Master’s Thesis in Industrial and Applied Mathematics

Homogenisation of a Linear Reaction-Diffusion System Modelling Smouldering Combustion in Perforated Media

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Abstract

The aim of the master thesis is to study the passage to the homogenisation limit in a semi-linear reaction-diffusion (RD) system modelling the smouldering combustion in two-dimensional media with microstructures (a paper sheet). The starting point of this investigation is a microscopic model proposed by Kagan and Sivashinsky in 2008. We assume our piece of paper to be perforated periodically with discs as repeating microstructure. We start by proving the well-posedness of our RD system. Then we use two-scale convergence to study the asymptotic behaviour of the solutions to the micro problem as the parameter $\varepsilon$ goes to zero. Here $\varepsilon$ is a small parameter related to the precise choice of the microstructure. As working plan we wish to derive rigorously an upscaled RD system (that we refer to as macro problem) as well as explicit formulae for the effective coefficients. On this way, we justify the macroscopic equations obtained earlier by [15] by means of a formal asymptotic homogenisation approach. Finally, we derive via two-scale asymptotic expansions a distributed microstructure model for combustion in high-contrast materials.
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Chapter 1

Introduction

Smouldering combustion is a flameless form of burning. It occurs when burning for instance coals, cotton, dust and smoking cigars. There are many ways to understand this natural phenomenon. One of these ways is by looking at this problem via a mathematical multi-scale approach. The reason why we wish to deal with this topic from a multiscale perspective is that we expect, in general, the precise composition and geometry of the pore structure (here, this is called microstructure or perforation) play an important role in the macroscopic burning of heterogeneous materials. The method used in this thesis for this is called ‘mathematical homogenisation’. It is an averaging method designed to derive macroscopic partial differential equations from equations posed at a micro-scale level. We are especially interested in understanding the smouldering combustion of periodically perforated (porous) media. Mathematically, this is the simplest combustion setting for a material that one can imagine.

The second chapter introduces the reader to basic aspects of modelling combustion. The role of chapters three and four is to prepare the formulation of the microscopic model ($P^\varepsilon$) posed in a medium with periodically distributed microstructures. Does there exist a solution to the microscopic model? If a solution exists, is it unique? These questions play an important role in chapter five in which the well-posedness of the microscopic model is tested.

The sixth chapter is devoted to two-scale convergence, a necessary tool for periodic homogenisation. We apply this rigorous approach to the microscopic model. By passing to $\varepsilon \to 0$ the homogenisation limit is studied, that is we look at the asymptotic behaviour when $\varepsilon \to 0$ of the solutions of the microscopic model. On this way, the macroscopic equations are justified. How to get averaged model equations and effective coefficients forms the main topic of chapter seven. Here, a formal homogenisation of the microscopic model is studied.

In chapter eight, the microscopic combustion is assumed to take place in high-contrast media. The result of the averaging procedure is a two-scale (upscaled) model. The main working ideas are still the formal homogenisation techniques presented in [13].

A short summary and ideas for future work form the final chapter. We add appendix A, which contains the inequalities used throughout this paper. They are especially useful for chapter five where the well-posedness of the microscopic model is tested.
2 Introduction
Chapter 2

Modelling Smouldering Combustion

This chapter introduces the smouldering combustion process for heterogeneous media.

2.1 Background

According to the ‘Oxford Advanced Learner’s Dictionary’ there are two descriptions of combustion:

- process of burning,
- chemical process in which substances combine with oxygen in air, producing heat and light.

Combustion processes have been occurring daily. Examples are the sun, cooking, electricity and heating. The latter meaning of combustion implies chemical reactions. Thermodynamics and fluid mechanics are the main subjects to deal with combustion. In this text, ‘smoulder’ will be treated mathematically.

Smouldering is a slow, flameless form of combustion that derives its heat from heterogeneous reactions occurring on the surface of a solid fuel when heated in the presence of oxygen. To be more informal, smoulder means burning slowly without flame. The basic difference between smouldering and flaming is that, in the firstly mentioned one, the oxidation reaction and the heat release occur on the surface of the solid, while in the secondly mentioned one, these occur in the gas phase above the fuel. Statistics about smouldering are approximately 770-970 K, 6-12 kJ/g and 10-30 mm/h for the peak temperature, average heat and spread velocity respectively. Compared to smouldering the same quantities for flaming are approximately 1770-2070 K, 16-30 kJ/g 1000 mm/h. A consequence of smouldering’s low temperature is that it emits a higher rate of combinations of toxic gases, for instance CO. Materials that can sustain a smouldering reaction are e.g. coals (Figure 2.2), cotton, dust, paper, peat (Figure 2.1), tobacco and charring polymers. The pictures are typical examples of smouldering combustion. Smouldering fuels consist of an aggregate and permeable medium formed by particulate, grains, fibres or having a porous structure. These aggregates facilitate the surface reaction with oxygen by giving a large surface area per unit volume. On the one hand, they permit oxygen transport to the reaction sites by convection and diffusion. On the other hand, they act as thermal insulation by reducing heat loss. Supply of heat flux to the solid fuel is needed for smouldering ignition. The subsequent temperature increase of the solid starts with the thermal degradation reactions (mainly endothermic pyrolysis). Then it leads to oxidation until the net heat released is high enough to balance the heat required for propagation. This net heat release is partially transferred by conduction, advection and radiation ahead of the reaction and partially lost to the environment. Oxygen is transported to the reaction zone by diffusion and convection such that it feeds the oxidation reactions. Once ignition happens, the smoulder reaction goes through the material gradually.
Another example of smouldering is paper. In this case, a model of the smouldering combustion of a sheet of paper is studied. The process of smouldering combustion of this material exposed to a flow of air confined in a narrow gap above the paper is modelled. The physical experiment is a geometry in which the sheet is ignited on one side rising the local temperature by means of a heat pulse. Furthermore, the combustion front travels from the ignition \( x = 0 \) to the opposite side \( x = L \). An assumption is made for the air flow which is parallel to the sheet paper, orthogonal to the combustion and in the opposite direction. An overview of this model is in Figure 2.4. Physical experiments on smouldering combustion have been carried out in the past. The papers by for instance [37], [35] and [36] are studied. Mathematical models for combustion are e.g. [24], [18] and [11]. The latter papers form the main ideas behind this study.

For simplicity, consider air to consist of twenty-one percents oxygen and seventy-nine percents nitrogen. Most of the air consists of nitrogen, then the combustion temperature and intensity are reduced because of the use of the thermal energy used to heat up during the way of burning.
2.1 Background

Figure 2.3: Smoking cigarette. (source: WikiPedia)

Figure 2.4: Model sketch of the setup. (1) glass top, (2) variable gap between top and bottom plates $h$, (3) outflow of combustion products, (4) spacers for controlling $h$, (5) ignition wire, (6) heat conducting boundaries, (7) flame front, (8) fuel, (9) interchangeable bottom plate, (10) uniform flow of $O_2$ and $N_2$, (11) gas diffuser, (12) gas inlet. See [11] for more information.

The combustion intensity between a fuel and an oxidiser depends on their relative concentrations. If their concentration ratio is chemically correct, i.e. all reactants are totally consumed in the reaction, then the combustion intensity is close the highest and this kind of burning is called stoichiometric combustion.

Taking into account the geometry of 2.4, the following assumptions only hold for this specific geometry. For a detailed explanation, please look in [11]. The chemical reaction takes place between cellulose and oxygen:

$$C_6H_{10}O_5 + 6O_2 \rightarrow 6CO_2 + 5H_2O.$$  

Note that the above reaction is called stoichiometrically balanced.

The reaction happens at the gas-solid interface, in which the local regression of the solid part is negligibly small (so no free boundaries occur). The char is assumed to be a reaction product. The reaction does not go through the whole depth of the solid layer. Then the reaction is oxygen limited. This implies a two-layer gas-solid system (Figure 2.5) with two heat equations, one for the gaseous part ($Y_g^3$) and the other for the solid part ($Y_s^3$). Also involved are the two equations for the volumetric mass friction of oxygen and surface mass fraction of the solid product.

The challenging issue here is to understand the transition from smouldering to flaming combustion. Figure 2.6 shows a picture of this issue.
2.2 Description of the governing equations

In order to understand the involved equalities it might be handy to give some background information about the terminology used concerning combustion.
2.2 Description of the governing equations

The reaction-diffusion system with corresponding initial and boundary conditions is:

\[(2.1) \quad \frac{\partial \Theta}{\partial t} + \text{div}(P\dot{\Theta} - L\nabla \Theta) = f(\Theta, \Psi) \quad \text{in} \quad \Omega, \]
\[(2.2) \quad \frac{\partial \Psi}{\partial t} + \text{div}(P\dot{\Psi} - D\nabla \Psi) = g(\Theta, \Psi) \quad \text{in} \quad \Omega_g, \]
\[(2.3) \quad \frac{\partial R}{\partial n} = h(\Theta, \Psi) \quad \text{at} \quad \Gamma. \]

The system of initial conditions is:

\[(2.4) \quad \Theta(0, x) = \Theta_0(x), x \in \bar{\Omega}, \]
\[(2.5) \quad \Psi(0, x) = \Psi_0(x), x \in \bar{\Omega}_g, \]
\[(2.6) \quad R(0, x) = R_0(x), x \in \Gamma. \]

The system of boundary conditions is:

\[(2.7) \quad n \cdot \dot{\Psi} = p(\Theta, \Psi) \quad \text{at} \quad \Gamma, \]
\[(2.8) \quad \Theta = 0 \quad \text{at} \quad \partial \Omega, \]
\[(2.9) \quad \Psi = b \quad \text{at} \quad \partial \Omega, \]
\[(2.10) \quad n \cdot (P\dot{\Theta} - L\nabla \Theta) = B(\Theta, \Psi) \quad \text{at} \quad \Gamma. \]

Equation (2.1) is called conservation of energy, while equations (2.2) and (2.3) are called conservations of mass. If the convection terms are cancelled, then equations (2.1) and (2.2) become parabolic or heat equations. They can be derived from Fourier’s law and conservation of energy. Fourier’s law says that the flow of heat energy through a surface is proportional to the negative temperature gradient across the surface,

\[(2.11) \quad q = -k\nabla T, \]

where \(k\) is the thermal conductivity and \(T\) is the temperature.

According to WikiPedia there are two Fick’s laws of diffusion. They describe diffusion and can compute the diffusion coefficient \(D\). An introduction to Fick’s laws is explained below.

Fick’s first law says there is a relation between the diffusive flux and the concentration. This is done by positing that the flux goes from regions of high concentration to regions of low concentration, with a size that is proportional to the concentration gradient. In one dimension this is

\[(2.12) \quad J = -D\nabla \phi, \]

where

- \(J\) is the diffusion flux, i.e. (amount of substance) per unit area per unit time. It measures the number of substance that goes through a small area during a small time interval,
- \(D\) is the diffusion coefficient in squared length per time unit,
- \(\phi\) is the concentration in (amount of substance) per cubic length unit.

Note that \(D\) is either a scaler or a tensor. Fick’s second law tells how diffusion leads that the concentration changes with time, i.e.

\[\frac{\partial \phi}{\partial t} = \text{div}(D\nabla \phi), \]

where \(t\) is the time variable and \(\Delta\) is the Laplace operator. This is a consequence of Fick’s first law combined with the mass balance, because

\[\frac{\partial \phi}{\partial t} = -\text{div}J = -\text{div}(-D\nabla \phi) = \text{div}(D\nabla \phi). \]

If \(D\) is a constant, then \(\frac{\partial \phi}{\partial t} = D\Delta \phi\), where \(\Delta\) is the Laplace operator. This is the so-called heat conduction equation. Equation (2.2) is an example that follows from Fick’s law of diffusion.
Chapter 3

Geometry. Data and Unknowns

Here we introduce our basic geometry together with the data and unknowns of the involved reaction-diffusion system.

3.1 Basic geometry

Figure 3.1: Micro scale geometry (3.1a) and reference unit cell (3.1b).

Figure 3.1 shows the concept of the pore used throughout this paper, which is also called here microstructure, perforation or inclusion. Inside $Y_g$ (the pores) and $Y_s$ (the solid) oxygen goes through the porous medium and reacts at the gas-solid boundary. Heat is carried by the oxygen diffusing through the pore parts of the porous medium. This section mainly follows [15].
Consider the following reference microscopic geometry (microstructure):

\[
S = (0, T) \quad \text{- time interval of interest,}
\]

\[
e_i \quad \text{- ith unit vector in } \mathbb{R}^2 \ (i = 1 \text{ or } 2),
\]

\[
Y \quad \text{- representative unit cell in } \mathbb{R}^2 \ (Y = (0, 1)^2),
\]

\[
Y_s \quad \text{- solid part (closed subset of } Y),
\]

\[
Y_g \quad \text{- gaseous (pore) part } (Y_g = Y \setminus Y_s),
\]

\[
\partial Y_s = \Gamma \quad \text{- piecewise smooth boundary of } Y_s,
\]

\[
\Gamma^{\varepsilon,D} \quad \text{- Dirichlet boundary of } Y,
\]

\[
\Gamma^{\varepsilon,N} \quad \text{- Neumann boundary of } Y,
\]

\[
n_g \quad \text{- outer normal on } \partial Y_s, \text{ pointing outside the solid part,}
\]

\[
Y := Y_g \cup Y_s \quad \text{- representative unit cell},
\]

where \( \Gamma^{\varepsilon,D} \cap \Gamma^{\varepsilon,N} := \emptyset \) and \( \Gamma^{\varepsilon,D} \cup \Gamma^{\varepsilon,N} := \partial Y \).

Denote \( \varepsilon > 0 \) by the scale factor - the ratio between the typical microscopic length scale \( l \) and the macroscopic length scale \( L \), i.e. \( \varepsilon = \frac{l}{L} \). Assume that the paper sample occupies a bounded region \( \Omega := (0, L)^2 \) in \( \mathbb{R}^2 \). The unit cell \( Y \) and the scale factor \( \varepsilon \) are chosen such that \( \Omega \) is covered by a finite union of sets \( \varepsilon Y^k = \varepsilon (k + Y) \), \( k \in \mathbb{Z}^n \).

Let the domain \( \Omega \subset \mathbb{R}^2 \) be covered by regular mesh of size \( \varepsilon \) as in Figure 3.1. Then a lattice of copies of cells \( \varepsilon Y^\varepsilon \) is generated for any \( \varepsilon > 0 \). Each cell is denoted by \( Y_i^\varepsilon = (0, \varepsilon)^2 \) with \( 1 \leq i \leq N(\varepsilon) \), and \( N(\varepsilon) \) denotes the number of cells along the planar directions. Each cell is homoeomorphic to \( Y^\varepsilon \) by a linear homoeomorphism \( \Pi_i^\varepsilon \) with ratio of magnitude \( 1/\varepsilon \), i.e. the cell is re-scaled by \( \varepsilon \). Hence,

\[
Y_{s,i}^\varepsilon := (\Pi_i^\varepsilon)^{-1} Y_{s,i}^\varepsilon,
\]

\[
Y_{g,i}^\varepsilon := (\Pi_i^\varepsilon)^{-1} Y_{g,i}^\varepsilon
\]

denote the solid and gaseous parts of the unit cell \( Y^\varepsilon \) respectively. The gas domain \( \Omega_{g,i}^\varepsilon \subset \Omega \) is obtained by removing the periodically distributed solid parts, i.e. \( \Omega_{g,i}^\varepsilon := \Omega \setminus \bigcup_{i=1}^{N(\varepsilon)} Y_{s,i}^\varepsilon \).

Now, the periodic microscopic geometry can be written as:

\[
Y_{g,i}^\varepsilon := \mathcal{U} \{ \varepsilon Y^k \mid k \in \mathbb{Z}^n : \varepsilon Y^k \subset \Omega^\varepsilon \} \quad \text{- total pore volume},
\]

\[
Y_{s,i}^\varepsilon := \Omega^\varepsilon \setminus Y_{g,i}^\varepsilon \quad \text{- total solid structure (matrix)},
\]

\[
\Omega^\varepsilon := Y_{g,i}^\varepsilon \cup Y_{s,i}^\varepsilon \quad \text{- inside reference cell},
\]

\[
\partial Y_{g,i}^\varepsilon := \mathcal{U} \{ \varepsilon \partial Y^k \mid k \in \mathbb{Z}^n : \varepsilon Y^k \subset \Omega^\varepsilon \} \quad \text{- total solid surface},
\]

\[
\Gamma^\varepsilon := \varepsilon (k + \partial Y_s) \quad \text{- total boundary of the solid},
\]

\[
n_g^\varepsilon \quad \text{- outer normal on } \Gamma^\varepsilon \text{ with respect to } Y_{g,i}^\varepsilon.
\]
The unknowns of the microscopic model are defined as:

\begin{align*}
\Theta^\varepsilon : \mathbb{S} \times \Omega^\varepsilon &\to \mathbb{R} \quad \text{temperature [K]}, \\
\Psi^\varepsilon : \mathbb{S} \times \Omega_g^\varepsilon &\to \mathbb{R} \quad \text{gas concentration [kg m^{-3}]}, \\
R^\varepsilon : \mathbb{S} \times \Gamma^\varepsilon &\to \mathbb{R} \quad \text{concentration of solid product [kg m^{-3}]}. 
\end{align*}

The parameters of the microscopic model are given by:

\begin{align*}
D^\varepsilon : \mathbb{S} \times \Omega^\varepsilon &\to \mathbb{R} \quad \text{diffusion coefficient for oxygen [m^2 s^{-1}]}, \\
L^\varepsilon : \mathbb{S} \times \Omega^\varepsilon &\to \mathbb{R} \quad \text{dimensionless Lewis number \(\lambda^\varepsilon\)}, \\
P^\varepsilon : \mathbb{S} \times \Omega^\varepsilon &\to \mathbb{R} \quad \text{transport coefficient (connected to the Lewis number) \(\frac{\mathcal{U}^\varepsilon L^\varepsilon}{\mathcal{D}^\varepsilon}\)}, \\
\lambda^\varepsilon &\in (0, \infty) \quad \text{thermal conductivity [W m^{-1} K^{-1}]}, \\
\rho^\varepsilon &\in (0, \infty) \quad \text{mass density [kg m^{-3}]}, \\
c^\varepsilon &\in (0, \infty) \quad \text{specific heat capacity [J kg^{-1} K^{-1}]}, \\
u^\varepsilon &\in (0, \infty) \quad \text{flow velocity [m s^{-1}]}, \\
l &\in (0, \infty) \quad \text{length of unit cell [m]}. 
\end{align*}

Initial and boundary data \((\Theta_0^\varepsilon, \Psi_0^\varepsilon, R_0^\varepsilon, b)\) to complete the model equations, see Section 4.3 for more information.
Chapter 4

Setting of the Microscopic Model Equations

This brief chapter contains the function spaces used in this thesis, a few mathematical (technical) assumptions on the objects we are working with. Finally, we list the model equations of the microscopic model.

4.1 Function spaces

Let \( \varepsilon > 0 \) be a scale parameter or more precisely: \( \varepsilon := \frac{1}{x} \), see figures 3.1a and 3.1b for more information about this parameter. For any \((t, x) \in S \times \mathbb{R}^2\), let \( D(t, \frac{x}{\varepsilon}) := D^\varepsilon(t, x) \), \( P(t, \frac{\cdot}{\varepsilon}) := P^\varepsilon(t, x) \), \( L(t, \frac{\cdot}{\varepsilon}) := L^\varepsilon(t, x) \) and \( B := B(t, x) \). The following notations have been used throughout this paper:

\[
\begin{align*}
(\alpha f, \beta g)_{L^2(\Omega^\varepsilon)} &:= \alpha \beta \int_{\Omega^\varepsilon} f(x)g(x)dx & \text{inner product in } L^2(\Omega^\varepsilon) \text{ for all } \alpha, \beta \in \mathbb{R}, \\
\langle f, g \rangle_{H^{-1}(\Omega^\varepsilon), H^1(\Omega^\varepsilon)} &:= \text{dual pairing between } H^{-1}(\Omega^\varepsilon) \text{ and } H^1(\Omega^\varepsilon), \\
\|f\|_{L^2(\Omega^\varepsilon)} &:= \left( \int_{\Omega^\varepsilon} |f(x)|^2dx \right)^{\frac{1}{2}} & \text{ } L^2(\Omega^\varepsilon)\text{-norm of } f, \\
C^\infty_\#(Y) &:= \text{the space of infinitely differentiable functions in } \mathbb{R}^2 \text{ that are periodic of period } Y, \\
L^2_\#(Y) &:= \text{ } L^2(Y) \text{ functions that are } Y\text{-periodic,} \\
H^1_\#(Y) &:= \text{closure of } (C^\infty_\#(Y))_{\text{equivalence}}, \\
H^1_\#/\mathbb{R} &:= \text{quotient space}, \text{i.e. the space of equivalent class with respect to the relation } u \simeq v \iff u - v = \text{constant for all } u, v \in H^1_\#, \\
L^\infty(\Omega) &:= \{ f : \Omega \to \mathbb{R}, f \text{ measurable such that there exists a } C \in \mathbb{R} \text{ with } |f| \leq C, \text{ almost everywhere on } \Omega \}, \\
\|u\|_{L^p(\Omega; L^2(\Omega^\varepsilon))}^p &:= \int_{\Omega} \|u\|_{L^2(\Omega^\varepsilon)}^p dx, p \in [1, \infty).
\end{align*}
\]
4.2 Technical assumptions

One assumes the following:

\[ S := (0,T), T \in (0,\infty), \]
\[ D^\varepsilon, P^\varepsilon, L^\varepsilon \in L^\infty(S \times \Omega^\varepsilon), \]
\[ f \in L^2(S \times \Omega^\varepsilon), \]
\[ B \in L^2(S \times \partial\Omega^\varepsilon), \]
\[ g \in L^2(S \times \Omega^\varepsilon), \]
\[ h \in L^2(S \times \Gamma^\varepsilon), \]
\[ p \in L^2(S \times \Gamma^\varepsilon). \]

Assume \( h \) to be a linearisation of the surface reaction term \( W \). If \( h \) is non-linear, then other averaging techniques need to be applied, see for instance [6] and [7] (especially regarding the periodic unfolding technique). In what follows, we do not take special structures for \( f, g \) and \( h \), but we rather prefer to assume that they are given functions with the regularity prescribed above.

4.3 Microscopic model

The reaction-diffusion system with the corresponding boundary and initial boundary value is the following:

\[ \partial_t \Theta^\varepsilon + \text{div}(P^\varepsilon \Theta^\varepsilon - L^\varepsilon \nabla \Theta^\varepsilon) = f(\Theta^\varepsilon, \Psi^\varepsilon) \text{ in } \Omega, \]
\[ \partial_t \Psi^\varepsilon + \text{div}(P^\varepsilon \Psi^\varepsilon - D^\varepsilon \nabla \Psi^\varepsilon) = g(\Theta^\varepsilon, \Psi^\varepsilon) \text{ in } \Omega^\varepsilon_y, \]
\[ \partial_\varepsilon \overline{\eta}^\varepsilon = h(\Theta^\varepsilon, \Psi^\varepsilon) \text{ at } \Gamma^\varepsilon, \]

where \( D \) is the diffusion coefficient and (4.1), (4.2) and (4.3) are the ‘heat equation’, the ‘oxygen equation’ and the ‘burning rate’ respectively. The terms \( P^\varepsilon \Theta^\varepsilon \) and \( L^\varepsilon \Theta^\varepsilon \) are called ‘convection’ and ‘diffusion transport’ respectively. Furthermore the boundary conditions are:

\[ n \cdot D^\varepsilon \nabla \Psi^\varepsilon = \varepsilon p(\Theta^\varepsilon, \Psi^\varepsilon) \text{ at } \Gamma^\varepsilon, \]
\[ \Theta^\varepsilon = 0 \text{ at } \partial\Omega, \]
\[ \Psi^\varepsilon = b \text{ at } \partial\Omega, \]
\[ n \cdot (P^\varepsilon \Theta^\varepsilon - L^\varepsilon \nabla \Theta^\varepsilon) = \varepsilon B(\Theta^\varepsilon, \Psi^\varepsilon) \text{ at } \Gamma^\varepsilon, \]

where \( b \) is a non-negative constant. The initial conditions are:

\[ \Theta^\varepsilon(0, x) = \Theta^\varepsilon_0(x), \ x \in \Omega, \]
\[ \Psi^\varepsilon(0, x) = \Psi^\varepsilon_0(x), \ x \in \Omega^\varepsilon_y, \]
\[ R^\varepsilon(0, x) = R^\varepsilon_0(x), \ x \in \Gamma^\varepsilon. \]
4.3 Microscopic model

Note that $R^\varepsilon$ can be (if needed) completely decoupled from the whole system.

In order to homogenise the Dirichlet boundary condition (4.6), do the following transformation:

\begin{equation}
\Lambda^\varepsilon := \Psi^\varepsilon - b.
\end{equation}

Since $\Psi^\varepsilon = \Lambda^\varepsilon + b$, the system (4.1)-(4.10) becomes:

\begin{align*}
\partial_t \Theta^\varepsilon + \text{div}(P^\varepsilon \Theta^\varepsilon - L^\varepsilon \nabla \Theta^\varepsilon) &= f(\Theta^\varepsilon, \Lambda^\varepsilon + b) \text{ in } \Omega^\varepsilon, \\
\partial_t \Lambda^\varepsilon + \text{div}(P^\varepsilon (\Lambda^\varepsilon + b) - D^\varepsilon \nabla \Lambda^\varepsilon) &= g(\Theta^\varepsilon, \Lambda^\varepsilon + b) \text{ in } \Omega^\varepsilon_g, \\
\partial_t R^\varepsilon &= h(\Theta^\varepsilon, \Lambda^\varepsilon + b) \text{ at } \Gamma^\varepsilon, \\
n \cdot D^\varepsilon \nabla \Lambda^\varepsilon &= \varepsilon p(\Theta^\varepsilon, \Lambda^\varepsilon + b) \text{ at } \Gamma^\varepsilon, \\
\Lambda^\varepsilon &= 0 \text{ at } \partial \Omega, \\
n \cdot (P^\varepsilon \Theta^\varepsilon - L^\varepsilon \nabla \Theta^\varepsilon) &= \varepsilon B(\Theta^\varepsilon, \Lambda^\varepsilon + b) \text{ at } \Gamma^\varepsilon, \\
\Theta^\varepsilon(0, x) &= 0, \ x \in \bar{\Omega}^\varepsilon, \\
\Lambda^\varepsilon(0, x) &= 0, \ x \in \bar{\Omega}^\varepsilon_g, \\
R^\varepsilon(0, x) &= R^\varepsilon_0(x), \ x \in \Gamma^\varepsilon.
\end{align*}

We refer to the system (4.12) to (4.20) as the microscopic problem $(P^\varepsilon)$. 
Setting of the Microscopic Model Equations
Chapter 5

Well-posedness of \((P^\varepsilon)\)

In this chapter, we present our concept of weak solution to the microscopic model \((P^\varepsilon)\). We prove \(\varepsilon\)-independent energy bounds which show that the existence of weak solutions to \((P^\varepsilon)\) is expected to hold. Finally, we prove, using a Gronwall-like argument, that such solutions are unique.

5.1 Weak formulation of the microscopic problem \((P^\varepsilon)\)

Our concept of weak solutions of \((P^\varepsilon)\) is as follows:

**Definition 1.** A triplet of functions

\[
\begin{align*}
\Theta^\varepsilon & \in H^1(S;L^2(\Omega^\varepsilon)) \cap L^2(S;H^1(\Omega^\varepsilon)), \\
\Lambda^\varepsilon & \in H^1(S;L^2(\Omega_g^\varepsilon)) \cap L^2(S;H^1(\Omega_g^\varepsilon)), \\
R^\varepsilon(\cdot,x) & \in W^{1,\infty}(S) \text{ almost every } x \in \Gamma^\varepsilon,
\end{align*}
\]

is called a weak solution to \((4.12)-(4.20)\) if \((4.1)\) holds and for almost every \(t \in S\) the following identities hold:

\[
\begin{align*}
(5.1) & \quad \partial_t \Theta^\varepsilon, \varphi)_{L^2(\Omega^\varepsilon)} + \varepsilon(\nabla \Theta^\varepsilon, \nabla \varphi)_{L^2(\Omega^\varepsilon)} - (P^\varepsilon \Theta^\varepsilon, \nabla \varphi)_{L^2(\Omega^\varepsilon)} = (f, \varphi)_{L^2(\Omega^\varepsilon)}, \\
(5.2) & \quad \partial_t \Lambda^\varepsilon, \phi)_{L^2(\Omega_g^\varepsilon)} + (D^\varepsilon \nabla \Lambda^\varepsilon, \nabla \phi)_{L^2(\Omega_g^\varepsilon)} = (g, \phi)_{L^2(\Omega_g^\varepsilon)} + \varepsilon(p, \phi)_{L^2(\Gamma^\varepsilon)} + (P^\varepsilon(\Lambda^\varepsilon + b), \nabla \phi)_{L^2(\Omega_g^\varepsilon)},
\end{align*}
\]

for all \((\varphi, \phi) \in H^1(\Omega) \times H^1(\Omega_g)\), and for almost every \(x \in \Gamma^\varepsilon\) the ordinary differential equation

\[
\partial_t R^\varepsilon(\cdot,x) = h(\Theta^\varepsilon(\cdot,x), \Lambda^\varepsilon(\cdot,x) + b).
\]

Additionally, the following initial data are given:

\[
\begin{align*}
\Theta^\varepsilon(0,x) & := \Theta^\varepsilon_0(x), \ x \in \bar{\Omega}^\varepsilon, \\
\Lambda^\varepsilon(0,x) & := \Lambda^\varepsilon_0(x), \ x \in \bar{\Omega}_g^\varepsilon, \\
R^\varepsilon(0,x) & := R^\varepsilon_0(x), \ x \in \Gamma^\varepsilon.
\end{align*}
\]

In the following, we discuss the well-posedness of \((P^\varepsilon)\).
5.2 Basic estimates

5.2.1 Energy estimates

Lemma 5.2.1. There exist constants $C_1 > 0$ and $C_2 > 0$, independent of $\epsilon$, such that

\begin{align}
(5.3) & \quad \|\Theta^\epsilon\|_{L^2(0, T; H^1(\Omega^\epsilon))} + \|\partial_t \Theta^\epsilon\|_{L^2(0, T; L^2(\Omega^\epsilon))} \leq C_1, \\
(5.4) & \quad \|\Lambda^\epsilon\|_{L^2(0, T; H^1(\Omega^\epsilon^c))} + \|\partial_t \Lambda^\epsilon\|_{L^2(0, T; L^2(\Omega^\epsilon^c))} \leq C_2.
\end{align}

Proof. First of all, prove the energy estimate of the heat equation (5.3). Make use of the following estimate:

\[
\epsilon(B, \varphi)_{L^2(\Gamma^\epsilon)} \\
\leq \epsilon([B, \varphi]_{L^2(\Gamma^\epsilon)}) \\
\leq \epsilon\|B\|_{L^2(\Gamma^\epsilon)} \|\varphi\|_{L^2(\Gamma^\epsilon)} \\
\leq \epsilon\|B\|_{L^2(\Gamma^\epsilon)} \|\varphi\|_{H^1(\Omega^\epsilon)} \\
\leq \epsilon\delta\|\varphi\|^2_{H^1(\Omega^\epsilon)} + \epsilon c(\delta)\|B\|_{L^2(\Gamma^\epsilon)}^2 \\
\leq \epsilon\delta\|\varphi\|^2_{H^1(\Omega^\epsilon)} + c_1\epsilon(\delta)\|B\|_{L^2(\Gamma^\epsilon)}^2,
\]

for almost every $x \in \Gamma^\epsilon$. Young’s inequality (A.7) and the trace inequality (A.9) have been used into the last two inequalities. Then

\[
(5.5) \quad (\partial_t \Theta^\epsilon, \varphi) + \epsilon(B, \varphi)_{L^2(\Gamma^\epsilon)} + (L^\epsilon \nabla \Theta^\epsilon, \nabla \varphi) = (f, \varphi) + (P^\epsilon \Theta^\epsilon, \nabla \varphi).
\]

The term $\epsilon(B, \varphi)_{L^2(\Gamma^\epsilon)}$ is finite by the above estimate. Take $\varphi := \Theta^\epsilon$. Then

\[
(\partial_t \Theta^\epsilon, \Theta^\epsilon) + (L^\epsilon \nabla \Theta^\epsilon, \nabla \Theta^\epsilon) = (f, \Theta^\epsilon) + (P^\epsilon \Theta^\epsilon, \nabla \Theta^\epsilon).
\]

Use ellipticity for the term $(L^\epsilon \nabla \Theta^\epsilon, \nabla \Theta^\epsilon)$.

\[
\frac{d}{dt}\|\Theta^\epsilon\|^2 + l_0 \|\nabla \Theta^\epsilon\|^2 \leq \|f\|\|\Theta^\epsilon\| + \|P^\epsilon \Theta^\epsilon\|\|\nabla \Theta^\epsilon\|
\leq \frac{\|f\|^2}{2} + \|\Theta^\epsilon\|^2 + \delta \|\nabla \Theta^\epsilon\|^2 + c(\delta)\|P^\epsilon\|^2_{L^\infty} \|\Theta^\epsilon\|^2,
\]

So,

\[
(5.6) \quad \frac{d}{dt}\|\Theta^\epsilon\|^2 + (l_0 - \delta) \|\nabla \Theta^\epsilon\|^2 \leq \frac{\|f\|^2}{2} + (\frac{1}{2} + c(\delta)\|P^\epsilon\|^2_{L^\infty})\|\Theta^\epsilon\|^2.
\]

Choose $\delta \in (0, l_0)$, with $l_0 > 0$. Apply (A.6) to the last equation. Then Gronwall’s inequality \(\eta'(t) \leq \phi(t)\eta(t) + \psi(t)\), with $\eta(t) = \|\Theta^\epsilon(t)\|^2 \geq 0$, $\phi(t) = 1 + 2c(\delta)\|P^\epsilon\|^2_{L^\infty}$ and $\psi(t) = \|f\|^2$. Then

\[
\|\Theta^\epsilon\|^2 = \eta(t) \leq \exp\left(\int_0^t 1 + 2c(\delta)\|P^\epsilon\|^2_{L^\infty} d\tau\right) \left(\|\Theta^\epsilon(0)\|^2 + \int_0^t \|f\|^2 d\tau\right)
\leq \exp\left(\int_0^t (T + 2c(\delta)\|P^\epsilon\|^2_{L^\infty}) d\tau\right) \left(\|\Theta^\epsilon(0)\|^2 + \int_0^T \|f\|^2 d\tau\right)
\leq C(T, c(\delta), P^\epsilon, f),
\]

where $C(T, c(\delta), P^\epsilon, f)$ is a constant depending on $T$, $c(\delta)$, $P^\epsilon$ and $f$. This completes the proof of the energy estimate (5.3).
5.2 Basic estimates

with \( T > 0, P^\varepsilon, L^\varepsilon \in L^\infty(S \times \Omega^\varepsilon), f \in L^2(S; L^2(\Omega^\varepsilon)) \).

Then \( \Theta^\varepsilon \in L^2(S; L^2(\Omega^\varepsilon)) \) and \( \partial_t \Theta^\varepsilon \in L^2(S; L^2(\Omega^\varepsilon)) \).

Integration of (5.6) on \((0, t)\) yields

\[
\int_0^t \| \nabla \Theta^\varepsilon \|^2 \, dt \leq c_1,
\]

where \( c_1 = \frac{1}{2(a_0 - \delta)} \left( \int_0^t \| f \|^2 \, dt + \left( \frac{1}{2} + c(\delta) \| P^\varepsilon \|_{L^\infty} \right) \right) < \infty \), because \( f, \Theta^\varepsilon \in L^2(S; L^2(\Omega^\varepsilon)) \) and \( P^\varepsilon \in L^\infty(S \times \Omega^\varepsilon) \).

Second, prove the energy estimate of the oxygen equation 5.4. Make use of the following estimate:

\[
\varepsilon(p, \phi)_{L^2(\Gamma^\varepsilon)} + (D^\varepsilon \nabla \Lambda^\varepsilon, \nabla \phi) = (g, \phi) + \varepsilon(p, \phi)_{L^2(\Gamma^\varepsilon)} + (P^\varepsilon(\Lambda^\varepsilon + b), \nabla \phi).
\]

The term \( \varepsilon(p, \phi)_{L^2(\Gamma^\varepsilon)} \) is finite by the above estimate. Take in (5.2) as test function \( \phi := \Lambda^\varepsilon \). Then one has the following:

\[
(\partial_t \Lambda^\varepsilon, \Lambda^\varepsilon) + (D^\varepsilon \nabla \Lambda^\varepsilon, \nabla \Lambda^\varepsilon) = (g, \Lambda^\varepsilon) + (P^\varepsilon(\Lambda^\varepsilon + b), \nabla \Lambda^\varepsilon).
\]

Using the ellipticity of the term \( (D^\varepsilon \nabla \Lambda^\varepsilon, \nabla \Lambda^\varepsilon) \), this leads to

\[
\frac{\eta(\varepsilon)}{2^{a_0}} \| \nabla \Lambda^\varepsilon \|^2 + d_0 \| \nabla \Lambda^\varepsilon \|^2 \leq \| g \| \| \Lambda^\varepsilon \| + \| P^\varepsilon \| (\| \Lambda^\varepsilon \| + \| b \|) \| \nabla \Lambda^\varepsilon \| \leq \frac{\| g \|^2 + \| b \|^2 + \| P^\varepsilon \|_{L^\infty}^2 \| \nabla \Lambda^\varepsilon \|^2 + \delta \| \nabla \Lambda^\varepsilon \|^2 + c(\delta) \| P^\varepsilon \|_{L^\infty}^2 \| \Lambda^\varepsilon \|^2, \]

or

\[
\frac{\eta(\varepsilon)}{2^{a_0}} \| \nabla \Lambda^\varepsilon \|^2 + (d_0 - \delta - \frac{1}{2}) \| \nabla \Lambda^\varepsilon \|^2 \leq \frac{\| g \|^2 + \| b \|^2 + \| P^\varepsilon \|_{L^\infty}^2}{2^{a_0}} \left( \frac{1}{2} + c(\delta) \| P^\varepsilon \|_{L^\infty}^2 \right) \| \Lambda^\varepsilon \|^2.
\]

Choose \( \delta \in (0, d_0 - \frac{1}{2}) \), with \( d_0 > \frac{1}{2} \). Apply (A.6) to the last equation. Then \( \eta'(t) \leq \phi(t) \eta(t) + \psi(t) \), with

\[
\begin{align*}
\eta(t) & := \frac{\| \Lambda^\varepsilon(0) \|^2}{2}, \\
\phi(t) & := \frac{1}{2} + c(\delta) \| P^\varepsilon \|_{L^\infty}^2, \\
\psi(t) & := \frac{\| g \|^2 + \| b \|^2 + \| P^\varepsilon \|_{L^\infty}^2}{2^{a_0}}.
\end{align*}
\]

Then

\[
\frac{\| \Lambda^\varepsilon \|^2}{2} = \eta(t) \leq \exp\left( \frac{\int_0^t + c(\delta) \| P^\varepsilon \|_{L^\infty}^2 \, dt}{} \right) \left( \| \Lambda^\varepsilon(0) \|^2 + \frac{1}{2} \int_0^t \| g \|^2 + \| b \|^2 + \| P^\varepsilon \|_{L^\infty}^2 \, dt \right) = \frac{1}{2} \exp\left( \frac{\int_0^t + c(\delta) \| P^\varepsilon \|_{L^\infty}^2 \, dt}{} \right) \left( \| b \|^2 t + \frac{1}{2} \int_0^t \| g \|^2 + \| P^\varepsilon \|_{L^\infty}^2 \, dt \right)
\]
So, by choosing $\gamma$ in Proposition 5.2.2. Note that here we need $20$ well-posedness of $\int_0^T \|g\|^2 + \|\Lambda^e\|^2 + 2\gamma(T) \|\Lambda^e\|d\tau - \|\Lambda^e(t)\|^2 < \infty$,

Integration of (5.8) on $(0,t)$ gives

$$
\int_0^t \|\nabla \Lambda^e\|^2 d\tau \leq c_2,
$$

where

$$
c_2 = \frac{1}{8d_0^2-2\delta -1} \left( \|b\|^2 + \int_0^t \|g\|^2 + \|P^e\|_{L^\infty}^2 + \|\Lambda^e\|^2 + \|P^e\|_{L^\infty}^2 \|\Lambda^e\| d\tau - \|\Lambda^e(t)\|^2 \right) < \infty,
$$

since $\Lambda^e \in L^2(S; L^2(\Omega^e))$, $0 \leq b < \infty$ and $P^e \in L^\infty(S \times \Omega^e)$. \qed

Note that here we need $2d_0 - 2\delta - 1 \geq 0$.

### 5.2.2 Uniqueness of weak solutions

**Proposition 5.2.2.** If (3.1)-(3.11) hold, then the weak solution to (4.12)-(4.20) is unique.

**Proof.** First of all, start with the heat equation. Let $(\Theta^e_1, \Lambda^e_1, R^e_1)$ and $(\Theta^e_2, \Lambda^e_2, R^e_2)$ be two weak solutions to $(P^e)$ in the sense of Definition 1. Subtracting them gives

$$
\partial_t (\Theta^e_2 - \Theta^e_1, \varphi)_{L^2(\Omega^e)} + \varepsilon(B(\Theta^e_2, \Lambda^e_2 + b) - B(\Theta^e_1, \Lambda^e_1 + b), \varphi)_{L^2(\Gamma^e)}
+ (L^e \nabla (\Theta^e_2 - \Theta^e_1), \nabla \varphi)_{L^2(\Gamma^e)} - (P^e(\Theta^e_2 - \Theta^e_1), \nabla \varphi) = 0.
$$

Take $\varphi := \Theta^e_2 - \Theta^e_1$. Then

$$
\frac{1}{2}\partial_t \|\Theta^e_2 - \Theta^e_1\|_{L^2(\Omega^e)}^2 + L^e(\nabla (\Theta^e_2 - \Theta^e_1), \nabla (\Theta^e_2 - \Theta^e_1))_{L^2(\Gamma^e)} = -\varepsilon(B(\Theta^e_2, \Lambda^e_2 + b) - B(\Theta^e_1, \Lambda^e_1 + b))_{L^2(\Gamma^e)} + (P^e(\Theta^e_2 - \Theta^e_1), \nabla (\Theta^e_2 - \Theta^e_1))_{L^2(\Gamma^e)}.
$$

Take $v := \Theta^e_2 - \Theta^e_1$ and $z := \Lambda^e_2 - \Lambda^e_1$. Then

$$
\frac{1}{2}\partial_t \|v\|_{L^2(\Gamma^e)}^2 + l_0 \|\nabla v\|_{L^2(\Gamma^e)}^2
\leq \varepsilon \left( \int \left( |v|^2 + |z|^2 \right) d\sigma \right)^{1/2} + \delta_1 \|\nabla v\|_{L^2(\Gamma^e)}^2 + c(\delta_1) \|P^e\|_{L^\infty(\Gamma^e)} \|v\|_{L^2(\Gamma^e)}^2
\leq c_1 \varepsilon \left( \int \left( |v|^2 + |z|^2 \right) d\sigma \right)^{1/2} + \delta_1 \|\nabla v\|_{L^2(\Gamma^e)}^2 + c(\delta_1) \|P^e\|_{L^\infty(\Gamma^e)} \|v\|_{L^2(\Gamma^e)}^2
\leq c_1 \varepsilon \left( \int \left( |v|^2 + |z|^2 \right) d\sigma \right)^{1/2} + \delta_1 \|\nabla v\|_{L^2(\Gamma^e)}^2 + c(\delta_1) \|P^e\|_{L^\infty(\Gamma^e)} \|v\|_{L^2(\Gamma^e)}^2
\leq c_1 \varepsilon \left( \int \left( |v|^2 + |z|^2 \right) d\sigma \right)^{1/2} + \delta_1 \|\nabla v\|_{L^2(\Gamma^e)}^2 + c(\delta_1) \|P^e\|_{L^\infty(\Gamma^e)} \|v\|_{L^2(\Gamma^e)}^2
\leq (c_1 + c(\delta_1)) \|P^e\|_{L^\infty(\Gamma^e)} \|v\|_{L^2(\Gamma^e)}^2 + (c_1 + c(\delta_1)) \|\nabla v\|_{L^2(\Gamma^e)}^2 + c(\delta_1) \|\nabla v\|_{L^2(\Gamma^e)}^2 + \delta_1 \|\nabla v\|_{L^2(\Gamma^e)}^2
$$

So, by choosing $\delta_1 \in (0, l_0 - \delta_1)$ we have:

$$
\left( \frac{1}{2} \partial_t \|v\|_{L^2(\Gamma^e)}^2 + (l_0 - \delta_1) \|\nabla v\|_{L^2(\Gamma^e)}^2 \right)
\leq \left( c_1 + c(\delta_1) \right) \frac{1}{2} \|P^e\|_{L^\infty(\Gamma^e)} \|v\|_{L^2(\Gamma^e)}^2 + c(\delta_1) \|\nabla v\|_{L^2(\Gamma^e)}^2 + \delta_1 \|\nabla v\|_{L^2(\Gamma^e)}^2.
$$
5.2 Basic estimates

The last term in 5.9 compensates with the elliptic term from the inequality for \( \Lambda_2^\circ - \Lambda_1^\circ =: z \), which can be treated similarly (playing with the Cauchy-Schwarz (A.1) and Young’s inequality (A.7) as well as the above inequality for perforated media).

In a similar way as above, but now taking \( \phi := z = \Lambda_2^\circ - \Lambda_1^\circ \) and choosing \( \delta_2 \in (0, d_0 - \tilde{e}_2) \), we have for the oxygen equation the following:

\[
\begin{align*}
\frac{1}{2} \partial_t \|z\|_{L^2(\Omega_t)}^2 + (d_0 - \tilde{e}_2 - \delta_2) &\|\nabla z\|_{L^2(\Omega_t)}^2 \\
\leq &\ (\tilde{e}_2 + c(\delta_2) \|P^c\|_{L^\infty(\Omega_t)}^2) \|z\|_{L^2(\Omega_t)}^2 + \tilde{e}_2 \|v\|_{L^2(\Omega_t)}^2 + \|\nabla v\|_{L^2(\Omega_t)}^2.
\end{align*}
\]

Adding up (5.9) and (5.10) gives

\[
\frac{1}{2} \partial_t (\|v\|_{L^2(\Omega_t)}^2 + \|z\|_{L^2(\Omega_t)}^2) + \min \{c_3, c_4\} (\|\nabla v\|_{L^2(\Omega_t)}^2 + \|\nabla z\|_{L^2(\Omega_t)}^2)
\leq \max \{c_5, c_6\} (\|v\|_{L^2(\Omega_t)}^2 + \|z\|_{L^2(\Omega_t)}^2),
\]

where

\[
\begin{align*}
c_3 &:= l_0 - \tilde{e}_1 - \delta_1 - \tilde{e}_2 \geq 0, \\
c_4 &:= d_0 - \tilde{e}_2 - \delta_2 - \tilde{e}_1 \geq 0, \\
c_5 &:= \tilde{e}_1 + \tilde{e}_2 + c(\delta_1) \|P^c\|_{L^\infty(\Omega_t)}^2 \geq 0, \\
c_6 &:= \tilde{e}_1 + \tilde{e}_2 + c(\delta_2) \|P^c\|_{L^\infty(\Omega_t)}^2 \geq 0.
\end{align*}
\]

Finally, taking \( \eta(t) := \|v\|_{L^2(\Omega_t)}^2 + \|z\|_{L^2(\Omega_t)}^2 \), and applying Gronwall’s inequality (A.6) to the above structure (5.11), we get

\[
\eta(t) = \|v\|_{L^2(\Omega_t)}^2 + \|z\|_{L^2(\Omega_t)}^2 \leq \exp \left( \int_0^t \phi_1(\tau) d\tau \right) \left( \eta(0) + \int_0^t \psi(\tau) d\tau \right).
\]

Then

\[
0 \leq \|v\|_{L^2(\Omega_t)}^2 + \|z\|_{L^2(\Omega_t)}^2 \leq 0
\]

and, consequently,

\[
\|v\|_{L^2(\Omega_t)}^2 + \|z\|_{L^2(\Omega_t)}^2 = 0
\]

for all \( t \in S \). Hence, (5.12) leads to \( \Theta_2^\circ = \Theta_1^\circ \) and \( \Lambda_2^\circ = \Lambda_1^\circ \) almost everywhere in space, which shows the uniqueness of weak solutions to (4.12)-(4.20).
Well-posedness of $(P^x)$
Chapter 6

Rigorous Passing to the Homogenisation Limit $\varepsilon \to 0$

In this chapter, we use the concept of weak solutions presented in chapter five to pass to the homogenisation limit $\varepsilon \to 0$. The main results of this chapter include existence and uniqueness of solutions to the macroscopic (homogenised) problem ($P^0$).

6.1 Introduction

Studying homogenisation limits for parabolic equations requires the concept of two-scale convergence including the time dependence. We start off with giving definitions and theorems about the two-scale convergence. We mainly use [23] and [2].

**Definition 2.** A sequence of functions $u^\varepsilon$ in $L^2(S; L^2(\Omega))$ is said to two-scale converge to a limit function $u^0(t,x,y)$ in $L^2(S; L^2(\Omega \times Y))$ if and only if

$$\lim_{\varepsilon \to 0} \int_s \int_u u^\varepsilon(t,x)\rho(t,x,\frac{x}{\varepsilon})dxdt = \frac{1}{\varepsilon^2} \int_s \int_y u^0(t,x,y)\rho(t,x,y)dydxdt$$

for all $\rho$ in $C^\infty_0(S \times \Omega; C^\infty_0(Y))$. If ($u^\varepsilon$) satisfies (6.1), then this is denoted by $u^\varepsilon \Rightarrow u^0$ and one says that the convergence takes place in $L^2(S;\Omega \times Y)$.

**Theorem 6.1.1** (Two-scale compactness). Let $u^\varepsilon$ be a uniformly bounded sequence in $L^2(S; L^2(\Omega))$. Then there exists a subsequence such that it two-scale converges to a limit function $u^0$ in $L^2(S; L^2(\Omega \times Y))$.

**Theorem 6.1.2.** The following statements hold:

- (i) Let $u^\varepsilon$ be a bounded sequence of functions in $L^2(S; H^1(\Omega))$ which converges weakly to a limit function $u$ in $L^2(S; H^1(\Omega))$. Then there exists a subsequence $u^{\varepsilon}_r$ which two-scale converges to $u$ and $\nabla u^{\varepsilon}$ two-scale converges to $\nabla x u(t,x) + \nabla_y u^1(t,x)$ with $u^1$ in $L^2(S \times \Omega; H^1_0(Y)/\mathbb{R})$.

- (ii) Let $u^\varepsilon$ be a bounded sequence in $L^2(S \times \Omega)$ such that $\varepsilon \nabla u^\varepsilon$ is also bounded in $[L^2(S \times \Omega)]^n$. Then there exists a two-scale limit $u^0(x,y)$ in $L^2(S \times \Omega; H^1_0(Y))$ such that at least a subsequence $u^{\varepsilon}_r$ of $u^\varepsilon$ converges to $u^0(t,x,y)$ and $\varepsilon \nabla u^{\varepsilon}_r$ converges to $\nabla_y u^0(t,x,y)$.

Remark: $\nabla u^{\varepsilon}_r \Rightarrow \nabla u^0 + \nabla_y u^1$ in $[L^2(S;\Omega \times Y)]^n$. 
Lemma 6.2.1 (Extension Lemma). The proof of part 1 comes from [14].

1. Assume that the data are sufficiently smooth. Then a neighborhood also holds. So, the first part of the Lemma has been proved.

2. On the one hand the inequality holds. On the other hand, the equality also holds. So, the first part of the Lemma has been proved.
2. Consider summation over \( k \) such that \( \varepsilon Y^k \subset \Omega \). Take \( d \in \{2, 3\} \). Then the following estimates hold:

\[
\|\tilde{\varepsilon}\|^2_{H^1(\Omega)} \leq \sum_{k \in \mathbb{Z}^d} \int |(\tilde{\varepsilon}(x)|^2 + |\nabla \tilde{\varepsilon}(x)|^2| dx
\]

\[
= \sum_{k \in \mathbb{Z}^d} c^d \int ||\varepsilon(\varepsilon z)|^2 + \frac{1}{\varepsilon^2} |\nabla \varepsilon(\varepsilon z)|^2| dz
\]

\[
\leq c \sum_{k \in \mathbb{Z}^d} \int |\varepsilon(\varepsilon z)|^2 + \frac{1}{\varepsilon^2} |\nabla \varepsilon(\varepsilon z)|^2| dz
\]

\[
= c\|\varepsilon\|^2_{H^1(\Omega)},
\]

where the substitutions \( z = \frac{x}{\varepsilon} \) and \( dx = \varepsilon^d dz \) have been used in (6.5). Inequality (6.6) is due to Lemma 6.2.1 part 1 (extension into the interior).

From now on, we denote both functions and their extensions by \( \Theta^\varepsilon \), \( \Lambda^\varepsilon \) and \( R^\varepsilon \).

### 6.3 Passing to \( \varepsilon \to 0 \) via two-scale convergence

The weak formulation of (5.1) is

\[
\int \int \partial_t \Theta^\varepsilon \varphi dx dt + \varepsilon \int \int B \varphi d\sigma dt + \int \int L^\varepsilon \nabla \Theta^\varepsilon \nabla \varphi dx dt - \int \int P^\varepsilon \Theta^\varepsilon \nabla \varphi dx dt
\]

(6.7) \[
= \int \int f \varphi dx dt.
\]

for all \( \varphi \in H^1(S \times \Omega) \). Choose in (6.7) the test function \( \varphi(t, x) = \varphi^0(t, x) + \varepsilon \varphi^1(t, x, \frac{x}{\varepsilon}) \), where \( \varphi^0 \in C^\infty_0(S \times \Omega) \) and \( \varphi^1 \in C^\infty_0(S \times \Omega; C^\infty_0(Y)) \). So (6.7) becomes

\[
\int \int \partial_t \Theta^\varepsilon(\varphi^0(t, x) + \varepsilon \varphi^1(t, x, \frac{x}{\varepsilon}))dx dt + \varepsilon \int \int B(\varphi^0(t, x) + \varepsilon \varphi^1(t, x, \frac{x}{\varepsilon}))d\sigma dt
\]

\[
+ \varepsilon \int \int L^\varepsilon \nabla \Theta^\varepsilon(\nabla \varphi^0(t, x) + \nabla \varphi^1(t, x, \frac{x}{\varepsilon}) + \varepsilon \nabla \varphi^1(t, x, \frac{x}{\varepsilon}))dx dt
\]

\[
- \varepsilon \int \int P^\varepsilon \Theta^\varepsilon(\nabla \varphi^0(t, x) + \nabla \varphi^1(t, x, \frac{x}{\varepsilon}) + \varepsilon \nabla \varphi^1(t, x, \frac{x}{\varepsilon}))dx dt
\]

\[
= \int \int f(\varphi^0(t, x) + \varepsilon \varphi^1(t, x, \frac{x}{\varepsilon}))dx dt.
\]

Arranging the terms in the above expression with respect to the orders of \( \varepsilon \), this gives

\[
\varepsilon^2 \int \int B \varphi^1(t, x, \frac{x}{\varepsilon})d\sigma dt + \varepsilon \int \int \partial_t \Theta^\varepsilon \varphi^1(t, x, \frac{x}{\varepsilon})dx dt + \varepsilon \int \int B \varphi^0(t, x)d\sigma dt
\]

\[
+ \varepsilon \int \int L^\varepsilon \nabla \Theta^\varepsilon(\nabla \varphi^0(t, x) + \nabla \varphi^1(t, x, \frac{x}{\varepsilon}))dx dt - \varepsilon \int \int P^\varepsilon \Theta^\varepsilon(\nabla \varphi^0(t, x) + \nabla \varphi^1(t, x, \frac{x}{\varepsilon}))dx dt
\]

\[
+ \varepsilon \int \int L^\varepsilon \nabla \Theta^\varepsilon(\nabla \varphi^0(t, x) + \nabla \varphi^1(t, x, \frac{x}{\varepsilon}))dx dt - \varepsilon \int \int P^\varepsilon \Theta^\varepsilon(\nabla \varphi^0(t, x)dx dt
\]

\[
- \varepsilon \int \int P^\varepsilon \Theta^\varepsilon(\nabla \varphi^1(t, x, \frac{x}{\varepsilon}))dx dt - \varepsilon \int \int f \varphi^0(t, x)dx dt - \varepsilon \int \int f \varphi^1(t, x, \frac{x}{\varepsilon})dx dt = \sum_{i=1}^{12} I_i = 0.
\]
The terms $I_1, \cdots, I_{12}$ are defined as follows:

\[
I_1 := \varepsilon^2 \int_\Gamma^* \int S \sigma^\varepsilon(t, x, \frac{\mathbf{z}}{\varepsilon}) d\sigma^\varepsilon dt;
\]

\[
I_2 := \varepsilon \int_\Gamma^* \int S \sigma^\varepsilon(t, x, \frac{\mathbf{z}}{\varepsilon}) d\sigma^\varepsilon dt;
\]

\[
I_3 := \varepsilon \int_\Gamma^* \int S \sigma^\varepsilon(t, x) d\sigma^\varepsilon dt;
\]

\[
I_4 := \varepsilon \int_\Omega \int S \sigma^\varepsilon(t, x, \frac{\mathbf{z}}{\varepsilon}) d\sigma^\varepsilon dt;
\]

\[
I_5 := -\varepsilon \int_\Omega \int S \sigma^\varepsilon(t, x, \frac{\mathbf{z}}{\varepsilon}) d\sigma^\varepsilon dt;
\]

\[
I_6 := \int_\Gamma^* \int S \sigma^\varepsilon(t, x) d\sigma^\varepsilon dt;
\]

\[
I_7 := \int_\Gamma^* \int S \sigma^\varepsilon(t, x, \frac{\mathbf{z}}{\varepsilon}) d\sigma^\varepsilon dt;
\]

\[
I_8 := \int_\Omega \int S \sigma^\varepsilon(t, x, \frac{\mathbf{z}}{\varepsilon}) d\sigma^\varepsilon dt;
\]

\[
I_9 := -\int_\Omega \int S \sigma^\varepsilon(t, x, \frac{\mathbf{z}}{\varepsilon}) d\sigma^\varepsilon dt;
\]

\[
I_{10} := -\int_\Omega \int S \sigma^\varepsilon(t, x, \frac{\mathbf{z}}{\varepsilon}) d\sigma^\varepsilon dt;
\]

\[
I_{11} := -\int_\Omega \int S \sigma^\varepsilon(t, x, \frac{\mathbf{z}}{\varepsilon}) d\sigma^\varepsilon dt;
\]

\[
I_{12} := -\varepsilon \int_\Omega \int S \sigma^\varepsilon(t, x, \frac{\mathbf{z}}{\varepsilon}) d\sigma^\varepsilon dt.
\]

**Lemma 6.3.1.** The terms $I_1, \cdots, I_{12}$ are bounded.

**Proof.** It is sufficient to only prove over $\Omega$ or $\Gamma^\varepsilon$, because $S = (0, T)$ with $0 < T < \infty$. Suppose $0 < \varepsilon < \infty$.

To ensure the boundedness of $I_1$, first observe that

\[
|\varepsilon \int_\Gamma^* \int S \sigma^\varepsilon(t, x, \frac{\mathbf{z}}{\varepsilon}) d\sigma^\varepsilon| \leq \sqrt{\varepsilon} \| B \|_{L^2(\Gamma^\varepsilon)} \sqrt{\varepsilon} \| \varphi^\varepsilon \|_{L^2(\Gamma^\varepsilon)} \leq \frac{\varepsilon}{2} (\| B \|_{L^2(\Gamma^\varepsilon)}^2 + \| \varphi^\varepsilon \|_{L^2(\Gamma^\varepsilon)}^2)
\]

\[
\leq \frac{1}{2} (\| B \|_{H^1(\Gamma^\varepsilon)}^2 + \| \varphi^\varepsilon \|_{H^1(\Gamma^\varepsilon)}) < \infty.
\]

Then $|I_1| \leq \frac{\varepsilon}{2} (\| B \|_{H^1(\Gamma^\varepsilon)}^2 + \| \varphi^\varepsilon \|_{H^1(\Gamma^\varepsilon)}) < \infty$. Note that the first inequality is a straightforward consequence of the trace inequality for perforated media (see Lemma A.8).

$I_3$ has been dealt in a similar way, i.e.

\[
|I_3| = |\varepsilon \int_\Gamma^* \int S \sigma^\varepsilon(t, x) d\sigma^\varepsilon| \leq \frac{1}{2} (\| B \|_{H^1(\Gamma^\varepsilon)}^2 + \| \varphi^\varepsilon \|_{H^1(\Gamma^\varepsilon)}) < \infty.
\]
To bound $I_2$ from above, inequalities (A.1), (A.2) and (A.4) have been used. Then it follows that
\[
\frac{1}{\varepsilon} |I_2| = \frac{1}{\varepsilon} \left| \int \partial_t \Theta \varphi^0(t, x, \frac{x}{\varepsilon}) \, dx \right| \leq \| \partial_t \Theta \|_{L^2(\Omega)} \| \varphi^0 \|_{L^2(\Omega)} + \frac{1}{\varepsilon} \| \Theta \|_{L^2(\Omega)}^2 \leq \frac{1}{\varepsilon} \left( \| \partial_t \Theta \|_{H^1(\Omega)} + \| \varphi^1 \|_{L^2(\Omega)} \right).
\]
Then, it follows that $|I_2| \leq \frac{1}{\varepsilon} \left( \| \partial_t \Theta \|_{H^1(\Omega)} + \| \varphi^1 \|_{H^1(\Omega)} \right) < \infty$.

For $I_4$, the following estimates have been obtained:
\[
\frac{1}{\varepsilon} |I_4| = \frac{1}{\varepsilon} \left| \int L^\varepsilon \nabla \Theta \nabla_x \varphi^1(t, x, \frac{x}{\varepsilon}) \, dx \right| \leq \| L^\varepsilon \|_{L^\infty(\Omega)} \| \nabla \Theta \|_{L^2(\Omega)} \| \nabla_x \varphi^1 \|_{L^2(\Omega)} \leq \frac{1}{\varepsilon} \left( \| L^\varepsilon \|_{H^1(\Omega)} + \| \varphi^1 \|_{H^1(\Omega)} \right) < \infty,
\]
because $L^\varepsilon \in L^\infty(S \times \Omega)$ has been given. This gives
\[
|I_4| \leq \frac{1}{\varepsilon} \max_{x \in \Omega} |L^\varepsilon(t, \frac{x}{\varepsilon})| \left( \| \Theta \|_{H^1(\Omega)}^2 + \| \varphi^1 \|_{H^1(\Omega)}^2 \right) < \infty.
\]

For $I_5$, it follows that
\[
|I_5| = \varepsilon \left| \int P^\varepsilon \Theta \varphi^1(t, x, \frac{x}{\varepsilon}) \, dx \right| \leq \frac{1}{\varepsilon} \max_{x \in \Omega} |P^\varepsilon(t, \frac{x}{\varepsilon})| \left( \| \Theta \|_{H^1(\Omega)}^2 + \| \varphi^1 \|_{H^1(\Omega)}^2 \right) < \infty,
\]
which is similar to $I_4$.

The terms $I_6$, $I_7$ and $I_8$, and $I_9$ and $I_{10}$ are similar to $I_2$, $I_4$ and $I_5$, respectively.

For $I_{11}$, it follows that
\[
\frac{1}{\varepsilon} |I_{11}| = \frac{1}{\varepsilon} \left| \int \frac{f \varphi^0(t, x) \, dx}{\Omega} \right| \leq \| f \|_{L^2(\Omega)} \| \varphi^0 \|_{L^2(\Omega)} \leq \frac{1}{\varepsilon} \left( \| f \|_{L^2(\Omega)} + \| \varphi^0 \|_{L^2(\Omega)} \right)
\]
\[
\leq \frac{1}{2} \left( \| f \|_{H^1(\Omega)}^2 + \| \varphi^0 \|_{H^1(\Omega)} \right) < \infty.
\]
Then it follows that $|I_{11}| \leq \frac{1}{2} \left( \| f \|_{H^1(\Omega)}^2 + \| \varphi^0 \|_{H^1(\Omega)}^2 \right) < \infty$.

The term $I_{12}$ is similar to $I_{11}$.

Based on Lemma 6.3.1, the following two-scale limits can be deduced:
\[
\lim_{\varepsilon \to 0} I_1 = \lim_{\varepsilon \to 0} I_2 = \lim_{\varepsilon \to 0} I_4 = \lim_{\varepsilon \to 0} I_5 = \lim_{\varepsilon \to 0} I_{12} = 0;
\]
\[
\lim_{\varepsilon \to 0} I_3 = \frac{1}{\varepsilon} \int \int \int B(x, y) \varphi^0(t, x) \, d\sigma \, dx \, dt;
\]
\[
\lim_{\varepsilon \to 0} I_6 = \frac{1}{\varepsilon} \int \int \int \partial_t \Theta \varphi^0(t, x, y) \, d\gamma \, dx \, dt;
\]
\[
\lim_{\varepsilon \to 0} (I_7 + I_8) = \frac{1}{\varepsilon} \int \int \int L(t, y)(\nabla_x \Theta + \nabla_y \Theta)(t, x, y) \varphi^0(t, x) + \nabla^y \varphi^1(t, x, y) \, dy \, dx \, dt;
\]
\[
\lim_{\varepsilon \to 0} (I_9 + I_{10}) = \frac{1}{\varepsilon} \int \int \int P(t, y) \Theta \varphi^0(t, x) + \nabla^y \varphi^1(t, x, y) \, dy \, dx \, dt;
\]
\[
\lim_{\varepsilon \to 0} I_{11} = \int f \varphi^0(t, x) \, dx.
\]
where the asymptotic expansion $\Theta^\varepsilon(t, x) := \Theta^0(t, x, y) + \varepsilon \Theta^1(t, x, y) + \varepsilon^2 \Theta^2(t, x, y) + \ldots$ has been used.

The treatment via the two-scale convergence for the oxygen equation is similar to the one of the heat equation. Integrating (5.2), then it follows that

\begin{align*}
(6.8) &\quad \int \int_{\Omega_g} \partial_t \Lambda^\varepsilon \phi \, dx \, dt - \varepsilon \int \int_{\Gamma^*} p \phi \sigma^* \, dt + \int \int_{\Omega_g} D^\varepsilon \nabla \Lambda^\varepsilon \nabla \phi \, dx \, dt - \int \int_{\Omega_g} P^\varepsilon(\Lambda^\varepsilon + b) \nabla \phi \, dx \, dt \\
&= \int \int_{\Omega_g} g \phi \, dx \, dt
\end{align*}

for all $\phi \in H^1(S \times \Omega_g)$. In (6.8) the test function $\phi(t, x) = \phi^0(t, x) + \varepsilon \phi^1(t, x, \tilde{z})$ has been chosen, where $\phi^0 \in C_0^\infty(S \times \Omega_g)$ and $\phi^1 \in C_0^\infty(S \times \Omega_g, C_0^\infty(Y))$. So, (6.8) becomes

\begin{align*}
&\int \int_{\Omega_g} \partial_t \Lambda^\varepsilon (\phi^0(t, x) + \varepsilon \phi^1(t, x, \tilde{z})) \, dx \, dt - \varepsilon \int \int_{\Gamma^*} p(\phi^0(t, x) + \varepsilon \phi^1(t, x, \tilde{z})) \, dt \\
&+ \int \int_{\Omega_g} D^\varepsilon \nabla \Lambda^\varepsilon (\nabla x \phi^0(t, x) + \nabla_y \phi^1(t, x, \tilde{z}) + \varepsilon \nabla_x \phi^1(t, x, \tilde{z})) \, dx \, dt \\
&- \int \int_{\Omega_g} P^\varepsilon(\Lambda^\varepsilon + b)(\nabla x \phi^0(t, x) + \nabla_y \phi^1(t, x, \tilde{z}) + \varepsilon \nabla_x \phi^1(t, x, \tilde{z})) \, dx \, dt \\
&= \int \int_{\Omega_g} g(\phi^0(t, x) + \varepsilon \phi^1(t, x, \tilde{z})) \, dx \, dt.
\end{align*}

Arranging the terms in the above expression with respect to the corresponding orders of $\varepsilon$, it follows that

\begin{align*}
&- \varepsilon^2 \int \int_{\Omega_g} p \phi^1(t, x, \tilde{z}) \, dx \, dt + \varepsilon \int \int_{\Omega_g} \partial_t \Lambda^\varepsilon \phi^1(t, x, \tilde{z}) \, dx \, dt - \varepsilon \int \int_{\Omega_g} p \phi^0(t, x) \, dx \, dt \\
&+ \varepsilon \int \int_{\Omega_g} D^\varepsilon \nabla \Lambda^\varepsilon \nabla_x \phi^1(t, x, \tilde{z}) \, dx \, dt - \varepsilon \int \int_{\Omega_g} P^\varepsilon(\Lambda^\varepsilon + b) \nabla x \phi^1(t, x, \tilde{z}) \, dx \, dt \\
&+ \int \int_{\Omega_g} \partial_t \Lambda^\varepsilon \phi^0(t, x) \, dx \, dt + \int \int_{\Omega_g} D^\varepsilon \nabla \Lambda^\varepsilon \nabla x \phi^0(t, x) \, dx \, dt + \int \int_{\Omega_g} D^\varepsilon \nabla \Lambda^\varepsilon \nabla_y \phi^1(t, x, \tilde{z}) \, dx \, dt \\
&- \int \int_{\Omega_g} P^\varepsilon(\Lambda^\varepsilon + b) \nabla x \phi^0(t, x) \, dx \, dt - \int \int_{\Omega_g} P^\varepsilon(\Lambda^\varepsilon + b) \nabla_y \phi^1(t, x, \tilde{z}) \, dx \, dt - \int \int_{\Omega_g} g \phi^0(t, x) \, dx \, dt \\
&- \varepsilon \int \int_{\Omega_g} g \phi^1(t, x, \tilde{z}) \, dx \, dt = \sum_{j=1}^{12} J_j = 0,
\end{align*}
6.3 Passing to $\varepsilon \to 0$ via two-scale convergence

where the terms $J_1, \cdots, J_{12}$ are defined as follows:

\[
J_1 := -\varepsilon^2 \iint_{S \Gamma^\varepsilon} p \phi^1(t, x, \frac{x}{\varepsilon}) d\sigma^\varepsilon dt;
\]

\[
J_2 := \varepsilon \int \int_{S \Omega^\varepsilon_y} \partial_t \Lambda^\varepsilon \phi^1(t, x, \frac{x}{\varepsilon}) dx dt;
\]

\[
J_3 := -\varepsilon \int \int_{S \Gamma^\varepsilon} p \phi^0(t, x) d\sigma^\varepsilon dt;
\]

\[
J_4 := \varepsilon \int \int_{S \Omega^\varepsilon_y} D^\varepsilon \nabla \Lambda^\varepsilon \nabla_x \phi^1(t, x, \frac{x}{\varepsilon}) dx dt;
\]

\[
J_5 := -\varepsilon \int \int_{S \Omega^\varepsilon_y} P^\varepsilon (\Lambda^\varepsilon + b) \nabla_x \phi^1(t, x, \frac{x}{\varepsilon}) dx dt;
\]

\[
J_6 := \int \int_{S \Omega^\varepsilon_y} \partial_t \Lambda^\varepsilon \phi^0_0(t, x) dx dt;
\]

\[
J_7 := \int \int_{S \Omega^\varepsilon_y} D^\varepsilon \nabla \Lambda^\varepsilon \nabla_x \phi^0_0(t, x) dx dt;
\]

\[
J_8 := \int \int_{S \Omega^\varepsilon_y} D^\varepsilon \nabla \Lambda^\varepsilon \nabla_y \phi^1(t, x, \frac{x}{\varepsilon}) dx dt;
\]

\[
J_9 := -\int \int_{S \Omega^\varepsilon_y} P^\varepsilon (\Lambda^\varepsilon + b) \nabla_y \phi^0(t, x) dx dt;
\]

\[
J_{10} := -\int \int_{S \Omega^\varepsilon_y} P^\varepsilon (\Lambda^\varepsilon + b) \nabla_y \phi^1(t, x, \frac{x}{\varepsilon}) dx dt;
\]

\[
J_{11} := -\int \int_{S \Omega^\varepsilon_y} g \phi^0(t, x) dx dt;
\]

\[
J_{12} := -\varepsilon \int \int_{S \Omega^\varepsilon_y} g \phi^1(t, x, \frac{x}{\varepsilon}) dx dt.
\]

**Lemma 6.3.2.** The terms $J_1, \cdots, J_{12}$ are bounded.

**Proof.** The proof of this result is similar to the proof of Lemma 6.3.1. \qed

Based on Lemma 6.3.2, the following two-scale limits hold:

\[
\lim_{\varepsilon \to 0} J_1 = \lim_{\varepsilon \to 0} J_2 = \lim_{\varepsilon \to 0} J_4 = \lim_{\varepsilon \to 0} J_5 = \lim_{\varepsilon \to 0} J_{12} = 0;
\]

\[
\lim_{\varepsilon \to 0} J_3 = -\frac{1}{\varepsilon^2} \int \int_{S \Omega^\varepsilon_y \Gamma^\varepsilon} p(x, y) \phi^0(t, x) d\sigma dx dt;
\]

\[
\lim_{\varepsilon \to 0} J_6 = \frac{1}{\varepsilon^2} \int \int_{S \Omega^\varepsilon_y Y} \partial_t \Lambda^0(t, x, y) \phi^0(t, x) dy dx dt;
\]
\[
\begin{align*}
\lim_{\varepsilon \to 0} (J_7 + J_8) &= \frac{1}{|T|} \int \int \int \int_D (t, y) \left( \nabla_x \Lambda^0 + \nabla_y \Lambda^1 (t, x, y) \right) \left( \nabla_x \phi^0 (t, x) + \nabla_y \phi^1 (t, x, y) \right) dy dx dt; \\
\lim_{\varepsilon \to 0} (J_9 + J_{10}) &= -\frac{1}{|T|} \int \int \int \int_D (t, y) \left( \Lambda^0 + b \right) \left( \nabla_x \phi^0 (t, x) + \nabla_y \phi^1 (t, x, y) \right) dy dx dt; \\
\lim_{\varepsilon \to 0} J_{11} &= -\int \int g \phi^0 (t, x) dx dt,
\end{align*}
\]

where the asymptotic expansion \( \Lambda^\varepsilon (t, x) := \Lambda^0 (t, x, y) + \varepsilon \Lambda^1 (t, x, y) + \varepsilon^2 \Lambda^2 (t, x, y) + \ldots \) has been used.

Now, collecting all the previous results mentioned in this section, the weak formulation of the limit two-scale problem, say \((P^0)\), can be stated.

**Definition 4** (Weak formulation of \((P^0)\)). Find the vector \((\Theta^0, \Theta^1, \Lambda^0, \Lambda^1)\), with \(\Theta^0 \in L^2 (S; H_0^1 (\Omega))\), \(\Theta^1 \in L^2 (S; L^2 (\Omega; H_0^1 (Y)))\), \(\Lambda^0 \in L^2 (S; H_0^1 (\Omega_y))\), \(\Lambda^1 \in L^2 (S; L^2 (\Omega_y; H_0^1 (Y)))\) such that

\[
\begin{align*}
(6.9) \quad &\frac{1}{|T|} \int \int \int \int_\Omega \left( (D \Theta^0 (t, x, y), \varphi^0 (t, x)) + (L (t, y) \nabla \Theta^0 + \nabla_y \Theta^1 (t, x, y) ) - P (t, y) \Theta^0 \right) \\
&\cdot \left( \nabla_x \varphi^0 (t, x) + \nabla_y \varphi^1 (t, x, y) \right) dy dx + \frac{1}{|T|} \int \int \int_\Gamma B (\Theta^0, \Lambda^0 + b) \varphi^0 (t, x) dx d\sigma \\
&= \int \int g \varphi^0 (t, x) dx \\
&\text{and}
\end{align*}
\]

\[
(6.10) \quad \frac{1}{|T|} \int \int \int \int_{\Omega_y} \left( (D \Lambda^0 (t, x, y), \phi^0 (t, x)) + (D (t, y) \nabla \Lambda^0 + \nabla_y \Lambda^1 (t, x, y) ) - P (t, y) (\Lambda^0 + b) \right) \\
&\cdot \left( \nabla_x \phi^0 (t, x) + \nabla_y \phi^1 (t, x, y) \right) dy dx - \frac{1}{|T|} \int \int \int_{\Gamma_y} p (x, y) \phi^0 (t, x) dx d\sigma dx
\]

hold for all \( \varphi^0 \in L^2 (S; H_0^1 (\Omega))\), \( \phi^0 \in L^2 (S; H_0^1 (\Omega_y))\), \( \varphi^1 \in L^2 (S \times \Omega; H_0^1 (Y) / \mathbb{R})\) and \( \phi^1 \in L^2 (S \times \Omega_y; H_0^1 (Y) / \mathbb{R})\).

**Theorem 6.3.3** (Existence of weak solution to \((P^0)\)). Problem \((P^0)\) has at least a weak solution in the sense of Definition 4.

**Proof.** Since \((P^\varepsilon)\) is well-posed and the passing to the limit \(\varepsilon \to 0\) via two-scale convergence is done rigorously, then as a consequence of the Compactness Theorem (Theorem 6.1.1), it can be deduced that there exists at least a subsequence \((\Theta^\varepsilon, \Lambda^\varepsilon)\) converging to the weak solution \((\Theta^0, \Lambda^0, \Theta^1, \Lambda^1)\) of \((P^0)\).

**Proposition 6.3.4.**

\[
\begin{align*}
\| \Theta^0 \|_{L^\infty (\Omega)} &\leq M_0^0, \\
\| \Lambda^0 \|_{L^\infty (\Omega)} &\leq M_0^0.
\end{align*}
\]

**Proof.** Using the same argument as in the proof of Theorem (6.3.3), the uniform \(L^\infty\)-bounds on \(\Theta^\varepsilon\) and \(\Lambda^\varepsilon\) are kept during the homogenisation procedure.
Theorem 6.3.5 (Uniqueness of weak solutions to \((P^0)\)). Problem \((P^0)\) has at most one weak solution in the sense of Definition 4.

Proof. Step 1. This step is inspired by the formal homogenisation of \((P^e)\), which is done in Chapter 7. So, the following cell problems are defined:

\begin{align}
\theta^g(y) &:= (\theta_1^g(y), \ldots, \theta_d^g(y))^T \in \mathbb{R}^d, \\
\theta^a(y) &:= (\theta_1^a(y), \ldots, \theta_d^a(y))^T \in \mathbb{R}^d,
\end{align}

where \(d \in \{2, 3\}\). Step 2. The cell functions \(\theta^g\) and \(\theta^a\) are used to eliminate partial differential equations for \(\theta_k^g(y)\) and \(\theta_k^a(y)\) \((k \in \{1, 2\})\), i.e.

\[
\begin{align}
\Theta^1(x, y) &= \theta^g(y) \cdot \nabla \Theta^0(x), \\
\Lambda^1(x, y) &= \theta^a(y) \cdot \nabla \Lambda^0(x),
\end{align}
\]

where \(\theta^g\) and \(\theta^a\) are solutions of the cell problems from Step 1. Thus the choice of the solution \((\Theta^0, \Theta^1, \Lambda^0, \Lambda^1)\) can be expressed in terms of \((\Theta^0, \Lambda^0)\) only. So, (6.9) and (6.10) become

\[
\begin{align}
\frac{1}{|\Omega|} \int_{\Omega} \int_{\Omega} \left( (\partial_t \theta^g(t, x, y) \phi^0(x) + (L(t, y)(\nabla_x \theta^0(x) + \nabla_y \theta^g(y) \nabla_x \theta^0(x))) \\
- P(t, y) \theta^0(\nabla_x \phi^0(x) + \nabla_y \phi^1(x, y))) \right) dx dy + \frac{1}{|\Gamma|} \int_{\Gamma} B(\theta^0, \Lambda^0 + b) \phi^0(x) d\sigma_g dx = \int_{\Omega} f \phi^0(x) dx
\end{align}
\]

and

\[
\begin{align}
\frac{1}{|\Omega|} \int_{\Omega} \int_{\Omega} \left( (\partial_t \theta^a(t, x, y) \phi^0(x) + (D(t, y)(\nabla_x \theta^0(x) + \nabla_y \theta^a(y) \nabla_x \theta^0(x))) \\
- P(t, y) (\Lambda^0 + b) (\nabla_x \phi^0(x) + \nabla_y \phi^1(x, y))) \right) dx dy - \frac{1}{|\Gamma_s|} \int_{\Gamma_s} p(\theta^0, \Lambda^0 + b) \phi^0(x) d\sigma_g dx = \int_{\Omega} g \phi^0(x) dx.
\end{align}
\]

Step 3. Use the working strategy from the proof of Proposition 5.2.2 to show the uniqueness of weak solutions in the sense of Definition 4. First of all, start with the equation for \(\Theta^0\). Let \((\Theta^0_1, \Lambda^0_1)\) and \((\Theta^0_2, \Lambda^0_2)\) be distinct two weak solutions to \((P^0)\) in the sense of Definition 4 satisfying \(\Theta^0_1(0, x, y) = \Theta^0_2(0, x, y)\) for all \((x, y) \in \Omega \times \bar{Y}\). For simplicity, assume \(|\bar{Y}| = 1\). Subtracting (6.13) and (6.14) formulated respectively for \((\Theta^0_1, \Lambda^0_1)\) and \((\Theta^0_2, \Lambda^0_2)\), and taking

\[
\begin{align}
(\phi^0, \phi^1) &= (\Theta^0_2 - \Theta^0_1, \Theta^0_2 - \Theta^0_1) \\
&= (\Theta^0_2 - \Theta^0_1, \nabla_y \theta^0(y) \nabla (\Theta^0_2 - \Theta^0_1)) = (v^0(x), \nabla_y \theta^0(y) \nabla v^0(x))
\end{align}
\]

and

\[
\begin{align}
(\phi^0, \phi^1) &= (\Lambda^0_2 - \Lambda^0_1, \Lambda^0_2 - \Lambda^0_1) \\
&= (\Lambda^0_2 - \Lambda^0_1, \nabla_y \theta^a(y) \nabla (\Lambda^0_2 - \Lambda^0_1)) = (z^0(x), \nabla_y \theta^a(y) \nabla z^0(x))
\end{align}
\]

as test functions yield

\[
\begin{align}
\frac{1}{|\Omega|} \int_{\Omega} \int_{\Omega} \left( (\partial_t v^0(t, x, y) \cdot v^0(t, x, y) + L(t, y)(\Lambda_1 + \nabla_y \theta^0(y))^2(\nabla_x v^0(t, x))^2) \\
- P(t, y) v^0(\Lambda_1 + \nabla_y \theta^0(y)) \nabla_x v^0(t, x)) \right) dx dy \\
+ \frac{1}{|\Gamma|} \int_{\Gamma} (B(\Theta^0_2, \Lambda^0_2 + b) \Theta^0_2 - B(\Theta^0_1, \Lambda^0_1 + b) \Theta^0_1) d\sigma_g dx = 0,
\end{align}
\]
or
\[
\int \int (\partial_t v^0(t, x, y) \cdot v^0(t, x, y) + D(t, y)(l_2 + \nabla_y w^0(y))(\nabla_x z^0(t, x))^2) \, dx \, dy
- \int \int ((p(\Theta^0_2, \Lambda^0_2 + b) - p(\Theta^0_1, \Lambda^0_1 + b))\Theta^0_2
+ B(\Theta^0_1, \Lambda^0_1 + b)v^0(\frac{dx}{\sigma_y}dy = 0,
\]
where \(l_1\) is an identity operator. Next the following estimates hold:

\[
\frac{d}{dt} ||v^0||_{L^2(\Omega)} + \inf_{S \times Y} L(t, y)c_1|Y|||\nabla v^0||_{L^2(\Omega)}^2
\leq ||P(t, y)||_{L^2(\Omega \times \Gamma)}c_2 Y ||v^0||_{L^2(\Omega)} \nabla v^0||_{L^2(\Omega)}^2
+ c_B \left( \left| \Theta^0_2 \right|_{L^2(\Omega \times \Gamma)} + \left| \Theta^0_1 \right|_{L^2(\Omega \times \Gamma)} \right) ||v^0||_{L^2(\Omega \times \Gamma)}^2
\leq \delta_1 ||v^0||_{L^2(\Omega)}^2 + c(\delta_1)c_2|Y|^2 ||P(t, y)||_{L^2(\Omega \times \Gamma)} v^0||_{L^2(\Omega)}^2
+ c_B (M^\Theta_2 + M^\Theta_2 + M^\Lambda_2) ||v^0||_{L^2(\Omega \times \Gamma)}^2
\leq \delta_1 ||v^0||_{L^2(\Omega)}^2 + c(\delta_1)c_2|Y|^2 ||P(t, y)||_{L^2(\Omega \times \Gamma)} v^0||_{L^2(\Omega)}^2
+ cB(M^\Theta_2 + M^\Theta_2 + M^\Lambda_2) ||v^0||_{L^2(\Omega \times \Gamma)}^2
\leq \delta_1 ||v^0||_{L^2(\Omega)}^2 + c(\delta_1)c_2|Y|^2 ||P(t, y)||_{L^2(\Omega \times \Gamma)} v^0||_{L^2(\Omega)}^2
+ cB(M^\Theta_2 + M^\Theta_2 + M^\Lambda_2) ||v^0||_{L^2(\Omega \times \Gamma)}^2
\]

Choose \(\delta_1 \in (0, \inf_{S \times Y} L(t, y)c_1|Y|)\). Then

\[
\frac{d}{dt} ||v^0||_{L^2(\Omega)}^2 + c_5 ||v^0||_{L^2(\Omega)}^2 \leq c_6 ||v^0||_{L^2(\Omega)}^2 + c_7 ||z^0||_{L^2(\Omega)}^2,
\]
where

\[
c_5 := \inf_{S \times Y} L(t, y)c_1|Y| - \delta_1,
\]
\[
c_6 := c(\delta_1)c_2|Y|^2 ||P(t, y)||_{L^2(\Omega \times \Gamma)} v^0||_{L^2(\Omega \times \Gamma)}^2 + cB(M^\Theta_2 + M^\Theta_2 + M^\Lambda_2),
\]
\[
c_7 := cB(M^\Theta_2 + M^\Theta_2 + M^\Lambda_2)||v^0||_{L^2(\Omega \times \Gamma)}^2.
\]
The treatment of the equation for \(A^0\) is similar to the one of the heat equation. In general \(|\frac{\partial}{\partial t}\frac{\nabla}{\nabla}\frac{dx}{\sigma_y}dy \leq 1\), with the simplification \(|Y| = 1\), this implies \(|Y'_{\sigma_y} | \leq 1\). Note that the quantity \(|\frac{\partial}{\partial t}\frac{\nabla}{\nabla}\frac{dx}{\sigma_y}dy \) plays the role of the porosity of the material. Due to Step 3 it follows that

\[
\int \int (\partial_t z^0(t, x, y) \cdot z^0(t, x, y) + D(t, y)(l_2 + \nabla_y w^\Lambda(y))(\nabla_x z^0(t, x))^2) \, dx \, dy
- \int \int ((p(\Theta^0_2, \Lambda^0_2 + b) - p(\Theta^0_1, \Lambda^0_1 + b))\Lambda^0
+ p(\Theta^0_1, \Lambda^0_1 + b)z^0) \, dx \, dy = 0,
\]
where \(l_2\) is an identity operator.
Analogously to the equation for $\Theta^0$, the following estimates hold for the equation for $\Theta^0$:

\[
\frac{d}{dt} \| z^0 \|_{L^2(\Omega)}^2 + \inf_{S \times Y} D(t, y) c_S |Y_g| \| \nabla z^0 \|_{L^2(\Omega)}^2 \\
\leq \| P(t, y) \|_{L^\infty(\Omega \times Y_g)} c_0 |Y_g| \| z^0 \|_{L^2(\Omega)} \| \nabla z^0 \|_{L^2(\Omega)} \\
+ c_p (\| A^0_2 \|_{L^\infty(\Omega \times Y_g)} (\| v^0 \|_{L^1(\Omega \times Y_g)} + \| z^0 \|_{L^1(\Omega \times Y_g)}) \\
+ (\| \Theta^0_1 \|_{L^\infty(\Omega \times Y_g)} + \| A^0_1 \|_{L^\infty(\Omega \times Y_g)}) \| z^0 \|_{L^2(\Omega \times Y_g)}) \\
+ c_p^2 (\| A^0_2 \|_{L^\infty(\Omega \times Y_g)} + \| A^0_1 \|_{L^\infty(\Omega \times Y_g)}) \| z^0 \|_{L^2(\Omega \times Y_g)}^2 \\
+ c_p (\| A^0_2 \|_{L^\infty(\Omega \times Y_g)} + \| A^0_1 \|_{L^\infty(\Omega \times Y_g)}) \| z^0 \|_{L^2(\Omega \times Y_g)} \\
\leq \delta_2 \| \nabla z^0 \|_{L^2(\Omega)}^2 + c_1 c_{12} |Y_g|^2 \| P(t, y) \|_{L^\infty(\Omega \times Y_g)} \| z^0 \|_{L^2(\Omega)}^2 \\
+ c_p (M^{A^0_2} + M^{A^0_1} + M^{\Theta^0_1}) \| z^0 \|_{L^1(\Omega \times Y_g)} + c_p M^{A^0_2} \| v^0 \|_{L^1(\Omega \times Y_g)} \\
\leq \delta_2 \| \nabla z^0 \|_{L^2(\Omega)}^2 + c_1 c_{12} |Y_g|^2 \| P(t, y) \|_{L^\infty(\Omega \times Y_g)} \| z^0 \|_{L^2(\Omega)}^2 \\
+ c_p (M^{A^0_2} + M^{A^0_1} + M^{\Theta^0_1}) \| z^0 \|_{L^1(\Omega \times Y_g)} + c_{10} c_p |Y_g| (M^{A^0_2} + M^{A^0_1} + M^{\Theta^0_1}) \| z^0 \|_{L^2(\Omega)}^2 \\
+ c_{11} c_p M^{A^0_2} \| \Gamma_g \| \| v^0 \|_{L^2(\Omega)}^2,
\]

Choose $\delta_2 \in (0, \inf_{S \times Y} D(t, y) c_S |Y_g|)$. Then

\[
\frac{d}{dt} \| z^0 \|_{L^2(\Omega)}^2 + c_{12} \| \nabla z^0 \|_{L^2(\Omega)}^2 \leq c_{13} \| z^0 \|_{L^2(\Omega)}^2 + c_{14} \| v^0 \|_{L^2(\Omega)}^2,
\]

where

\[
c_{12} := \inf_{S \times Y} D(t, y) c_S |Y_g| - \delta_2,
\]

\[
c_{13} := c_1 c_{12} |Y_g|^2 \| P(t, y) \|_{L^\infty(\Omega \times Y_g)} + c_{10} c_p |Y_g| (M^{A^0_2} + M^{A^0_1} + M^{\Theta^0_1}),
\]

\[
c_{14} := c_{11} c_p M^{A^0_2} \| \Gamma_g \| \| v^0 \|_{L^2(\Omega)}^2.
\]

Summing up (6.15) and (6.16) gives

\[
\frac{1}{2} \partial_t (\| v^0 \|_{L^2(\Omega)}^2 + \| z^0 \|_{L^2(\Omega)}^2) + \min\{c_{15}, c_{16}\} (\| \nabla v^0 \|_{L^2(\Omega)}^2 + \| \nabla z^0 \|_{L^2(\Omega)}^2) \\
\leq \max\{c_{17}, c_{18}\} (\| v^0 \|_{L^2(\Omega)}^2 + \| z^0 \|_{L^2(\Omega)}^2),
\]

where

\[
c_{15} := c_5 - c_{14} \geq 0,
\]

\[
c_{16} := c_{12} - c_7 \geq 0,
\]

\[
c_{17} := c_6 + c_{14} \geq 0,
\]

\[
c_{18} := c_7 + c_{13} \geq 0.
\]

Neglect the term $\min\{c_{15}, c_{16}\} (\| \nabla v^0 \|_{L^2(\Omega)}^2 + \| \nabla z^0 \|_{L^2(\Omega)}^2)$, because it is non-negative. Take $\eta(t) := \| v^0 \|_{L^2(\Omega)}^2 + \| z^0 \|_{L^2(\Omega)}^2$, $\phi(t) := 2 \max\{c_{17}, c_{18}\}$ and $\psi(t) := 0$, and apply Gronwall’s inequality (A.6) to (6.17). This gives

\[
\eta(t) = \| v^0 \|_{L^2(\Omega)}^2 + \| z^0 \|_{L^2(\Omega)}^2 \\
\leq \exp \left( \int_0^t \phi(\tau) \psi(\tau) \right) \eta(0) + \int_0^t \psi(\tau) \right) \\
= \exp \left( \int_0^t 2 \max\{c_{17}, c_{18}\} \right) \left( \| v^0 \|_{L^2(\Omega)}^2 + \| z^0 \|_{L^2(\Omega)}^2 \right) = 0.
\]
Consequently, this implies that for all \( t \in S \) the following result holds:

\[
0 \leq \|v^0\|_{L^2(\Omega)}^2 + \|z^0\|_{L^2(\Omega)}^2 \leq 0,
\]

and hence, this gives

\[
\|v^0\|_{L^2(\Omega)}^2 + \|z^0\|_{L^2(\Omega)}^2 = 0.
\]

This shows that \( \Theta^0 = \Theta^1 \) almost everywhere in \( \Omega \times Y \) and \( \Lambda^0 = \Lambda^1 \) almost everywhere in \( \Omega \times Y_g \) for all \( t \in S \). This shows the desired uniqueness of weak solutions to \((P^0)\).

In order to be able to use two-scale convergence concepts in perforated media, a necessary condition is to construct extensions for \((\Theta^0, \Lambda^0, R^0)\).
Chapter 7

Formal Homogenisation of Problem \((P^\varepsilon)\)

Within the framework of this chapter, we rediscover based on formal homogenisation the strong formulation of the model \((P^0)\) (obtained previously by means of two-scale convergence).
For convenience, the formal homogenisation procedure is performed separately for each equation.
It is important to note that the reaction-diffusion system \((P^\varepsilon)\) is only weakly coupled and this fact is used essentially in the following equations.

7.1 Asymptotic homogenisation assumption

Throughout this chapter, the following asymptotic homogenisation assumptions are:

\begin{align*}
(7.1) \quad & \Theta^\varepsilon(t, x) = \Theta^0(t, x, y) + \varepsilon \Theta^1(t, x, y) + \varepsilon^2 \Theta^2(t, x, y) + \ldots, \\
(7.2) \quad & \Lambda^\varepsilon(t, x) = \Lambda^0(t, x, y) + \varepsilon \Lambda^1(x, y) + \varepsilon^2 \Lambda^2(t, x, y) + \ldots, \\
(7.3) \quad & f(\Theta^\varepsilon, \Lambda^\varepsilon + b) = f(\Theta^0, \Lambda^0 + b) + \varepsilon \Theta^1 \partial_1 f(\Theta^0, \Lambda^0 + b) + \varepsilon^2 \partial_2 f(\Theta^0, \Lambda^0 + b) + \ldots, \\
(7.4) \quad & g(\Theta^\varepsilon, \Lambda^\varepsilon + b) = g(\Theta^0, \Lambda^0 + b) + \varepsilon \Theta^1 \partial_1 g(\Theta^0, \Lambda^0 + b) + \varepsilon^2 \partial_2 g(\Theta^0, \Lambda^0 + b) + \ldots, \\
(7.5) \quad & B(\Theta^\varepsilon, \Lambda^\varepsilon + b) = B(\Theta^0, \Lambda^0 + b) + \varepsilon \Theta^1 \partial_1 B(\Theta^0, \Lambda^0 + b) + \varepsilon^2 \partial_2 B(\Theta^0, \Lambda^0 + b) + \ldots, \\
(7.6) \quad & p(\Theta^\varepsilon, \Lambda^\varepsilon + b) = p(\Theta^0, \Lambda^0 + b) + \varepsilon \Theta^1 \partial_1 p(\Theta^0, \Lambda^0 + b) + \varepsilon^2 \partial_2 p(\Theta^0, \Lambda^0 + b) + \ldots,
\end{align*}

where \(y = \tfrac{x}{\varepsilon}\). Note the following calculation rule: if \(\alpha(t, x) = \beta(t, x, y)|_{y=x/\varepsilon}\) is sufficiently smooth, then \(\nabla \alpha := \nabla_x \beta + \tfrac{1}{\varepsilon} \nabla_y \beta|_{y=x/\varepsilon}\). For simplicity, the divergence operator with respect to \(x\) is denoted by \(\nabla_x\) and the one with respect to \(y\) is denoted by \(\nabla_y\).

7.2 Derivation of the cell problems for the heat equation

First, look at \(\Theta^\varepsilon\).

\[ \partial_t \Theta^\varepsilon + \text{div}(P^\varepsilon \Theta^\varepsilon - L^\varepsilon \nabla \Theta^\varepsilon) = f(\Theta^\varepsilon, \Lambda^\varepsilon + b) \text{ in } \Omega, \]
\[ n \cdot (P^\varepsilon \Theta^\varepsilon - L^\varepsilon \nabla \Theta^\varepsilon) = \varepsilon B(\Theta^\varepsilon, \Lambda^\varepsilon + b) \text{ at } \Gamma^\varepsilon. \]

Substituting the homogenisation assumptions (7.1)-(7.6) into the above partial differential equations gives

\[ \partial_t (\Theta^0 + \varepsilon \Theta^1 + \varepsilon^2 \Theta^2 + \ldots) + (\nabla_x \beta + \tfrac{1}{\varepsilon} \nabla_y \beta)(P(t, y) - L(t, y)(\nabla_x + \tfrac{4}{\varepsilon} \nabla_y))((\Theta^0 + \varepsilon \Theta^1 + \varepsilon^2 \Theta^2 + \ldots) = f(\Theta^0, \Lambda^0 + b) + \varepsilon \Theta^1 \partial_1 f(\Theta^0, \Lambda^0 + b) + \varepsilon^2 \partial_2 f(\Theta^0, \Lambda^0 + b) + \ldots, \]
Lemma 7.2.1. Consider the boundary value problem

\[
\text{Lemma 7.2.1. Consider the boundary value problem}
\]

or

\[
\begin{align*}
\varepsilon^{-2} \nabla_y \cdot (L(t,y) \nabla_y \Theta^0) + \varepsilon^{-1} (\nabla_y \cdot (P(t,y) \Theta^0) - \nabla_y \cdot (L(t,y)(\nabla_x \Theta^0 + \nabla_y \Theta^1)) \\
- \nabla_x \cdot (L(t,y) \nabla_y \Theta^0) + \partial_t \Theta^0 + \nabla_x \cdot (P(t,y) \Theta^0) - \nabla_x \cdot (L(t,y)(\nabla_x \Theta^0 + \nabla_y \Theta^1)) \\
+ \nabla_y \cdot (P(t,y) \Theta^1) - \nabla_y \cdot (L(t,y)(\nabla_x \Theta^1 + \nabla_y \Theta^2)) + \ldots
\end{align*}
\]

or

\[
\begin{align*}
(7.7) \quad & (\varepsilon^{-2} A^0_{-2} + \varepsilon^{-1} A^0_{-1} + A_0^0 + \ldots)(\Theta^0, \Theta^1, \Theta^2, \ldots) = f(\Theta^0, \Lambda^0 + b) + \varepsilon \Theta^1 \partial_1 f(\Theta^0, \Lambda^0 + b) \\
& + \varepsilon \Lambda^1 \partial_2 f(\Theta^0, \Lambda^0 + b) + O(\varepsilon^2),
\end{align*}
\]

where

\[
\begin{align*}
A^0_{-2} & := -\nabla_y \cdot (L(t,y) \nabla_y), \\
A^0_{-1} & := -\nabla_x \cdot (L(t,y) \nabla_y + \nabla_y \cdot P(t,y) - \nabla_y \cdot (L(t,y)(\nabla_x + \nabla_y))), \\
A_0^0 & := \partial_t + \nabla_x \cdot P(t,y) - \nabla_y \cdot (L(t,y)(\nabla_x + \nabla_y)) + \nabla_y \cdot P(t,y) - \nabla_y \cdot (L(t,y)(\nabla_x + \nabla_y)).
\end{align*}
\]

Do the same for the above boundary condition. This gives

\[
\begin{align*}
- \varepsilon^{-1} n(y) \cdot L(t,y) \nabla_y \Theta^0 + n(y) \cdot (P(t,y) \Theta^0 - L(t,y)(\nabla_x \Theta^0 + \nabla_y \Theta^1)) \\
+ \varepsilon n(y) \cdot (P(t,y) \Theta^1 - L(t,y)(\nabla_x \Theta^1 + \nabla_y \Theta^2)) + \ldots
\end{align*}
\]

In order to ensure the existence and uniqueness of periodic solutions to (7.7), the following lemma is needed:

**Lemma 7.2.1.** Let \( F \in L^2(Y) \) and take \( A \) to be a uniformly second order elliptic operator. Consider the boundary value problem

\[
A \Phi = F \text{ in } Y,
\]

where \( \Phi \) is \( Y \)-periodic. Then the following statements hold:

1. There exists a \( Y \)-periodic solution \( \Phi \) if and only if \( \frac{1}{m(Y)} \int_Y f \, dy = 0 \).

2. If there exists a solution, then it is unique up to an additive constant.

Here, \( \frac{1}{m(Y)} \) denotes the Lebesgue measure of \( Y \). See e.g. [31] for more information about this.

**Proof.** The proof of Lemma (7.2.1) can be found in e.g. [28]. \( \square \)

Let \( d \in \{2, 3\} \). The equations from the previous section lead to the cell problems \( (P^0_{-2}) \), \( (P^0_{-1}) \) and \( (P^0_0) \).

\[
\begin{align*}
(P^0_{-2}) \quad \begin{cases} 
-\nabla_y \cdot (L(t,y) \nabla_y \Theta^0) = 0 & \text{in } Y, \\
-\partial_t + \nabla_y \cdot (P(t,y) \Theta^0 - L(t,y)(\nabla_x \Theta^0 + \nabla_y \Theta^1)) = 0 & \text{on } \Gamma,
\end{cases}
\end{align*}
\]

Then this implies that \( \Theta^0 \) is independent of \( y \), i.e. \( \Theta^0(t,x,y) = \Theta^0(t,x) \).

\[
\begin{align*}
(P^0_{-1}) \quad \begin{cases} 
\nabla_y \cdot (P(t,y) \Theta^0) - \nabla_y \cdot (L(t,y)(\nabla_x \Theta^0 + \nabla_y \Theta^1)) = 0 & \text{in } Y, \\
\partial_t + \nabla_y \cdot (P(t,y) \Theta^0 - L(t,y)(\nabla_x \Theta^0 + \nabla_y \Theta^1)) = 0 & \text{on } \Gamma,
\end{cases}
\end{align*}
\]

\( \Theta^1 \) is \( Y \)-periodic.
Since \((P_{-1}^0)\) is linear, it is expected that \(\Theta^1\) is a linear function of \(\nabla_x \Theta^0\). The calculation of \(\Theta^1\) in terms of the gradient of \(\Theta^0\) is

\[
\Theta^1(t, x, y) := \tilde{\Theta}^1(t, x) + \sum_{j=1}^d \partial_{x_j} \Theta^0 w^\Theta_j(y) = \tilde{\Theta}^1(t, x) + w^\Theta(y) \nabla_x \Theta^0,
\]

where

\[
w^\Theta(y) = \begin{pmatrix} w^\Theta_1(y) \\ \vdots \\ w^\Theta_d(y) \end{pmatrix} \in \mathbb{R}^d \text{ (} d \in \{2, 3\} \text{)}
\]

are the so-called cell functions. Some remarks:

\[
\nabla_x \Theta^0(t, x) = \sum_{j=1}^d \partial_{x_j} \Theta^0(t, x) \bar{e}_j,
\]

\[
\nabla_y \Theta^1(t, x, y) = \sum_{j=1}^d \partial_{y_j} \Theta^0(t, x) \nabla_y w^\Theta_j(y).
\]

Then for all \(j \in \{1, \ldots, d\}\), the following cell problem is

\[
(P_{-1,j}^\Theta) \begin{cases}
\nabla_y \cdot (P(t, y) \Theta^0) - \nabla_y \cdot (L(t, y)(\bar{e}_j + \nabla_y w^\Theta_j(y))) = 0 & \text{in } Y, \\
n(y) \cdot (P(t, y) \Theta^0 - L(t, y)(\bar{e}_j + \nabla_y w^\Theta_j(y)) = 0 & \text{on } \Gamma,
\end{cases}
\]

If one assumes as in \([15]\) that \(\nabla_y \cdot P(t, y) = 0\), then \((P_{-1,j}^\Theta)\) satisfies a little (in particular, \(\nabla_y \cdot (P(t, y) \Theta^0) = 0\)). A remark is that the cell function \(w^\Theta\) is the unique solution of the cell problem \((P_{-1,j}^\Theta)\). This is due to Lemma 7.2.1.

Remark: if \(\Theta^0 = \Theta^0(t, x)\) and \(B(\Theta^0, \Lambda^0 + b)\) is independent of \(y\), then \(\Lambda^0(t, x, y) = \Lambda^0(t, x)\).

In order to obtain the homogenised equation from \((P_{-1}^\Theta)\), take the integral over the \(Y\)-cell, i.e.

\[
\frac{1}{|Y|} \int_Y \partial_t \Theta^0 \, dy + \frac{1}{|Y|} \int_Y \nabla_x \cdot (P(t, y) \Theta^0) \, dy - \frac{1}{|Y|} \int_Y \nabla_x \cdot (L(t, y)(\nabla_x \Theta^0 + \nabla_y \Theta^1)) \, dy
\]

\[
+ \frac{1}{|Y|} \int_Y \nabla_y \cdot (P(t, y) \Theta^1) \, dy - \frac{1}{|Y|} \int_Y \nabla_y \cdot (L(t, y)(\nabla_x \Theta^1 + \nabla_y \Theta^2)) \, dy = \frac{1}{|Y|} \int_Y f(\Theta^0, \Lambda^0 + b) \, dy,
\]

where

\[
T_{0}^\Theta := \frac{1}{|Y|} \int_Y \partial_t \Theta^0 \, dy = \partial_t \Theta^0,
\]

\[
T_{1}^\Theta := \frac{1}{|Y|} \int_Y \nabla_x \cdot (P(t, y) \Theta^0) \, dy = \frac{\bar{\partial} \Theta^0}{|Y|} \int_Y P(t, y) \, dy = \bar{P} \nabla \cdot \Theta^0, \text{ with } \bar{P} := \frac{1}{|Y|} \int_Y P(t, y) \, dy
\]

and \(\Theta^0\) is independent of \(y\),
\[
T_2^\Theta := -\frac{1}{|\gamma|} \int_Y \nabla_x \cdot (L(t,y)(\nabla_x \Theta^0 + \nabla_y \Theta^1)) dy \\
= -\frac{1}{|\gamma|} \sum_{i=1}^d \sum_{j=1}^d \int_Y L(t,y)(\delta_{ij} + \partial_{y_j} w_{y_j}^\Theta(y)) dy \partial_{x_i} \Theta^0 = -\nabla_x \cdot (\bar{A}^\Theta \nabla_x \Theta^0),
\]

with
\[
\bar{A}^\Theta = (a_{ij}^\Theta)_{i,j=1}^d,
\]
\[
a_{ij}^\Theta = \frac{1}{|\gamma|} \int_Y L(t,y)(\delta_{ij} + \partial_{y_j} w_{y_j}^\Theta(y)) dy = \frac{1}{|\gamma|} \int_Y L(t,y)(\epsilon_j + \nabla_y w_{y_j}^\Theta(y)) \epsilon_i dy,
\]
\[
T_3^\Theta := \frac{1}{|\gamma|} \int_Y \nabla_y \cdot (P(t,y)\Theta^1) dy = \frac{1}{|\gamma|} \int_Y \sum_{j=1}^d \partial_{x_j} \Theta^0(t,x) \nabla_y \cdot (P(t,y)w_{y_j}^\Theta(y)) dy,
\]
\[
= \frac{1}{|\gamma|} \sum_{j=1}^d \int_Y \nabla_y \cdot (P(t,y)w_{y_j}^\Theta(y)) dy = \frac{1}{|\gamma|} \sum_{j=1}^d \partial_{x_j} \Theta^0(t,x) \int_B n(y) \cdot (P(t,y)w_{y_j}^\Theta(y)) d\sigma_y
\]
\[
= 0 \text{ (due to the periodicity of } P(t,y)w_{y_j}^\Theta(y)),
\]
\[
T_4^\Theta := -\frac{1}{|\gamma|} \int_Y \nabla_y \cdot (L(t,y)(\nabla_x \Theta^1 + \nabla_y \Theta^2)) dy = -\frac{1}{|\gamma|} \int_{\partial Y_y} n(y) \cdot (L(t,y)(\nabla_x \Theta^1 + \nabla_y \Theta^2)) d\sigma_y
\]
\[
= -\frac{1}{|\gamma|} \int_{\partial Y_y} B(\Theta^0, \Lambda^0 + b) d\sigma_y
\]
\[
= -\frac{1}{|\gamma|} B(\Theta^0, \Lambda^0 + b),
\]
\[
T_5^\Theta := \frac{1}{|\gamma|} \int_Y f(\Theta^0, \Lambda^0 + b) dy = f(\Theta^0, \Lambda^0 + b).
\]

The quantity \(\frac{|\tilde{\gamma}|}{|\gamma|}\) is called the surface porosity. This is the area of the solid part \(Y_s\) of the total area of the cell. The quantity \(\frac{|\tilde{\gamma}|}{|\gamma|}\) is called the volumetric porosity.

### 7.3 Derivation of the cell problems for the oxygen equation

The homogenisation assumption for the oxygen equation is the same as the homogenisation assumption for the heat equation. Look at the heat equation for further details.

Now, look at the partial differential equations for \(\Lambda\):
\[
\partial_t \Lambda^\varepsilon + \text{div}(P^\varepsilon(\Lambda^\varepsilon + b) - D^\varepsilon \nabla \Lambda^\varepsilon) = g(\Theta^\varepsilon, \Lambda^\varepsilon + b) \text{ in } \Omega_y^\varepsilon,
\]
\[
n \cdot D^\varepsilon \nabla \Lambda^\varepsilon = \varepsilon p(\Theta^\varepsilon, \Lambda^\varepsilon + b) \text{ at } \Gamma_y^\varepsilon,
\]
\[
\Lambda^\varepsilon = 0 \text{ at } \partial \Omega.
\]

Substituting the homogenisation assumptions (7.1)-(7.6) into the above partial differential equations gives
\[
\partial_t (\Lambda^0 + \varepsilon \Lambda^1 + \varepsilon^2 \Lambda^2 + \ldots) + (\nabla_x + \frac{1}{\varepsilon} \nabla_y) \cdot (P(t,y)(\Lambda^0 + \varepsilon \Lambda^1 + \varepsilon^2 \Lambda^2 + \ldots))
\]
\[
= D(t,y)(\nabla_x + \frac{1}{\varepsilon} \nabla_y)(\Lambda^0 + \varepsilon \Lambda^1 + \varepsilon^2 \Lambda^2 + \ldots)
\]
\[
= g(\Theta^0, \Lambda^0 + b) + \varepsilon \Theta^1 \partial_t g(\Theta^0, \Lambda^0 + b) + \varepsilon \Lambda^1 \partial_x g(\Theta^0, \Lambda^0 + b) + O(\varepsilon^2)
\]

...
\[\begin{align*}
&\varepsilon^{-2}\nabla_y \cdot (D(t, y)\nabla_y A^0) + \varepsilon^{-1}(\nabla_y \cdot (P(t, y)\Lambda^0 + b)) - \nabla_y \cdot (D(t, y)\nabla_y A^0) \\
&\varepsilon^{-2}(\nabla_y \cdot (D(t, y)(\nabla_x A^0 + \nabla_y A^1))) + \partial_t A^0 + \nabla_y \cdot (P(t, y)\Lambda^0 + b)) \\
&\varepsilon^{-2}(\nabla_x \cdot (D(t, y)(\nabla_x A^0 + \nabla_y A^1)) + \nabla_y \cdot (P(t, y)\Lambda^1) - \nabla_y \cdot (D(t, y)(\nabla_x A^1 + \nabla_y A^2)) + \ldots \\
&= g(\Theta^0, \Lambda^0 + b) + \varepsilon \Theta^1 \partial_1 g(\Theta^0, \Lambda^0 + b) + \varepsilon \Lambda^1 \partial_2 g(\Theta^0, \Lambda^1 + b) + O(\varepsilon^2),
\end{align*}\]

or
\[
\begin{align*}
&\varepsilon^{-2}A_{-2} + \varepsilon^{-1}A_{-1} + A_0 + \ldots (A^0, b, A^1, A^2, \ldots) = g(\Theta^0, \Lambda^0 + b) + \varepsilon \Theta^1 \partial_1 g(\Theta^0, \Lambda^0 + b) \\
&+ \varepsilon \Lambda^1 \partial_2 g(\Theta^0, \Lambda^0 + b) + O(\varepsilon^2),
\end{align*}\]

where
\[
\begin{align*}
A_{-2} := & -\nabla_y \cdot (D(t, y)\nabla_y y), \\
A_{-1} := & -\nabla_x \cdot (D(t, y)\nabla_y y) + \nabla_y \cdot P(t, y) - \nabla_y \cdot (D(t, y)(\nabla_x + \nabla_y)), \\
A_0 := & \partial_t + \nabla_x \cdot (P(t, y)(\Lambda^0 + b)) - \nabla_x \cdot (D(t, y)(\nabla_x + \nabla_y)) + \nabla_y \cdot P(t, y) \\
& - \nabla_y \cdot (D(t, y)(\nabla_x + \nabla_y)).
\end{align*}\]

Do the same for the above boundary condition. This gives
\[
\begin{align*}
&\varepsilon^{-1}n(y) \cdot D(t, y)\nabla_y A^0 + n(y) \cdot D(t, y)(\nabla_x A^0 + \nabla_y A^1) \\
&\varepsilon n(y) \cdot D(t, y)(\nabla_x A^1 + \nabla_y A^2) + \ldots \\
&= \varepsilon p(\Theta^0, \Lambda^0 + b) + \varepsilon^2 \Theta^1 \partial_1 p(\Theta^0, \Lambda^0 + b) + \varepsilon^2 \Lambda^1 \partial_2 p(\Theta^0, \Lambda^0 + b) + \ldots.
\end{align*}\]

These equations lead to the cell problems \((P_{A_{-2}}), (P_{A_{-1}})\) and \((P_0)\).

Then this implies that \(A^0\) is independent of \(y\), i.e. \(A^0(t, x, y) = A^0(t, x)\).

Since \((P_{A_{-1}})\) is linear, it is expected that \(\Lambda^1\) is a linear function of \(\nabla_x A^0\). The calculation of \(\Lambda^1\) in terms of the gradient of \(A^0\) is
\[
\Lambda^1(t, x, y) := \tilde{\Lambda}^1(t, x) + \sum_{l=1}^d \partial_{x_l} A^0 w_l^A(y) = \tilde{\Lambda}^1(t, x) + w^A(y) \nabla_x A^0,
\]

where
\[
w^A(y) = \begin{pmatrix} w_1^A(y) \\ \vdots \\ w_d^A(y) \end{pmatrix} \in \mathbb{R}^d
\]

are cell functions. The following remarks are denoted:
\[
\nabla_x A^0(t, x) = \sum_{l=1}^d \partial_{x_l} A^0(t, x) \bar{e}_l,
\]
\[
\nabla_y A^0(t, x) = \sum_{l=1}^d \partial_{x_l} A^0(t, x) \nabla_y w_l^A(y).
\]
Then for all \( l \in \{1, \ldots, d\} \), the following cell problem is

\[
\begin{align*}
(P^\Lambda_{-1,l}) & \quad \begin{cases}
\vec{\nabla}_y \cdot (P(t,y)(\Lambda^0 + b)) - \vec{\nabla}_y \cdot (D(t,y)(\vec{c}_l + \nabla_y w^\Lambda_l(y))) = 0 & \text{in } Y_g, \\
n(\vec{y}) \cdot D(t,y)(\vec{c}_l + \nabla_y w^\Lambda_l(y)) = 0 & \text{on } \partial Y_g,
\end{cases}
\end{align*}
\]

\( w^\Lambda_l \) is \( Y \)-periodic.

If one assumes as in [15] that \( \vec{\nabla}_y \cdot P(t,y) = 0 \), then \( P^\Lambda_{-1,l} \) satisfies a little. If \( \int_{Y_g} w^\Lambda_l \, dy = 0 \), then the cell function \( w^\Lambda \) is the unique solution of the cell problem \((P^\Theta_{-1,l})\). This is essentially due to Lemma 7.2.1.

\[
(P^\Lambda_0) \quad \begin{cases}
\partial_t \Lambda^0 + \vec{\nabla}_x \cdot (P(t,y)(\Lambda^0 + b)) - \vec{\nabla}_x \cdot (D(t,y)(\nabla_x \Lambda^0 + \nabla_y \Lambda^1)) + \vec{\nabla}_y \cdot (P(t,y)\Lambda^1) = 0 & \text{in } Y_g, \\
n(\vec{y}) \cdot D(t,y)(\nabla_x \Lambda^1 + \nabla_y \Lambda^2) = g(\Theta^0, \Lambda^0 + b) & \text{on } \partial Y_g, \\
\Lambda^0 = 0 & \text{at } \Gamma_g,
\end{cases}
\]

\( \Lambda^2 \) is \( Y \)-periodic.

Note that if both \( \Lambda^0 = \Lambda^0(t,x) \) and \( p(\Theta^0, \Lambda^0 + b) \) are independent of \( y \), then \( \Theta^0(t,x) = \Theta^0(t,x) \) for all \( (t,x) \in S \times Y \).

Analogously to the treatment of the heat equation, the integral is taken over the \( Y_g \)-cell in order to get the homogenised equation for \((P^\Lambda_0)\), i.e.

\[
(7.8) \quad \frac{1}{|T|} \int_{Y_g} \partial_t \Lambda^0 \, dy + \frac{1}{|T|} \int_{Y_g} \vec{\nabla}_x \cdot (P(t,y)(\Lambda^0 + b)) \, dy - \frac{1}{|T|} \int_{Y_g} \vec{\nabla}_x \cdot (D(t,y)(\nabla_x \Lambda^0 + \nabla_y \Lambda^1)) \, dy
\]

\[
+ \frac{1}{|T|} \int_{Y_g} \vec{\nabla}_y \cdot (P(t,y)\Lambda^1) \, dy - \frac{1}{|T|} \int_{Y_g} \vec{\nabla}_y \cdot (D(t,y)(\nabla_x \Lambda^1 + \nabla_y \Lambda^2)) \, dy = \frac{1}{|T|} \int_{Y_g} g(\Theta^0, \Lambda^0 + b) \, dy,
\]

where

\[
T^\Lambda_0 := \frac{1}{|T|} \int_{Y_g} \partial_t \Lambda^0 \, dy = \frac{1}{|Y_g|} |\partial_t \Lambda^0|,
\]

\[
T^\Lambda_1 := \frac{1}{|T|} \int_{Y_g} \vec{\nabla}_x \cdot (P(t,y)(\Lambda^0 + b)) \, dy = \frac{\vec{\nabla}_x^\Lambda \Lambda^0}{|Y_g|} \int_{Y_g} P(t,y) \, dy = P^\Lambda \vec{\nabla}_x \Lambda^0,
\]

with \( P^\Lambda := \frac{1}{|T|} \int_{Y_g} P(t,y) \, dy \),

\[
T^\Lambda_2 := -\frac{1}{|T|} \int_{Y_g} \vec{\nabla}_x \cdot (D(t,y)(\nabla_x \Lambda^0 + \nabla_y \Lambda^1)) \, dy
\]

\[
= -\vec{\nabla}_x \cdot \left( \frac{1}{|T|} \int_{Y_g} D(t,y)(\nabla_x \Lambda^0 + \sum_{l=1}^d \partial_{x_l} \Lambda^0 \nabla_y w^\Lambda_l(y)) \, dy \right)
\]

\[
- \frac{1}{|T|} \sum_{k=1}^d \sum_{l=1}^d \partial_{x_k} \left( \frac{1}{|T|} \int_{Y_g} D(t,y)(\delta_{kl} + \partial_{y_l} w^\Lambda_l(y)) \, dy \right) \partial_{x_l} \Lambda^0
\]

\[
= -\vec{\nabla}_x \cdot (A^\Lambda \nabla_x \Lambda^0),
\]
The quantities contain the following system of equations. Combining the homogenised equations from the previous sections, the effective (upscaled) model contains the following system of equations.

\[ \begin{align*}
\bar{A}^k &= (\bar{\alpha}^k)_k, \quad\text{for all } k = 1, \ldots, d, \\
\bar{a}^k_{kl} &= \frac{1}{Y_g} \int_{Y_g} D(t, y)(\delta_{kl} + \partial_{y_k} w^k_0(y))dy = \frac{1}{Y_g} \int_{Y_g} D(t, y)(\bar{e}_l + \nabla_y w^k_0(y))\bar{e}_k dy, \\
T^3_3 &= \frac{1}{Y_g} \int_{Y_g} \bar{\nabla}_y \cdot (P(t, y)\Lambda^3)dy = \frac{1}{Y_g} \int_{Y_g} \sum_{i=1}^d \partial_x \Lambda^0(t, x) \bar{\nabla}_y \cdot (P(t, y)w^i_0(y))dy \\
&= \frac{1}{Y_g} \sum_{i=1}^d \partial_x \Lambda^0(t, x) \int_{\partial Y_g} n(y) \cdot (P(t, y)w^i_0(y))d\sigma_y \\
&= 0 \text{ (due to the periodicity of } P(t, y)w^i_0(y)), \\
T^4_4 &= -\frac{1}{Y_g} \int_{Y_g} \bar{\nabla}_y \cdot (\nabla_x \Lambda^4 + \nabla_y \Lambda^4)dy = -\frac{1}{Y_g} \int_{\partial Y_g} n(y) \cdot (\nabla_x \Lambda^4 + \nabla_y \Lambda^4)d\sigma_y \\
&- \frac{1}{Y_g} \int_{\partial Y_g} \partial^p(\Theta^0, \Lambda^0 + b)d\sigma_y \\
&= \frac{|\partial Y_g|}{Y_g} p(\Theta^0, \Lambda^0 + b), \\
T^5_5 &= \frac{1}{Y_g} \int_{Y_g} g(\Theta^0, \Lambda^0 + b)dy = \frac{|\partial Y_g|}{Y_g} g(\Theta^0, \Lambda^0 + b).
\end{align*} \]

The quantities \( |\partial Y_g| \) and \( \frac{|\partial Y_g|}{Y_g} \) are called the volumetric and surface porosities respectively, i.e. the gaseous volume of the total volume of the cell and the gaseous area of the total area of the cell respectively.

### 7.4 Averaged model equations and derivation of effective coefficients. Part 1

Combining the homogenised equations from the previous sections, the effective (upscaled) model contains the following system of equations.

- **Macroscopic unknowns:**
  \[ \begin{align*}
  \bar{\Theta}(t, x) &= \Theta^0(t, x) = \frac{1}{Y_g} \int_{Y_g} \Theta^0(t, x, \cdot )dy, \\
  \bar{\Lambda}(t, x) &= \Lambda^0(t, x) = \frac{1}{Y_g} \int_{Y_g} \Lambda^0(t, x, \cdot )dy, \\
  \bar{R}(t, x) &= R^0(t, x).
  \end{align*} \]

- **Macroscopic equations:**
  \[ \begin{align*}
  \partial_t \bar{\Theta} + P^\Theta \bar{\nabla}_y \cdot \bar{\Theta} - \bar{\nabla}_y \cdot (\bar{\Lambda}^\Theta \nabla \bar{\Theta}) - B(\bar{\Theta}, \bar{\Lambda} + b) &= \bar{f}(\Theta, \Lambda + b) \text{ in } S \times \Omega, \\
  \partial_t \bar{\Lambda} + P^\Lambda \bar{\nabla}_y \cdot \bar{\Lambda} - \bar{\nabla}_y \cdot (\bar{\Lambda}^\Lambda \nabla \bar{\Lambda}) - \bar{p}(\Theta, \bar{\Lambda} + b) &= \bar{g}(\Theta, \Lambda + b) \text{ in } S \times \Omega, \\
  \partial_t \bar{R}(t, x) &= \bar{h}(\Theta, \Lambda + b) \text{ in } S \text{ and almost every } \Omega.
  \end{align*} \]
Effective coefficients:

\[
\bar{A}^\Theta = \left( \frac{1}{\Omega} \int_Y L(t, y) (\hat{\varepsilon}_j + \nabla_y \hat{w}_j(y)) \hat{e}_i \, dy \right)_{i,j=1,\ldots,\text{dim}},
\]

\[
\bar{A}^\Lambda = \left( \frac{1}{\Omega} \int_Y D(t, y) (\hat{\varepsilon}_i + \nabla_y \hat{w}_i(y)) \hat{e}_k \, dy \right)_{k,l=1,\ldots,\text{dim}},
\]

\[
\bar{B}(\hat{\Theta}, \bar{\Lambda} + b) = \frac{1}{\partial Y_g} \int_{\partial Y_g} B(\Theta^0, \Lambda^0 + b) \, d\sigma_y,
\]

\[
\bar{\rho}(\hat{\Theta}, \bar{\Lambda} + b) = \frac{1}{\Omega} \int_Y p(\Theta^0, \Lambda^0 + b) \, dy,
\]

\[
\bar{P}^\Theta = \frac{1}{\Omega} \int_Y P(t, y) \, dy,
\]

\[
\bar{P}^\Lambda = \frac{1}{\Omega} \int_Y P(t, y) \, dy,
\]

\[
\bar{f}(\hat{\Theta}, \bar{\Lambda} + b) := f(\Theta^0, \Lambda^0 + b) = \frac{1}{\Omega} \int_Y f(\Theta^0, \Lambda^0 + b) \, dy,
\]

\[
\bar{g}(\hat{\Theta}, \bar{\Lambda} + b) := \frac{1}{\Omega} \int_Y g(\Theta^0, \Lambda^0 + b) \, dy,
\]

\[
\hat{\Theta}(0, x) = 0, \quad x \in \Omega,
\]

\[
\hat{\Lambda}(0, x) = 0, \quad x \in \Omega_g,
\]

\[
\hat{R}(0, x) = R_0(x), \quad x \in \Gamma.
\]

The effective coefficients describe a homogeneous medium that is independent of \( f, g \) and \( h \) and the boundary conditions. The averaged model is weakly coupled. The coupling enters the boundary conditions and the right hand sides of the partial differential equations. The method of (periodic) homogenisation was able to translate the microscopic model into a macroscopic one.

### 7.5 Averaged model equations and derivation of effective coefficients. Part 2

There is another (shorter) way of obtaining the averaged model equations and effective coefficients. See [22] for more details. The basic idea comes from using the so-called solvability condition, which says

\[
\frac{1}{\Omega} \int_Y \text{RHS} \, dy = \frac{1}{\partial Y} \int_{\partial Y} \text{boundary terms} \, d\sigma_y,
\]

where RHS is an abbreviation for ‘right hand side’.

In this section, the so-called double dot product is introduced. Given are two matrices \( A \) and \( B \). Then the inner product between these matrices is denoted by

\[
A : B := \text{tr}(A^T B) = \sum_{i,j} a_{ij} b_{ij},
\]

which one calls here the \textit{double dot product}. In other references, (7.10) is called Frobenius product. It is used to represent multiplying and summing across two indices as indicated in (7.10). See
for instance [22] or [27] for more details about the use of this inner product in the context of homogenisation.

First of all, look at problem \((P_0^\Theta)\). Find the equations for \(\Theta^2\), i.e.

\[
\begin{align*}
-\nabla_y \cdot (L(t,y) \nabla_y \Theta^2) &= -\partial_t \Theta^0 - \nabla_x \cdot (P(t,y) \Theta^0) + \nabla_x \cdot (L(t,y)(\nabla_x \Theta^0 + \nabla_y \Theta^1)) \\
- n(y) \cdot L(t,y) \nabla_y \Theta^2 &= -n(y) \cdot P(t,y) \Theta^1 + n(y) \cdot L(t,y) \nabla_x \Theta^1 + \Lambda^0 \Theta^0 + B(\Theta^0, \Lambda^0 + b) \text{ on } \partial Y_x,
\end{align*}
\]

\(\Theta^2\) is \(Y\)-periodic.

Apply (7.9) and this gives

\[
I_0^\Theta + I_1^\Theta + I_2^\Theta + I_3^\Theta + I_4^\Theta + I_5^\Theta + I_6^\Theta = I_7^\Theta + I_8^\Theta + I_9^\Theta,
\]

where

\[
I_0^\Theta := -\frac{1}{|Y|} \int_Y \partial_t \Theta^0 \, dy = -\partial_t \Theta^0,
\]

\[
I_1^\Theta := -\frac{1}{|Y|} \int_Y \nabla_x \cdot (P(t,y) \Theta^0) \, dy = -\frac{\nabla_x \Theta^0}{|Y|} \int_Y P(t,y) \, dy = -\tilde{P}^\Theta \nabla_x \cdot \Theta^0,
\]

with \(\tilde{P}^\Theta := \frac{1}{|Y|} \int_Y P(t,y) \, dy,
\]

\[
I_2^\Theta := \frac{1}{|Y|} \int_Y \nabla_x \cdot (L(t,y) \nabla_x \Theta^0) \, dy = \frac{\nabla_x \cdot (\int_Y L(t,y) \, dy \nabla_x \Theta^0)}{|Y|}
\]

\[
= \frac{1}{|Y|} \int_Y L(t,y) \, dy \cdot \nabla_x \cdot \nabla_x \Theta^0, \text{ where (7.10) has been applied},
\]

\[
I_3^\Theta := \frac{1}{|Y|} \int_Y \nabla_x \cdot (L(t,y) \nabla_x \Theta^1) \, dy = \frac{1}{|Y|} \int_Y L(t,y) \, dy \cdot \nabla_x \cdot \nabla_y (w^\Theta(y) \nabla_x \Theta^0) \, dy
\]

\[
= \frac{1}{|Y|} \int_Y L(t,y) \nabla_y w^\Theta(y) \, dy \cdot \nabla_x \cdot \nabla_x \Theta^0 \, dy,
\]

\[
I_4^\Theta := -\frac{1}{|Y|} \int_Y \nabla_y \cdot (P(t,y) \Theta^1) \, dy = -\frac{1}{|Y|} \int_{\partial Y_x} n(y) \cdot P(t,y) \Theta^1 \, d\sigma_y
\]

\[
= \frac{1}{|Y|} \int_{\partial Y_x} n(y) \cdot P(t,y) \Theta^1 \, d\sigma_y =: I_7^\Theta,
\]

\[
=0 \text{ by the periodicity of } P(t,y) \Theta^1
\]

\[
I_5^\Theta := \frac{1}{|Y|} \int_Y \nabla_y \cdot (L(t,y) \nabla_x \Theta^1) \, dy = \frac{1}{|Y|} \int_{\partial Y_x} n(y) \cdot L(t,y) \nabla_x \Theta^1 \, d\sigma_y
\]

\[
+ \frac{1}{|Y|} \int_{\partial Y_x} n(y) \cdot L(t,y) \nabla_x \Theta^1 \, d\sigma_y =: I_8^\Theta,
\]

\[
=0 \text{ by the periodicity of } Y
\]

\[
I_6^\Theta := \frac{1}{|Y|} \int_Y f(\Theta^0, \Lambda^0 + b) \, dy = f(\Theta^0, \Lambda^0 + b),
\]

\[
I_9^\Theta := \frac{1}{|Y|} \int_{\partial Y_x} B(\Theta^0, \Lambda^0 + b) \, d\sigma_y = \frac{|\partial Y_x|}{|Y|} B(\Theta^0, \Lambda^0 + b).
\]
Combining the above results gives the following macroscopic equation:

\[
\frac{1}{|\Omega|} \int_{\Omega} L(t,y)(\mathbb{I}^\Theta + \nabla_y w^\Theta(y))dy : \nabla_x \nabla_x \Theta^0 = \partial_t \Theta^0 + \nabla^\Theta \cdot \Theta^0 - f(\Theta^0, \Lambda^0 + b) \\
+ \frac{\partial Y_s}{\partial y} B(\Theta^0, \Lambda^0 + b),
\]

where $\mathbb{I}^\Theta$ is an identity operator, and $\nabla_x \nabla_x \Theta^0$ is a tensor.
Computing the cell functions $w^\Theta$ gives the effective coefficient tensor $A^\Theta$, which has been calculated in the previous section.

In a similar way, apply this technique to $(P^\Lambda_0)$. Find the equations for $\Lambda^0$, i.e.

\[
-\nabla_y \cdot (D(t,y)\nabla_y \Lambda^2) = -\partial_t \Lambda^0 - \nabla_x \cdot (P(t,y)(\Lambda^0 + b)) + \nabla_x \cdot (D(t,y)(\nabla_x \Lambda^0 + \nabla_y \Lambda^1)) \\
- n(y) \cdot \nabla_y \Lambda^2 = n(y) \cdot D(t,y)\nabla_y \Lambda^1 - p(\Theta^0, \Lambda^0 + b) \text{ on } \partial Y_s,
\]

$\Lambda^2$ is $Y$-periodic.

Apply (7.9), but do this in $Y_s$ instead of $Y$ and this gives

\[
I_0^\Lambda + I_1^\Lambda + I_2^\Lambda + I_3^\Lambda + I_4^\Lambda + I_5^\Lambda + I_6^\Lambda = I_7^\Lambda + I_8^\Lambda,
\]

where

\[
\begin{align*}
I_0^\Lambda &:= -\frac{1}{|\Omega|} \int_{\Omega_y} \partial_t \Lambda^0 dy = -\frac{|\Omega|}{|\Omega_y|} \partial_t \Theta^0, \\
I_1^\Lambda &:= -\frac{1}{|\Omega|} \int_{\Omega_y} \nabla_x \cdot (P(t,y)(\Lambda^0 + b))dy = -\frac{\nabla_x^\Lambda}{|\Omega_y|} \int_{\Omega_y} P(t,y)dy = -\nabla^\Lambda \cdot \Lambda^0, \\
\text{with } P^\Lambda &:= \frac{1}{|\Omega|} \int_{\Omega_y} P(t,y)dy, \\
I_2^\Lambda &:= \frac{1}{|\Omega|} \int_{\Omega_y} \nabla_x \cdot (D(t,y)\nabla_x \Lambda^0)dy = \frac{1}{|\Omega^\prime|} \nabla^\prime \cdot (\int_{\Omega_y} D(t,y)dy \nabla_x \Lambda^0) \\
&= \frac{1}{|\Omega|} \int_{\Omega_y} D(t,y)dy : \nabla_x \nabla_x \Lambda^0, \text{ where (7.10) has been applied,} \\
I_3^\Lambda &:= \frac{1}{|\Omega|} \int_{\Omega_y} \nabla_x \cdot (D(t,y)\nabla_y \Lambda^1)dy = \frac{1}{|\Omega|} \int_{\Omega_y} D(t,y)dy : \nabla_x \cdot \nabla_y (w^\Lambda(y)\nabla_x \Lambda^0)dy \\
&= \frac{1}{|\Omega|} \int_{\Omega_y} D(t,y)\nabla_y w^\Lambda \, \nabla_x \Lambda^0 dy, \\
I_4^\Lambda &:= -\frac{1}{|\Omega|} \int_{\Omega_y} \nabla_y \cdot (P(t,y)\Lambda^1)dy = -\frac{1}{|\partial Y_s|} \int_{\partial Y_s} n(y) \cdot P(t,y)\Lambda^1 d\sigma_y \\
&= 0 \text{ on } \partial Y_s \\
&= 0 \text{ by the periodicity of } P(t,y)\Lambda^1
\end{align*}
\]
7.5 Averaged model equations and derivation of effective coefficients. Part 2

\[ I_0^\Lambda := \frac{1}{|\mathcal{Y}|} \int_{\mathcal{Y}_0} \nabla_y \cdot (D(t,y) \nabla_x \Lambda^1) dy = \frac{1}{|\mathcal{Y}|} \int n(y) \cdot D(t,y) \nabla_x \Lambda^1 d\sigma_y + \frac{1}{|\mathcal{Y}|} \int n(y) \cdot D(t,y) \nabla_x \Lambda^1 d\sigma_y =: I_7^\Lambda, \]

\[ = 0 \text{ by the periodicity of } \mathcal{Y} \]

\[ I_0^\Lambda := \frac{1}{|\mathcal{Y}|} \int_{\mathcal{Y}_0} g(\Theta^0, \Lambda^0 + b) dy = \frac{|\mathcal{Y}_0|}{|\mathcal{Y}|} g(\Theta^0, \Lambda^0 + b), \]

\[ I_0^\Lambda := -\frac{1}{|\mathcal{Y}|} \int_{\mathcal{Y}_0} p(\Theta^0, \Lambda^0 + b) d\sigma_y = -\frac{|\mathcal{Y}_0|}{|\mathcal{Y}|} p(\Theta^0, \Lambda^0 + b). \]

Combining the above results gives the following macroscopic equation:

\[
\frac{1}{|\mathcal{Y}|} \int_{\mathcal{Y}_0} D(t,y)(\mathbb{I}^\Lambda + \nabla_y \mathbb{u}^{\Lambda,T}(y)) dy : \nabla_x \nabla_x \Lambda^0
\]

\[ = \partial_t \Lambda^0 + \bar{P}^\Lambda \nabla_x \cdot \Lambda^0 - \frac{|\mathcal{Y}_0|}{|\mathcal{Y}|} g(\Theta^0, \Lambda^0 + b) - \frac{|\mathcal{Y}_0|}{|\mathcal{Y}|} p(\Theta^0, \Lambda^0 + b), \]

where \( \mathbb{I}^\Lambda \) is an identity operator.

Computing the cell functions \( \mathbb{u}^{\Lambda} \) gives the effective coefficient tensor \( \bar{A}^\Lambda \), which has been calculated in the previous section. Note that (7.11) is the same as the corresponding macroscopic equation for \( \Theta^0 \) (7.8) derived in Section 7.3.
Formal Homogenisation of Problem ($P^c$)
Chapter 8

Formal Derivation of a Distributed Microstructure Smouldering Combustion Model

In this chapter, one is interested in a combustion scenario where the perforations (microstructures) play a very important role in what heat conduction is concerned. Namely, the solid grains need much more time to get heated compared to the Ys parts of the pores. One models this situation by taking into account a special scaling in the heat conduction coefficients. This scaling, \( \frac{L}{\varepsilon} \), is of order of \( \varepsilon^2 \). Such a case \( \frac{L}{\varepsilon} = \mathcal{O}(\varepsilon^2) \) is called ‘diffusion’ in high-contrast media. This chapter relies on the previous chapter and section 1.5 of [13].

8.1 Combustion in high-contrast media

The microstructure model in this chapter reads as

\[
\begin{align*}
(P^*) & \quad \begin{cases}
\partial_t \tau^\varepsilon + \text{div}(P^\varepsilon \tau^\varepsilon - L^\varepsilon \nabla \tau^\varepsilon) = f(\tau^\varepsilon, \Psi^\varepsilon) & \text{in } \Omega^\varepsilon, \\
\partial_t \Psi^\varepsilon + \text{div}(P^\varepsilon \Psi^\varepsilon - D^\varepsilon \nabla \Psi^\varepsilon) = g(\tau^\varepsilon, \Psi^\varepsilon) & \text{in } \Omega^\varepsilon, \\
\partial_t \Pi^\varepsilon - \varepsilon^2 \text{div}(D_s^\varepsilon \nabla \Pi^\varepsilon) = 0 & \text{in } \Omega^\varepsilon, \\
n^\varepsilon \cdot D^\varepsilon \nabla \Psi^\varepsilon = \varepsilon p(\tau^\varepsilon, \Psi^\varepsilon) & \text{at } \Gamma^\varepsilon, \\
\tau^\varepsilon = a & \text{at } \partial \Omega, \\
\Psi^\varepsilon = b & \text{at } \partial \Omega, \\
n^\varepsilon \cdot (P^\varepsilon T^\varepsilon - L^\varepsilon \nabla T^\varepsilon) = \varepsilon^2 n^\varepsilon \cdot D_s^\varepsilon \nabla \Pi^\varepsilon = \varepsilon B(\tau^\varepsilon, \Psi^\varepsilon) & \text{at } \Gamma^\varepsilon,
\end{cases}
\end{align*}
\]

where \( a > 0 \) and \( b > 0 \). Let \( T^\varepsilon := \tau^\varepsilon - a \) and \( \Lambda^\varepsilon := \Psi^\varepsilon - b \). Then problem \((P^*)\) becomes

\[
\begin{align*}
(P^\#) & \quad \begin{cases}
\partial_t T^\varepsilon + \text{div}(P^\varepsilon (T^\varepsilon + a) - L^\varepsilon \nabla T^\varepsilon) = f(T^\varepsilon + a, \Lambda^\varepsilon + b) & \text{in } \Omega^\varepsilon, \\
\partial_t \Lambda^\varepsilon + \text{div}(P^\varepsilon (\Lambda^\varepsilon + b) - D^\varepsilon \nabla \Lambda^\varepsilon) = g(T^\varepsilon + a, \Lambda^\varepsilon + b) & \text{in } \Omega^\varepsilon, \\
\partial_t \Pi^\varepsilon - \varepsilon^2 \text{div}(D_s^\varepsilon \nabla \Pi^\varepsilon) = 0 & \text{in } \Omega^\varepsilon, \\
n^\varepsilon \cdot D^\varepsilon \nabla \Psi^\varepsilon = \varepsilon p(T^\varepsilon + a, \Lambda^\varepsilon + b) & \text{at } \Gamma^\varepsilon, \\
T^\varepsilon = 0 & \text{at } \partial \Omega, \\
\Lambda^\varepsilon = 0 & \text{at } \partial \Omega, \\
n^\varepsilon \cdot (P^\varepsilon (T^\varepsilon + a) - L^\varepsilon \nabla T^\varepsilon) = \varepsilon^2 n^\varepsilon \cdot D_s^\varepsilon \nabla \Pi^\varepsilon = \varepsilon B(T^\varepsilon + a, \Lambda^\varepsilon + b) & \text{at } \Gamma^\varepsilon.
\end{cases}
\end{align*}
\]
Throughout this chapter, one uses the following asymptotic expansions:

\begin{align}
(8.1) & \quad T^\varepsilon(t, x) = T^0(t, x, y) + \varepsilon T^1(t, x, y) + \varepsilon^2 T^2(t, x, y) + \ldots, \\
(8.2) & \quad \Lambda^\varepsilon(t, x) = \Lambda^0(t, x, y) + \varepsilon \Lambda^1(t, x, y) + \varepsilon^2 \Lambda^2(t, x, y) + \ldots, \\
(8.3) & \quad f(T^\varepsilon + \alpha, \Lambda^\varepsilon + b) = f(T^0 + \alpha, \Lambda^0 + b) + \varepsilon T^1 \partial_1 f(T^0 + \alpha, \Lambda^0 + b) \\
& \quad + \varepsilon \Lambda^1 \partial_2 f(T^0 + \alpha, \Lambda^0 + b) + \ldots, \\
(8.4) & \quad g(T^\varepsilon + \alpha, \Lambda^\varepsilon + b) = g(T^0 + \alpha, \Lambda^0 + b) + \varepsilon T^1 \partial_1 g(T^0 + \alpha, \Lambda^0 + b) \\
& \quad + \varepsilon \Lambda^1 \partial_2 g(T^0 + \alpha, \Lambda^0 + b) + \ldots, \\
(8.5) & \quad B(T^\varepsilon + \alpha, \Lambda^\varepsilon + b) = B(T^0 + \alpha, \Lambda^0 + b) + \varepsilon T^1 \partial_1 B(T^0 + \alpha, \Lambda^0 + b) \\
& \quad + \varepsilon \Lambda^1 \partial_2 B(T^0 + \alpha, \Lambda^0 + b) + \ldots, \\
(8.6) & \quad p(T^\varepsilon + \alpha, \Lambda^\varepsilon + b) = p(T^0 + \alpha, \Lambda^0 + b) + \varepsilon T^1 \partial_1 p(T^0 + \alpha, \Lambda^0 + b) \\
& \quad + \varepsilon \Lambda^1 \partial_2 p(T^0 + \alpha, \Lambda^0 + b) + \ldots,
\end{align}

where \( y = \frac{x}{\varepsilon} \). Again, as discussed in Chapter 7, note the following calculation rule: if \( \alpha(t, x) = \beta(t, x, y)|_{y=x/\varepsilon} \) is sufficiently smooth, then \( \nabla \alpha := \nabla_x \beta + \frac{1}{\varepsilon} \nabla_y \beta|_{y=x/\varepsilon} \). For simplicity, the divergence operator with respect to \( x \) is denoted by \( \nabla_x \), and the one with respect to \( y \) is denoted by \( \nabla_y \).

### 8.2 Derivation of the cell problems for the heat equation

First, look at \( T^\varepsilon \).

\[
(P^T) \quad \begin{cases}
\partial_t T^\varepsilon + \text{div}(P^\varepsilon(T^\varepsilon + a) - \varepsilon \nabla T^\varepsilon) = f(T^\varepsilon + a, \Lambda^\varepsilon + b) & \text{in } \Omega^\varepsilon, \\
n \cdot (P^\varepsilon(T^\varepsilon + a) - \varepsilon \nabla T^\varepsilon) = \varepsilon B(T^\varepsilon + a, \Lambda^\varepsilon + b) & \text{at } \Gamma^\varepsilon, \\
T^\varepsilon = 0 & \text{at } \partial \Omega.
\end{cases}
\]

Substituting the homogenisation assumptions (8.1)-(8.6) into problem \((P^T)\) gives

\[
\begin{align*}
\partial_t (T^0 + \varepsilon T^1 + \varepsilon^2 T^2 + \ldots) + (\nabla_x + \varepsilon \nabla_y) \cdot (P(t, y) - L(t, y)(\nabla_x + \varepsilon \nabla_y))(T^0 + a \\
+ \varepsilon T^1 + \varepsilon^2 T^2 + \ldots) = f(T^0 + a, \Lambda^0 + b) + \varepsilon T^1 \partial_1 f(T^0 + a, \Lambda^0 + b) \\
+ \varepsilon \Lambda^1 \partial_2 f(T^0 + a, \Lambda^0 + b) + \ldots,
\end{align*}
\]
or
\[
\begin{align*}
- \varepsilon^{-2} \nabla_x \cdot (L(t, y) \nabla_y T^0) & + \varepsilon^{-1} (\nabla_x \cdot (P(t, y)(T^0 + a)) - \nabla_y \cdot (L(t, y)(\nabla_x T^0 + \nabla_y T^1))) \\
- \nabla_x \cdot (L(t, y) \nabla_y T^0) & + \partial_1 T^0 + \nabla_x \cdot (P(t, y)(T^0 + a)) - \nabla_x \cdot (L(t, y)(\nabla_x T^0 + \nabla_y T^1)) \\
+ \nabla_y \cdot (P(t, y) T^0 - \nabla_y \cdot (L(t, y)(\nabla_x T^1 + \nabla_y T^2))) & + \ldots = f(T^0 + a, \Lambda^0 + b) \\
+ \varepsilon T^1 \partial_1 f(T^0 + a, \Lambda^0 + b) & + \varepsilon \Lambda^1 \partial_2 f(T^0 + a, \Lambda^0 + b) + O(\varepsilon^2),
\end{align*}
\]
or
\[
(\varepsilon^{-2} \tilde{A}^2 + \varepsilon^{-1} \tilde{A}^1 + A^0 + \ldots)(T^0 + a, T^1, T^2, \ldots) = f(T^0 + a, \Lambda^0 + b) \\
+ \varepsilon T^1 \partial_1 f(T^0 + a, \Lambda^0 + b) + \varepsilon \Lambda^1 \partial_2 f(T^0 + a, \Lambda^0 + b) + O(\varepsilon^2),
\]

where
\[
\begin{align*}
\tilde{A}^2 & := -\nabla_y \cdot (L(t, y) \nabla_y), \\
\tilde{A}^1 & := -\nabla_x \cdot (L(t, y) \nabla_y) + \nabla_y \cdot P(t, y) - \nabla_y \cdot (L(t, y)(\nabla_x + \nabla_y)), \\
A^0 & := \partial_1 + \nabla_x \cdot P(t, y) - \nabla_x \cdot (L(t, y)(\nabla_x + \nabla_y)) + \nabla_y \cdot P(t, y) - \nabla_y \cdot (L(t, y)(\nabla_x + \nabla_y)).
\end{align*}
\]
Now one wishes to get the boundary condition. Collecting the terms, one gets
\[
- \varepsilon^{-1} n(y) \cdot L(t,y) \nabla_y T^0 + n(y) \cdot (P(t,y)(T^0 + a) - L(t,y)(\nabla_x T^0 + \nabla_y T^1)) \\
+ \varepsilon n(y) \cdot (P(t,y)T^1 - L(t,y)(\nabla_x T^1 + \nabla_y T^2)) + \ldots \\
= \varepsilon B(T^0 + a, \Lambda^0 + b) + \varepsilon^2 T^3 \partial_t B(T^0 + a, \Lambda^0 + b) + \ldots.
\]

Let \( d \in \{2, 3\} \). One is led to the cell problems \((P_{T_2}^T)\), \((P_{T_1}^T)\) and \((P_0^T)\). In order to ensure the existence and uniqueness of periodic solutions \((P_{T_2}^T)\), \((P_{T_1}^T)\) and \((P_0^T)\), Lemma 7.2.1 is needed. One starts with
\[
(P_{T_2}^T) \begin{cases}
- \tilde{\nabla}_y \cdot (L(t,y) \nabla_y T^0) = 0 & \text{in } \Omega_y, \\
n(y) \cdot L(t,y) \nabla_y T^0 = 0 & \text{at } \Gamma,
\end{cases}
\]
\( T^0 \) is \( Y \)-periodic.

Furthermore, one sees that \( T^0 \) is independent of \( y \), i.e. \( T^0(t,x,y) = T^0(t,x) \).
\[
(P_{T_1}^T) \begin{cases}
\tilde{\nabla}_y \cdot (P(t,y)(T^0 + a)) - \tilde{\nabla}_y \cdot (L(t,y)(\nabla_x T^0 + \nabla_y T^1)) = 0 & \text{in } \Omega_y, \\
n(y) \cdot (P(t,y)(T^0 + a) - L(t,y)(\nabla_x T^0 + \nabla_y T^1)) = 0 & \text{at } \Gamma,
\end{cases}
\]
\( T^1 \) is \( Y \)-periodic.

Since \((P_{T_1}^T)\) is linear, it is expected that \( T^1 \) is a linear function of \( \nabla_x T^0 \). The calculation of \( T^1 \) in terms of the gradient of \( T^0 \) is
\[
T^1(t,x,y) := \tilde{T}^1(t,x) + \sum_{j=1}^d \partial_{x_j} T^0 w_j^T(y) = \tilde{T}^1(t,x) + w^T(y) \nabla_x T^0,
\]
where
\[
w^T(y) = \begin{pmatrix}
w_1^T(y) \\
\vdots \\
w_d^T(y)
\end{pmatrix} \in \mathbb{R}^d
\]
are the so-called cell functions. Some remarks:
\[
\nabla_x T^0(t,x) = \sum_{j=1}^d \partial_{x_j} T^0(t,x) \tilde{e}_j,
\]
\[
\nabla_y T^1(t,x,y) = \sum_{j=1}^d \partial_{x_j} T^0(t,x) \nabla_y w_j^T(y).
\]

Then for all \( j \in \{1, \ldots, d\} \), one has the following cell problem:
\[
(P_{T_{1,j}}^T) \begin{cases}
\tilde{\nabla}_y \cdot (P(t,y)(T^0 + a)) - \tilde{\nabla}_y (L(t,y)(\tilde{e}_j + \nabla_y w_j^T(y))) = 0 & \text{in } \Omega_y, \\
n(y) \cdot (P(t,y)T^0 - L(t,y)(\tilde{e}_j + \nabla_y w_j^T(y))) = 0 & \text{at } \Gamma,
\end{cases}
\]
\( w_j^T \) is \( Y \)-periodic.

If one assumes as in [15] that the term \( \tilde{\nabla}_y \cdot P(t,y) = 0 \), then \((P_{T_{1,j}}^T)\) satisfies a little. This means incompressible fluxes with respect to \( y \). A remark is that the cell function \( w^T \) is the unique solution of the cell problem \((P_{T_{1,j}}^T)\). This is due to Lemma 7.2.1.
\[
(P_0^T) \begin{cases}
\partial_t T^0 + \tilde{\nabla}_x \cdot (P(t,y)(T^0 + a)) - \tilde{\nabla}_x (L(t,y)(\nabla_x T^0 + \nabla_y T^1)) + \tilde{\nabla}_y \cdot (P(t,y)T^1) \\
- \tilde{\nabla}_y \cdot (L(t,y)(\nabla_x T^1 + \nabla_y T^2)) = f(T^0 + a, \Lambda^0 + b) & \text{in } \Omega_y, \\
n(y) \cdot (P(t,y)T^1 - L(t,y)(\nabla_x T^1 + \nabla_y T^2)) = B(T^0 + a, \Lambda^0 + b) & \text{on } \Gamma,
\end{cases}
\]
\( T^2 \) is \( Y \)-periodic.
Remark: if \( T^0 = T^0(t, x) \) and \( B(T^0, \Lambda^0 + b) \) is independent of \( y \), then \( \Lambda^0(t, x, y) = \Lambda^0(t, x) \).

In order to obtain the homogenised equation from \((P^0_\Omega)\), integrating the \( \Omega_y \)-cell, i.e.

\[
\frac{1}{|\Omega_y|} \int_{\Omega_y} \partial_t T^0 dy + \frac{1}{|\Omega_y|} \int_{\Omega_y} \vec{\nabla}_x \cdot (P(t, y)T^0) dy = \frac{1}{|\Omega_y|} \int_{\Omega_y} \vec{\nabla}_x \cdot (L(t, y)(\nabla_x T^0 + \nabla_y T^1)) dy
\]

\[
+ \frac{1}{|\Omega_y|} \int_{\Omega_y} \vec{\nabla}_y \cdot (P(t, y)T^0) dy - \frac{1}{|\Omega_y|} \int_{\Omega_y} \vec{\nabla}_y \cdot (L(t, y)(\nabla_x T^1 + \nabla_y T^2)) dy
\]

\[
= \frac{1}{|\Omega_y|} \int_{\Omega_y} f(T^0 + a, \Lambda^0 + b) dy,
\]

where

\[
I^T_0 := \frac{1}{|\Omega_y|} \int_{\Omega_y} \partial_t T^0 dy = \partial_t T^0,
\]

\[
I^T_1 := \frac{1}{|\Omega_y|} \int_{\Omega_y} \vec{\nabla}_x \cdot (P(t, y)T^0) dy = \frac{\vec{\nabla} \cdot T^0}{|\Omega_y|} \int_{\Omega_y} P(t, y) dy = \vec{\nabla}^T \cdot T^0, \text{ with } \vec{\nabla}^T := \frac{1}{|\Omega_y|} \int_{\Omega_y} P(t, y) dy
\]

and \( T^0 \) is independent of \( y \).

\[
I^T_2 := - \frac{1}{|\Omega_y|} \int_{\Omega_y} \vec{\nabla}_x \cdot (L(t, y)(\nabla_x T^0 + \nabla_y T^1)) dy = - \vec{\nabla}_x \cdot \left( \frac{1}{|\Omega_y|} \int_{\Omega_y} L(t, y)(\nabla_x T^0 + \sum_{j=1}^{d} \partial_{x_j} T^0 \partial_{y_j} T^j(y)) dy \right)
\]

\[
= - \frac{1}{|\Omega_y|} \sum_{i=1}^{d} \partial_{x_i} \left( \int_{\Omega_y} L(t, y)(\delta_{ij} + \partial_{y_j} w^T_j(y)) \partial_{x_j} T^0 \right) = - \vec{\nabla}_x \cdot (\vec{A} \cdot \nabla_x T^0),
\]

with

\[
\vec{A} := (\vec{a}^T_{ij})_{i,j=1,d},
\]

\[
\vec{a}^T_{ij} := \frac{1}{|\Omega_y|} \int_{\Omega_y} L(t, y)(\delta_{ij} + \partial_{y_j} w^T_j(y)) dy = \frac{1}{|\Omega_y|} \int_{\Omega_y} L(t, y)(\vec{e}_j + \nabla_y w^T_j(y)) \vec{e}_i dy,
\]

\[
I^T_3 := \frac{1}{|\Omega_y|} \int_{\Omega_y} \vec{\nabla}_y \cdot (P(t, y)T^1) dy = \frac{1}{|\Omega_y|} \int_{\Omega_y} \sum_{j=1}^{d} \partial_{x_j} T^0(t, x) \vec{\nabla}_y \cdot (P(t, y)w^T_j(y)) dy,
\]

\[
= \frac{1}{|\Omega_y|} \sum_{j=1}^{d} \partial_{x_j} T^0 \int_{\Omega_y} \vec{\nabla}_y \cdot (P(t, y)w^T_j(y)) dy = \frac{1}{|\Omega_y|} \sum_{j=1}^{d} \partial_{x_j} T^0(t, x) \int_{\partial\Omega_y} n(y) \cdot (P(t, y)w^T_j(y)) d\sigma_y
\]

\[
= 0 \text{ (due to the periodicity of } P(t, y)w^T_j(y)),
\]

\[
I^T_4 := - \frac{1}{|\Omega_y|} \int_{\Omega_y} \vec{\nabla}_y \cdot (L(t, y)(\nabla_x T^1 + \nabla_y T^2)) dy = - \frac{1}{|\Omega_y|} \int_{\partial\Omega_y} n(y) \cdot (L(t, y)(\nabla_x T^1 + \nabla_y T^2)) d\sigma_y
\]

\[
- \frac{1}{|\Omega_y|} \int_{\partial\Omega_y} n(y) \cdot (L(t, y)(\nabla_x T^1 + \nabla_y T^2)) d\sigma_y = - \frac{1}{|\Gamma|} \int_{\Gamma} B(T^0 + a, \Lambda^0 + b) d\sigma_y
\]

\[
= - \frac{|\Gamma|}{|\Omega_y|} B(T^0 + a, \Lambda^0 + b),
\]
8.3 Derivation of the cell problems for the oxygen equation

The homogenisation assumption for the oxygen equation is essentially the same as the homogenisation assumption for the heat equation. Now one looks at the partial differential equations for $\Lambda^\varepsilon$.

\[
(P^\Lambda) \begin{cases} 
\partial_t (\Lambda^0 + \varepsilon \Lambda^1 + \varepsilon^2 \Lambda^2 + \ldots) \\
\div (P^\Lambda (\Lambda^0, \Lambda^0)) + \partial_t \Lambda^0 + \nabla_x \cdot (P(t,y)(\Lambda^0) - \nabla_x \cdot (D(t,y)\nabla_y \Lambda^0)) \\
- \varepsilon^{-2} \nabla_y \cdot (D(t,y)\nabla_y \Lambda^0) + \varepsilon^{-1} \nabla_y \cdot (P(t,y)(\Lambda^0) - \nabla_x \cdot (D(t,y)\nabla_y \Lambda^0)) \\
- \varepsilon^{-1} \nabla_y \cdot (P(t,y)(\Lambda^0) - \nabla_x \cdot (D(t,y)\nabla_y \Lambda^0)) \\
= g(T^0 + a, \Lambda^0 + b) + \varepsilon T^1 \partial_1 g(T^0 + a, \Lambda^0 + b) + \varepsilon \Lambda^1 \partial_2 g(T^0 + a, \Lambda^0 + b) + \mathcal{O}(\varepsilon^2),
\end{cases}
\]

or

\[
(P^\Lambda) \begin{cases} 
(\varepsilon^{-2} A_{-2} + \varepsilon^{-1} A_{-1} + A_0 + \ldots)(\Lambda^0) + h, \Lambda^1, \Lambda^2, \ldots) \\
= g(T^0 + a, \Lambda^0 + b) + \varepsilon T^1 \partial_1 g(T^0 + a, \Lambda^0 + b) + \varepsilon \Lambda^1 \partial_2 g(T^0 + a, \Lambda^0 + b) + \mathcal{O}(\varepsilon^2),
\end{cases}
\]

where

\[
A_{-2} := -\nabla_y \cdot (D(t,y)\nabla_y), \\
A_{-1} := -\nabla_x \cdot (D(t,y)\nabla_y) + \nabla_y \cdot P(t,y) - \nabla_y \cdot (D(t,y)\nabla_x + \nabla_y)), \\
A_0 := \partial_t + \nabla_x \cdot (P(t,y)(\Lambda^0 + b)) - \nabla_y \cdot (D(t,y)\nabla_x + \nabla_y)) + \nabla_y \cdot P(t,y) - \nabla_y \cdot (D(t,y)(\nabla_x + \nabla_y)).
\]

To get the boundary condition, one proceeds as follows:

\[
\varepsilon n(y) \cdot D(t,y)\nabla_y \Lambda^0 - n(y) \cdot D(t,y)(\nabla_x \Lambda^0 + \nabla_y \Lambda^1) \\
\varepsilon n(y) \cdot D(t,y)(\nabla_x \Lambda^1 + \nabla_y \Lambda^2) + \ldots = \varepsilon p(T^0 + a, \Lambda^0 + b) + \varepsilon T^1 \partial_1 p(T^0 + a, \Lambda^0 + b) + \varepsilon \Lambda^1 \partial_2 p(T^0 + a, \Lambda^0 + b) + \ldots.
\]

These equations lead to the cell problems $\Lambda_{-2}$, $\Lambda_{-1}$ and $\Lambda_0$ (having the same structure as the cell problems for $T^\varepsilon$).

\[
(P_{-2}^\Lambda) \begin{cases} 
-\nabla_y \cdot (D(t,y)\nabla_y \Lambda^0) = 0 \quad \text{in } \Omega_y, \\
n(y) \cdot D(t,y)\nabla_y \Lambda^0 = 0 \quad \text{at } \Gamma,
\end{cases}
\]

\[
\Lambda_0 \text{ is } Y\text{-periodic.}
\]
Then this implies that $\Lambda^0$ is independent of $y$, i.e. $\Lambda^0(t, x, y) = \Lambda^0(t, x)$.

\[
(P^\Lambda_{\Lambda^1}) \begin{cases}
\widetilde{\nabla}_y \cdot (P(t, y)(\Lambda^0 + b)) - \widetilde{\nabla}_y \cdot (D(t, y)(\nabla_x \Lambda^0 + \nabla_y \Lambda^1)) = 0 & \text{in } \Omega_g, \\
n(y) \cdot D(t, y)(\nabla_x \Lambda^0 + \nabla_y \Lambda^1) = 0 & \text{at } \Gamma,
\end{cases}
\]

where $\Lambda^1$ is $Y$-periodic.

Since $(P^\Lambda_{\Lambda^1})$ is linear, it is expected that $\Lambda^1$ is a linear function of $\nabla_x \Lambda^0$. The calculation of $\Lambda^1$ in terms of the gradient of $\Lambda^0$ is

\[
\Lambda^1(t, x, y) := \Lambda^1(t, x) + \sum_{i=1}^{d} \partial_{x_i} \Lambda^0 w_i^\Lambda(y) = \Lambda^1(t, x) + w^\Lambda(y) \nabla_x \Lambda^0,
\]

where

\[
w^\Lambda(y) = \begin{pmatrix} w_1^\Lambda(y) \\ \vdots \\ w_d^\Lambda(y) \end{pmatrix} \in \mathbb{R}^d
\]

are cell functions.
One denotes:

\[
\nabla_x \Lambda^0(t, x) = \sum_{i=1}^{d} \partial_{x_i} \Lambda^0(t, x) \vec{e}_i,
\]

\[
\nabla_y \Lambda^1(t, x, y) = \sum_{i=1}^{d} \partial_{y_i} \Lambda^0(t, x) \nabla_y w_i^\Lambda(y).
\]

For all $l \in \{1, \ldots, d\}$, the following cell problem is

\[
(P^\Lambda_{\Lambda^1,l}) \begin{cases}
\nabla_y \cdot (P(t, y)(\Lambda^0 + b)) - \nabla_y \cdot (D(t, y)(\vec{e}_l + \nabla_y w_l^\Lambda(y))) = 0 & \text{in } \Omega_g, \\
n(y) \cdot D(t, y)(\vec{e}_l + \nabla_y w_l^\Lambda(y)) = 0 & \text{at } \Gamma,
\end{cases}
\]

are $Y$-periodic.

If one assumes as in [15] that $\widetilde{\nabla}_y \cdot P(t, y) = 0$, then $(P^\Lambda_{\Lambda^1,l})$ satisfies a little. Finally, one has the following partial differential equations in terms of $\Lambda^2$:

\[
(P^\Lambda_{\Lambda^2}) \begin{cases}
\partial_t \Lambda^0 + \nabla_x \cdot (P(t, y)(\Lambda^0 + b)) - \nabla_x \cdot (D(t, y)(\nabla_x \Lambda^0 + \nabla_y \Lambda^1)) + \nabla_y \cdot (P(t, y)\Lambda^1) = 0 & \text{in } \Omega_g, \\
n(y) \cdot D(t, y)(\nabla_x \Lambda^1 + \nabla_y \Lambda^2) = p(T^0 + a, \Lambda^0 + b) & \text{at } \Gamma,
\end{cases}
\]

$\Lambda^2$ is $Y$-periodic.

Note that if both $\Lambda^0 = \Lambda^0(t, x)$ and $p(T^0 + a, \Lambda^0 + b)$ are independent of $y$, then $T^0(t, x, y) = T^0(t, x)$ for all $(t, x) \in S \times Y$. Analogously to the treatment of the heat equation, the integral is taken over the $\Omega_g$-cell in order
to get the homogenised equation for \((P_0^\Lambda)\), i.e.
\[
\frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \partial_t \Lambda^0 dy + \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \vec{\nabla}_x \cdot (P(t, y) (\Lambda^0 + b)) dy - \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \vec{\nabla}_x \cdot (D(t, y) (\nabla_x \Lambda^0 + \nabla_y \Lambda^1)) dy
\]
\[
+ \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \vec{\nabla}_y \cdot (P(t, y) \Lambda^1) dy - \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \vec{\nabla}_y \cdot (D(t, y) (\nabla_x \Lambda^1 + \nabla_y \Lambda^2)) dy
\]
\[
= \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} g(T^0 + a, \Lambda^0 + b) dy.
\]
where
\[
I_0^\Lambda := \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \partial_t \Lambda^0 dy = \partial_t \Lambda^0,
\]
\[
I_1^\Lambda := \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \vec{\nabla}_x \cdot (P(t, y) (\Lambda^0 + b)) dy = \vec{\nabla}_x \cdot \frac{\tilde{\rho}}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} P(t, y) dy = \tilde{P} \Lambda \vec{\nabla} \cdot \Lambda^0
\]
with \(\tilde{P} := \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} P(t, y) dy.
\]
\[
I_2^\Lambda := -\frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \vec{\nabla}_x \cdot (D(t, y) (\nabla_x \Lambda^0 + \nabla_y \Lambda^1)) dy
\]
\[
= -\vec{\nabla}_x \cdot \left( \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} D(t, y) (\nabla_x \Lambda^0 + \sum_{i=1}^d \partial_{x_i} \Lambda^0 \nabla_y w_{i}^\Lambda(y)) dy \right)
\]
\[
= -\frac{1}{|\Omega_\Lambda|} \sum_{k=1}^d \partial_{x_k} \sum_{i=1}^d \left( \int_{\Omega_\Lambda} D(t, y) (\delta_{kl} + \partial_{y_k} w_{i}^\Lambda(y)) dy \right) \partial_{x_i} \Lambda^0 = -\vec{\nabla}_x \cdot (\tilde{\Lambda} \nabla_x \Lambda^0),
\]
with
\[
\tilde{\Lambda} := (\tilde{\rho}_{kl})_{k, l = 1}^{d},
\]
\[
\tilde{\rho}_{kl} := \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} D(t, y) (\delta_{kl} + \partial_{y_k} w_{i}^\Lambda(y)) dy = \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} D(t, y) (\vec{c}_l + \nabla_y w_{i}^\Lambda(y)) \vec{c}_l dy,
\]
\[
I_3^\Lambda := \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \vec{\nabla}_y \cdot (P(t, y) \Lambda^1) dy = \frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \sum_{i=1}^d \partial_{x_i} \Lambda^0(t, x) \vec{\nabla}_y \cdot (P(t, y) w_{i}^\Lambda(y)) dy
\]
\[
= \frac{1}{|\Omega_\Lambda|} \sum_{i=1}^d \partial_{x_i} \Lambda^0 \int_{\Omega_\Lambda} \vec{\nabla}_y \cdot (P(t, y) w_{i}^\Lambda(y)) dy = \frac{1}{|\Omega_\Lambda|} \sum_{i=1}^d \partial_{x_i} \Lambda^0(t, x) \int_{\partial \Omega} n(y) \cdot (P(t, y) w_{i}^\Lambda(y)) d\sigma_y
\]
\[
= 0 \text{ (due to the periodicity of } P(t, y) w_{i}^\Lambda(y) \text{)},
\]
\[
I_4^\Lambda := -\frac{1}{|\Omega_\Lambda|} \int_{\Omega_\Lambda} \vec{\nabla}_y \cdot (D(t, y) (\nabla_x \Lambda^1 + \nabla_y \Lambda^2)) dy = -\frac{1}{|\Omega_\Lambda|} \int_{\Gamma} n(y) \cdot (D(t, y) (\nabla_x \Lambda^1 + \nabla_y \Lambda^2)) d\sigma_y
\]
For getting the boundary conditions in this case, one proceeds as follows:

\[
- \frac{1}{|\Omega|} \int_{\Omega} n(y) \cdot (D(t, y)(\nabla_x \Lambda^1 + \nabla_y \Lambda^2)) \, d\sigma_y = - \frac{1}{|\Gamma|} \int_{\Gamma} p(T^0 + a, \Lambda^0 + b) \, d\sigma_y
\]

= \frac{-1}{|\Omega|} p(T^0 + a, \Lambda^0 + b),

\[I_0^\Lambda := \frac{-1}{|\Omega|} \int_{\Omega} g(T^0 + a, \Lambda^0 + b) \, dy = g(\Theta^0 + a, \Lambda^0 + b).\]

Here, $|\Omega_g|$ means the volume of $\Omega_g$. The quantities $\frac{|\Omega_0|}{|\Omega|}$ and $\frac{|\Gamma|}{|\Gamma|}$ are called the volumetric and surface porosities respectively.

### 8.4 Derivation of the cell problems for the $\Pi^\varepsilon$-equation

The high-contrast scaling $\frac{L^\varepsilon}{\varepsilon^2} = \mathcal{O}(\varepsilon^2)$ plays an important role at this stage. The following partial differential equation for $\Pi^\varepsilon$ is called the heat equation in the solid:

\[
(P_\Pi^\varepsilon) \begin{cases}
\partial_t \Pi^\varepsilon - \varepsilon^2 \text{div}(D^\varepsilon \nabla \Pi^\varepsilon) = 0 & \text{in } \Omega^\varepsilon, \\
\varepsilon n^\varepsilon \cdot D^\varepsilon \nabla \Pi^\varepsilon = \varepsilon B(T^\varepsilon, \Lambda^\varepsilon + b) & \text{at } \Gamma^\varepsilon.
\end{cases}
\]

Substituting the homogenisation assumptions (8.1)-(8.6) into problem $(P_\Pi^\varepsilon)$ gives

\[
\partial_t (\Pi^0 + \varepsilon \Pi^1 + \varepsilon^2 \Pi^2 + \ldots) - \varepsilon^2 (\nabla_x + \frac{1}{\varepsilon} \nabla_y) \cdot (D_s(t, y)(\nabla_x + \frac{1}{\varepsilon})(\Pi^0 + \varepsilon \Pi^1 + \varepsilon^2 \Pi^2 + \ldots)) = 0,
\]

or

\[
\partial_t \Pi^0 - \nabla_y \cdot (D_s(t, y)\nabla_y \Pi^0) + \varepsilon (\partial_t \Pi^1 - \nabla_x \cdot (D_s(t, y)\nabla_y \Pi^0)) \\
- \nabla_y \cdot (D_s(t, y)(\nabla_x \Pi^0 + \nabla_y \Pi^1))) + \varepsilon^2 (\partial_t \Pi^2 - \nabla_x \cdot (D_s(t, y)(\nabla_x \Pi^0 + \nabla_y \Pi^1))) \\
- \nabla_y \cdot (D_s(t, y)(\nabla_x \Pi^1 + \nabla_y \Pi^2))) + \mathcal{O}(\varepsilon^3) = 0.
\]

Note that in the $\varepsilon^0$ order terms of this partial differential equation, the variable $x$ plays the role of a parameter. Here the variable $y$ is the important one. Essentially, one has $\Pi = \Pi(t, x, y)$ for $(t, x, y) \in S \times \Omega \times \Omega_y$.

For getting the boundary conditions in this case, one proceeds as follows:

\[
\varepsilon^2 n(y) \cdot D_s(t, y)(\nabla_x + \frac{1}{\varepsilon} \nabla_y)(\Pi^0 + \varepsilon \Pi^1 + \varepsilon^2 \Pi^2 + \ldots) = \varepsilon B(T^0 + a, \Lambda^0 + b) \\
+ \varepsilon^2 T^1 \partial_1 B(T^0 + a, \Lambda^0 + b) + \varepsilon^2 \Lambda^1 \partial_2 B(T^0 + a, \Lambda^0 + b) + \ldots,
\]

or

\[
\varepsilon n(y) \cdot D_s(t, y)\nabla_y \Pi^0 + \varepsilon^2 n(y) \cdot D_s(t, y)(\nabla_x \Pi^0 + \nabla_y \Pi^1) \\
+ \varepsilon^3 n(y) \cdot D_s(t, y)(\nabla_x \Pi^1 + \nabla_y \Pi^2) + \ldots \\
= \varepsilon B(T^0 + a, \Lambda^0 + b) + \varepsilon^2 T^1 \partial_1 B(T^0 + a, \Lambda^0 + b) + \varepsilon^2 \Lambda^1 \partial_2 B(T^0 + a, \Lambda^0 + b) + \ldots.
\]

These equations lead to the cell problem

\[
(P_\Pi^\varepsilon) \begin{cases}
\partial_t \Pi^0 - \nabla_y \cdot (D_s(t, y)\nabla_y \Pi^0) = 0 & \text{in } \Omega^\varepsilon, \\
n(y) \cdot D_s(t, y)\nabla_y \Pi^0 = B(T^0 + a, \Lambda^0 + b) & \text{at } \Gamma^\varepsilon,
\end{cases}
\]

$\Pi^0$ is $Y$-periodic.
8.5 Averaged model equations and derivation of effective coefficients

The homogenisation of the initial boundary value problem \((P^\#)\) is given by

\[
\begin{align*}
(p^\#) & \quad \left\{ \begin{array}{l}
\partial_t T + PT \hat{\nabla}_x \cdot T - \hat{\nabla}_x \cdot (A^T \nabla_x T) \\
- \left[ \begin{array}{c} \frac{[T]}{[T]} \\ \frac{[\Gamma]}{[\Gamma]} \end{array} \right] B(T + a, \Lambda + b) + f(T + a, \Lambda + b), & t \in S, x \in \Omega, \\
\partial_t \Lambda + P^\Lambda \hat{\nabla}_x \cdot \Lambda - \hat{\nabla}_x \cdot (A^\Lambda \nabla_x \Lambda) & \\
- \left[ \begin{array}{c} \frac{[T]}{[T]} \\ \frac{[\Gamma]}{[\Gamma]} \end{array} \right] p(T + a, \Lambda + b) + g(T + a, \lambda + b), & t \in S, x \in \Omega, \\
\partial_t \Pi - \hat{\nabla}_y \cdot (D_s(t, y) \nabla_y \Pi) = 0, & t \in S, x \in \Omega, y \in \Omega_s, \\
T = 0 & \text{at } \partial \Omega, \\
\Lambda = 0 & \text{at } \partial \Omega, \\
\nu(y) \cdot D_s(t, y) \nabla_y \Pi = B(T + a, \Lambda + b), & t \in S, x \in \Omega, y \in \Gamma.
\end{array} \right.
\end{align*}
\]
Chapter 9

Conclusion

The thesis has started off with a (microscopic) reaction-diffusion system of equations modelling smouldering combustion in domains with periodically distributed microstructure. We have proved good energy bounds and uniqueness of weak solutions to the microscopic model, preparing in this way the background for the main task of this thesis, that is passing with $\varepsilon \to 0$ in a weakly coupled partly dissipative system of reaction-diffusion equations. Here $\varepsilon$ is scale parameter related to the choice of the microstructure.

We have accomplished this averaging task via two different methods:

- formal asymptotic homogenisation and
- rigorous two-scale convergence.

(9) also ensures the existence of weak solutions of the macroscopic problem.

We have also shown the uniqueness solutions as $\varepsilon = 0$. Furthermore, we have studied a combustion scenario taking place in high contrast media.

As averaged equations, we have obtained here the classical structure of double-porosity-type models.

There are several topics which have been discussed in this paper. First of all, periodicity of the geometry has been assumed. This assumption has been done in order to apply the formal homogenisation expansion technique properly. However, it would be interesting if the periodicity assumption has been dropped. Can the microscopic still be solved with formal asymptotic expansions? If the answer is negative, what other techniques are able to handle this problem?

Related to this problem is the geometry of the microscopic model. The structure of a grain has been assumed to be circular, see the pictures 3.1a and 3.1b. What happens if the grains are not circular, for instance an auxiliary form without any patterns? This question is also linked to the periodicity of the cells.

Another interesting thing is the partly dissipative system, due to the presence of the ordinary differential equation

\[
\begin{align*}
\frac{\partial R'}{\partial t}(t, x) &= b(\Theta(t, x), \Psi(t, x)) \quad \text{at } \Gamma^v, \\
R'(0, x) &= R'_0(x),
\end{align*}
\]

(9.1)

The periodic unfolding technique can be applied to the above ordinary differential equation if (9.1) has a monotonic non-linear structure. Typical questions in this context are: how can this technique be applied to (9.1) and what are the results?

Interesting open problems in this context are:

- How far can we deviate from the periodicity assumption?
- How can we extend the current model and averaging technique to capture flaming combustion in periodic media?
Appendix A

Basic Inequalities

This appendix contains a few elementary inequalities used in this thesis.

A.1 Cauchy-Schwarz inequality

\[ |x \cdot y| \leq |x||y|, \]

for all \( x, y \in \mathbb{R}^d, d \in \{1, 2, 3, \ldots\} \).

A.2 Hölder’s inequality

Assume \( 1 \leq p, q \leq \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \). Then if \( u \in L^p(X), v \in L^q(X) \), it follows that

\[ \int_X |uv|dx \leq \|u\|_{L^p(X)}\|v\|_{L^q(X)}. \]

A.3 Ellipticity

The differential operator \( L \) is (uniformly) elliptic if there exists a constant \( \alpha > 0 \) such that

\[ \sum_{i,j=1}^n a^{ij}(x)\xi_i \xi_j \geq \alpha |\xi|^2 \]

for almost every \( x \in \Omega \) and all \( \xi \in \mathbb{R}^d, d \in \{1, 2, 3, \ldots\} \).

A.4 Arithmetic mean-geometric mean inequality

For any non-negative numbers \( a_1, a_2, \ldots, a_d, d \in \{1, 2, 3, \ldots\} \), the inequality

\[ \sqrt[d]{a_1a_2\cdots a_d} \leq \frac{a_1 + a_2 + \cdots + a_d}{d} \]

holds. The sign of equality holds if and only if \( a_1 = a_2 = \ldots = a_d \).
A.5  Gronwall’s inequality (differential form)

Let \( \eta(\cdot) \) be a non-negative, absolutely continuous function on \((0, T)\), which satisfies for a.e. \( t \in (0, T) \) the differential inequality

\[
\eta'(t) \leq \phi(t)\eta(t) + \psi(t),
\]

where \( \phi(t) \) and \( \psi(t) \) are non-negative, summable functions on \((0, T)\). Then it holds that

\[
\eta(t) \leq e^{\int_0^t \phi(s)ds} \left( \eta(0) + \int_0^t \psi(s)ds \right)
\]

for all \( 0 < t < T \).

A.6  A variant of Young’s inequality

Let \( 1 < p, q < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Then for \( \delta > 0 \) the inequality

\[
ab \leq \delta a^p + C(\delta)b^q,
\]

with \( C(\delta) = (\delta p)^{-a/p} q^{-1} \) and for all \( a, b > 0 \) holds. Note that \( C(\delta) < \infty \).

A.7  Trace inequality

Trace Theorem A.7.1. Let \( p \in (1, \infty) \). Assume \( \Omega \) is bounded and \( \partial \Omega \) is \( C^1 \). Then there exists a bounded operator

\[
T : W^{1,p}(\Omega) \to L^p(\partial \Omega)
\]

such that

1. \( Tu = u|_{\partial \Omega} \) if \( u \in W^{1,p}(\Omega) \cap C(\bar{\Omega}) \)

and

2. \[
\|Tu\|_{L^p(\partial \Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}
\]

for each \( u \in W^{1,p}(\Omega) \), with the constant \( C \) depending on \( p \) and \( \Omega \).

A.8  Trace inequality for perforated media

Lemma A.8.1. Let \( \psi \in H^1(\Omega^\varepsilon) \). Then

\[
\varepsilon\|\psi\|_{L^2(\Gamma^\varepsilon)}^2 \leq c\|\psi\|_{H^1(\Omega^\varepsilon)}^2,
\]

where \( c \) is independent of \( \varepsilon \).
A.8 Trace inequality for perforated media

Proof. Define on \( \partial \Omega \) a smooth extension \( c = (c_1, \cdots, c_d) \) of the normal unit vector \( n \) by setting
\[
c_j(y) = n_j(y) \quad \text{on } \partial \Omega, \ j = 1, \ldots, d,
\]
and choose \( c_j \) to have their support in a neighbourhood of \( \partial \Omega \). Moreover, choose \( c_j \in C^1(\bar{Y}) \).

Then due to periodicity, it follows that
\[
\varepsilon \int_{\Gamma^e} \psi^2 \, ds = \varepsilon \int_{\Gamma^e} \psi^2 c_j(\bar{\tau})n_j \, ds
\]
\[
= \varepsilon \int_{\Omega^e} \text{div}(\psi^2 c_j(\bar{\tau})n_j) \, dx
\]
\[
= \varepsilon \int_{\Omega^e} c_j(\bar{\tau}) \frac{\partial \psi^2}{\partial x_j} \, dx + \int_{\Omega^e} \frac{\partial c_j(\bar{\tau})}{\partial x_j} \psi^2 \, dx
\]
\[
= 2\varepsilon \int_{\Omega^e} c_j(\bar{\tau}) \psi \frac{\partial \psi}{\partial x_j} \, dx + \int_{\Omega^e} \frac{\partial c_j(\bar{\tau})}{\partial x_j} \psi^2 \, dx.
\]

Hence,
\[
\varepsilon \|\psi^2\|_{L^2(\Gamma^e)}^2 \leq 2\varepsilon C \|\psi\|_{L^2(\Omega^e)} \|\nabla \psi\|_{(L^2(\Omega^e))^d} + C \|\psi\|_{L^2(\Omega^e)}^2
\]
\[
\leq C \|\psi\|_{H^1(\Omega^e)}^2,
\]
which was to be shown. \( \square \)
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