\kappa_j} \left( \frac{\partial y_\alpha}{\partial x_j} \frac{\partial H}{\partial y_j} - \frac{\partial y_\alpha}{\partial y_j} \frac{\partial H}{\partial x_j} \right)
\]
which are Hamilton’s canonical equations. Substituting the Hamiltonian (B.1) gives
\[
\dot{x}_\alpha = -\frac{1}{8\pi\kappa_\alpha} \sum_{j=1}^{N} \sum_{j \neq \alpha} \kappa_i \kappa_j \frac{2(y_i - y_j)}{(x_i - x_j)^2 + (y_i - y_j)^2} \delta_{i\alpha} + \kappa_i \kappa_j \frac{-2(y_i - y_j)}{(x_i - x_j)^2 + (y_i - y_j)^2} \delta_{j\alpha}
\]
\[
\dot{y}_\alpha = -\frac{1}{2\pi} \sum_{j=1}^{N} \sum_{j \neq \alpha} \frac{y_i - y_j}{(x_i - x_j)^2 + (y_i - y_j)^2}
\]
A similar derivation for \(\dot{y}_\alpha\) gives the following coupled set of equations for the motion of the vortices:
\[
\dot{x}_i = -\frac{1}{2\pi} \sum_{j=1}^{N} \sum_{j \neq i} \frac{y_i - y_j}{(x_i - x_j)^2 + (y_i - y_j)^2},
\]
\[
\dot{y}_i = \frac{1}{2\pi} \sum_{j=1}^{N} \frac{x_i - x_j}{(x_i - x_j)^2 + (y_i - y_j)^2}.
\]
The differential equations (B.2) can be captured in one equation in the complex variable \(z = x + iy\):
\[
\dot{z}_\alpha = \dot{x}_\alpha - i \dot{y}_\alpha = \frac{1}{2\pi} \sum_{j=1}^{N} \kappa_j \frac{(y_i - y_j) - i(x_i - x_j)}{(x_i - x_j)^2 + (y_i - y_j)^2}
\]
B.2. Passive tracers

The motion of a passive tracer, i.e. a vortex with zero strength, is governed by the following equations:
\[
\dot{x} = \frac{\partial \Psi}{\partial y}, \quad \dot{y} = -\frac{\partial \Psi}{\partial x},
\]
where the stream function \(\Psi\) satisfies
\[
\Psi(x, y, t) = -\frac{1}{4\pi} \sum_{j=1}^{N} \kappa_j \log \left( (x - x_j)^2 + (y - y_j)^2 \right). \tag{B.5}
\]

Substituting the stream function \(\Psi\) of (B.5) into (B.4) gives an equation for the velocity field \((\dot{x}, \dot{y})\) of the passive particles:
\[
\dot{x} = -\frac{1}{2\pi} \sum_{j=1}^{N} \kappa_j \frac{y - y_j}{(x - x_j)^2 + (y - y_j)^2}, \tag{B.6}
\]
\[
\dot{y} = \frac{1}{2\pi} \sum_{j=1}^{N} \kappa_j \frac{x - x_j}{(x - x_j)^2 + (y - y_j)^2}.
\]

The stream function \(\Psi\) depends explicitly on time via the positions \((x_j(t), y_j(t))\) of the point vortices. Therefore, we have a nonautonomous system of equations, which is not integrable generally.

Analogous to the derivation of (B.3), which describes the vortex motion in complex coordinates, the system (B.6) can be assembled in one equation:
\[
\dot{z}^* = \frac{1}{2\pi i} \sum_{j=1}^{N} \kappa_j \frac{1}{z^* - z_j}, \tag{B.7}
\]
The set of equations (B.4) can be rewritten in complex coordinates:

\[
\frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial \Psi}{\partial z^*} \frac{\partial z^*}{\partial y} = i \left( \frac{\partial \Psi}{\partial z} - \frac{\partial \Psi}{\partial z^*} \right),
\]

\[
\frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial \Psi}{\partial z^*} \frac{\partial z^*}{\partial x} = \frac{\partial \Psi}{\partial z} + \frac{\partial \Psi}{\partial z^*},
\]

(B.8)

such that we find for the velocity field \( \dot{z}^* = \{z^*, \Psi\} \):

\[
\dot{z}^* = \frac{\partial \Psi}{\partial y} + i \frac{\partial \Psi}{\partial x} = 2i \frac{\partial \Psi}{\partial z},
\]

(B.9)

where * indicates complex conjugation and \( \{\cdot\} \) denotes the Poisson bracket, defined in equation (3.7) [44]. The fundamental Poisson bracket is

\[ \{z, z^*\} = -2i. \]

Similar to (B.9), we can derive from (B.8) that

\[
\dot{z} = \frac{\partial \Psi}{\partial y} - i \frac{\partial \Psi}{\partial x} = -2i \frac{\partial \Psi}{\partial z^*},
\]

(B.10)

where \( \dot{z} \) follows from (B.7)

\[
\dot{z} = -\frac{1}{2\pi i} \sum_{j=1}^{N} \kappa_j \frac{1}{z^* - z_j^*}.
\]

(B.11)

Notice that (B.9) with substitution of (B.7), and (B.10) in combination with (B.11) are of the form \( \frac{\partial \Psi}{\partial z} = \Delta(z), \frac{\partial \Psi}{\partial z^*} = \Theta(z^*). \) Integration then yields:

\[
\Psi(z, z^*, t) = \int \Delta(z) \, dz + f(z^*),
\]

\[
\Psi(z, z^*, t) = \int \Theta(z^*) \, dz^* + g(z).
\]

These equations can only be satisfied simultaneously if \( f(z^*) = \int \Theta(z^*) \, dz^* \) and \( g(z) = \int \Delta(z) \, dz \), which gives for \( \Psi \):

\[
\Psi(z, z^*, t) = \int \Delta(z) \, dz + \int \Theta(z^*) \, dz^*
\]

= \frac{1}{2i} \int \dot{z}^* \, dz - \frac{1}{2i} \int \dot{z} \, dz^*

= -\frac{1}{4\pi} \sum_{j=1}^{N} \kappa_j \log(z - z_j) - \frac{1}{4\pi} \sum_{j=1}^{N} \kappa_j \log(z^* - z_j^*)
\]

Notice that (B.9) with substitution of (B.7), and (B.10) in combination with (B.11) are of the form \( \frac{\partial \Psi}{\partial z} = \Delta(z), \frac{\partial \Psi}{\partial z^*} = \Theta(z^*). \) Integration then yields:

\[
\Psi(z, z^*, t) = \int \Delta(z) \, dz + f(z^*),
\]

\[
\Psi(z, z^*, t) = \int \Theta(z^*) \, dz^* + g(z).
\]

These equations can only be satisfied simultaneously if \( f(z^*) = \int \Theta(z^*) \, dz^* \) and \( g(z) = \int \Delta(z) \, dz \), which gives for \( \Psi \):

\[
\Psi(z, z^*, t) = \int \Delta(z) \, dz + \int \Theta(z^*) \, dz^*
\]

= \frac{1}{2i} \int \dot{z}^* \, dz - \frac{1}{2i} \int \dot{z} \, dz^*

= -\frac{1}{4\pi} \sum_{j=1}^{N} \kappa_j \log(z - z_j) - \frac{1}{4\pi} \sum_{j=1}^{N} \kappa_j \log(z^* - z_j^*)
\]
B.3. Location of critical points

In section 3.1, it has been shown that the system of three point vortices is integrable, since the three quantities $H, P_\phi$ and $P_x^2 + P_y^2$ mutually Poisson commute, i.e., the Poisson bracket of any combination is zero.

Here, we focus on locating the critical points, hoping that these can help us to find the hyperbolic trajectories later. It follows from the theory of Aref & Brøns [24], whose main results are from a paper by Siebeck [47], that the critical points of a flow induced by three point vortices are located in the foci of a conic section, i.e. a hyperbola, a parabola or an ellipse. The type of this critical point conic section depends on the vortex strengths $\kappa_i$: it is an ellipse if $\kappa_1 \kappa_2 \kappa_3 (\kappa_1 + \kappa_2 + \kappa_3) > 0$, a parabola if $\kappa_1 \kappa_2 \kappa_3 (\kappa_1 + \kappa_2 + \kappa_3) = 0$, a hyperbola if $\kappa_1 \kappa_2 \kappa_3 (\kappa_1 + \kappa_2 + \kappa_3) < 0$.

Note that the conic is no parabola for systems with equal vortex strengths and that it is a parabola only if the sum of the vortex strengths is zero. Figure B.1 shows the critical points for point vortices located at the vertices of a number of isosceles triangles, which have a constant base and varying height, and the streamline passing through these points. In the equilateral case, when the height $h = \sqrt{3}$, the two critical points coincide and a so-called monkey-saddle appears. These topological aspects of this situation are discussed in more detail in section 3.7.2. For $h < \sqrt{3}$ there is one streamline passing through both critical points, whereas in the the case that $h > \sqrt{3}$, there are two distinct single streamlines going through the two points.

The vortices in this calculation all have equal strength: $\kappa_i = \kappa = 1$. The polynomial $P(z)$ from equation (3.12) is given by

$$P(z) = (z - 1)(z + 1) + (z - 1)(z - ih) + (z + 1)(z - ih).$$

Solving $P(z) = 0$ yields the roots

$$z_{s1,2} = \frac{1}{3} \left( ih \pm \sqrt{3 - h^2} \right).$$
Figure B.1: The location of the critical points for point vortices at the vertices of six isosceles triangles and the streamline passing through them. These streamlines self-intersect at the location of the critical points. With a constant base of width 2, the height \( h \) of the triangle varies from 0.5 (left) to 1.5 (right), \( 1.7, \sqrt{3}, 1.75 \) and 2 (next page). In the equilateral case, where the height \( h = \sqrt{3} \), the two critical points coincide and a so-called monkey-saddle appears. For \( h < \sqrt{3} \) there is one streamline passing through both critical points, whereas in the case that \( h > \sqrt{3} \), there are two distinct streamlines going through the two points.

The two critical points coincide for \( h = \sqrt{3} \) and are located at \( z_{s_1} = z_{s_2} = \frac{1}{\sqrt{3}} \). If \( h \) is close to \( \sqrt{3} \), the equilateral triangle, we can write \( h = \sqrt{3} + \varepsilon \) for small \( \varepsilon \). A perturbation calculation shows that the critical points are located at

\[
\begin{align*}
z_{s_2} &= i \left( \frac{1}{\sqrt{3}} - \frac{\sqrt{2}}{3^{3/4}} \sqrt{\varepsilon} + \frac{\varepsilon}{3} + \mathcal{O}(\varepsilon^{3/2}) \right), \\
z_{s_2} &= i \left( \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{3^{3/4}} \sqrt{\varepsilon} + \frac{\varepsilon}{3} + \mathcal{O}(\varepsilon^{3/2}) \right).
\end{align*}
\]

The first order deviation from the equilateral triangle is \( \mathcal{O}(\sqrt{\varepsilon}) \), so a small deviation creates a significant distance between the critical points. For \( \varepsilon > 0 \) \( z_{s_1} \) and \( z_{s_2} \) are purely imaginary, so they lie on the vertical axis. Furthermore, if \( \varepsilon < 0 \), \( z_{s_1} \) and \( z_{s_2} \) have a real part (with the same size but different sign), while the imaginary parts are equal, so that they lie on a horizontal line. These observations are shown in figure B.1.
B.4. Corotating frame

The transformation to a frame rotating with frequency $\Omega$ gives for the trajectories of the vortices

$$\tilde{z}_m = z_m e^{-i\Omega t}, \quad m = 1, \ldots, N,$$

where the tilde indicates coordinates in the corotating frame. Differentiation of (B.12) with respect to $t$ yields

$$\dot{\tilde{z}}_m = (\dot{z}_m - i\Omega z_m)e^{-i\Omega t}.$$  \hspace{1cm} (B.13)

After taking the complex conjugate of (B.13), substitution of (B.3) and application of (B.12), the equation describing the motion of the vortices in the corotating frame
becomes
\[ \dot{\tilde{z}}_m = \tilde{z}_m^* e^{i\Omega t} + i\Omega \tilde{z}_m^* e^{i\Omega t} = \frac{1}{2\pi i} \sum_{j=1}^{N} \kappa_j \frac{1}{\tilde{z}_m - \tilde{z}_j} + i\Omega \tilde{z}_m. \]

A similar approach for the coordinates \( \tilde{z} = z e^{i\Omega t} \) of the passive tracers in the rotating frame gives
\[ \dot{\tilde{z}}^* = \frac{1}{2\pi i} \sum_{m=1}^{N} \kappa_m \frac{1}{\tilde{z}^* - \tilde{z}_m^*} + i\Omega \tilde{z}^*. \] (B.14)

The complex conjugate of (B.14) is
\[ \dot{\tilde{z}} = -\frac{1}{2\pi i} \sum_{m=1}^{N} \kappa_m \frac{1}{\tilde{z}^* - \tilde{z}_m^*} - i\Omega \tilde{z}. \] (B.15)

The stream function \( \tilde{\Psi} (\tilde{z}, \tilde{z}^*, t) \) in the moving frame follows from integration of (B.9) and (B.10) in the corotating frame with substitution of (B.14) and (B.15), respectively:
\[ \tilde{\Psi} (\tilde{z}, \tilde{z}^*, t) = \frac{1}{2i} \int \dot{\tilde{z}}^* d\tilde{z} + \tilde{f}(\tilde{z}^*) \]
\[ = \frac{1}{2i} \int \left( \frac{1}{2\pi i} \sum_{m=1}^{N} \kappa_m \frac{1}{\tilde{z}^* - \tilde{z}_m^*} + i\Omega \tilde{z}^* \right) d\tilde{z} + \tilde{f}(\tilde{z}^*), \]
and, at the same time,
\[ \tilde{\Psi} (\tilde{z}, \tilde{z}^*, t) = -\frac{1}{2i} \int \dot{\tilde{z}}^* d\tilde{z} + \tilde{g}(\tilde{z}) \]
\[ = -\frac{1}{2i} \int \left( -\frac{1}{2\pi i} \sum_{m=1}^{N} \kappa_m \frac{1}{\tilde{z}^* - \tilde{z}_m^*} - i\Omega \tilde{z} \right) d\tilde{z} + \tilde{g}(\tilde{z}), \]
which can only be true if
\[ \tilde{f}(\tilde{z}^*) = -\frac{1}{2i} \int -\frac{1}{2\pi i} \sum_{m=1}^{N} \kappa_m \frac{1}{\tilde{z}^* - \tilde{z}_m^*} d\tilde{z}^*, \]
\[ \tilde{g}(\tilde{z}) = \frac{1}{2i} \int \frac{1}{2\pi i} \sum_{m=1}^{N} \kappa_m \frac{1}{\tilde{z} - \tilde{z}_m} d\tilde{z}. \]
This then gives for \( \tilde{\Psi} \)
\[
\tilde{\Psi}(\tilde{z}, \tilde{z}^*, t) = \frac{1}{2i} \int \left( \frac{1}{2\pi i} \sum_{m=1}^{N} \kappa_m \frac{1}{\tilde{z} - \tilde{z}_m} + i\Omega \tilde{z}^* \right) d\tilde{z} - \frac{1}{2i} \int \frac{1}{2\pi i} \sum_{m=1}^{N} \kappa_m \frac{1}{\tilde{z}^* - \tilde{z}_m} d\tilde{z}^* \\
= -\frac{1}{4\pi} \sum_{m=1}^{N} \kappa_m \log(\tilde{z} - \tilde{z}_m) + \frac{1}{2} \Omega \tilde{z} \tilde{z}^* - \frac{1}{4\pi} \sum_{m=1}^{N} \kappa_m \log(\tilde{z}^* - \tilde{z}_m) \\
= -\frac{1}{4\pi} \sum_{m=1}^{N} \kappa_m \log |\tilde{z} - \tilde{z}_m|^2 + \frac{1}{2} \Omega \tilde{z} \tilde{z}^*.
\]

B.5. Point vortices on a circle

The motion of \( N \) point vortices located on a circle is considered in more detail. The vortex motion is examined from a more fundamental point of view, which means vectorial addition of the induced velocities. It is shown that the velocity induced in a vortex by the others, is tangent to the circle at any moment in time.

We again assume that the vortices \( p_i \) initially have coordinates \( x_{ki} = (x_{k0}, y_{k0}) = (r \cos \varphi_k, r \sin \varphi_k) \), with \( \varphi_{k0} = 2\pi \frac{k-1}{N}, k = 1, \ldots, N \). This means that vortex \( p_1 \) lies on the \( x \)-axis with \( x_1 = (r, 0) \). In complex notation, we have \( z_{k0} = r e^{i\varphi_{k0}} \). The induced velocity of \( p_1 \) is calculated; the induced velocities of the other vortices follow from rotational symmetry. For the trajectories of the vortices it then suffices to shows that this velocity is tangential to the circle, i.e., \( v_1 = (0, v_{1y}) \). The angular frequency then follows from \( \Omega = |v_1|/|x_1| \).

Generally, the velocity that a vortex \( p_k \) at position \( x_k \) of strength \( \kappa \) induced at vortex \( p_1 \) located at \( x_1 \) equals \( \kappa/(2\pi d) \), where the vortices are separated by a distance \( d \). The induced velocity is directed perpendicular to the line connecting the vortices and in such a way, that if \( \kappa > 0 \), then induced velocity points in the direction of \( (y_1 + y_k, x_1 - x_k) = (y_1 + y_k)e_x + (x_1 - x_k)e_y \), so that its direction \( e_{k1} \) is
\[
e_{k1} = \frac{(y_1 + y_k, x_1 - x_k)}{|(y_1 + y_k, x_1 - x_k)|} = \frac{(y_1 + y_k)e_x + (x_1 - x_k)e_y}{|x_1 - x_k|}.
\]

This gives for the velocity \( v_{k1} \) induced by vortex \( p_k \) at \( p_1 \):
\[
v_{k1} = \frac{\kappa}{2\pi |x_1 - x_k|} e_{k1} = \frac{\kappa}{2\pi} \frac{(y_1 + y_k)e_x + (x_1 - x_k)e_y}{|x_1 - x_k|^2}.
\]

This has been illustrated in figure B.2. Now we can substitute the coordinates of the points vortices on the circle and apply superposition of the \( N \) induced velocities:
\[
v_1 = \frac{\kappa}{2\pi} \sum_{i=2}^{N} \frac{1}{|x_1 - x_i|} e_{i1}.
\]
Figure B.2: The velocity induced in a vortex which is located on the horizontal axis. The other vortices (circles) induce a velocity (straight, solid arrows) at this vortex perpendicular to the line connecting them. The vertical component of the induced velocities is the same for all vortices, and the vectorial sum (dashed arrow) of the velocities has no horizontal component. This means that the net velocity is tangent to the circle.

\[ v_{1x} = -\frac{\kappa}{2\pi r} \sum_{i=2}^{N} \frac{\sin\left(\frac{2\pi i - 1}{N}\right)}{(1 - \cos\left(2\pi \frac{i - 1}{N}\right))^2 + \left(-\sin\left(2\pi \frac{i - 1}{N}\right)\right)^2} \]

For the components \((v_{1x}, v_{1y})\) we find for \(v_{1x}\)

\[ v_{1x} = -\frac{\kappa}{2\pi r} \sum_{i=2}^{N} \frac{\sin\left(\frac{2\pi i - 1}{N}\right)}{2 - 2\cos\left(2\pi \frac{i - 1}{N}\right)} \]

Now use the fact that sin\((2\pi - x) = -\sin x\) and cos\((2\pi - x) = \cos x\) for all \(x\). If \(N\) is
even, so the sum has an odd number of terms, the middle term in the sum has index $i = N/2 + 1$, which gives
\[
\sin \left(2\pi \frac{N/2 + 1 - 1}{N}\right) = \frac{\sin \pi}{2 - 2 \cos \pi} = 0.
\]
This is the contribution from the vortex on the opposite side of the circle, which has only a component in tangent to the circle, i.e., in the $y$-direction.

For the remaining contributions the sum can be divided into two parts:
\[
v_{1x} = -\frac{\kappa}{2\pi r} \left( \sum_{i=2}^{N/2} \frac{1}{2 - 2 \cos \left(2\pi \frac{i-1}{N}\right)} \sin \left(2\pi \frac{i-1}{N}\right) + \sum_{i=N/2+2}^{N} \frac{1}{2 - 2 \cos \left(2\pi \frac{i-1}{N}\right)} \sin \left(2\pi \frac{i-1}{N}\right) \right)
\]
\[
= -\frac{\kappa}{2\pi r} \left( \sum_{i=2}^{N/2} \frac{1}{2 - 2 \cos \left(2\pi \frac{i-1}{N}\right)} - \sum_{j=N-i+2}^{N/2} \frac{1}{2 - 2 \cos \left(2\pi \frac{j-1}{N}\right)} \sin \left(2\pi \frac{j-1}{N}\right) \right)
\]
\[
= 0.
\]
If $N$ is odd, there is an even number of terms in the sum. Consequently, there is no middle term and the derivation that $v_{1x} = 0$ is analogous with the sum split into a sum from $i = 2$ to $i = (N + 1)/2$ and a sum from $i = (N + 3)/2$ to $i = N$.

For the $y$-component $v_{1y}$ of $v_1$ we find
\[
v_{1y} = \frac{\kappa}{2\pi r} \sum_{i=2}^{N} \frac{1 - \cos \left(2\pi \frac{i-1}{N}\right)}{(1 - \cos \left(2\pi \frac{i-1}{N}\right))^2 + (-\sin \left(2\pi \frac{i-1}{N}\right))^2}
\]
\[
= \frac{\kappa}{2\pi r} \sum_{i=2}^{N} \frac{1 - \cos \left(2\pi \frac{i-1}{N}\right)}{2 - 2 \cos \left(2\pi \frac{i-1}{N}\right)}
\]
\[
= \frac{\kappa}{2\pi r} \sum_{i=2}^{N} \frac{1}{2}
\]
\[
= \frac{\kappa}{4\pi r} (N - 1).
\]
This means that the vertical component of the induced velocities is the same for all vortices. The angular frequency $\Omega$ is then found from
\[
\Omega = \frac{v_{1y}}{r} = \frac{(N - 1)\kappa}{4\pi r^2}.
\]
Note that the equations (B.17) for $v_{1x}$ and (B.18) for $v_{1y}$ can be captured in a single complex equation:

$$v_{1y} - iv_{1x} = \frac{\kappa}{2\pi r} \sum_{j=2}^{N} \frac{(1 - \cos (2\pi \frac{j-1}{N})) + i \sin (2\pi \frac{j-1}{N})}{(1 - \cos (2\pi \frac{j-1}{N}))^2 + (- \sin (2\pi \frac{j-1}{N}))^2}$$

$$= \frac{\kappa}{2\pi r} \sum_{j=2}^{N} \frac{1 - e^{-2\pi i(j-1)/N}}{1 - 2 - 2 \cos (2\pi \frac{j-1}{N})}$$

$$= \frac{\kappa}{2\pi r} \sum_{j=2}^{N} \frac{1 - e^{-2\pi i(j-1)/N}}{2 - e^{2\pi i(j-1)/N} - e^{-2\pi i(j-1)/N}}$$

$$= \frac{\kappa}{2\pi r} \sum_{j=2}^{N} \frac{1 - e^{-2\pi i(j-1)/N}}{(1 - e^{-2\pi i(j-1)/N})(1 - e^{2\pi i(j-1)/N})}$$

$$= \frac{\kappa}{2\pi r} \sum_{j=2}^{N} \frac{1}{1 - e^{2\pi i(j-1)/N}}$$

$$= \frac{\kappa}{4\pi r} (N - 1),$$

which was used in the derivation of $\Omega$ in section 3.6.

The proof that

$$\sum_{j=2}^{N} \frac{1}{1 - e^{2\pi i(j-1)/N}} = \frac{N - 1}{2}$$

can be derived more generally from the theory in section 3.7. We found that for arbitrary $z_k$

$$\sum_{k} \frac{1}{z - z_k} = \frac{d}{dz} \left( \prod_k (z - z_k) \right) \prod_k (z - z_k), \quad (B.19)$$

and that for $z_j = e^{2\pi i(j-1)/N}$,

$$\prod_{j=1}^{N} (z - z_j) = z^N - 1,$$

or

$$\prod_{j=2}^{N} (z - z_j) = \frac{z^N - 1}{z - 1},$$
With the sum in (B.19) running from \( j = 2 \) to \( j = N \), we have

\[
\sum_{j=2}^{N} \frac{1}{z - z_j} = \frac{\frac{d}{dz} \left( \prod_{j=2}^{N} (z - z_j) \right)}{\prod_{j=2}^{N} (z - z_N)}
\]

\[
= \frac{\frac{d}{dz} \left( \frac{z^N - 1}{z - 1} \right)}{\frac{z^N - 1}{z - 1}}
\]

\[
= \frac{(N - 1) z^{N+1} - N z^N + z}{z(z^N - 1)(z - 1)}
\]

(B.20)

Both the numerator and the denominator of (B.20) go to zero for \( z \to 1 \). Two times the application of l'Hôpital's rule then gives

\[
\sum_{j=2}^{N} \frac{1}{1 - z_j} = \frac{N(N - 1)}{2N} = \frac{N - 1}{2}.
\]
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C.1. Finite time approximation

It is very common in practice, e.g., with oceanographic data, that the data of the velocity field are only known for a finite amount of time. In this section we deal with approximate equations valid in this case [7]. Additional theory can be found in [15, 27, 33, 34].

The approximations of the hyperbolic trajectories that have to be made lead to the following definition.

**Definition C.1.** The approximate hyperbolic trajectory $z_a$ on the interval $[T_1, T_2]$ is defined as a solution of

$$z_a(t) = X(t) \int_{T_1}^{t} P X^{-1}(s) f(z_a(s), s) ds - X(t) \int_{t}^{T_2} Q X^{-1}(s) f(z_a(s), s) ds. \quad (C.1)$$

**Theorem C.2.** Under the conditions of theorem 4.3 holding in the domain $[T_1, T_2]$, there is a unique approximate hyperbolic trajectory defined as in definition C.1.

**Proof.** The proof is analogous to the proof of theorem 4.3 with obvious modifications for the finite time interval. In the derivation that $T z_a$ is a contraction mapping, an extra inequality after $(\dagger)$ is needed:

$$\int_{T_1}^{t} k \ e^{-\alpha(t-s)} \ ds + \int_{t}^{T_2} \ell \ e^{-\beta(s-t)} \ ds \leq \int_{-\infty}^{t} k \ e^{-\alpha(t-s)} \ ds + \int_{t}^{\infty} \ell \ e^{-\beta(s-t)} \ ds$$

because the integrands are positive. \hfill \blacksquare

**Theorem C.3.** Under the conditions of theorem 4.3:

$$\| z_a - z_h \|_{L^\infty([T_1, T_2]; \mathbb{R}^2)} \leq \left( 1 - \| \nabla f \|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2)} \eta \right)^{-1} \left( \max_{T_1 \geq t \geq T_2} \left| \int_{-\infty}^{T_1} X(t) P X^{-1}(s) f(z_h(s), s) ds \right| + \max_{T_1 \leq t \leq T_2} \left| \int_{T_2}^{\infty} X(t) Q X^{-1}(s) f(z_h(s), s) ds \right| \right), \quad (C.2)$$

where $z_a$ is the unique solution of (C.1) and $z_h$ is the unique solution of (4.14).
Proof. From (4.14) and (C.1) it follows that

\[ z_a(t) - z_h(t) = -X(t) \int_{-\infty}^{T_1} PX^{-1}(s) f(z_h(s), s)\, ds \\
+ X(t) \int_{T_1}^{t} PX^{-1}(s) (f(z_a(s), s) - f(z_h(s), s))\, ds \\
- X(t) \int_{t}^{T_2} PX^{-1}(s) (f(z_a(s), s) - f(z_h(s), s))\, ds \\
+ X(t) \int_{T_2}^{\infty} PX^{-1}(s) f(z_h(s), s)\, ds. \]

Taking the supremum over \( t \) gives

\[
\|z_a - z_h\|_{L^\infty([T_1,T_2];\mathbb{R}^2)} \leq \sup_{t \in [T_1,T_2]} \left| \int_{-\infty}^{T_1} X(t) PX^{-1}(s) f(z_h(s), s)\, ds \right| \\
+ \left| \int_{T_1}^{t} X(t) PX^{-1}(s) (f(z_h(s), s) - f(z_a(s), s))\, ds \right| \\
+ \left| \int_{t}^{T_2} X(t) QX^{-1}(s) (f(z_h(s), s) - f(z_a(s), s))\, ds \right| \\
+ \sup_{t \in [T_1,T_2]} \left| \int_{T_2}^{\infty} X(t) QX^{-1}(s) f(z_h(s), s)\, ds \right|.
\]

Then apply the definition of exponential dichotomy to the second and third term:

\[
\|z_a - z_h\|_{L^\infty([T_1,T_2];\mathbb{R}^2)} \leq \sup_{t \geq T_1} \left| \int_{-\infty}^{T_1} X(t) PX^{-1}(s) f(z_h(s), s)\, ds \right| \\
+ \left| \int_{T_1}^{t} X(t) PX^{-1}(s) \left( \frac{k}{\alpha} + \frac{\ell}{\beta} \right) \|z_a - z_h\|_{L^\infty([T_1,T_2];\mathbb{R}^2)}\, ds \right| \\
+ \sup_{t \leq T_2} \left| \int_{T_2}^{\infty} X(t) QX^{-1}(s) f(z_h(s), s)\, ds \right|. 
\]

Equation (C.2) now follows from rearranging the terms and (4.29), since then the correct inequality is conserved. Namely, if \( \|\nabla f\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} > 1 \), rearrangement of terms would yield a lower bound for \( \|z_a - z_h\|_{L^\infty([T_1,T_2];\mathbb{R}^2)} \) instead of an upper bound. ■

Under the assumptions that

\[
\lim_{T_1 \to -\infty} \sup_{T_1 \geq t} \left| \int_{-\infty}^{T_1} X(t) PX^{-1}(s) f(z_h(s), s)\, ds \right| = 0
\]
and

\[
\lim_{T_2 \to \infty} \max_{t \leq T_2} \left| \int_{T_2}^{\infty} X(t) Q X^{-1}(s) f(z_h(s), s) \, ds \right| = 0,
\]

it follows that

\[
\lim_{T_1 \to -\infty} \lim_{T_2 \to \infty} \|z_a - z_h\|_{L^\infty([T_1, T_2]; \mathbb{R}^2)} = 0,
\]

which means that in the infinite time limit the approximate hyperbolic trajectory converges to the exact hyperbolic trajectory.

### C.2. Transformation via singular value decomposition

In this section we take another approach by transforming the system of equations describing the velocity field. By performing a so-called singular value decomposition the system can be diagonalised, and therefore, decoupled. This theory is founded on two papers by Mancho et al. [5, 34]. Many equations and concepts can be transformed directly by this transformation and the equations are simplified significantly. However, it assumes the existence of the transformation, and moreover, some concepts and characteristics can be obscured by the enormous simplifications.

We start again with a hyperbolic curve in space \( \mathbf{x}_{cp} \). One possible guess for \( \mathbf{x}_{cp} \) is a curve of instantaneous critical points (assuming its existence) of (4.2), i.e. \( \mathbf{v}(\mathbf{x}_{cp}(t), t) = 0, \ t \in \mathbb{R} \). Hyperbolicity is introduced by requiring that for each fixed \( t, \mathbf{x}_{cp}(t) \) is a hyperbolic critical point of the frozen time velocity field. The first guess is \( \mathbf{x}_{cp} \), and the velocity field \( \dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}, t) \) is localised around \( \mathbf{z} = \mathbf{x} - \mathbf{x}_{cp} \), which gives

\[
\dot{\mathbf{z}}(t) = A(t)\mathbf{z}(t) + \mathbf{f}(\mathbf{z}(t), t), \tag{C.3}
\]

with \( A(t) \) and \( \mathbf{f}(t) \) as defined in (4.11) and (4.12).

The field localised around the hyperbolic path (C.3) is now transformed to a diagonal system, using the singular value decomposition of the fundamental solution matrix \( X \) of the linearised equation around \( \mathbf{x}_{cp} \), equation (4.9). This singular value decomposition is discussed in appendix D. Then the transformation \( T \) following from the singular value decomposition is executed, introducing the new variable \( \mathbf{w} \) as

\[
\mathbf{w}(t) = T(t)\mathbf{z}(t),
\]
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to obtain the transformed system

\[ \dot{w}(t) = Dw(t) + h(w(t), t). \]  
(C.4)

where \( D \) is a constant, diagonal matrix. Therefore, (C.4) describes a decoupled system:

\[ \dot{w}_1(t) = d_1 w_1(t) + h_1(t) \]  
(C.4-i)
\[ \dot{w}_2(t) = d_2 w_2(t) + h_2(t) \]  
(C.4-ii)

where \( h(t) \) is written instead of \( h(w_1(t), w_2(t), t) \) for readability.

Associated with these equations, we have the integral equations:

\[ w_1(t) = \int_{-\infty}^{t} e^{d_1(t-s)} h_1(s) \, ds \]  
\[ w_2(t) = -\int_{t}^{\infty} e^{d_2(t-s)} h_2(s) \, ds, \]  
(C.5)

where \( d_1 < 0 \) and \( d_2 > 0 \) are the eigenvalues of \( D \), and, since \( D \) is diagonal, the diagonal elements of \( D \).

In order to establish an equivalence between the integral equations (C.5) and the differential equations (C.4), the following limits need to hold for the solution \( w \) of the differential equations.

\[ e^{d_1 t} w_1(t) \to 0 \quad \text{for} \quad t \to \infty, \]  
(C.6a)
\[ e^{d_2 t} w_2(t) \to 0 \quad \text{for} \quad t \to -\infty. \]  
(C.6b)

**Theorem C.4.** A solution \( w \) of the differential equation (C.4) that satisfies (C.6), is equivalent to a solution of the integral equation (C.5).

**Proof.** Suppose we have a solution of (C.5). If we calculate the right hand side of (C.4) by substituting (C.5), this yields:

\[ w_1(t) = d_1 \int_{-\infty}^{t} e^{d_1(t-s)} h_1(s) \, ds + h_1(t) \]  
(C.7-i)
\[ w_2(t) = -d_2 \int_{t}^{\infty} e^{d_2(t-s)} h_2(s) \, ds + h_2(t). \]  
(C.7-ii)
Differentiation with respect to \( t \) gives, using the Leibniz integral rule for integration:

\[
\dot{w}_1(t) = \int_{-\infty}^{t} \frac{\partial}{\partial t} \left( e^{d_1(t-s)h_1(s)} \right) \, ds + e^{d_1(t-s)h_1(s)} \bigg|_{s=t} \frac{\partial t}{\partial t} - h_1(t)
\]

\[
\dot{w}_2(t) = -\int_{t}^{\infty} \frac{\partial}{\partial t} \left( e^{d_2(t-s)h_2(s)} \right) \, ds + e^{d_2(t-s)h_2(s)} \bigg|_{s=t} \frac{\partial t}{\partial t} + h_2(t)
\]

A comparison of (C.8-i) and (C.8-ii) with (C.7-i) and (C.7-ii), respectively, shows that a solution of (C.5) is a solution of (C.4).

For this part of the proof it remains to show that this solution satisfies (C.6). First take the first equation of (C.5). Multiply this equation with \( e^{-d_1t} \):

\[
e^{-d_1t}w_1(t) = \int_{-\infty}^{t} e^{-d_1s}h_1(s) \, ds,
\]

such that in the limit \( t \to -\infty \) the right hand side goes to zero, because \( e^{-d_1s}h_1 \) is an integrable function and the integration boundaries become equal. Since the left hand side then goes to zero as well, \( w_1 \) needs to satisfy (C.6a). Similarly, multiply the second equation of (C.5) with \( e^{-d_2t} \). Then, if \( t \to \infty \) the right hand side goes to zero. The left hand side goes to zero for \( t \to \infty \) only if \( w_2 \) satisfies (C.6b). This completes the first part of the proof.

Now suppose we have a solution of the differential equation (C.4). The first equation (C.4-i), \( \dot{w}_1(t) = d_1w_1(t) + h_1(t) \) has the general solution

\[
w_1(t) = e^{d_1t} \left( \int_{0}^{t} e^{-d_1s}h_1(s) \, ds + w_1(0) \right), \tag{C.9}
\]

which can be rewritten as

\[
e^{-d_1t}w_1(t) = \int_{0}^{t} e^{-d_1s}h_1(s) \, ds + w_1(0).
\]

\[\tag{1}\]

The Leibniz integral rule gives a formula for differentiation of a definite integral of which the limits are functions of the differential variable [12]:

\[
\frac{\partial}{\partial z} \int_{b(z)}^{a(z)} f(x, z) \, dx = \int_{b(z)}^{a(z)} \frac{\partial f}{\partial z} \, dx + f(b(z), z) \frac{\partial b}{\partial z} - f(a(z), z) \frac{\partial a}{\partial z}.
\]
In the limit $t \to -\infty$, the left hand side goes to zero if $w_1$ obeys (C.6a). Then we have for the right hand side

$$w_1(0) = -\int_{-\infty}^{0} e^{-d_1s}h_1(s) \, ds.$$ 

Then (C.9) becomes

$$w_1(t) = e^{d_1t} \int_{-\infty}^{t} e^{-d_1s}h_1(s) \, ds,$$

which equals (C.7-i). Similarly, the general solution of $\dot{w}_2(t) = d_2w_2(t) + h_2(t)$ is

$$w_2(t) = e^{d_2t} \left( \int_{0}^{t} e^{-d_2s}h_2(s) \, ds + w_2(0) \right),$$

which can be rewritten as

$$e^{-d_2t}w_2(t) = \int_{0}^{t} e^{-d_2s}h_2(s) \, ds + w_2(0).$$

In the limit $t \to \infty$, the left hand side goes to zero if $w_2$ obeys (C.6b). Then we have for the right hand side:

$$w_2(0) = -\int_{0}^{\infty} e^{-d_2s}h_2(s) \, ds.$$ 

Then (C.11) becomes

$$w_2(t) = -e^{d_2t} \int_{t}^{\infty} e^{-d_2s}h_2(s) \, ds,$$

matching (C.7-ii).

Summarising, a solution $w$ of (C.4), satisfying (C.6), is a solution of (C.5). The converse was also shown to be true, so the proof of theorem C.4 is complete. ■

The results here and the valid equations in the original system from section 4.2 can be summarised in the following diagram.

\[ \begin{array}{c}
\dot{z} = Az + f \\
|PX^{-1}z| \to 0 \\
\downarrow T
\end{array} \quad \begin{array}{c}
z = X \int_{-\infty}^{t} PX^{-1}f \, ds \\
-X \int_{t}^{\infty} QX^{-1}f \, ds \\
\downarrow T
\end{array} \quad \begin{array}{c}
\dot{w} = Dw + h \\
w_{1,2} \to 0
\end{array} \quad \begin{array}{c}
w_1 = \int_{-\infty}^{t} e^{d_1(t-s)}h_1 \, ds, \\
w_2 = -\int_{t}^{\infty} e^{d_2(t-s)}h_2 \, ds
\end{array} \]
C.3. Equilateral triangle: exponential dichotomy

In this section we want to apply the theory as presented in chapter 4 to the equilateral configuration. The reason is that this theory requires a known velocity field, which is the case in the autonomous field corresponding to the flow in the corotating frame with the triangle. The velocity field and the location of the six hyperbolic critical points are known.

We consider the triangle with vortex positions in the complex plane of the corotating frame (the tildes have been omitted for convenience):

\[ z_1 = i, \quad z_2 = -\frac{1}{2} \sqrt{3} - \frac{1}{2} i, \quad z_3 = \frac{1}{2} \sqrt{3} - \frac{1}{2} i, \]

so that the radius of the circle equals 1. The velocity field of the passive tracers is given by (3.17). with \( z = x + iy \) and \( \kappa \) is taken to be 1, such that \( \Omega = 1/(2\pi) \). The \( x \)- and \( y \)-component of the velocity field \( \mathbf{v} = (v_x, v_y) \) are found by taking the real and imaginary part of the complex conjugate of (3.17), respectively:

\[
\begin{align*}
v_x &= \frac{y^7 + 3x^2y^5 - 3y^5 - 2y^4 + 3x^4y^3 - 6x^2y^3 + 6x^2y^2 + 3y^2 + x^6y - 3x^4y + y - 3x^2}{2\pi ((x^3 - 3xy^2)^2 + (1 + 3x^2y - y^3)^2)} \\
v_y &= \frac{-xy^6 - 3x^3y^4 + 3xy^4 + 2xy^3 - 3x^5y^2 + 6x^3y^2 - 6x^3y + 6xy - x^7 + 3x^5 - x}{2\pi ((x^3 - 3xy^2)^2 + (1 + 3x^2y - y^3)^2)}. 
\end{align*}
\]

From these velocity components the Jacobian \( \nabla \mathbf{v} \) can be determined. This gives for the linearised field around the critical point \( \mathbf{x}_{cp} \)

\[ \dot{\mathbf{\xi}} = \nabla \mathbf{v}(\mathbf{x}_{cp}(t), t)\mathbf{\xi} \]

equations of the form

\[ \dot{\mathbf{\xi}} = \begin{pmatrix} \mu & \mu_1 \\ -\mu & \mu_2 \end{pmatrix} \mathbf{\xi}. \quad (C.13) \]

In order to find the fundamental solution matrix of such a system, the differential equations

\[
\begin{align*}
\dot{x}(t) &= \mu x + \mu_1 y \\
\dot{y}(t) &= -\mu y + \mu_2 x
\end{align*}
\]

need to be solved. This gives as fundamental solution matrix \( X \)

\[
X(t) = \begin{pmatrix} \cosh(\lambda t) + \frac{\mu}{\lambda} \sinh(\lambda t) & \frac{\mu_1}{\lambda} \sinh(\lambda t) \\ \frac{\mu_2}{\lambda} \sinh(\lambda t) & \cosh(\lambda t) - \frac{\mu}{\lambda} \sinh(\lambda t) \end{pmatrix},
\]

with

\[ \lambda \equiv \sqrt{\mu^2 + \mu_1 \mu_2}. \]
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Note that $\lambda_{1,2} = \pm \sqrt{\mu^2 + \mu_1 \mu_2}$ are the eigenvalues of the matrix (C.13):

$$\det \begin{pmatrix} \mu - \lambda & -\mu_1 \\ \mu_2 & -\mu - \lambda \end{pmatrix} = \lambda^2 - \mu^2 - \mu_1 \mu_2 = 0,$$

$$\lambda_{1,2} = \pm \sqrt{\mu^2 + \mu_1 \mu_2}.$$

These eigenvalues are real in the case of an hyperbolic critical point. So $\lambda$ is a positive parameter, which proves to be crucial.

The matrix $X$ satisfies $\dot{X} = AX$, because it is the fundamental solution matrix. The inverse $X^{-1}$ is given by

$$X^{-1}(t) = \begin{pmatrix} \cosh(\lambda t) - \frac{\mu_1}{\lambda} \sinh(\lambda t) & -\frac{\mu_1}{\lambda} \sinh(\lambda t) \\ -\frac{\mu_2}{\lambda} \sinh(\lambda t) & \cosh(\lambda t) + \frac{\mu_1}{\lambda} \sinh(\lambda t) \end{pmatrix},$$

We have to find a projection $P$ such that the requirements (4.4-i) and (4.4-ii) are met. It turns out that the matrices

$$P = \frac{1}{2\lambda} \begin{pmatrix} \lambda - \mu & -\mu_1 \\ -\mu_2 & \lambda + \mu \end{pmatrix},$$

$$Q = \frac{1}{2\lambda} \begin{pmatrix} \lambda + \mu & \mu_1 \\ \mu_2 & \lambda - \mu \end{pmatrix},$$

yield

$$X(t)PX^{-1}(s) = \frac{e^{\lambda(s-t)}}{2\lambda} \begin{pmatrix} \lambda - \mu & -\mu_1 \\ -\mu_2 & \lambda + \mu \end{pmatrix},$$

$$X(t)QX^{-1}(s) = \frac{e^{\lambda(t-s)}}{2\lambda} \begin{pmatrix} \lambda + \mu & \mu_1 \\ \mu_2 & \lambda - \mu \end{pmatrix}.$$

It can easily be verified that $P^2 = P$ and $Q^2 = Q$. Consequently, requirement (4.4-i) for exponential dichotomy is satisfied with $\alpha = \lambda$ and $k = \max\{|\lambda - \mu| + |\mu_2|, |\lambda + \mu| + |\mu_1|\}$; requirement (4.4-ii) is met with $\beta = \lambda$ and $\ell = k$. We see that the eigenvalues of the linearised matrix (C.13) around the hyperbolic points (which differ only in sign in incompressible flows) determine the characteristic time scale of separation from and attraction to the hyperbolic points of the trajectories.

For the three hyperbolic critical points around the origin, we find $\alpha = \beta = 0.253$; for the hyperbolic on the outer circle we have $\alpha = \beta = 0.034$. Note that it follows from symmetry that these sets of points need to have the same exponential dichotomy parameter.
D. Singular value decomposition

D.1. Theory

The singular value decomposition can be considered as a generalisation of the spectral theorem to arbitrary matrices, which are not necessarily square. The spectral theorem states that normal matrices can be unitarily diagonalised using a basis of eigenvectors. A normal matrix is a matrix \(A\) with \(A^*A = AA^*\), where \(A^*\) is the conjugate transpose of \(A\). The eigen-decomposition then reads

\[
A = U \Lambda U^*,
\]

(D.1)

where \(U\) is a unitary matrix and \(\Lambda\) is a diagonal matrix with the eigenvalues of \(A\) as its elements.

A nonnegative real number \(\sigma\) is a singular value of a matrix \(M \in \mathbb{C}^{m \times n}\) if and only if there exists vectors \(u \in \mathbb{C}^m\), with \(\|u\|_2 = 1\) and \(v \in \mathbb{C}^n\), with \(\|v\|_2 = 1\), such that

\[
Mv = \sigma u
\]

\[
M^*u = \sigma v,
\]

(D.2)

where \(M^*\) denotes the conjugate transpose of \(M\). The vectors \(u\) and \(v\) are called left-singular and right-singular vectors for \(\sigma\), respectively\(^1\). From (D.2) it follows that

\[
M^*Mv = M^*\sigma u = \sigma M^*u = \sigma^2 v
\]

\[
MM^*u = M\sigma v = \sigma Mv = \sigma^2 u,
\]

which shows that the squares of the singular values are the eigenvalues of \(MM^*\) and \(M^*M\).

The singular value decomposition of \(M\) is a factorization of the form

\[
M = U \Sigma V^*,
\]

\(^1\)This nomenclature comes from the fact that in the first equation of (D.2) \(M\) is multiplied from the right by \(v\) and the second equation can be rewritten as \(u^*M = \sigma v^*\), which is left multiplication of \(M\).
where $U$ is an $m \times m$ unitary\(^2\) matrix, the columns of which consist of the left-singular vectors for the corresponding singular values. $\Sigma$ is a diagonal $m \times n$ matrix with the singular values of $M$ on the diagonal, and $V^*$ is the conjugate transpose of $V$, which is an $n \times n$ unitary matrix with the right-singular vectors for the corresponding singular values as its columns.

In the special case that the matrix $M$ is Hermitian (or symmetric in the real case), i.e., $M^* = M$, which is positive semi-definite, i.e., a matrix with real nonnegative eigenvalues, then the singular values are equal to the eigenvalues and the singular vectors coincide with the eigenvectors of $M$. The singular value decomposition is then equal to the eigen decomposition (D.1).

### D.2. Application to velocity field

We are considering a velocity field of the form:

$$\dot{z}(t) = F(t)z(t) + f(t) \quad t \in [t_0, t_L]. \quad (D.3)$$

The fundamental solution matrix $X$ of (D.3), which satisfies

$$\dot{X}(t) = F(t)X(t), \quad X(0) = I, \quad (D.4)$$

is represented in a singular value decomposition:

$$X(t) = B(t) \exp(\Sigma(t)) R^T(t), \quad (D.5)$$

where $B$ and $R$ are orthogonal (unitary) matrices, i.e., $B^T(t) = B^{-1}(t)$ and $R^T(t) = R^{-1}(t)$. $\Sigma$ is a diagonal matrix with $\Sigma(0) = 0$, such that $\exp(\Sigma(0)) = I$, $I$ being the identity matrix. The matrix $\exp(\Sigma)$ has the singular values of the fundamental solution matrix $X$ on its diagonal. In order to obtain a decoupled set of differential equations, we then apply the transformation $w = Tz$, where $T$ is derived from the singular value decomposition (D.5) of $X$. The transformed differential equation then reads

$$\dot{w}(t) = T^\prime(t)z(t) + T(t)\dot{z}(t)$$

$$= T^\prime(t)z(t) + T(t)\left(A(t)z(t) + f(z(t), t)\right)$$

$$= Dw(t) + h(t), \quad (D.6)$$

\(^2\)A matrix $U$ is unitary if $U^H = U^{-1}$, where $U^H$ is the conjugate transpose of $U$ and $U^{-1}$ is the matrix inverse. The superscript $^H$ comes from Hermitian, a name given to a matrix $A$ if $A^H = A$. For real matrices, unitary is equivalent to orthogonal, i.e., $U^T = U^{-1}$, with $U^T$ the matrix transpose of $U$.\[\]
which means that
\[
\dot{T}(t) = DT(t) - T(t)A(t)
\]
\[
Tf(t) = h(t).
\]

Here \(D\) is a time independent diagonal matrix and it can be shown that the diagonal elements of \(D\) are the finite time Lyapunov exponents \([5]\), see section 4.1.2. The diagonalisation means that the system (D.3) has been decoupled. The transformed differential equation (D.6) has fundamental solution matrix \(Y\), which is governed by:
\[
\dot{Y}(t, t_0) = DY(t, t_0),
\]
with initial condition \(Y(t_0, t_0) = I\) and is given by
\[
Y_{ij}(t, t_0) = \begin{cases} 
eq 0, & \text{for } i \neq j. 
\end{cases}
\]

If \(t_0 \to -\infty\) and \(t_L \to \infty\), then the solution of the transformed differential equation (D.6) is given by \([48]\):
\[
w_i(t) = \begin{cases} 
\int_{-\infty}^{t} Y_{ii}(t, \tau)h_i(\tau) d\tau, & \text{for } d_i < 0 \\
- \int_{t}^{\infty} Y_{ii}(t, \tau)h_i(\tau) d\tau, & \text{for } d_i > 0.
\end{cases}
\]

This formulation is the analog of (4.14) in the transformed coordinates \([7]\). Note that, like (4.14) for the original problem, this formula for \(w\) is implicit if \(f = f(z(t), t)\) for all \(t\), because then \(g = g(w(t), t)\); therefore, it is suitable for iteration rather than direct solving generally.

It can readily be shown that (D.8) is a solution of transformed system (D.6). To that end, take the time derivative of (D.8), for instance for \(d_i < 0\), using the Leibniz integral rule (see section C.2):
\[
\dot{w}_i(t) = \frac{d}{dt} \int_{-\infty}^{t} Y_{ii}(t, \tau)h_i(\tau) d\tau
\]
\[
= \int_{-\infty}^{t} \frac{d}{dt} (Y_{ii}(t, \tau)h_i(\tau)) d\tau + Y_{ii}(t, t) \frac{dt}{dt} h_i(t)
\]
\[
= \int_{-\infty}^{t} d_i Y_{ii}(t, \tau)h_i(\tau) d\tau + h_i(t),
\]
so \( \dot{w} \) satisfies

\[
\dot{w}_i(t) = d_i w_i(t) h_i(t).
\]

A similar argument shows that this also holds for \( d_i > 0 \), so that

\[
\dot{w}(t) = D w(t) + h(t).
\]

There is no exponential time dependence in the expression for \( w \) in (D.8), since the exponential part disappears because of the integration boundaries.

There is an interesting link between the existence of a singular value decomposition and an exponential dichotomy: it has been shown by Ide \textit{et al.} that exponential dichotomy is a frame invariant property \cite{15}. In that paper an equivalent definition of hyperbolicity is given, which represents the fundamental solution matrix in the form of a singular value decomposition. This equivalence implies that in order to prove hyperbolicity of a trajectory, it suffices to consider the decoupled equations, which are obtained by transforming the original equations using the singular value decomposition. We then have hyperbolicity if \( d_1 < 0 \) and \( d_2 > 0 \).

\section*{D.3. Calculation of transformation matrices}

Information on \( D \) and \( T \) can be gained from the fundamental solution matrices \( X \) and \( Y \). A relation between \( T \) and \( X \) and \( Y \) can be found by considering the initial conditions \( w(t_0) = T(t_0)z(t_0) \) so that the solutions of the homogeneous equations are

\[
z(t) = X(t)z(t_0) \quad \quad \quad w(t) = Y(t, t_0)w(t_0),
\]

with \( w(t) = T(t)z \) for all \( t \). This gives

\[
T(t)X(t)z(t_0) = T(t)z(t)
\]

\[
= \dot{w}(t)
\]

\[
= Y(t, t_0)w(t_0)
\]

\[
= Y(t, t_0)T(t_0)z(t_0).
\]

This can be written as

\[
\left( Y(t, t_0)T(t_0) - T(t)X(t) \right) z(t_0),
\]

for arbitrary \( z(t_0) \), from which it follows that

\[
T(t) = Y(t, t_0)T(t_0)X^{-1}(t). \quad (D.9)
\]
D.3. Calculation of transformation matrices

Consequently, $T(t_0)$ needs to be specified. Evaluation of (D.9) at $t = t_L$ gives after rearrangement of terms:

$$X(t_L) = T^{-1}(t_L)Y(t_L, t_0)T(t_0). \tag{D.10}$$

We then choose $T(t_0)$ in such a way that the $X$ and $Y$ system are aligned at $t = t_L$ for convenience.$^3$ This leads to the choices $T(t_0) = R^T(t_L)$ and $T(t_L) = B^T(t_L)$. This gives for (D.10)

$$X(t_L) = B(t_L)Y(t_L, t_0)R^T(t_L)$$

$$= B(t_L)\exp((t_L - t_0)D)R^T(t_L). \tag{D.11}$$

Evaluation of (D.5) at $t = t_L$ yields

$$X(t_L) = B(t_L)\exp(\Sigma(t_L))R^T(t_L), \tag{D.12}$$

so that a comparison of (D.11) and (D.12) shows that

$$D = \frac{1}{t_L - t_0}\Sigma(t_L). \tag{D.13}$$

With (D.13) we find for the transformation matrix $T$ in (D.9), substituting (D.7), $T(t_0) = R^T(t_L)$ and (D.5):

$$T(t) = e^{D(t-t_0)}R^T(t_L)R(t)e^{-\Sigma(t)}B^T(t).$$

The matrices $B$, $R$ and $\Sigma$ can be obtained as follows. Substituting the singular value decomposition (D.5) for $X$ into (D.4), we have that$^4$

$$\dot{B}(t)\exp(\Sigma(t))R^T(t) + B(t)\dot{\Sigma}(t)\exp(\Sigma(t))R^T(t) + B(t)\exp(\Sigma(t))\dot{R}^T(t) = F(t)B(t)\exp(\Sigma(t))R^T(t).$$

Multiplying the terms in this equation from the left with $B^T(t)$ and from the right with $R(t)\exp(-\Sigma(t))$ then yields

$$B(t)^T\dot{B}(t) + \dot{\Sigma}(t) + \exp(\Sigma(t))\dot{R}(t)^T R(t) \exp(-\Sigma(t)) = H(t), \tag{D.14}$$

with $H \equiv B^T F B$.

$B$ and $R$ are orthogonal matrices and therefore, they can be parametrised by angles $\varphi$ and $\vartheta$ in the following way:

$$B = \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix}, \quad R = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}. \tag{D.15}$$

---

$^3$Aligning the system at $t = t_0$ is difficult since $X(t_0) = I$, which has no natural alignment [5].

$^4$Differentiation is carried out element-wise in the matrices, so differentiation and transposition can be interchanged.
D. Singular value decomposition

Substituting (D.15) into (D.14) then gives

\[
\begin{pmatrix}
0 & \dot{\vartheta} \\
-\dot{\vartheta} & 0
\end{pmatrix}
+ \begin{pmatrix}
\dot{\sigma_1} & 0 \\
0 & \dot{\sigma_2}
\end{pmatrix}
+ \begin{pmatrix}
e^{\sigma_1} & 0 \\
0 & e^{\sigma_2}
\end{pmatrix}
\begin{pmatrix}
0 & -\dot{\varphi} \\
\dot{\varphi} & 0
\end{pmatrix}
\begin{pmatrix}
e^{-\sigma_1} & 0 \\
0 & e^{-\sigma_2}
\end{pmatrix}
= \begin{pmatrix}
\dot{\vartheta} \\
-\dot{\vartheta} + \dot{\varphi} e^{\sigma_1 - \sigma_2}
\end{pmatrix}
\begin{pmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{pmatrix}.
\]  

(D.16)

Solving for \(\dot{\sigma_1}, \dot{\sigma_2}, \dot{\vartheta}\) and \(\dot{\varphi}\) gives

\[
\begin{aligned}
\dot{\sigma_1} &= H_{11} \\
\dot{\sigma_2} &= H_{22} \\
\dot{\vartheta} &= \frac{1}{2}(H_{12} - H_{21}) + \frac{1}{2}(H_{12} + H_{21}) \coth(\sigma_2 - \sigma_1) \\
\dot{\varphi} &= \frac{1}{2}(H_{12} + H_{21}) \text{cosech}(\sigma_2 - \sigma_1),
\end{aligned}
\]  

(D.17)

where \(H_{ij}\) are the matrix elements of \(H\) and \(\sigma_i\) are the diagonal elements of \(\Sigma\), i.e., \(\sigma_i = \Sigma_{ii}\) for \(i = 1\) and \(i = 2\). Since \(B\) is unknown beforehand, \(H_{ij}\) must be expressed in terms of \(\vartheta\):

\[
H = B^T F B =
\begin{pmatrix}
F_{11} \cos^2 \vartheta + F_{22} \sin^2 \vartheta - \tilde{F} \sin(2\vartheta) & F_{12} \cos^2 \vartheta - F_{21} \sin^2 \vartheta + \hat{F} \sin(2\vartheta) \\
F_{21} \cos^2 \vartheta - F_{12} \sin^2 \vartheta + \tilde{F} \sin(2\vartheta) & F_{22} \cos^2 \vartheta + F_{11} \sin^2 \vartheta + \hat{F} \sin(2\vartheta)
\end{pmatrix},
\]  

(D.18)

with

\[
\tilde{F} = \frac{1}{2}(F_{21} + F_{12}),
\]

\[
\hat{F} = \frac{1}{2}(F_{11} - F_{22}).
\]

Therefore, we can write

\[
\begin{aligned}
H_{12} - H_{21} &= F_{12} - F_{21} \\
H_{12} + H_{21} &= (F_{12} + F_{21}) \cos(2\vartheta) + (F_{11} - F_{22}) \sin(2\vartheta),
\end{aligned}
\]

(D.19)

so that the combination of (D.17), (D.18) and (D.19) yields a system of four differential equations for the four unknowns \(\sigma_1, \sigma_2, \dot{\vartheta}\) and \(\dot{\varphi}\).
Example flow

In order to clarify the theory from chapter 4, which is based on the theory as discussed in [7], we apply it to a simple set of analytically solvable differential equations in this chapter.

E.1. Problem definition

Consider the nonautonomous velocity field \( \dot{x}(t) = v(x(t), t) \) for all \( t \in \mathbb{R} \), given by the following set of equations:

\[
\begin{cases}
\dot{x}(t) = \frac{1}{2} y(t) - \frac{1}{2} \sin t, \\
\dot{y}(t) = \frac{1}{2} x(t) - \frac{3}{2} \cos t.
\end{cases}
\]  

(E.1)

The analytical solution of the system (E.1) is

\[
\begin{align*}
  x(t) &= (x_0 - 1) \cosh \left( \frac{t}{2} \right) + y_0 \sinh \left( \frac{t}{2} \right) + \cos t, \\
y(t) &= (x_0 - 1) \sinh \left( \frac{t}{2} \right) + y_0 \cosh \left( \frac{t}{2} \right) - \sin t,
\end{align*}
\]  

(E.2)

where \( x_0 = x(0) \) and \( y_0 = y(0) \) are the initial conditions. Note that \( \nabla \cdot v = 0 \), so this flow is incompressible. Furthermore, \( \omega_z = \partial_x v - \partial_y u = \frac{1}{2} - \frac{1}{2} = 0 \), so the flow is irrotational as well. A few trajectories are shown in figure E.1. The trajectories decay exponentially to the dashed lines \( y=x \) and \( y=-x \).

E.2. Exponential dichotomy

We aim to find hyperbolic trajectories of the system (E.1). To that end, we the concept of an exponential dichotomy, as discussed in section 4.1.3. A trajectory \( x_h \) is said to be hyperbolic if the fundamental solution matrix of linearised systems of equations

\[
\dot{\xi} = \nabla v(x_h(t), t) \xi
\]  

(E.3)
E. Example flow

Figure E.1: A few trajectories for $t \in (-2\pi, 2\pi)$ for various initial conditions: $(x_0, y_0) = (1, -1), (-1, -1)(3, -1), (1, 1)$. This collection does not cover the case $x_0 - 1 = \pm y_0$, which yields special trajectories as is discussed in section E.3. The trajectories decay exponentially to the dashed lines $y = x$ and $y = -x$, which are no hyperbolic trajectories, since $y = x$ and $y = -x$ are no solutions of the differential equations (E.1) and are therefore no particle trajectories. The arrows indicate forward time evolution.

has an exponential dichotomy, i.e., if there are a projection $P$ and positive constants $K, L, \alpha$ and $\beta$, such that

$$
\| X(t)PX^{-1}(s) \|_2 \leq k e^{-\alpha(t-s)} , \quad s \leq t \leq \infty \quad \text{(E.4-i)}
$$

$$
\| X(t)QX^{-1}(s) \|_2 \leq \ell e^{-\beta(s-t)} , \quad s \geq t \geq -\infty, \quad \text{(E.4-ii)}
$$

where $Q = I - P$ and $X$ is the fundamental solution matrix of the system (E.3).

Note that in the case of a velocity field that is linear in $x$, $\nabla v$ in (4.1.3) is constant so that the linearisation is the same for any trajectory. Consequently, if there exists one trajectory that is hyperbolic, then so are all trajectories. This makes that the chosen velocity field has special characteristics, which are absent in a velocity field generally. However, since this field is analytically solvable (which is not guaranteed in general), a linear example problem has been chosen in order to clarify the theory.

The curve of instantaneous critical points $x_{cp} = (x_{cp}, y_{cp})$ is found by solving
\( \mathbf{v}(x_{cp}(t), t) = 0 \) for all \( t \), so

\[
\begin{align*}
x_{cp}(t) &= 3 \cos t, \\
y_{cp}(t) &= \sin t.
\end{align*}
\] (E.5)

Now define \( \mathbf{z} = x_h - x_{cp} \). Following [7], \( \mathbf{z} \) is determined by

\[
\dot{\mathbf{z}}(t) = A(t)\mathbf{z}(t) + \mathbf{f}(\mathbf{z}(t), t),
\] (E.6)

with

\[
A(t) = \nabla \mathbf{v}(x_{cp}(t), t)
\]

and

\[
\mathbf{f}(\mathbf{z}, t) = \mathbf{v}(\mathbf{z} + x_{cp}(t), t) - \nabla \mathbf{v}(x_{cp}(t), t)\mathbf{z} - \dot{x}_{cp}(t).
\]

The goal is now to find a trajectory \( \mathbf{z}_h \) of (E.6), so that we can find a hyperbolic trajectory \( x_h \) of (E.1) using \( x_h = \mathbf{z}_h + x_{cp} \). Equation (E.6) leads to the following set of equations:

\[
\begin{pmatrix}
\dot{z}_1(t) \\
\dot{z}_2(t)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix} \begin{pmatrix}
z_1(t) \\
z_2(t)
\end{pmatrix} + \begin{pmatrix}
\frac{1}{2}z_2(t) \\
\frac{1}{2}z_1(t)
\end{pmatrix} - \begin{pmatrix}
0 & \frac{1}{2} \\
\frac{1}{2} & 0
\end{pmatrix} \begin{pmatrix}
z_1(t) \\
z_2(t)
\end{pmatrix} - \begin{pmatrix}
-3 \sin t \\
\cos t
\end{pmatrix}
\]

which simplifies to

\[
\begin{cases}
\dot{z}_1(t) = \frac{1}{2}z_2(t) + 3 \sin t, \\
\dot{z}_2(t) = \frac{1}{2}z_1(t) - \cos t.
\end{cases}
\] (E.7)

The solution of (E.7) is, with \( z_{10} = z_1(0) \) and \( z_{20} = z_2(0) \).

\[
\begin{align*}
z_1(t) &= -2 \cos t + (2 + z_{10}) \cosh \left( \frac{t}{2} \right) + z_{20} \sinh \left( \frac{t}{2} \right), \\
z_2(t) &= -2 \sin t + (2 + z_{10}) \sinh \left( \frac{t}{2} \right) + z_{20} \cosh \left( \frac{t}{2} \right).
\end{align*}
\] (E.8)

The goal is now to prove that (E.8) is a hyperbolic trajectory. Therefore, we need to show that the linearised equation

\[
\dot{\mathbf{\xi}} = \nabla \mathbf{v}(x_{cp}(t), t)\mathbf{\xi}
\] (E.9)

has an exponential dichotomy. The autonomous equation \( \dot{\mathbf{x}} = A_0\mathbf{x} \) has an exponential dichotomy if and only if no eigenvalue of the constant matrix \( A_0 \) has zero real part [32]. The matrix \( \nabla \mathbf{v}(x_{cp}(t), t) \) is taken from (E.7):

\[
\nabla \mathbf{v}(x_{cp}(t), t) = \begin{pmatrix}
0 & \frac{1}{2} \\
\frac{1}{2} & 0
\end{pmatrix}.
\]
The eigenvalues are $\lambda_{1,2} = \pm \frac{1}{2}$, so we know that (E.9) has an exponential dichotomy. Since we need the associated matrices $P$ and $X$ for later purposes, we calculate them nonetheless.

The matrix $X$ is the fundamental solution matrix of (E.9), so $\xi(t) = X(t)\xi_0$ is the solution passing through $\xi_0 = (\xi_{01}, \xi_{02})$ at $t = 0$, and $X(0)$ is the identity matrix $I$. Substituting this into (E.9) leads to the following set of equations:

$$
\begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix}
\begin{pmatrix}
\xi_{01} \\
\xi_{02}
\end{pmatrix} = 
\begin{pmatrix}
0 & \frac{1}{2} \\
\frac{1}{2} & 0
\end{pmatrix}
\begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix}
\begin{pmatrix}
\xi_{01} \\
\xi_{02}
\end{pmatrix},
$$

yielding

$$
\begin{cases}
\dot{X}_{11} = \frac{1}{2}X_{21}, \\
\dot{X}_{12} = \frac{1}{2}X_{22}, \\
\dot{X}_{21} = \frac{1}{2}X_{11}, \\
\dot{X}_{22} = \frac{1}{2}X_{12}.
\end{cases}
$$

The corresponding solution, obeying the initial condition $X(0) = I$, is

$$
X(t) = \begin{pmatrix}
\cosh \left( \frac{t}{2} \right) & \sinh \left( \frac{t}{2} \right) \\
\sinh \left( \frac{t}{2} \right) & \cosh \left( \frac{t}{2} \right)
\end{pmatrix}.
$$

(E.10)

This matrix also shows that all trajectories are hyperbolic, since the linearised equations (E.9) have exponentially growing and decaying solutions [15].

The inverse matrix $X^{-1}$ is

$$
X^{-1}(t) = \begin{pmatrix}
\cosh \left( \frac{t}{2} \right) & -\sinh \left( \frac{t}{2} \right) \\
-\sinh \left( \frac{t}{2} \right) & \cosh \left( \frac{t}{2} \right)
\end{pmatrix}.
$$

Now we try to prove the exponential dichotomy by checking (E.4-i) and (E.4-ii). First consider $t \geq s$. The norm must be taken of the following product matrices:

$$
A = \begin{pmatrix}
\cosh \left( \frac{t}{2} \right) & \sinh \left( \frac{t}{2} \right) \\
\sinh \left( \frac{t}{2} \right) & \cosh \left( \frac{t}{2} \right)
\end{pmatrix}
\begin{pmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{pmatrix}
\begin{pmatrix}
\cosh \left( \frac{s}{2} \right) & -\sinh \left( \frac{s}{2} \right) \\
-\sinh \left( \frac{s}{2} \right) & \cosh \left( \frac{s}{2} \right)
\end{pmatrix}.
$$

The definition of the exponential dichotomy uses the spectral norm $\| \cdot \|_2$. However, for the sake of convenience we take the maximum absolute row sum as matrix norm, $\|A\|_\infty = \max_j \sum_i |A_{ij}|$. We can then use the equivalence of norms:

$$
\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty
$$

(E.11)
for \( A \in \mathbb{R}^{m \times n} \) [49].

\[
\sum_j |A_{1j}| = \left| \sinh \left( \frac{s}{2} \right) \left( P_{11} \cosh \left( \frac{t}{2} \right) + P_{21} \sinh \left( \frac{t}{2} \right) \right) \right|
\]
\[
- \left| \cosh \left( \frac{s}{2} \right) \left( P_{12} \cosh \left( \frac{t}{2} \right) + P_{22} \sinh \left( \frac{t}{2} \right) \right) \right|
\]
\[
+ \left| \cosh \left( \frac{s}{2} \right) \left( P_{11} \cosh \left( \frac{t}{2} \right) + P_{21} \sinh \left( \frac{t}{2} \right) \right) \right|
\]
\[
- \sinh \left( \frac{s}{2} \right) \left( P_{12} \cosh \left( \frac{t}{2} \right) + P_{22} \sinh \left( \frac{t}{2} \right) \right) \right| \tag{E.12-i}
\]

\[
\sum_j |A_{2j}| = \left| \sinh \left( \frac{s}{2} \right) \left( P_{21} \cosh \left( \frac{t}{2} \right) + P_{11} \sinh \left( \frac{t}{2} \right) \right) \right|
\]
\[
- \cosh \left( \frac{s}{2} \right) \left( P_{22} \cosh \left( \frac{t}{2} \right) + P_{12} \sinh \left( \frac{t}{2} \right) \right) \right| 
\]
\[
+ \cosh \left( \frac{s}{2} \right) \left( P_{21} \cosh \left( \frac{t}{2} \right) + P_{11} \sinh \left( \frac{t}{2} \right) \right) \right|
\]
\[
- \sinh \left( \frac{s}{2} \right) \left( P_{22} \cosh \left( \frac{t}{2} \right) + P_{12} \sinh \left( \frac{t}{2} \right) \right) \right| \tag{E.12-ii}
\]

In order to eliminate the terms with \( e^{t-s} \), we need \( P_{11} + P_{22} = -(P_{12} + P_{21}) \). Furthermore, \( P \) has to satisfy \( P^2 = P \). A possible solution for the projection matrix is

\[
P = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{E.13}
\]

which then gives

\[
\sum_j |A_{1j}| = \sum_j |A_{2j}| = e^{\frac{1}{2}(s-t)}.
\]

Therefore, this fulfills the first requirement (E.4-i) for an exponential dichotomy with \( \alpha = \frac{1}{2} \) and \( \frac{1}{2} \leq K \leq 2 \) according to (E.11).

\( P \) maps a vector \((x_1, x_2)\) onto \((\frac{1}{2} x_1 - \frac{1}{2} x_2, -\frac{1}{2} x_1 + \frac{1}{2} x_2)\), so \( P \) maps points onto the line \( y = -x \). Generally, there is only one indeterminacy in the choice of the projection \( P \). Namely, if \( P' \) is a projection with the same range as \( P \), then (E.9) possesses an exponential dichotomy with projection \( P' \) [32]. Since the range of \( P \) is the line \( y = x \), \( P' \) must be a scalar multiple of \( P \). However, since the \( P'^2 = P' \), the only possibility is \( P' = P \) and therefore, \( P \) is unique. The matrix \( Q \) is given by

\[
Q = I - P = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]
Q maps a vector \((x_1, x_2)\) onto \((\frac{1}{2} x_1 + \frac{1}{2} x_2, \frac{1}{2} x_1 + \frac{1}{2} x_2)\), so \(Q\) maps points onto the line \(y = x\). The lines \(y = -x\) and \(y = x\) are the lines that the (hyperbolic) trajectories decay to exponentially in time, as is shown later. These lines cannot be the special hyperbolic trajectories, because \(y = x\) and \(y = -x\) are no solutions of the differential equations (E.1), and are therefore no trajectories. Consequently, they cannot be the invariant manifolds either.

The second requirement (E.4-ii) for exponential dichotomy is met as well, since the product of matrices gives

\[
B = X(t)QX^{-1}(s) = \frac{1}{2} e^{\frac{1}{2}(t-s)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
\]

so \(\|B\|_\infty = e^{\frac{1}{2}(t-s)}\) and the second requirement (E.4-ii) is met with \(\beta = \frac{1}{2}\) and \(\frac{1}{2} \leq L \leq 2\).

### E.3. Special hyperbolic trajectories

Now that we know that \(z_h\) in (E.8) is a hyperbolic trajectory of (E.7), we can find a hyperbolic trajectory \(x_h\) of (E.1) from \(z_h\) and the curve of instantaneous critical points \(x_{cp}\): \(x_h = z_h + x_{cp}\) [7].

\[
\begin{align*}
x_h(t) &= \cos t + (2 + z_{10}) \cosh \left(\frac{t}{2}\right) + z_{20} \sinh \left(\frac{t}{2}\right), \\
y_h(t) &= -\sin t + (2 + z_{10}) \sinh \left(\frac{t}{2}\right) + z_{20} \cosh \left(\frac{t}{2}\right).
\end{align*}
\]

(E.14)

This solution (E.14) is the trajectory (E.2) found from (E.1) directly, with \(z_{10} = x_0 - 3\) and \(z_{20} = y_0\).

Equation (E.14) shows that in general, the trajectories approach the lines \(y = x\) and \(y = -x\) for \(t \to \infty\) and \(t \to -\infty\), see figure E.1. Neither of these lines is a hyperbolic trajectory, because \(y = x\) and \(y = -x\) do not solve the differential equations (E.1) and are therefore no particle trajectories. They can be no stable or unstable invariant manifolds either, since trajectories cross both lines.

These lines are not approached in the special case that \(x_0 - 1 = \pm y_0\), because then the remaining exponential terms cancel. For \(x_0 - 1 = y_0\), the terms with \(e^{-t/2}\) in (E.2) eliminate. Therefore, the unit circle, i.e. \((x, y) = (\cos t, -\sin t)\), is a special hyperbolic trajectory, as the other contributions become exponentially small for \(t \to -\infty\). Note that this circle is a particular solution of the differential equations (E.1). The special hyperbolic trajectory is shown in the left graph of figure E.2. For \(x_0 - 1 = -y_0\), the terms with \(e^{t/2}\) in equation (E.2) cancel. The special hyperbolic trajectory is the
E.3. Special hyperbolic trajectories

Figure E.2: Left figure: the trajectories for $t \in (-5\pi, 0)$ for various initial conditions: $x_0 - 1 = y_0$ with $x_0 \in \{-0.5, 3, 10\}$. Right figure: $t \in (0, 5\pi)$ and $x_0 - 1 = -y_0$ with $x_0 \in \{-0.5, 3, 10\}$. For these three different initial values, the trajectories decay to the unit circle for $t \to -\infty$ (left) and $t \to \infty$ (right). This unit circle is a special hyperbolic trajectory, because the terms with $e^{-t/2}$ ($e^{t/2}$) in equation (E.2) eliminate and the other contribution becomes exponentially small for $t \to -\infty$ ($t \to \infty$).

Figure E.3 and show that the distance between the curve of instantaneous critical points $x_{cp}$ (the dashed line), as given by equation (E.5), and the special hyperbolic trajectory need not be small for the computation of the special hyperbolic trajectory, starting from $x_{cp}$.

As we have seen in chapter 4, the integral solution

$$z(t) = X(t) \int_{-\infty}^{t} PX^{-1}(s)f(z(s), s)ds - X(t) \int_{t}^{\infty} QX^{-1}(s)f(z(s), s)ds \quad (E.15)$$

selects special trajectories from the solutions of the differential equation (E.6).

In this case, we find for $f(z(t), t)$

$$f(z(t), t) = \mathbf{v}(z(t)) + x_{cp}(t), t - \nabla \mathbf{v}(x_{cp}(t), t)z(t) - \dot{x}_{cp}(t)$$

$$= \begin{pmatrix} \frac{1}{2} z_1 \\ \frac{1}{2} z_2 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} -3 \sin t \\ \cos t \end{pmatrix}$$

$$= \begin{pmatrix} 3 \sin t \\ -\cos t \end{pmatrix}.$$  \quad (E.16)
Figure E.3: The trajectory with $x_0 - 1 = y_0 = 1$ for $t \in (-5\pi, 0)$ (top) and $x_0 = -y_0 = 1$ and $t \in (-\frac{\pi}{4}, 5\pi)$ (bottom). The special hyperbolic trajectory is the circle as the other contributions become exponentially small for $t \to -\infty$ (top) and $t \to \infty$ (bottom). Because of the initial conditions, the terms with $e^{-t/2}$ (top) and $e^{t/2}$ (bottom) in (E.2) eliminate. The curve of instantaneous critical points $x_{cp}$ is given by the dashed line. Its distance to the special hyperbolic trajectory is not necessarily small for computation of the special hyperbolic trajectory, starting from $x_{cp}$.

In this example equation (E.16) for $f$ does not contain $z$. Consequently, equation (E.15) is explicit and we can use it to compute $z$ directly. Generally, this is not the case and one has to resort to other methods to solve this integral equation, e.g., via iteration (successive approximation).

In order to calculate the integral solution (E.15), equation (E.16) is substituted into (E.15):

$$
z(t) = X(t) \int_{-\infty}^{t} \frac{1}{2} \left( \begin{array}{ccc} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \begin{pmatrix} \cosh \left( \frac{s}{2} \right) & -\sinh \left( \frac{s}{2} \right) \\ -\sinh \left( \frac{s}{2} \right) & \cosh \left( \frac{s}{2} \right) \\ 1 & -\cos s \end{pmatrix} \begin{pmatrix} 3 \sin s \\ -3 \sin s \\ -\cos s \end{pmatrix} ds - \left( X(t) \int_{t}^{\infty} \frac{1}{2} \left( \begin{array}{ccc} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \begin{pmatrix} \cosh \left( \frac{s}{2} \right) & -\sinh \left( \frac{s}{2} \right) \\ -\sinh \left( \frac{s}{2} \right) & \cosh \left( \frac{s}{2} \right) \\ 1 & -\cos s \end{pmatrix} \begin{pmatrix} 3 \sin s \\ -3 \sin s \\ -\cos s \end{pmatrix} ds \right)$$
E.3. Special hyperbolic trajectories

\[ X(t) \int_{-\infty}^{t} \frac{1}{2} \left( \frac{e^{s/2} (\cos s + 3 \sin s)}{e^{s/2} (\cos s + 3 \sin s)} \right) \, ds - \]
\[ X(t) \int_{t}^{\infty} \frac{1}{2} \left( \frac{e^{-s/2} (\cos s - 3 \sin s)}{e^{-s/2} (\cos s - 3 \sin s)} \right) \, ds. \]

Solving the integral and substituting (E.10) for \( X \) then yields

\[ z(t) = \begin{pmatrix} \cosh \left( \frac{t}{2} \right) & \sinh \left( \frac{t}{2} \right) \\ \sinh \left( \frac{t}{2} \right) & \cosh \left( \frac{t}{2} \right) \end{pmatrix} \begin{pmatrix} e^{t/2} (\cos t + \sin t) \\ e^{t/2} (\cos t + \sin t) \end{pmatrix} - \]
\[ \begin{pmatrix} \cosh \left( \frac{t}{2} \right) & \sinh \left( \frac{t}{2} \right) \\ \sinh \left( \frac{t}{2} \right) & \cosh \left( \frac{t}{2} \right) \end{pmatrix} \begin{pmatrix} e^{-t/2} (\cos t + \sin t) \\ e^{-t/2} (\cos t + \sin t) \end{pmatrix} \]
\[ = \begin{pmatrix} -2 \cos t \\ -2 \sin t \end{pmatrix} = z_h(t). \quad \text{(E.17)} \]

The solution \( x_h \) of the original problem (E.1) is now obtained from \( x_h = z_h + x_{cp} \):

\[ x_h(t) = \begin{pmatrix} -2 \cos t \\ -2 \sin t \end{pmatrix} + \begin{pmatrix} 3 \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \quad \text{(E.18)} \]

which is the circle, the special hyperbolic trajectory as found before. Note that the hyperbolic terms of (E.2) are not present in (E.18), because these terms make the trajectories approach the lines \( y = x \) and \( y = -x \), which are no special hyperbolic trajectories.

It is now verified whether the solutions (E.8) of the differential equations (E.6), combined with the conditions

\[ |QX^{-1}(t)z(t)| \to 0 \quad \text{for} \quad t \to \infty, \]
\[ |PX^{-1}(t)z(t)| \to 0 \quad \text{for} \quad t \to -\infty. \quad \text{(E.19)} \]

yield the same result as (E.17). We find the following:

\[ QX^{-1}(t)z(t) = \begin{pmatrix} 1 + \frac{1}{2} (z_{10} + z_{20}) - e^{-t/2} (\cos t - \sin t) \\ 1 + \frac{1}{2} (z_{10} + z_{20}) - e^{-t/2} (\cos t - \sin t) \end{pmatrix}, \]
\[ PX^{-1}(t)z(t) = \begin{pmatrix} 1 + \frac{1}{2} (z_{10} - z_{20}) - e^{t/2} (\cos t - \sin t) \\ -1 - \frac{1}{2} (z_{10} - z_{20}) + e^{t/2} (\cos t - \sin t) \end{pmatrix}. \]
Then the conditions (E.19) give
\[\begin{align*}
1 + \frac{1}{2}(z_{10} + z_{20}) &= 0, \\
1 + \frac{1}{2}(z_{10} - z_{20}) &= 0.
\end{align*}\]
The result for \(z_{10}\) and \(z_{20}\) is
\[\begin{align*}
z_{10} &= -2, \\
z_{20} &= 0.
\end{align*}\]
The restrictions (E.19) select the following special trajectory from the solutions (E.8) of the differential equation:
\[z(t) = \begin{pmatrix} -2 \cos t \\ -2 \sin t \end{pmatrix},\]
matching the solution (E.18) of the integral equation. This confirms that the integral equation selects hyperbolic trajectories from the solutions of the differential equation that have smaller growth in time which than the other trajectories.

### E.4. Invariant manifolds

In order to find the stable and unstable invariant manifold, we return to the demands (4.6-i), (4.6-ii) and (4.6-iii) for an exponential dichotomy. Since the rank of \(P\) is 1, equation (4.6-i) says that there is a one-dimensional subspace \(E^s\) of \(\mathbb{R}^2\) of solutions that decay to zero exponentially as \(t \to \infty\). \(E^s\) is given by the line of initial values \(y_0 = 1 - x_0\), since the solution of the linearised (homogeneous) equation (E.3) is given by
\[(x(t), y(t)) = (x_0 e^{-t/2}, y_0 e^{-t/2}).\]
Equation (E.4-ii) says that there is a one-dimensional subspace \(E^u\) of \(\mathbb{R}^2\) of solutions that go to infinity exponentially as \(t \to \infty\), or, equivalently, that decay to zero for \(t \to -\infty\). \(E^u\) is the line of initial values \(y_0 = x_0 - 1\), since the solution of the linearised (homogeneous) equation (E.3) is given by
\[(x(t), y(t)) = (x_0 e^{t/2}, y_0 e^{t/2}).\]
It is easily verified that the third demand (4.6-iii) is satisfied, because
\[X(t)PX^{-1}(t) = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.\]
Now that \(E^s\) and \(E^u\) are known, we can apply the theory from section 4.1.4 to find
approximations for segments of $W^s_{\text{app}}$ of $W^u$ and $W^u_{\text{app}}$ of $W^s$ [34]. This is illustrated in figure E.4. $W^s_{\text{app}}$ is found from the line segment between $x_{\text{SHT}}(0) = (1, 0)$ and $x^s_W(t)(0) = x_{\text{SHT}}(0) + \lambda e_s(0)$ (the dashed line), $W^u_{\text{app}}$ is found from the line segment between $x_{\text{SHT}}(0)$ and $x^u_W(t)(0) = x_{\text{SHT}}(0) + \lambda e_u(0)$ (the dotted-dashed line). $\lambda$ has been chosen both positive and negative, to create the intersection of $W^s_{\text{app}}$ and $W^u_{\text{app}}$ as the hyperbolic trajectory $\gamma$.

$W^s_{\text{app}}$ is found from the line segment between $x_{\text{SHT}}$ and $x^s_W(t) = x_{\text{SHT}} + \lambda e_s$, $W^u_{\text{app}}$ is found from the line segment between $x_{\text{SHT}}$ and $x^u_W(t) = x_{\text{SHT}} + \lambda e_u$. Here, SHT is short for special hyperbolic trajectory. $\lambda$ has been chosen both positive and negative, to create the intersection of $W^s_{\text{app}}$ and $W^u_{\text{app}}$ as the hyperbolic trajectory $\gamma$.

The manifolds can be found for any time $t$ in this way. Since the SHT is the unit circle and $e^s$ and $e^u$ are constant (independent of time), the manifolds are the red and green line as in figure E.4, which rotate along the circle. Therefore, in the extended phase space, the manifolds describe a helix.

As can be seen from figure E.4, the local unstable manifolds $W^u_{\text{app}}$ for $t = 0$ is given

\[ W^s_{\text{app}} \text{ and } W^u_{\text{app}} \text{ of stable and unstable manifolds } W^u \text{ and } W^s \text{ for } t = 0, \text{ respectively.} \]

\[ W^s_{\text{app}} \text{ is found from the line segment between } x_{\text{SHT}}(0) = (1, 0) \text{ and } x^s_W(t)(0) = x_{\text{SHT}}(0) + \lambda e_s(0) \text{ (the dashed line), } \]

\[ W^u_{\text{app}} \text{ is found from the line segment between } x_{\text{SHT}}(0) \text{ and } x^u_W(t)(0) = x_{\text{SHT}}(0) + \lambda e_u(0) \text{ (the dotted-dashed line). } \lambda \text{ has been chosen both positive and negative, to create the intersection of } W^s_{\text{app}} \text{ and } W^u_{\text{app}} \text{ as the hyperbolic trajectory } \gamma. \]
by a line segment in the direction \((1, 1)\), passing through \((1, 0)\). We take \(x_0 = 1 - \varepsilon\)
and \(y_0 = -\varepsilon\) and substitute this into the solutions (E.2). This yields
\[
\begin{align*}
x(t) &= -\varepsilon \cosh \left(\frac{t}{2}\right) - \varepsilon \sinh \left(\frac{t}{2}\right) + \cos t \\
y(t) &= -\varepsilon \sinh \left(\frac{t}{2}\right) - \varepsilon \cosh \left(\frac{t}{2}\right) - \sin t.
\end{align*}
\]
This can be written as
\[
\begin{align*}
x(t) &= A(t) + \cos t \\
y(t) &= A(t) - \sin t,
\end{align*}
\]
with \(A(t) = -\varepsilon \sinh \left(\frac{t}{2}\right) - \varepsilon \cosh \left(\frac{t}{2}\right)\). This indicates a circular motion with varying
radius, which is determined by \(A(t)\). Since \(x\) and \(y\) depend both on the same factor \(A\),
it follows that the straight line segment \(W_{\text{app}}^u\) is not deformed by the time evolution
due to (E.1), but merely stretched or contracted. According to the theory from section
4.1.4, \(W^u\) is found from forward time evolution of trajectories starting on \(W_{\text{app}}^u\).

Analogously, the analysis for \(W_{\text{app}}^s\) yields the same results by the transformation
\(\varepsilon \rightarrow -\varepsilon\). \(W^s\) is found from backward time evolution of trajectories starting on \(W_{\text{app}}^s\).
This process is illustrated in figure E.5. The invariant manifolds are the surfaces
enclosed by the curved dashed lines. They contain the special hyperbolic trajectory,
which is represented by the bold line.
Figure E.5: The evolution of the approximations $W^u_{\text{app}}$ and $W^s_{\text{app}}$ of the unstable and stable manifold. $W^u_{\text{app}}$ at $t = -\pi$ is given by the green line segment (bottom left) in the direction $(x, y) = (1, 1)$, passing through $(x, y) = (-1, 0)$. $W^u(0)$ is found from forward time evolution (meaning a clockwise spiral motion in forward time) of $W^u_{\text{app}}$ at $t = -\pi$, which merely stretches it. Backward time evolution (counterclockwise spiral motion in forward time) of $W^s_{\text{app}}$ at $t = \pi$, which is the red line segment (top left) in the direction of $(x, y) = (1, -1)$, passing through $(x, y) = (1, 0)$, yields $W^s(0)$. The projections of both time evolutions on the plane $t = 0$ are shown as well, where the dots indicate intersections with the circle. The invariant manifolds are the coloured surfaces enclosed by the curved lines. They contain the special hyperbolic trajectory, which is represented by the bold line.
E.5. Singular value decomposition

We want to find the matrices $A$ and $D$ as described in appendix D in order to define the transformed problem (D.6). The matrix $F$ is easily found from (E.1):

$$F(t) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$  

This gives for $H$:

$$H(t) = B^T(t)F(t)B(t)$$

$$= \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \sin(2\vartheta) & \frac{1}{2} \cos(2\vartheta) \\ \frac{1}{2} \cos(2\vartheta) & \frac{1}{2} \sin(2\vartheta) \end{pmatrix}.$$  

Substituting these matrix elements of $H$ into (D.17) leads to the following equations for $\dot{\vartheta}, \dot{\varphi}, \dot{\sigma}_1$ and $\dot{\sigma}_2$

$$\dot{\sigma}_1 = -\frac{1}{2} \sin(2\vartheta)$$

$$\dot{\sigma}_2 = \frac{1}{2} \sin(2\vartheta)$$

$$\dot{\vartheta} = 0$$

$$\dot{\varphi} = 0.$$  

Integration with respect to $t$ gives

$$\sigma_1 = -\frac{1}{2} \sin(2\vartheta) \ t + c_1$$

$$\sigma_2 = \frac{1}{2} \sin(2\vartheta) \ t + c_2$$

$$\vartheta = c_3$$

$$\varphi = c_4,$$

where the $c_i$ are constants, for $i = 1, 2, 3, 4$. $\sigma_1(0) = \sigma_2(0) = 0$ demands that $c_1 = c_2 = 0$, and $c_3$ and $c_4$ are determined from

$$X(t) = B(t) \exp(\Sigma(t)) R^T(t) \equiv G(t),$$  \hspace{1cm} (E.20)  

with matrix elements $X_{ij}$ ($i,j \in \{1, 2\}$) of $X$ in (E.10) given by

$$X_{11} = X_{22} = \cosh \left( \frac{t}{2} \right) = \frac{1}{2} \left( e^{\frac{1}{2} t} + e^{-\frac{1}{2} t} \right)$$

$$X_{12} = X_{21} = \sinh \left( \frac{t}{2} \right) = \frac{1}{2} \left( e^{\frac{1}{2} t} - e^{-\frac{1}{2} t} \right).$$
The matrix elements of $G_{ij}$ of $G$ in (E.20) are equal to

$$
G_{11} = \cos \vartheta \cos \varphi \ e^{-\frac{1}{2} \sin(2\vartheta)t} + \sin \vartheta \sin \varphi \ e^{\frac{1}{2} \sin(2\vartheta)t}
$$

$$
G_{12} = \sin \vartheta \cos \varphi \ e^{\frac{1}{2} \sin(2\vartheta)t} - \cos \vartheta \sin \varphi \ e^{-\frac{1}{2} \sin(2\vartheta)t}
$$

$$
G_{21} = \cos \vartheta \sin \varphi \ e^{\frac{1}{2} \sin(2\vartheta)t} - \sin \vartheta \cos \varphi \ e^{-\frac{1}{2} \sin(2\vartheta)t}
$$

$$
G_{22} = \cos \vartheta \cos \varphi \ e^{\frac{1}{2} \sin(2\vartheta)t} + \sin \vartheta \sin \varphi \ e^{-\frac{1}{2} \sin(2\vartheta)t}
$$

In order to match the powers $e^{t/2}$ and $e^{-t/2}$ in $G$ and $X$, we must take $\sin(2\vartheta) = 1$, and therefore $\vartheta = \frac{\pi}{4}$. Matching the coefficients then gives $\varphi = \frac{\pi}{4}$. These values of $\vartheta$ and $\varphi$ lead to the following orthogonal matrices $B$ and $R$ and the diagonal matrices $\Sigma$ en $D$:

$$
B(t) = R(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
$$

$$
\Sigma(t) = \begin{pmatrix} -\frac{1}{2}t & 0 \\ 0 & \frac{1}{2}t \end{pmatrix}
$$

$$
D = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}
$$

(E.21)

(E.22)

It can be checked directly whether this matrix $\Sigma$ in (E.21) is correct, because $\exp(\Sigma)$ contains the singular values of $X$ on its diagonal. These singular values are the square roots of the eigenvalues of $X^T X$:

$$
X^T(t)X(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}
$$

which has eigenvalues $e^{t}$ and $e^{-t}$. Therefore, the square roots of these eigenvalues correspond to the diagonal elements of $\exp(\Sigma)$. The transformation matrix $T$ can now be found from

$$
T(t) = \exp(tD)R^T(t_L)R(t)\exp(-\Sigma(t))B^T(t),
$$

where in this case $R$ is independent of $t$ and $\Sigma$ is linear in $t$. Therefore, $R^T(t_L)R(t) = R^T R = I$ and $\exp(tD)I\exp(-\Sigma(t)) = \exp(\Sigma(t))\exp(-\Sigma(t)) = I$ for any $t$, so that

$$
T(t) = B^T(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
$$

(E.23)

The transformed differential equations now have the following form:

$$
\dot{w}(t) = D w(t) + h(t),
$$

(E.24)
with $D$ as in equation (E.22) and

$$h(t) = T(t)f(t) = \frac{1}{2\sqrt{2}} (3\cos t - \sin t, -3\cos t - \sin t)$$

To check whether this transformation matrix is correct, we calculate the special hyperbolic trajectory $w = (w_{SHT1}, w_{SHT2})$ in this system using (D.8) and transform it back. As is proved by Ide et al. [15], $w_{SHT} = T x_{SHT}$ is a SHT of (D.6) if $x_{SHT}$ is a SHT of the original problem (D.3). In this example $d_1 = -\frac{1}{2}$ and $d_2 = \frac{1}{2}$, so

$$w_{SHT1} = \int_{-\infty}^{t} Y_{11}(t, \tau) h_1(\tau) \, d\tau$$
$$= \int_{-\infty}^{t} \frac{1}{2\sqrt{2}} e^{-\frac{1}{2}(t-\tau)} (3\cos \tau - \sin \tau) \, d\tau$$
$$= \frac{1}{\sqrt{2}} (\cos t + \sin t)$$

$$w_{SHT2} = -\int_{t}^{\infty} Y_{22}(t, \tau) h_2(\tau) \, d\tau$$
$$= -\int_{t}^{\infty} \frac{1}{2\sqrt{2}} e^{\frac{1}{2}(t-\tau)} (-3\cos \tau - \sin \tau) \, d\tau$$
$$= \frac{1}{\sqrt{2}} (\cos t - \sin t)$$

Transforming back gives

$$x_{SHT}(t) = T^{-1}(t)w_{SHT}(t)$$
$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \frac{1}{2\sqrt{2}} \begin{pmatrix} \cos t + \sin t \\ \cos t - \sin t \end{pmatrix}$$
$$= (\cos t, -\sin t),$$

which matches the SHT that was found in section E.3.
F. Technology assessment

Many large scale geophysical fluid flows are to a good approximation incompressible and two-dimensional, for which a number of reasons have been given. These two-dimensional flows can be better understood by studying two-dimensional turbulence. Knowledge of turbulence and turbulent transport in the oceans and the atmosphere is crucial for weather predictions and climate models. Furthermore, insight into transport in turbulent flows is important for industrial applications of mixing and stirring, e.g., for efficient mixing of paint or chemical reactants.

From a mathematical point of view, the application of dynamical systems theory to fluid dynamics is very challenging, and many aspects are still poorly understood. For example, the relation between critical points and the corresponding particles trajectories, when going from a time independent to a time dependent flow is difficult to understand. In the applications, the dynamical systems approach has the advantage that flows with arbitrary time dependence as well as flows given by discrete experimental data can be analysed.