Osmotic Cell Swelling in the Fast Diffusion Limit

Local Well-posedness and Stability Analysis around Equilibria

Master's Thesis Industrial and Applied Mathematics

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Abstract

By osmotic cell swelling, we mean the movement of an organic cell with semipermeable membrane. From a mathematical point of view, this problem is still not fully understood. In this thesis, we present a simple model. We prove that this model is locally well-posed (i.e. there is a unique solution), and that there is an attractive manifold of equilibria. By the latter, we mean that for initial values close to this manifold, the solution converges to some point of this manifold.

In our model, we consider a general class of initial configurations of the cell, from which it is going to evolve by osmosis (which makes the cell swell) and surface tension (which makes the cell shrink while trying to minimize the area of the membrane). The main simplification in our model which is not present in most other models, is that we assume that the solute inside the cell is uniformly distributed at each time. This models fast diffusion of the solute.

Because of the modelling of osmosis, we need global (and not just local) information of the membrane to describe its motion at an arbitrary point. Therefore, our problem belongs to a class of hard problems for which a lot of research is going on. However, our problem will be one of the easiest of such.

Our model is described by a free boundary problem. We can get rid of the free boundary by fixing a hypersurface (i.e. a certain compact Riemannian manifold) in time, from which we parameterize our moving membrane. This yields a nonlinear evolution equation on a fixed function space on this hypersurface.

We use little Hölder spaces for showing local existence and uniqueness (see Theorem 7.3) for a classical solution to our problem. Furthermore, we use Sobolev spaces to show that the manifold of equilibria is attractive (see Theorem 7.4). The other chapters of this thesis contain well-known and less well-known theory which is required to prove these theorems.
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1 Introduction

1.1 Biological background

Xenopus is a genus of frogs. Its oocytes (egg cells) happen to be quite useful in biochemical research. These are very large (its diameter is about 1 mm), are relatively easy to culture and easy to use in experiments. Their use in experiments is that mRNA from other organisms can be injected in the oocyte. Gene expression may be induced because of this mRNA, which yield in a so-called gene-product. This means that the oocyte produces either rRNA (other RNA produced by ribosomes) or proteins.

One of these experiments that have been carried out, consisted of injecting xenopus oocytes with mRNA of which it is known to produce aquaporins (in [23], the author refers to [24] and [31] for the articles about these experiments). Aquaporins (which are assumed to be proteins) facilitate only the flow of H$_2$O molecules through the cell membrane. They prevent all other molecules, that are for example dissolved in water, to flow through the membrane. This causes the cell wall to be semipermeable. After a while, in that experiment, the oocytes were exposed to a hypo-osmotic environment, i.e. they were placed in a fluid with a low concentration of solute. It appeared that the oocytes injected with the concerning mRNA swelled notably more rapidly than those that were not injected with this mRNA.

1.2 Modelling

Various mathematical models have been made to describe this cell swelling (see e.g. [23]). The importance of an accurate model is that it can be used together with experiments to test hypotheses about cell membranes and to estimate values of related quantities. For example, one would be able to test the assumption that aquaporins are proteins. Furthermore, one can accurately estimate the increase of permeability of the membrane when a certain amount of mRNA is injected. Hence, an accurate model will be very useful for biochemical research.

However, the accuracy of a model often has a positive correlation to the difficulty of showing mathematical properties of this model. In order to understand more accurate models, it often proves useful to consider more simplified models first. This is the state in which one of the aspects of mathematical research about cell swelling is now; trying to prove properties of simplified models.

Besides creating a better understanding of aquaporins, considering the models described above is also of interest for other research fields in mathematics. To explain this, we first need to describe the basics of the model that we are going to consider. In this model,
we consider the evolution of a cell wall in time. The cell is modelled by a semipermeable membrane, which only allows transport of fluid molecules, but not of solvent molecules (like salt). This evolution is described by surface tension (this is related to the curvature of the membrane) and osmotic effects (the concentration of solvent in the cell). The simplification in our model in comparison to most other models, is that we assume that the rate at which the solvent diffuses, is very fast in comparison to the rate at which the membrane moves. As a result, we assume that the solvent is uniformly distributed at each moment of the evolution.

This model is of interest to other fields of mathematical research, because it is a nonlocal free boundary problem. The global dependence is due to osmosis, i.e. the concentration of the solute inside the cell. Our model is one of the most easiest among such problems. Understanding its properties may help understanding models that are more involved. This includes models from other fields of research.

The properties of our model that we investigate, are well-posedness and that the manifold of equilibria is attractive. By well-posedness, we mean that the equation describing the evolution of the cell, will have a unique solution for a certain time period. This time period will be small, but greater than 0. By the manifold of equilibria being attractive, we mean the following. It is not hard to show (see the end of Section 7.1) that there are configurations of the cell for which it is in equilibrium (i.e. it will not move from this configuration as time passes). Such configurations are called equilibria of the evolution equation. Together, these equilibria form a manifold. By this manifold being attractive, we mean that if we start the evolution of the cell wall with a configuration that is close enough to this manifold, that then the cell wall will converge to some configuration in this manifold. However, we do not know (in advance) the configuration to which it will evolve.

1.3 Contents of the thesis

The main results of this thesis are Theorem 7.3 and Theorem 7.4. These theorems state respectively that our model is well-posed, and what the attractive manifold of equilibria is. To understand why these theorems are true, one needs to understand the theory given in the other chapters and sections of this thesis. Therefore, we will give an outline of this thesis.

Section 2.1, Chapters 3 and 4, and Appendix D describe basic concepts of differential geometry, interpolation spaces, semigroup theory, and spherical harmonics. One needs this theory to understand the other sections and chapters in this thesis. These parts of the thesis contain well-known theory that can be found in various books. We will hardly give any proofs. At the beginning of these parts, the interested reader is referred to literature in which these proofs can be found. For the experienced reader, it will suffice to scan through these parts merely to get used to some of the notation.

Sections 2.2 and 2.3, and Chapters 5 and 6 contain theory that is less commonly known. This theory is stronger related to our problem. Sections 2.2 and 2.3 are about showing a technique that allows us to describe the free boundary problem by a nonlinear second order evolution equation defined in a fixed Banach space. Chapter 5 gives existence and uniqueness results for solutions to a general class of fully nonlinear problems. Chapter 6 gives stability results for a general class of quasi-linear problems. We will not give all the proofs, but for those that we do not present here, references are given.
Chapter 7 describes our model as well as its main properties. Most of it is done by ourselves. In Section 7.1, we derive our model. In the end, we give some insight in how the membrane evolves, and we motivate why we can expect stability of equilibria. Section 7.2 is self-explanatory. For understanding both sections, the reader does not have to understand the theory of the previous chapters. In Section 7.3 we transform the description of our model from a free boundary problem into an evolution equation on a function space with fixed domain, in which there is no boundary condition. Chapter 2 contains the required theory to understand this section. Sections 7.4 and 7.5 contain our main results. In the final section of this chapter we show how one can extend some of the results to other or more general cases.

The first part of Appendix A is about (little) Hölder spaces on bounded domains of $\mathbb{R}^n$. Our focus will be on the extension of Hölder continuous functions to smooth hypersurfaces. This extension is done by localization of the hypersurface. Most properties of (little) Hölder spaces on domains of $\mathbb{R}^n$ also hold for (little) Hölder spaces on smooth hypersurfaces, but we have not found this explicitly in the literature. Proposition A.10 is the key for showing that most properties are preserved. Corollary 3.11 and Corollary B.5 are examples of preserved properties. We have given the proofs ourselves.

Appendices B and C contain regularity results for operators on (little) Hölder spaces on smooth hypersurfaces. Because these calculations are quite involved, they have not been included in Section 2.2.

We close the thesis with a discussion chapter, where we evaluate the results of our two main theorems. We also present our current ideas for further research.

1.4 Remarks before reading

We have written this thesis in a common mathematical style, except for commenting on certain statements. We show this by an example. Let $A$, $B$ and $C$ be some expressions. It may happen that the following relation holds

$$A = B \star C,$$

$\star$: because of Theorem 4.9 and (2.17).

The asterisk is used to attend the reader that the concerning equality is not trivial. Directly under the equation, an explanation is given. So in this case, this explanation tells the reader why $B = C$ holds. From the absence of an asterisk above the first equality sign, the reader is told that it is easy to see that $A = B$ holds true. This equality then follows either from a straightforward calculation, or from the text directly above the equation.

We generalize this concept to inequalities, inclusions of sets etc. If more than one equality, inclusion etc. needs special attention, we will number the asterisks by a number in their subscript, and give comments on each of the corresponding statements separately. If the explanation is given by another single equation (for example, the equation labeled by (1.4)), we will write (1.4) instead of an asterisk.

Some of the remarks in this thesis are quite long. We denote the end of a remark by ■. For the end of a proof, we use the symbol □.
Last, because of time issues, we will be brief about introducing Sobolev spaces on compact Riemannian manifolds and about some of their properties.
2 Differential geometry and hypersurfaces

The purpose of this chapter is to present the tools that we need for casting the equation which describes the evolution of the cell wall (which is a free boundary problem), into a non-linear differential equation with an additional non-linear integral term. In this equation, there will be no boundary conditions, let alone a moving boundary, which gives a way more convenient description of the evolution of the cell wall for showing local existence, uniqueness and stability around the equilibrium.

In the first section, we start with constructing tools from differential geometry that we need in the subsequent two sections. Since this theory is well-known, we refer for proofs to the literature. The subsequent two sections are directly related to our evolution equation.

2.1 Differential geometry in $\mathbb{R}^n$

In this section, we present the differential geometry that we need later on. It is our purpose to show the tools we are going to need; not to give proofs about whether these tools are well-defined or why they work. Proofs can be found in numerous books, e.g. in at least one of the following three works. For this section, we have followed the same set up as in [7]. We have also used some theory from [16] and [8].

Differential geometry is a vast and abstract field in mathematics. We, however, restrict ourselves to the geometry of subsets of $\mathbb{R}^n$. Therefore, we do not need the abstract definition of a manifold. The following definition of a $k$-regular manifold will be convenient.

**Definition 2.1.** ($k$-regular manifold). Let $k \in \mathbb{N}_+ \cup \{\infty\}$. $M \subset \mathbb{R}^n$ is called a $k$-regular manifold of dimension $m$ with $1 \leq m \leq n$ iff for every $p \in M$ there exists an open neighbourhood $W$ of $p$ in $\mathbb{R}^n$, a $U \subset \mathbb{R}^m$ open and a $\phi \in C^k(U; \mathbb{R}^n)$ such that:

(i) $\phi(U) = M \cap W =: V$,

(ii) the mapping $\varphi : U \to V$ given by $\varphi(u) = \phi(u)$ is invertible, and $\varphi^{-1} \in C(V; U)$, and

(iii) for all $u \in U$ the Jacobian matrix $J_{\varphi}(u) \in \mathbb{R}^{n \times m}$ has full rank.

The triple $(U, \varphi, V)$ is called a local representation of $M$ around $p$.

**Remark 2.2.** In the following, we denote by $M$ any $k$-regular manifold of dimension $m$. So by fixing an $M$, we automatically fix the numbers $k$ and $m$ too, unless mentioned otherwise.

It is more common to use charts instead of local representations. If $(U, \varphi, V)$ is a local representation, then $(\varphi^{-1}, V)$ denotes the related chart. Since we often use the set $U$, we...
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prefer to use local representations. Sometimes we will use the word 'chart' to indicate the set \( U \).

Points (i) and (ii) state that a \( k \)-regular manifold can locally be described by a subset of the Euclidean space \( \mathbb{R}^m \). This is the property that allows us to translate differential operators, which are initially defined on an Euclidean space, onto differential operators on a \( k \)-regular manifold. The use of point (iii) will become clear later on.

Since we regard \( M \) as a subset of \( \mathbb{R}^n \), we can use the usual definition for continuity for mappings \( f : M \to \mathbb{R}^d \). However we cannot apply the usual definition of the directional derivative in \( \mathbb{R}^n \) to such mappings \( f \), since for \( p \in M \), \( v \in \mathbb{R}^n \) and \( h > 0 \), there is no guarantee that \( p + hv \) is an element of \( M \). If it is not, \( f(p + hv) \) is not defined, which shows that the usual definition of the directional derivative is not applicable.

We are now going to introduce a differentiable structure on \( M \), which we need to define differentiability of a function on \( M \). Let \((U, \varphi, V)\) and \((\tilde{U}, \tilde{\varphi}, \tilde{V})\) be two local representations such that \( \tilde{V} := V \cap \tilde{V} \neq \emptyset \). Then the transition map from \( \varphi^{-1}(\tilde{V}) \subset U \) to \( \tilde{\varphi}^{-1}(\tilde{V}) \subset \tilde{U} \) is given by

\[
\tilde{\varphi}^{-1} \circ \varphi \in C(\varphi^{-1}(\tilde{V}), \tilde{\varphi}^{-1}(\tilde{V})).
\]  

(2.1)

From (i) and (ii) of Definition 2.1, it follows that this transition map is invertible.

For defining a differentiable structure, we need more regularity of this transition map. We call \((U, \varphi, V)\) and \((\tilde{U}, \tilde{\varphi}, \tilde{V})\) \(C^k\)-compatible if either \( \tilde{V} = \emptyset \), or the transition map satisfies

\[
\tilde{\varphi}^{-1} \circ \varphi \in C^k(\varphi^{-1}(\tilde{V}), \tilde{\varphi}^{-1}(\tilde{V})),
\]

(2.2)

\[
\varphi^{-1} \circ \tilde{\varphi} \in C^k(\tilde{\varphi}^{-1}(\tilde{V}), \varphi^{-1}(\tilde{V})).
\]  

(2.3)

Elements of the set \( \text{Diff}^k \) are called diffeomorphisms of order \( k \). If \( k = 0 \), these functions are called homeomorphisms (however, \( M \) is a \( k \)-regular manifold, so \( k \geq 1 \)). If \( k = \infty \), we simply write \( \text{Diff} \) and call its elements simply diffeomorphisms.

Let \( \{p_\alpha | \alpha \in A\} \subset M \) (\( A \) is an index set, which may be uncountable) be a set of points such that

\[
\bigcup_{\alpha \in A} V_\alpha = M,
\]

where \( (U_\alpha, \varphi_\alpha, V_\alpha) \) are local representations of \( M \) around \( p_\alpha \), which are all \( C^k \)-compatible with each other. Then, \( A := \{(U_\alpha, \varphi_\alpha, V_\alpha) | \alpha \in A\} \) is called a \( C^k \)-atlas of \( M \). \( A \) is called maximal when every local representation \((U, \varphi, V)\) of \( M \) which is \( C^k \)-compatible with every local representation of \( A \), is contained in \( A \). It can be shown that every \( k \)-regular manifold \( M \) has a maximal \( C^k \)-atlas \( A \), which we call a \( C^k \)-structure on \( M \). We call the pair \((M, A)\) a \( C^k \)-manifold. In the following, if we take a local representation of \( M \), we mean that we take an element of \( A \). Since we rarely use \( A \) explicitly, we just write \( M \) for a \( C^k \)-manifold.

Because of our \( C^k \)-structure on \( M \), we can define differentiability of functions defined on \( M \). Remark 2.4 shows what this structure implies.
Definition 2.3. (Differentiability, partial derivative). For any \( d \in \mathbb{N}_+ \), \( \ell \in \mathbb{N} \cup \{\infty\} \) such that \( \ell \leq k \), we say for an \( f \in C(M, \mathbb{R}^d) \), that \( f \in C^\ell(M, \mathbb{R}^d) \) iff for every local representation \((U, \varphi, V)\) it holds that \( f \circ \varphi \in C^\ell(U, \mathbb{R}^d) \). For any \( i \in \{1, \ldots, m\} \), \( p \in M \), we define for any local representation \((U, \varphi, V)\) of \( M \) around \( p \):

\[
\frac{\partial}{\partial u_i} f(p) := \frac{\partial(f \circ \varphi)}{\partial u_i}(\varphi^{-1}(p)).
\]

Remark 2.4. Let \( f \in C^\ell(M, \mathbb{R}^d) \), and let \((U, \varphi, V)\) and \((\tilde{U}, \tilde{\varphi}, \tilde{V})\) be two local representations of \( M \) around \( p \in M \). Let \( \tilde{V} := V \cap \tilde{V} \). Then by Definition 2.3, we know that \( f \circ \varphi \in C^\ell(U, \mathbb{R}^d) \) and \( f \circ \tilde{\varphi} \in C^\ell(\tilde{U}, \mathbb{R}^d) \). But then we also have that

\[
\hat{f} \circ \varphi|_{\varphi^{-1}(\tilde{V})} = (f \circ \tilde{\varphi})|_{\tilde{\varphi}^{-1}(\tilde{V})}.
\]

By the chain rule, the right hand side is still a \( C^\ell \) function because of our \( C^k \)-structure. Formally speaking, the \( C^k \)-structure enables us to switch between charts (via transition maps) without influencing the regularity of \( f \). ■

Although the partial derivative given by Definition 2.3 is useful in practice, it depends on the local representation. We are now going to extend this definition of a derivative to one that is independent of a local representation. For this purpose, we start by introducing an important object, called the tangent space, which is independent of our choice of a local representation.

Definition 2.5. (Tangent space). Fix \( p \in M \). Then the tangent space of \( M \) at \( p \) is given by

\[
T_p M := \{ v \in \mathbb{R}^m \mid \exists \epsilon > 0 \exists \gamma \in C^\epsilon((-\epsilon, \epsilon); M) : \gamma(0) = p \text{ and } \gamma'(0) = v \}.
\]

Remark 2.6. To be more precise; for an interval \( I \subset \mathbb{R} \), we define

\[
C^k(I; M) := \{ \gamma \in C^k(I; \mathbb{R}^n) \mid \gamma(I) \subset M \}.
\]

Then it is immediately clear what \( \gamma' \) in Definition 2.5 means.

Given \( p \in M \) and a local representation \((U, \varphi, V)\) around \( p \). Set \( u := \varphi^{-1}(p) \). Then one can show that

\[
T_p M = \left\{ \frac{\partial \varphi}{\partial u_1}(u), \ldots, \frac{\partial \varphi}{\partial u_m}(u) \right\}.
\]

Point (iii) of Definition 2.1 implies that

\[
\left\{ \frac{\partial \varphi}{\partial u_1}(u), \ldots, \frac{\partial \varphi}{\partial u_m}(u) \right\}
\]

is an independent set of vectors in \( \mathbb{R}^m \). Therefore, it spans an \( m \)-dimensional vector field. ■

With the help of the tangent space, we can define the differential of a function.

Definition 2.7. (Differential). Let for \( i \in \{1, 2\} \), \( M_i \subset \mathbb{R}^{n_i} \) an \( m_i \)-dimensional \( C^k \)-manifold, and let

\[
f \in C^1(M_1, M_2) := \{ f \in C^1(M_1; \mathbb{R}^{n_2}) \mid f(M_1) \subset M_2 \}.
\]

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Then the differential of \( f \) at \( p \in M_1 \) is defined by the mapping

\[
d_p f : T_p M_1 \rightarrow T_{f(p)} M_2,
\]

which assigns to any \( v \in T_p M_1 \) the value

\[
d_p f (v) := (f \circ \gamma)'(0),
\]

where the curve \( \gamma \in C^k ((-\varepsilon, \varepsilon); M_1) \) is chosen such that it satisfies \( \gamma(0) = p \) and \( \gamma'(0) = v \).

**Remark 2.8.** The existence of such a curve \( \gamma \) in Definition 2.7 is ensured by Definition 2.5. One can show that the differential is independent on the choice of the curve \( \gamma \). So the concept of a differential is well-defined. Furthermore, one can show that the differential is a linear mapping.

An important special case is when \( M_2 = \mathbb{R}^d \) for \( d \in \mathbb{N}^+ \). Since \( T_x \mathbb{R}^d = \mathbb{R}^d \), we then have \( d_p f (v) \in \mathbb{R}^d \).

Now it is easy to define the directional derivative in a coordinate independent way.

**Definition 2.9.** (Directional derivative). Let \( d \in \mathbb{N}_+ \) and \( f \in C^1 (M, \mathbb{R}^d) \). Then for any \( p \in M \), we define the directional derivative of \( f \) in any direction \( v \in T_p M \) by

\[
\partial_v f (p) := d_p f (v).
\]

**Remark 2.10.** Note that Definition 2.9 is independent on the choice of a local representation. To see whether Definition 2.9 really is an extension of Definition 2.3, let \( p \in M \) and choose a local representation \( (U, \varphi, V) \) around \( p \). If we pick \( v = \frac{\partial \varphi}{\partial u_i} (u) \) (where \( u := \varphi^{-1} (p) \)), it is not difficult to show that \( \partial_i f (p) = \partial_v f (p) \). One can see this by choosing the curve \( \gamma \) in Definition 2.7 as \( \gamma(\tau) := \varphi(u + \tau e_i) \).

We continue by defining an inner product on the tangent space of \( M \) at \( p \). One of its purposes is that it is going to give us an coordinate independent definition of the gradient on \( M \), which is an extension of the usual gradient.

**Definition 2.11.** (First fundamental form). Let \( p \mapsto g_p \) be the mapping from \( M \) where

\[
g_p := (\cdot, \cdot)|_{T_p M \times T_p M},
\]

where \((\cdot, \cdot)\) denotes the Euclidean inner product on \( \mathbb{R}^n \). This mapping is called the first fundamental form of \( M \).

For a local representation \( (U, \varphi, V) \) of \( M \) around \( p \in M \) (set \( u := \varphi^{-1} (p) \)), we define for \( i, j \in \{1, \ldots, m\} \)

\[
g_{ij} (u) := \frac{\partial \varphi}{\partial u_i} (u) \cdot \frac{\partial \varphi}{\partial u_j} (u),
\]

where the dot denotes the Euclidean inner product. We denote by \( G(u) \) the \( m \times m \)-matrix of which its elements are given by \( g_{ij} (u) \). Furthermore, we define \( g^{ij} (u) \) to be the elements of \( G^{-1} (u) \), and \( g(u) := \det G(u) \).
Remark 2.12. Note that $g_{ij} \in C^{k-1}(U)$ (because $\varphi \in C^k(U, V)$) and $G(u)$ symmetric. Furthermore, since the Jacobian matrix of $\varphi$ is given by

$$J_\varphi(u) = \left(\frac{\partial \varphi_i}{\partial u_j}\right)_{i,j=1}^{m+1,m} \in \mathbb{R}^{n \times m},$$

we see that $G(u) = J_\varphi(u)^T J_\varphi(u)$. Thanks to point (iii) of Definition 2.1, it follows that $G(u)$ is positive definite, which implies that $G^{-1}(u)$ exists and is positive definite too. Note that we then also have that $g^{ij} \in C^{k-1}(U)$.

Related to the first fundamental form, are the Christoffel symbols. They will prove to be useful later on. For their definition, we need $k \geq 2$.

**Definition 2.13.** (Christoffel symbols). Let $(U, \varphi, V)$ be a local representation around $p$, and set $u := \varphi^{-1}(p)$. For any $i, j, k \in \{1, \ldots, m\}$, the **Christoffel symbols** are defined by

$$\Gamma^i_{jk}(u) := g^{ia}(u)\frac{\partial^2 \varphi}{\partial u_j \partial u_k}(u) \cdot \frac{\partial \varphi}{\partial u_a}(u).$$

(Note that the summation is only over the index $\alpha$, running from 1 tot $m$).

**Remark 2.14.** One can show that another representation of the Christoffel symbols is given by

$$\Gamma^i_{jk}(u) = \frac{1}{2} g^{ia}(u) \left( \frac{\partial g_{ia}}{\partial u_j}(u) + \frac{\partial g_{ia}}{\partial u_k}(u) - \frac{\partial g_{jk}}{\partial u_a}(u) \right).$$

From either representation, it follows easily that $\Gamma^i_{jk} \in C^{k-2}(U)$.

For the definition of the gradient on $M$, we consider $k \geq 1$ again.

**Definition 2.15.** (Gradient). Let $f \in C^1(M)$ and $p \in M$. Then we define the **gradient** on $M$ of $f$ at $p$ (denoted by $\nabla_M f(p)$) as the unique vector $v(p) \in T_p M$ satisfying

$$\forall w \in T_p M : d_p f(w) = v(p) \cdot w.$$

**Remark 2.16.** Since $d_p f$ is a linear map, the uniqueness of such a $v(p)$ follows immediately. Another equivalent definition of the gradient, is given with the help of a local representation $(U, \varphi, V)$ around $p$ (set $u := \varphi^{-1}(p)$). Then one can show that

$$\nabla_M f(p) = g^{ij}(u) \frac{\partial (f \circ \varphi)}{\partial u_i}(u) \frac{\partial \varphi}{\partial u_j}(u)$$

$$\begin{align*}
\hat{=} & g^{ij}(\varphi^{-1}(p)) \partial_i f(p) \frac{\partial \varphi}{\partial u_j}(\varphi^{-1}(p)),
\end{align*}$$

* : use $u := \varphi^{-1}(p)$ and Definition 2.3,

where we have used the summation convention. If $f$ can be smoothly extended to an $\tilde{f} \in C^k(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open neighbourhood of $M$ (so that $f = \tilde{f}|_M$), one can show that for any $p \in M$, $\nabla_M f(p)$ is just the orthogonal projection of $\nabla \tilde{f}(p)$ on $T_p \Gamma$, where $\nabla$ is the usual gradient on $\mathbb{R}^n$.

It is not directly clear that these representations of the gradient are equivalent, nor that they are well-defined (i.e. independent of the choice of a local representation, or independent of the extension of $f$ to $\Omega$). But again, it is not our goal to prove this.
From now on, unless mentioned otherwise, we will use the summation convention whenever an index appears twice. This happens to be quite useful when writing differential operators in terms of local representations.

We are now going to define integration on a $C^\infty$-manifold $M \subset \mathbb{R}^n$ of dimension $m$ ($1 \leq m \leq n$), which is compact as a subset of $\mathbb{R}^n$. A more common name for a $C^\infty$-manifold is a smooth manifold (it does not include compactness in its definition). The problem is that we only have local descriptions of $M$, while we need a global description to define the integral over $M$ of a function defined on $M$. The idea is to smoothly fit all local representations together. This is done by a smooth partition of unity:

**Definition 2.17.** (Smooth partition of unity). Let $M$ be a smooth manifold, $A$ an index set, $\mathcal{X} = \{X_\alpha \subset M \mid \alpha \in A\}$ any open cover of $M$. A smooth partition of unity subordinate to $\mathcal{X}$ is a set $\{\psi_\alpha \in C^\infty(M) \mid \alpha \in A\}$ which satisfies

(i) $\forall p \in M \forall \alpha \in A : \psi_\alpha(p) \in [0, 1]$,

(ii) $\forall \alpha \in A : \text{supp} \psi_\alpha \subset X_\alpha$,

(iii) $\{\text{supp} \psi_\alpha \mid \alpha \in A\}$ is locally finite, i.e. for all $p \in M$ there exists a neighbourhood $V$ of $p$ in $M$ such that there are at most finitely many $\alpha \in A$ such that $\text{supp} \psi_\alpha \cap V \neq \emptyset$,

(iv) $\forall p \in M : \sum_{\alpha \in A} \psi_\alpha(p) = 1$.

In [16], Theorem 2.25, it is proven that such a smooth partition of unity exists for any smooth manifold and any open cover of it.

**Remark 2.18.** A useful covering $\mathcal{X}$ of a smooth manifold $M$ which is compact as a subset of $\mathbb{R}^n$, is to take $A = M$ as a subset of $\mathbb{R}^n$ for the index set. Then for any $\alpha \in A$, choose a local representation $(U_\alpha, \varphi_\alpha, V_\alpha)$ such that $\alpha \in V_\alpha$. Then $\mathcal{X} = \{V_\alpha \subset M \mid \alpha \in A\}$ provides an open covering for $M$. The compactness of $M$ guarantees the existence of a finite subcover $\{V_\alpha \mid \alpha \in A\}$, which has a partition of unity $\{\psi_\alpha \mid \alpha \in A\}$ subordinate to it. We call

$$\{(U_\alpha, \varphi_\alpha, V_\alpha, \psi_\alpha) \mid \alpha \in A\}$$

a localizing system on $M$. By its construction, we have shown that a smooth manifold has a localizing system if it is compact as a subset of $\mathbb{R}^n$.

A localizing system is the main tool for defining objects on $M$ which need a global description of $M$. Integration is one of such.

**Definition 2.19.** (Integral) Let $\{(U_\alpha, \varphi_\alpha, V_\alpha, \psi_\alpha) \mid \alpha \in A\}$ be a localizing system on a smooth manifold $M$ which is compactly embedded in $\mathbb{R}^n$. Then for any $f \in C(M)$, we define the integral by

$$\int_M f := \sum_{i=1}^N \int_{U_i} \psi_i(\varphi_i(u)) f(\varphi_i(u)) \sqrt{g_i(u)} \, du.$$
Remark 2.20. At first sight, this definition might look hard. Essentially, it is just an extension of the theory of multi-variable integration. To see this, we write

$$\int_M f = \int_M f \left( \sum_{i=1}^N \psi_i \chi_i \right) = \sum_{i=1}^N \int_{V_i} \psi_i f.$$  

Suppose that $V_i$ is a subset of $\mathbb{R}^m$. Then $\varphi_i : U_i \to V_i$ is an invertible, smooth mapping between open subsets of $\mathbb{R}^m$, for which we know that

$$\int_{V_i} \psi_i f = \int_{U_i} (\psi_i f) \circ \varphi_i \left| \det J_{\varphi_i} \right| = \int_{U_i} (\psi_i f) \circ \varphi_i \sqrt{g_i},$$  

$\ast$: see Remark 2.12.  

This is all we need to know about $C^k$-manifolds. Our cell wall will be described by a special class of $C^k$-manifolds, called closed $C^k$-hypersurfaces:

Definition 2.21. (Closed $C^k$-hypersurface). Let $m, k \in \mathbb{N}_+$. Then $\Gamma \subset \mathbb{R}^{m+1}$ will be called a closed $C^k$-hypersurface if there exists a domain $\Omega \subset \mathbb{R}^{m+1}$ such that $\Gamma := \partial \Omega$ is a connected $C^k$-manifold.

Remark 2.22. The usual definition of a closed $C^k$-hypersurface contains a larger class of $C^k$-manifolds. For example, the Klein bottle is a closed, smooth hypersurface according to the usual definition, but it is not a closed, smooth hypersurface according to Definition 2.21. So, whenever we consider closed $C^k$-hypersurfaces, we actually mean a subclass of the ‘usual’ closed $C^k$-hypersurfaces.

From now on, we denote by $\Gamma$ such a closed $C^k$-hypersurface of dimension $m$.

The restriction of general $C^k$-manifolds to closed $C^k$-hypersurface where $k \geq 2$, allows us to define the unique normal vector and the mean curvature, which is, formally speaking, a measure of how much a surface is curved. It generalizes the curvature of a curve. We will make this more precise, but first:

Definition 2.23. (Gauß mapping, normal space). Let $S_m$ denote the unit sphere in $\mathbb{R}^{m+1}$. Then any $n \in C^{k-1}(\Gamma, S_m)$ satisfying $n(p) \cdot v = 0$ for all $v \in T_p \Gamma$ and $\|n(p)\| = 1$ for all $p \in \Gamma$, is called a Gauß mapping. We call $N_p \Gamma := (T_p \Gamma)^\perp$ the normal space of $\Gamma$.

Remark 2.24. We use the norm $\| \cdot \|$ that is induced by the first fundamental form (see Definition 2.11).

Since our hypersurface is regular enough, $n$ is well-defined. The fact that $n$ is not unique, is because we can take $n$ to be the inner or outer normal of $\Gamma$. From now on, we take $n$ to be the outer normal.  

Note that we can regard $S_m$ as a closed $C^\infty$-hypersurface. Then, by Definition 2.7, we have for any $p \in \Gamma$ and $v \in T_p \Gamma$, that

$$d_p n(v) \in T_n(p) S_m \cong (n(p))^\perp \cong T_p \Gamma,$$
Definition 2.25. (Weingarten map). For any \( p \in \Gamma \), the Weingarten map is defined by

\[
W_p : T_p \Gamma \to T_p \Gamma, \quad W_p(v) := -d_p n(v).
\]

Remark 2.26. Note that \( W_p \) is a linear mapping on a finite dimensional vector space. ■

The Weingarten map will be the key for defining curvature, but first, we need the following definition.

Definition 2.27. (Second and third fundamental form). The mappings \( p \mapsto b_p \) and \( p \mapsto c_p \) from \( \Gamma \) defined by

\[
b_p(v, w) := W_p(v) \cdot w, \quad c_p(v, w) := W_p(v) \cdot W_p(w),
\]

are called, respectively, the second and third fundamental form of \( \Gamma \).

For a local representation \((U, \varphi, V)\) of \( \Gamma \) around \( p \in \Gamma \) (set \( u := \varphi^{-1}(p) \)), we define for \( i, j \in \{1, \ldots, m\} \)

\[
b_{ij}(u) := b_p \left( \frac{\partial \varphi}{\partial u_i}(u), \frac{\partial \varphi}{\partial u_j}(u) \right),
\]

\[
c_{ij}(u) := c_p \left( \frac{\partial \varphi}{\partial u_i}(u), \frac{\partial \varphi}{\partial u_j}(u) \right).
\]

We denote by \( B(u) \), \( C(u) \) the \( m \times m \)-matrices of which their elements are given by \( b_{ij}(u) \), \( c_{ij}(u) \) respectively.

Remark 2.28. It is not hard to show that other representations of \( b_{ij}(u) \) and \( c_{ij}(u) \) are given by

\[
b_{ij}(u) = \frac{\partial (n \circ \varphi)}{\partial u_i}(u) \cdot \frac{\partial \varphi}{\partial u_j}(u)
\]

\[
= (n \circ \varphi)(u) \cdot \frac{\partial^2 \varphi}{\partial u_i \partial u_j}(u),
\]

\[
c_{ij}(u) = \frac{\partial (n \circ \varphi)}{\partial u_i}(u) \cdot \frac{\partial (n \circ \varphi)}{\partial u_j}(u).
\]

The second representation of \( b_{ij}(u) \) clearly implies that \( B(u) \) is symmetric. Therefore, \( W_p \) is self-adjoint. Furthermore, it is clear that \( C(u) \) is symmetric too. Also, since \( \varphi \in C^k(U) \) and \( n \in C^{k-1}(\Gamma, S_m) \), we have that \( b_{ij}, c_{ij} \in C^{k-2}(U) \). ■

\( W_p \) being self-adjoint, implies that there is a complete basis of eigenvectors. Now we are ready to define a concept of curvature.
Definition 2.29. (Principal curvatures, Gauß curvature, mean curvature). Let $p \in \Gamma$. The eigenvalues of $W_p$, denoted by $\kappa_i(p)$, are called the principal curvatures of $\Gamma$ at $p$ (note that there is no summation in the right hand side). The Gauß curvature of $\Gamma$ at $p$ is defined as

$$K(p) := \det(W_p) = \prod_{i=1}^{m} \kappa_i,$$

and the mean curvature of $\Gamma$ at $p$ is defined as

$$H(p) := \frac{1}{m} \text{tr}(W_p) = \frac{1}{m} \sum_{i=1}^{m} \kappa_i.$$

Remark 2.30. The principal curvatures are equal to the curvature at $p$ of the curve which describes the intersection of $\Gamma$ with the plane spanned by $n(p)$ and an eigenvector of $\kappa_i(p)$.

The factor $1/m$ in the definition of $H$ makes sure that the mean curvature of a sphere of radius $R$ in $\mathbb{R}^{m+1}$ equals $-1/R$ at any point of the sphere.

The mean curvature $H$ will have an important role in our model. However, for calculations, Definition 2.29 does not give a convenient representation for $H$. In order to derive other representations, we have to introduce more differential operators on $\Gamma$ (other than the gradient defined in Definition 2.15).

Definition 2.31. (Surface divergence). Let $\Gamma$ be a bounded, closed, smooth hypersurface. For any $f \in C^1(\Sigma ; T\Sigma)$ := $\{ f \in C^1(\Sigma ; \mathbb{R}^{m+1}) | \forall p \in \Gamma : f(p) \in T_p\Gamma \}$, we define the surface divergence of $f$ as the unique solution $g \in C(\Sigma)$ to

$$\forall v \in C^1(\Sigma) : \int_{\Sigma} f \cdot \nabla \Gamma v = - \int_{\Sigma} g v.$$ (2.4)

We denote $g = \text{div}_\Gamma f$.

Remark 2.32. Definition 2.31 is not the usual definition of the surface divergence that one finds in literature, nor is it the one given in [7]. The usual definition is in terms of the abstract framework that one usually considers while studying differential geometry, but which we do not discuss here. For that definition of the surface divergence, Definition 2.31 appears as a theorem (see for example [17], Problem 3-3 on page 43).

We omit the proof of the existence of a unique $g \in C(\Sigma)$. Note that $\nabla \Gamma v \in C(\Sigma ; T\Sigma)$, and that the expression in Definition 2.31 is the equivalence of the integration by parts formula in Euclidean space.

We now give a local description of the surface divergence (see [8], equation (15) on p. 43). Let $f \in C^1(\Gamma ; T\Gamma)$, and let $(U, \varphi, V)$ be a local representation around $p$ (set $u := \varphi^{-1}(p)$). Since $f(p) \in T_p\Gamma$, there exists $f_i(u)$ such that

$$f(p) = f_i(u) \frac{\partial \varphi}{\partial u_i}(u).$$
Then it holds that
\[
\text{div}_\Gamma f(p) = \frac{1}{\sqrt{g(u)}} \frac{\partial (\sqrt{g} f_i)}{\partial u_i}(u)
\]  \hspace{1cm} (2.5)

One can extend the definition of the surface divergence to general closed $C^k$-hypersurfaces such that (2.5) still holds. We omit the details.

**Definition 2.33.** (Laplace-Beltrami operator). Let $f \in C^2(\Gamma)$. Then we define the Laplace-Beltrami operator of $f$ at $p \in \Gamma$ by
\[
\Delta_\Gamma f(p) := \text{div}_\Gamma \left( \nabla_\Gamma f(p) \right).
\]

**Remark 2.34.** With the help of a local representation $(U, \varphi, V)$ around $p$ (set $u := \varphi^{-1}(p)$), one can show that
\[
\Delta_\Gamma f(p) = \frac{1}{\sqrt{g(u)}} \frac{\partial}{\partial u_i} \left( \sqrt{g(u)} g^{ij}(u) \frac{\partial (f \circ \varphi)}{\partial u_j}(u) \right)
\]
\[
= g^{ij}(u) \left( \frac{\partial^2 (f \circ \varphi)}{\partial u_i \partial u_j}(u) - \Gamma^k_{ij}(u) \frac{\partial (f \circ \varphi)}{\partial u_k}(u) \right).
\]

Since $G^{-1}(u)$ is positive definite (see Remark 2.12), it follows from the second expression of $\Delta_\Gamma f(p)$ that $\Delta_\Gamma$ is an elliptic operator.

The next Theorem, of which the proof can be found in [7], Lemma 2.27, relates $H$ to our differential operators.

**Theorem 2.35.** Let $p \in \Gamma$. Then the following statements are true:

(i) Let $I : \Gamma \to \mathbb{R}^{m+1}$ be the identity mapping, and let $\Delta_\Gamma I(p) := (\Delta_\Gamma I_i(p))_{i=1}^{m+1}$. Then
\[
H(p) n(p) = \frac{1}{m} \Delta_\Gamma I(p),
\]

(ii) Let $(U, \varphi, V)$ be a local representation around $p$ (set $u := \varphi^{-1}(p)$). Then
\[
H(p) n(\varphi(u)) = \frac{1}{\sqrt{g(u)}} \frac{\partial}{\partial u_i} \left( \sqrt{g(u)} g^{ij}(u) \frac{\partial \varphi}{\partial u_j}(u) \right).
\]

By the local descriptions of the surface gradient and surface divergence, it is not hard to see that statements (i) and (ii) are equivalent.

This finishes the theory of differential geometry that we need. Especially in the following section we will need a lot of the definitions and tools that we have presented in this section.

### 2.2 Hypersurfaces parametrized by a reference hypersurface

This section is more related to our problem than the previous section. However, since it deals with a well-known strategy for solving free boundary problems, it can be considered as general theory. The idea is to fix a smooth hypersurface $\Gamma$, and to parametrize the hypersurface.
of interest $\Gamma_\rho$, by a function $\rho$ defined on $\Gamma$. Then one would expect that certain objects related to $\Gamma_\rho$, like the volume of the area it encloses, its normal vector and its mean curvature, can be expressed in terms of $\rho$ and $\Gamma$. This is true, and the related expressions will be given by non-linear (differential) operators acting on $\rho$. Since $\Gamma$ is fixed, the dependence on $\Gamma$ is not important.

This construction is presented, for example, in [10], Section 2. Since the authors do not give a detailed derivation, we will give this derivation ourselves.

We start by making the setting more precise. Let $m \in \mathbb{N}_+$ and $\Gamma$ be a closed $C^\infty$-hypersurface of $\mathbb{R}^{m+1}$ as in Definition 2.21, such that $\Gamma$ is bounded. $\Gamma$ is going to be our reference hypersurface. The boundedness of $\Gamma$ implies that $\mathbb{R}^{m+1} \setminus \Gamma$ is a disconnected set consisting of two components, $\Omega$ (bounded) and $\Omega'$ (unbounded), which are both open with respect to $\mathbb{R}^{m+1}$. For a schematic overview of the hypersurfaces and regions that we consider together with some other quantities, see Figure 2.1.

![Figure 2.1: A schematic overview of our setting, together with some related objects. We have chosen to display just a piece of $\Gamma_\rho$ and $\Omega_\delta$.](image)

We continue by defining a specific neighbourhood of $\Gamma$, in which $\Gamma_\rho$ is going to be defined later on. Let

$$\Omega_\delta := \{ x \in \mathbb{R}^{m+1} | d(x, \Gamma) < \delta \}$$

be this neighbourhood, where $d$ is the usual distance function in $\mathbb{R}^{m+1}$. We choose $\delta > 0$ such that

$$\forall x \in \Omega_\delta \exists! \xi \in \Gamma : d(x, \Gamma) = |x - \xi|,$$

where $|\cdot|$ is defined as the 2-norm in $\mathbb{R}^{m+1}$. The existence of such $\delta$ is given by the tubular neighbourhood theorem (see [16], Theorem 10.19). This theorem also states that the following map is a diffeomorphism:

$$X : \Gamma \times (-\delta, \delta) \to \Omega_\delta, \quad X(\xi, \eta) := \xi + \eta n(\xi).$$
\( n \) denotes the outer normal vector of \( \Gamma \) with respect to \( \Omega \) (see also Definition 2.23). We denote the inverse of \( X \) by \( (S, \Lambda) \). Note that

\[
S(x) = \arg \min_{\xi \in \Gamma} |x - \xi|, \\
\Lambda(x) = \begin{cases} 
  d(x, \Gamma) & \text{if } x \in \Omega' \\
  0 & \text{if } x \in \Gamma \\
  -d(x, \Gamma) & \text{if } x \in \Omega.
\end{cases}
\]  

(2.9)

For an overview of this mapping and all other mappings that we are going to use, see Figure 2.2.

Figure 2.2: Schematic view of the smooth diffeomorphisms that we use. The vertical lines indicate the interval \((-\delta, \delta)\). The region with the oval-shaped, dashed boundary illustrates the Cartesian product of the inner two sets.

It is easy to see that \( \Omega\delta \) is a \( C^\infty \)-manifold. Just take the single local representation \((\Omega\delta, I, \Omega\delta)\) which describes \( \Omega\delta \) completely. However, it will be more convenient to work with those charts on \( \Omega\delta \) that are induced by \( \Gamma \). We define these charts in the sequel.

Let \((U, \varphi, V)\) be a local representation around \( \xi \) with \( u := \varphi^{-1}(\xi) \). Then \((U^*, \varphi^*, V^*)\), defined
We set
\[ U^* := U \times (-\delta, \delta), \]
\[ \varphi^*(u^*) := X(\varphi(u), u_{m+1}), \]
\[ V^* := \varphi^*(U^*), \tag{2.10} \]

\* : we write \( U^* \ni u^* = (u_1, \ldots, u_{m+1}) \), where \( (u_1, \ldots, u_m) = u \in U \) and \( u_{m+1} \in (-\delta, \delta) \),
is a local representation of \( \Omega_\xi \) around \( \xi \). Note that \( \varphi^* = X \circ (\varphi, id) : U^* \to V^* \), from which it easily follows that it is a diffeomorphism. From the tubular neighbourhood theorem, it follows that \( (U^*, \varphi^*, V^*) \) is contained in the canonical \( C^\infty \)-structure on \( \Omega_\xi \) (i.e. the structure that is induced by the single local representation \((\Omega_\xi, I, \Omega_\xi)\)).

To distinguish between mathematical objects that are related to local representations of \( \Gamma \) or \( \Omega_\xi \), we will denote the objects related to \( \Omega_\xi \) by an asterisk in its superscript (as we have done already for the local representations themselves). In this section, we do not use push forwards or pull backs, so there is no danger for confusion when we use \( \varphi^* \) for the mapping of a local representation of \( \Omega_\xi \).

Later on, we are going to need the matrix representation of the first fundamental form (see Definition 2.11) on \( \Omega_\xi \) with respect to local representations as in (2.10). By Definition 2.11, we have for any \( u^* \in U^* \) and \( i, j \in \{1, \ldots, m+1\} \)
\[ g_{ij}^*(u^*) = \frac{\partial \varphi^*}{\partial u_i}(u^*) \cdot \frac{\partial \varphi^*}{\partial u_j}(u^*). \tag{2.11} \]

From (2.10) and (2.8), it is easy to see that
\[ \frac{\partial \varphi^*}{\partial u_i}(u^*) = \frac{\partial \varphi}{\partial u_i}(u) + u_{m+1} \frac{\partial (n \circ \varphi)}{\partial u_i}(u) \in T_{\xi} \Gamma, \quad \text{for } i \in \{1, \ldots, n\}, \tag{2.12} \]
\[ \frac{\partial \varphi^*}{\partial u_{m+1}}(u^*) = n(\varphi(u)) \in N_{\xi} \Gamma, \]
where the spaces \( T_{\xi} \Gamma \) and \( N_{\xi} \Gamma \) are defined in Definition 2.5 and Definition 2.23. Combining (2.11) and (2.12) yields
\[ g_{ij}^*(u^*) = \begin{cases} g_{ij}(u) - 2u_{m+1}b_{ij}(u) + u_{m+1}^2c_{ij}(u) & \text{if } i, j \in \{1, \ldots, m\} \\ 1 & \text{if } i = j = m + 1 \\ 0 & \text{else,} \end{cases} \tag{2.13} \]
where \( b_{ij}(u) \) and \( c_{ij}(u) \) are defined in Definition 2.27. In matrix form, this reads
\[ G^*(u^*) = (g_{ij}^*(u^*))^{m+1}_{i,j=1} = \begin{pmatrix} \frac{G(u) - 2u_{m+1}B(u) + u_{m+1}^2C(u)}{0^1} \\ 0 \end{pmatrix}, \tag{2.14} \]
and for its inverse, we get
\[ \left( (g^*)^{ij}(u^*) \right)^{m+1}_{i,j=1} = \begin{pmatrix} \left( \frac{G(u) - 2u_{m+1}B(u) + u_{m+1}^2C(u)}{0^1} \right)^{-1} \\ 0 \end{pmatrix}. \tag{2.15} \]

We set \( g^*(u^*) = \det G^*(u^*) \). Note that \( g_{ij}^*, (g^*)^{ij}, g^* \in C^\infty(U^*) \).
It will be convenient to write the integral over \( \Omega_i \) (which is just the usual integral over an open set of \( \mathbb{R}^{m+1} \)) in terms of its local representations. Let \( \{(U_i, \varphi_i, V_i, \psi_i) \mid i \in \{1, \ldots, N\}\} \) be a localizing system of \( \Gamma \) (see Remark 2.18. The existence of such a localizing system is guaranteed by the compactness of \( \Gamma \)). For any \( i \in \{1, \ldots, N\} \), let

\[
\psi^*_i : V^*_i \to \mathbb{R}, \quad \psi^*_i(x) := \psi_i(S(x)).
\]  

(2.16)

Note that \( \{(U^*_i, \varphi^*_i, V^*_i, \psi^*_i) \mid i \in \{1, \ldots, N\}\} \) is not a localizing system on \( \Omega_i \), because \( \text{supp} \psi^*_i \not\subseteq V^*_i \). However, we do have that \( \psi^*_i|_{\Omega_i \setminus V_i} \equiv 0 \) and that \( \sum_{i=1}^{N} \psi^*_i \equiv 1 \) on \( \Omega_i \). Then, by the usual transformation formula for integrals (see Remark 2.20), we have for any \( f \in C(\Omega_i) \) that

\[
\int_{\Omega_i} f = \sum_{i=1}^{N} \int_{V^*_i} \psi^*_i f = \sum_{i=1}^{N} \int_{U^*_i} (\psi^*_i f \circ \varphi^*_i) \sqrt{g^*_i}. 
\]  

(2.17)

We continue by constructing our parametrized hypersurface. We do this by considering elements of the set

\[
\mathcal{U} := \left\{ \rho \in h^{2+\alpha}(\Gamma) \mid \max_{\xi \in \Gamma} |\rho(\xi)| < \delta \right\}, 
\]  

(2.18)

where \( \alpha \in (0, 1) \) is fixed and \( h^{2+\alpha}(\Gamma) \) denotes the little Hölder space of exponent \( 2 + \alpha \) (see Appendix A). The motivation for this particular set will become clear later on. For now, we define for \( \rho \in \mathcal{U} \),

\[
\Gamma_\rho := \left\{ X(\xi, \rho(\xi)) \mid \xi \in \Gamma \right\}.
\]  

(2.19)

Figure 2.3 shows a typical example of a piece of such a \( \Gamma_\rho \). It is a close-up of Figure 2.1. The condition that the maximum of \( |f| \) is smaller than \( \delta \), ensures that \( \xi \mapsto X(\xi, \rho(\xi)) \) is invertible as a function mapping \( \Gamma \) to \( \Gamma_\rho \).

\( \Gamma_\rho \) is going to be the hypersurface of interest. Just like \( \Gamma \), it is bounded and closed, but it is not smooth. To see this, let \( (U, \varphi, V) \) be a local representation of \( \Gamma \) around \( \xi \). Then \( (\tilde{U}, \tilde{\varphi}, \tilde{V}) \) as given by

\[
\tilde{U} := U, \\
\tilde{\varphi}(u) := X(\varphi(u), \rho(\varphi(u))), \\
\tilde{V} := \tilde{\varphi}(\tilde{U}),
\]  

(2.20)

is a local representation of \( \Gamma_\rho \) around \( X(\xi, \rho(\xi)) \). From (2.20) and Proposition A.2 it follows that \( \tilde{\varphi} \in h^{2+\alpha}(\tilde{U} ; \mathbb{R}^{m+1}) \). Then from Definition 2.1 we see that \( \Gamma_\rho \) is a \( C^2 \)-manifold. Furthermore, the \( C^\infty \)-structure on \( \Gamma \) induces a differentiable structure on \( \Gamma_\rho \). To see this, consider the local representations \( (U, \varphi, V) \) of the \( C^\infty \)-structure on \( \Gamma \). Then (2.20) gives local representations \( (\tilde{U}, \tilde{\varphi}, \tilde{V}) \) for \( \Gamma_\rho \). All these local representations together form an atlas on \( \Gamma_\rho \) (one needs that these local representations are compatible and cover \( \Gamma_\rho \), but this is not hard to show). This induces a \( C^2 \)-structure on \( \Gamma_\rho \).

In Section 7.3, we need to have a description of the following three quantities related to \( \Gamma_\rho \) as operators acting on \( \rho \in \mathcal{U} \): the volume enclosed by \( \Gamma_\rho \), the normal vector on \( \Gamma_\rho \) and the mean curvature of \( \Gamma_\rho \).

We start with the volume enclosed by \( \Gamma_\rho \). We do not use any literature for deriving its representation in terms of \( \rho \). For \( \Gamma \), the volume of the enclosed domain is the \( (m + 1) \)-dimensional Lebesgue measure of \( \Omega \), denoted by \( |\Omega| \). We define \( \Omega_\rho \) in the same way as we
defined $\Omega$ with respect to $\Gamma$, but now with respect to $\Gamma_\rho$. Of course, we denote by $|\Omega_\rho|$ its volume.

It will be convenient to write $|\Omega_\rho|$ as a sum of $|\Omega|$ and the signed area between $\Gamma_\rho$ and $\Gamma$, i.e.

$$|\Omega_\rho| = |\Omega| + |\Omega_\rho \setminus \Omega| - |\Omega \setminus \Omega_\rho|.$$ 

$|\Omega|$ is known, and the other two domains are subsets of $\Omega_\delta$, for which (2.17) will be useful. Let $(U, \varphi, V)$ be a local representation of $\Gamma$. Then (2.10) gives a local representation for $\Omega_\delta$, and we obtain

$$|{(\Omega_\rho \setminus \Omega) \cap V^s}| = \int_{V^s} \chi_{(\Omega_\rho \setminus \Omega)}$$

$$= \int_{U^s} \chi_{\Omega_\rho \setminus \Omega}(\varphi^s(u^s)) \sqrt{g^s(u^s)} \, du^s$$

$$= \int_{U} \chi_{\{\rho(\varphi(u)) > 0\}} \int_{\delta} \chi_{\{0 < u_{m+1} < \rho(\varphi(u))\}} \sqrt{g^s(u, u_{m+1})} \, du_{m+1} \, du$$

$$= \int_{U} \chi_{\{\rho(\varphi(u)) > 0\}} \int_{0}^{\rho(\varphi(u))} \sqrt{g^s(u, u_{m+1})} \, du_{m+1} \, du.$$

$^*_1$ : use the usual change of variables formula. For details, see Remark 2.20,

$^*_2$ : $\varphi^s(u^s) \in \Omega_\rho \setminus \Omega \Leftrightarrow \left(\rho(\varphi(u)) > 0, \text{ and } (0, \rho(\varphi(u))) \ni \Lambda(\varphi^s(u^s)) = u_{m+1}\right).$

Similarly, one can derive

$$|{(\Omega \setminus \Omega_\rho) \cap V^s}| = \int_{U} \chi_{\{\rho(\varphi(u)) < 0\}} \int_{\rho(\varphi(u))}^{0} \sqrt{g^s(u, u_{m+1})} \, du_{m+1} \, du.$$
Observe that
\[
\left| (\Omega_{\rho} \setminus \Omega) \cap V^* \right| - \left| (\Omega \setminus \Omega_{\rho}) \cap V^* \right| = \int_U \int_0^{\rho(u)} \sqrt{g^*(u, u_{m+1})} \, du_{m+1} \, du.
\]

To make this result global (i.e. get rid of \(V^*\), we need a localizing system \((U_i, \varphi_i, V_i, \psi_i) \mid i \in \{1, \ldots, N\}\) on \(\Gamma\). Let \(\psi_i^*\) as in (2.16), then
\[
|\rho| = |\rho| + |\rho| - |\rho|_{\rho}^* \left( \int_U \psi_i^*(\varphi_i^*(u)) \sqrt{g^*(u, u_{m+1})} \, du_{m+1} \, du \right).
\]

\(*: \text{use Definition 2.19. This is an abuse of notation, because } g^* \text{ and } g \text{ depend on the local representation. However, it is common to write the integral like this.}*

For our application, we are more interested in the operator
\[
\mathcal{V}^* \in C^\infty(U; h^{2+\alpha}(\Gamma)), \quad \mathcal{V}(\rho) := \frac{\omega_{m+1}}{|\Omega_{\rho}|},
\]

\(*: \text{see Appendix C,}\)

where \(\omega_{m+1}\) denotes the volume of the unit ball in \(\mathbb{R}^{m+1}\) (for its explicit expression, see (7.8)). Note that \(\mathcal{V}(\rho)\) is a real number, though we regard it as the corresponding constant function in \(h^{2+\alpha}(\Gamma)\). Since \(\mathcal{V}(\rho) > 0\) for any \(\rho \in U\), the operator \(\rho \mapsto \mathcal{V}(\rho)\) has the same regularity as \(\mathcal{V}\).

We now turn our attention to the normal vector \(n_{\rho}\) of \(\Gamma_{\rho}\). The idea of the following derivation is presented in [10], Section 2, in [7], the proof of Lemma 2.33, and in [19], Section 3.2. However, we provide additional details here. We start by defining an auxiliary function \(N_{\rho} \in C^2(\Omega_{\rho})\) by
\[
N_{\rho}(x) := \Lambda(x) - \rho(S(x)).
\]

From (2.9) it is easy to see that
\[
\begin{cases}
N_{\rho}(x) < 0 & \text{if } x \in \Omega_{\rho} \\
N_{\rho}(x) = 0 & \text{if } x \in \Gamma_{\rho} \\
N_{\rho}(x) > 0 & \text{else.}
\end{cases}
\]

Since \(\nabla N_{\rho}\) is orthogonal to level sets (note that (2.23) ensures that the zero level set of \(N\) is equal to \(\Gamma_{\rho}\)), it holds that
\[
\forall x \in \Gamma_{\rho} : n_{\rho}(x) = \frac{\nabla N_{\rho}(x)}{|\nabla N_{\rho}(x)|}.
\]
To calculate the gradient of $N_\rho$, it will be convenient to use the local expression of the gradient as in Remark 2.16. So, let $(U^*, \varphi^*, V^*)$ be a local representation of $\Omega$ around $x$, and set $u^* = (u, u_{m+1}) := (\varphi^*)^{-1}(x)$. Then

$$\nabla N_\rho(x) = (g^*)^{ij}(u^*) \frac{\partial (N_\rho \circ \varphi^*)}{\partial u_i} (u^*) \frac{\partial \varphi^*}{\partial u_j}(u^*).$$  \hfill (2.25)

Observe that

$$(N_\rho \circ \varphi^*)(u^*) = \Lambda(\varphi^*(u^*)) - \rho(S(\varphi^*(u^*))) = u_{m+1} - \rho(\varphi(u)).$$  \hfill (2.26)

Set $\xi := S(x)$. For $i, j \in \{1, \ldots, m\}$, let

$$\tilde{g}^{ij}(\rho)(\xi) := (g^*)^{ij}(\varphi^{-1}(\xi), \rho(\xi)), \quad \tilde{X}(\rho)(\xi) := X(\xi, \rho(\xi)).$$

Using (2.26) in (2.25) together with the notation introduced in Definition 2.3, we obtain

$$\begin{align*}
\nabla N_\rho(\tilde{X}(\rho)(\xi)) &\overset{\ast_1}{=} \tilde{g}^{ij}(\rho)(\xi) \left(-\partial_i \rho(\xi) \frac{\partial \varphi^*}{\partial u_j}(\varphi^{-1}(\xi), \rho(\xi)) + n(\xi) \right) \\
&\overset{\ast_2}{=} -\tilde{g}^{ij}(\rho)(\xi) \partial_i \rho(\xi) \left(\frac{\partial \varphi}{\partial u_j}(\varphi^{-1}(\xi)) + \rho(\xi) \partial_j n(\xi) \right) + n(\xi).
\end{align*}$$  \hfill (2.27)

$\ast_1$ : the summation over $i$ and $j$ is from 1 to $m$, while the summation in (2.25) is from 1 to $m + 1$. If $i = j = m + 1$, the expression in the right hand side of (2.25) reduces to $n(\xi)$, which explains the second term. If either $i$ or $j$ equals $m + 1$, and the other is smaller, then $(g^*)^{ij}(u^*) = 0$ because of (2.15), so this yields no contribution,

$\ast_2$ : see (2.10).

For the norm of $\nabla N_\rho$, we obtain

$$\left| \nabla N_\rho(\tilde{X}(\rho)) \right|^2 = \nabla N_\rho(\tilde{X}(\rho)) \cdot \nabla N_\rho(\tilde{X}(\rho))$$

$$\overset{\ast}{=} \tilde{g}^{ij}(\rho) \tilde{g}^{ij}(\rho) \partial_i \rho \partial_j \rho \tilde{g}_{ij}(\rho) + 1$$

$$\overset{\ast}{=} \tilde{g}^{ij}(\rho) \partial_i \rho \partial_j \rho + 1.$$

$\ast$ : use (2.27), \(\left(\frac{\partial \varphi}{\partial u_j} \circ \varphi^{-1} + \rho \partial_j n\right) \perp n\), and \(\frac{\partial \varphi^*}{\partial u_j} = (\rho^{-1})^*_{ij}\).

We define $L(\rho) : \Gamma \rightarrow \mathbb{R}$ locally by

$$L(\rho)|_V := \left| \nabla N(\tilde{X}(\rho)) \right| = \sqrt{\tilde{g}^{ij}(\rho) \partial_i \rho \partial_j \rho + 1}. \hfill (2.29)$$

A global definition of $L$ is given in Appendix B. Proposition B.6 states that $L \in C^\infty(\bar{U}; \mathcal{H}^{1+s}(\Gamma))$.

From now on, we will regard $n_\rho$ as a function on $\Gamma$. Then, by (2.24), (2.27) and (2.29), we obtain the following local description of $n_\rho$:

$$n_\rho|_V = \frac{1}{L(\rho)} \left(-\tilde{g}^{ij}(\rho) \partial_i \rho \left(\frac{\partial \varphi}{\partial u_j} \circ \varphi^{-1} + \rho \partial_j n\right) + n \right). \hfill (2.30)$$
The last quantity we want to express in terms of $\rho$, is the mean curvature of $\Gamma_\rho$, denoted by $H(\rho) : \Gamma \to \mathbb{R}$. This is done, for example, in [10], Lemma 3.1. We will only state the main result here, together with an overview of the basic steps of the proof.

**Theorem 2.36.** There exist operators

$$P \in C^\infty(\mathcal{U}, \mathcal{L}(h^{2+\alpha}(\Gamma), h^\alpha(\Gamma))), \text{ and } Q \in C^\infty(\mathcal{U}, h^\alpha(\Gamma))$$

such that for $\rho \in \mathcal{U}$, it holds that $-H(\rho) = P(\rho)\rho + Q(\rho)$.

**Remark 2.37.** Unlike [10], Lemma 3.1, we have put a minus sign in front of $H$, simply because in [10], the mean curvature is defined with the opposite sign.

The proof starts by using Theorem 2.35.(i), and by expressing this formula in terms of differential operators acting on $\Gamma$. Then, local representations are used to express these differential operators explicitly. This yields the following expressions for $P(\rho)$ and $Q(\rho)$:

$$P(\rho) = \frac{1}{m}(p_{ij}(\rho)\partial_i\partial_j + p_i(\rho)\partial_i), \quad Q(\rho) = \frac{1}{m}q(\rho),$$

$$p_{ij}(\rho) = \frac{1}{L(\rho)^3}\left(-L(\rho)^2\tilde{g}^{ij} + \tilde{g}^{ik}\tilde{g}^{j\ell}\partial_k\rho \partial_\ell\rho\right),$$

$$p_i(\rho) = \frac{1}{L(\rho)^2}\left(L(\rho)^2\tilde{g}^{ik}\tilde{\Gamma}^i_{jk} + \tilde{g}^{i\ell}\tilde{g}^{k\ell}\tilde{\Gamma}^{m+1}_{jk}\partial_k\rhoight.\left.\partial_\ell\rho + 2\tilde{g}^{ij}\tilde{\Gamma}^i_{m+1,j}\partial_k\rho - \tilde{g}^{i\ell}\tilde{g}^{k\ell}\tilde{\Gamma}^i_{jk}\partial_\ell\rho \partial_n\rho\right),$$

$$q(\rho) = -\frac{1}{L(\rho)}\tilde{g}^{ij}\tilde{\Gamma}^{m+1}_{ij}.$$  

*: We have abbreviated the right hand side by removing the argument $\rho$ from $\tilde{g}^{ij}$ and $\tilde{\Gamma}^{i}_{jk}$.

Furthermore, $L(\rho)$ is given by (2.29), and $\tilde{\Gamma}^{i}_{jk}(\rho)$ can be expressed in terms of $\tilde{g}_{ij}(\rho)$ by Remark 2.14 (although the Christoffel symbols involve higher order derivatives, we still have $\tilde{\Gamma}^{i}_{jk}(\rho) \in h^{2+\alpha}(\Gamma)$, because these derivatives act on the dependence of $u^* \in U^*$).

By looking carefully at the proof of [10], Lemma 3.1, one notices that one can easily extend Theorem 2.36 to $h^{k+\alpha}(\Gamma)$ for general $k \in \mathbb{N}$. This basically already follows from the representations of $P$ and $Q$ given by (2.31) and basic properties of little Hölder spaces. To see how the more general statement looks like, let

$$\mathcal{U}_k := \left\{ \rho \in h^{k+\alpha}(\Gamma) \mid \max_{\xi \in \Gamma} |\rho(\xi)| < \delta \right\}, \quad k \in \mathbb{N}.$$  

(2.32)

Note that this is an extension of (2.18), and that $\mathcal{U} = \mathcal{U}_2$. The extension of Theorem 2.36 is that we have for any $k \in \mathbb{N}$:

$$P \in C^\infty(\mathcal{U}_{k+1}, \mathcal{L}(h^{k+2+\alpha}(\Gamma), h^{k+\alpha}(\Gamma))), \quad Q \in C^\infty(\mathcal{U}_{k+1}, h^{k+\alpha}(\Gamma)).$$

Note that $k = 1$ (this corresponds exactly to [10], Lemma 3.1) yields a stronger statement than (a part of) the statement in Theorem 2.36. For our purposes, we only need this extension for $k = 0$. Furthermore, for proving local existence and uniqueness (see Section 7.4), the statement in Theorem 2.36 suffices.

In [10], it is remarked that although the hypersurface is a sphere, the proof holds for any smooth hypersurface.
The goal of this section is now achieved. If one fixes a bounded, closed \( C^\infty \)-hypersurface \( \Gamma \), then (2.19) shows how one can construct a closed \( C^2 \)-hypersurface \( \Gamma_\rho \) on it, depending on \( \rho \in \Omega \) (see (2.18)). Formulas (2.21), (2.30) and Theorem 2.36 show how various quantities related to \( \Gamma_\rho \), can be explicitly expressed in terms of \( \rho \).

2.3 Normal velocity

This section is an extension of the theory of the previous section. We use the same setting as in the previous section without mentioning it any further. Since our cell wall will evolve in time, we will have a different \( \rho \) describing the cell wall by \( \Gamma_\rho \) at each time \( t \geq 0 \). The only quantity of interest of \( \Gamma_\rho \) as it evolves in time, will be the normal velocity \( V \). This section is about defining \( V \) properly, and deriving an expression for \( V \) which explicitly shows the dependence on \( \rho \). This is done, for example, in [7], Section 2.2. We will give a selective summary of that section here.

Fix \( T > 0 \) and let
\[
\rho \in C^1([0, T]; h^u(\Gamma)) \cap C([0, T]; \Omega),
\]
(2.33)
where \( \Omega \) is defined in (2.18). Then \( \rho \) induces a family of hypersurfaces, given by
\[
\{\Gamma_{\rho(t)} \mid t \in [0, T]\},
\]
where \( \Gamma_{\rho(t)} \) is defined in (2.19).

The idea is to look at the \((m + 1)\)-dimensional closed \( C^1 \)-hypersurface given by
\[
M_T := \bigcup_{t \in (0, T)} \Gamma_{\rho(t)} \times \{t\},
\]
which is embedded in \( \mathbb{R}^{m+1} \times (0, T) \). That its regularity is just \( C^1 \), is due to the fact that \( \rho \in C^1([0, T]; h^u(\Gamma)) \). We have to be very careful here, because in [7], the authors only consider the case in which \( M_T \) is smooth.

**Definition 2.38.** (Normal velocity) Fix \((x, t) \in M_T\), set \( \xi := S(x) \in \Gamma \), and choose \( \varepsilon > 0 \) and \( \gamma \in C^1((t - \varepsilon, t + \varepsilon); \mathbb{R}^{m+1}) \) such that \( \gamma(t) = x \) and \( \forall \tau \in (t-\varepsilon, t+\varepsilon) : \gamma(\tau) \in \Gamma_{\rho(t)} \). Then we define the normal velocity \( V \) of \( \Gamma_{\rho(t)} \) at \( x \) by
\[
V(\xi, t) := n_{\rho(t)}(\xi) \cdot \gamma'(t).
\]

**Remark 2.39.** To show the existence of such a curve \( \gamma \) for a given pair \((x, t) \in M_T\), let \( \xi := S(x) \). Then the curve defined by \( \gamma(\tau) = X(\xi, \rho(\tau)(\xi)) \) satisfies all the conditions in 2.38 if one chooses \( \varepsilon > 0 \) such that \((t - \varepsilon, t + \varepsilon) \subset [0, T]\).

In [7], Lemma 2.33, it is shown that Definition 2.38 is independent of the choice of \( \gamma \). ■

We continue by deriving an expression for \( V \) which explicitly depends on \( \rho \). From Definition 2.38 and its Remark, we already have
\[
V(\xi, \rho(t)) = n_{\rho(t)}(\xi) \cdot \frac{d}{dt}X(\xi, \rho(t)(\xi)).
\]
(2.34)
Observe that
\[
\frac{d}{dt} X(\xi, \rho(t)(\xi)) = \frac{\partial X}{\partial \eta}(\xi, \rho(t)(\xi)) \frac{\partial \rho}{\partial t}(\xi) = n(\xi) \frac{\partial \rho}{\partial t}(\xi),
\]

*: a direct consequence of (2.8).

Inserting this in (2.34) yields
\[
V(\xi, \rho(t)) = n_{\rho(t)}(\xi) \cdot n(\xi) \frac{d}{dt} \rho(t)(\xi) \overset{*}{=} \frac{1}{L(\rho(t))(\xi)} \rho'(t)(\xi),
\]

*: use (2.30) and the orthogonality relations used in (2.28). From (2.33) it is clear that \(\rho'\) denotes the derivative of \(\rho\) with respect to time, such that \(\rho'(t) \in h^\alpha(\Gamma)\).

With this expression, we have achieved the goal of this section.
3 Interpolation theory

This chapter is a selective summary of the first chapter of [18]. Additionally, we extend one of its results to the setting of bounded, closed, smooth hypersurfaces. We also present a theorem based on [2], Chapter 5. It is our goal to present only those results that are important to us, together with the needed mathematical framework.

**Definition 3.1.** (Intermediate space, interpolation space). Let $X, Y, E$ be Banach spaces. $E$ is called an intermediate space between $X$ and $Y$ if

$$Y \hookrightarrow E \hookrightarrow X.$$  

If moreover

$$\forall T \in \mathcal{L}(X) : T|_Y \in \mathcal{L}(Y) \Rightarrow T|_E \in \mathcal{L}(E),$$

then $E$ is called an interpolation space between $X$ and $Y$.

Although the definition of $\mathcal{L}$ is well-known, we want to remark that for two vector spaces $X$ and $Y$, $\mathcal{L}(X, Y)$ denotes the space of all bounded, linear operators mapping from $X$ to $Y$. In case $X = Y$, we simply write $\mathcal{L}(X)$.

In practice, one starts with a given Banach space $X$ and an unbounded closed operator $A$ on $X$ defined on $D(A)$. Then $Y := D(A)$ is a Banach space when it is endowed with the graph norm of $A$, and $Y \hookrightarrow X$. It is not directly clear how an interpolation space $E$ between $X$ and $Y$ looks like.

Let us start by defining a class of intermediate spaces. In the following, $X$ and $Y$ are Banach space such that $Y \hookrightarrow X$.

**Definition 3.2.** Let $\sigma \in [0, 1]$ and $E$ an intermediate space between $X$ and $Y$. Then

$$E \in J_{\sigma}(X, Y) \iff \exists c > 0 \forall y \in Y : \|y\|_E \leq c \|y\|_X^{1-\sigma} \|y\|_Y^\sigma.$$  

An important example of such an intermediate space is given in the next Proposition:

**Proposition 3.3.** Let $0 < \theta < \alpha$ and $\Omega \subset \mathbb{R}^n$ open with uniform $C^\alpha$ boundary. Then $C^\theta(\bar{\Omega}) \subset J_{\theta/\alpha}(C^0(\bar{\Omega}), C^\alpha(\bar{\Omega})).$

For the definition of the Hölder space $C^\theta(\bar{\Omega})$, see Appendix A.

We move on to interpolation spaces. The $K$-method gives us an abstract construction of a certain class of interpolation spaces, called real interpolation spaces. It works as follows. Let $X$ and $Y$ Banach space such that $Y \hookrightarrow X$. For $t \in (0, \infty)$ and $x \in X$, we define

$$K(t, x; X, Y) := \inf \{ \|a\|_X + t\|b\|_Y \mid \exists a \in X, b \in Y : a + b = x \}.$$
We want to remark that in [18], for any \( \theta \in (0, 1), p \in [1, \infty], \) the following spaces are interpolation spaces:

\[
\begin{align*}
\{ (X, Y)_\theta, p :& = \{ x \in X \mid t \mapsto t^{-\theta - \frac{1}{p}} K(t, x) \in L_p(0, \infty) \} \\
\| x \|_{(X, Y)_\theta, p} :& = \| t \mapsto t^{-\theta - \frac{1}{p}} K(t, x) \|_{L_p(0, \infty)}, \\
\{ (X, Y)_\theta :& = \{ x \in X \mid \lim_{t \to 0} t^{-\theta} K(t, x) = 0 \} \\
\| x \|_{(X, Y)_\theta} :& = \| x \|_{(X, Y)_\theta, \infty}. \tag{3.1}
\end{align*}
\]

It is not at all obvious that these spaces are Banach spaces, let alone interpolation spaces. That this is true, is proved in [18]. We want to remark that we are only interested in certain properties of these real interpolation spaces, so that we skip the proofs here.

The following Proposition and Theorems happen to be useful. Again, \( X \) and \( Y \) are Banach space such that \( Y \hookrightarrow X \).

**Proposition 3.4.** For any \( \theta \in (0, 1), 1 \leq p_1 \leq p_2 < \infty, \) it holds that

\[
Y \hookrightarrow (X, Y)_{\theta, p_1} \hookrightarrow (X, Y)_{\theta, p_2} \hookrightarrow (X, Y)_{\theta} \hookrightarrow (X, Y)_{\theta, \infty} \hookrightarrow X.
\]

We want to remark that in [18], \( p_2 = \infty \) is also allowed in Proposition 3.4. This would imply that for any \( \theta \in (0, 1) \), one has \( (X, Y)_{\theta} = (X, Y)_{\theta, \infty} \), which is not true in general. Therefore, we regard this as a typo in [18], which yields in excluding the case \( p_2 = \infty \) in Proposition 3.4.

**Theorem 3.5.** For \( i \in \{1, 2\}, \) let \( X_i, Y_i \) be Banach spaces such that \( Y_i \hookrightarrow X_i \), and let \( T \in \mathcal{L}(X_1, X_2) \cap \mathcal{L}(Y_1, Y_2) \). Then for all \( \theta \in (0, 1), p \in [1, \infty], \) we have

\[
\max \left\{ \| T \|_{\mathcal{L}(X_1, X_2)}, \| T \|_{\mathcal{L}(Y_1, Y_2)} \right\} \leq \| T \|^{1-\theta}_{\mathcal{L}(X_1, X_2)} \| T \|^{\theta}_{\mathcal{L}(Y_1, Y_2)}. \]

**Theorem 3.6.** (Reiteration Theorem). Let \( 0 < \theta_0 < \theta_1 < 1, \theta \in (0, 1) \) and set \( \nu = (1 - \theta)\theta_0 + \theta \theta_1. \) Then for any \( p \in [1, \infty], \) the following two statements are true:

(i) If for all \( i \in \{0, 1\}, E_i \) is an intermediate space between \( (X, Y)_{\theta, \infty} \) and \( Y \), then

\[
(E_0, E_1)_{\theta, p} \hookrightarrow (X, Y)_{\nu, p}, \quad (E_0, E_1)_{\nu} \hookrightarrow (X, Y)_{\nu}.
\]

(ii) If for all \( i \in \{0, 1\}, E_i \in J_0(X, Y), \) then

\[
(X, Y)_{\nu, p} \hookrightarrow (E_0, E_1)_{\theta, p}, \quad (X, Y)_{\nu} \hookrightarrow (E_0, E_1)_{\nu}.
\]

**Proposition 3.7.** For \( \theta \in (0, 1), p \in [1, \infty), Y \) is dense in both \( (X, Y)_{\theta, p} \) and \( (X, Y)_{\theta} \).
The following proposition is not stated in [18]. Its proof is not hard. We will give a sketch of it.

**Proposition 3.8.** For \( i \in \{1, 2\} \), let \( X_i, Y_i \) be Banach spaces such that \( Y_i \hookrightarrow X_i \). Then for any \( \theta \in (0, 1) \), \( p \in [1, \infty) \), it holds that

\[
(X_1, Y_1)_{\theta, p} \times (X_2, Y_2)_{\theta, p} = (X_1 \times X_2, Y_1 \times Y_2)_{\theta, p},
\]

\[
(X_1, Y_1)_{\theta} \times (X_2, Y_2)_{\theta} = (X_1 \times X_2, Y_1 \times Y_2)_{\theta}.
\]

**Proof.** Let \( x = (x_1, x_2) \in (X_1, Y_1)_{\theta} \times (X_2, Y_2)_{\theta} \). It is easy to show that

\[
K(t, x; X_1 \times X_2, Y_1 \times Y_2) = K(t, x; X_1, Y_1) + K(t, x; X_2, Y_2).
\]

Then we have by (3.2) that \( x \in (X_1 \times X_2, Y_1 \times Y_2)_{\theta} \).

Let \( x \in (X_1 \times X_2, Y_1 \times Y_2)_{\theta} \). For \( i \in \{1, 2\} \), consider the projection \( P_i : X_1 \times X_2 \rightarrow X_i \). It is easy to see that we can apply Theorem 3.5 to \( P_i \). This yields

\[
P_i \in \mathcal{L}((X_1 \times X_2, Y_1 \times Y_2)_{\theta}, (X_i, Y_i)_{\theta}).
\]

Hence, \( x_i = P_i x \in (X_i, Y_i)_{\theta} \). Therefore, \( x = (x_1, x_2) \in (X_1, Y_1)_{\theta} \times (X_2, Y_2)_{\theta} \).

An important example of an interpolation space is given by the following Theorem ([18], Corollary 1.2.19). It deals with (little) Hölder spaces (see Appendix A), and it is the main reason why we consider little Hölder spaces in Chapter 7.

**Theorem 3.9.** Let \( 0 \leq \theta_1 < \theta_2, \sigma \in (0, 1), \) and \( \Omega \in \mathbb{R}^n \) open with uniform \( C^{\theta_2} \) boundary. If \( \theta_1 + \sigma(\theta_2 - \theta_1) \notin \mathbb{N} \), then

\[
\left( C^{\theta_1}(\overline{\Omega}), C^{\theta_2}(\overline{\Omega}) \right)_{\sigma} = h^{\theta_1 + \sigma(\theta_2 - \theta_1)}(\overline{\Omega}).
\]

We have not found the following Corollary in [18], so we prove it ourselves.

**Corollary 3.10.** Let \( \theta_1, \theta_2 \in \mathbb{R}_+ \setminus \mathbb{N} \) such that \( \theta_1 < \theta_2 \), let \( \sigma \in (0, 1) \), and \( \Omega \in \mathbb{R}^n \) open with uniform \( C^{\theta_2} \) boundary. If \( \theta_1 + \sigma(\theta_2 - \theta_1) \notin \mathbb{N} \), then

\[
\left( h^{\theta_1}(\overline{\Omega}), h^{\theta_2}(\overline{\Omega}) \right)_{\sigma} = h^{\theta_1 + \sigma(\theta_2 - \theta_1)}(\overline{\Omega}).
\]

**Proof.** Let \( \theta_1, \theta_2, \sigma \) and \( \Omega \) as in Corollary 3.10. To shorten our notation, we remove \( (\overline{\Omega}) \) from the notation of the function spaces. Let \( n := \lfloor \theta_2 \rfloor \). By Theorem 3.9 we have

\[
\forall i \in \{1, 2\} : (C^0, C^n)_{\theta_i/n} = h^{\theta_i}. \tag{3.3}
\]

Now we want to apply both parts of Theorem 3.6 to the interpolation spaces \( h^{\theta_i} \) between \( C^0 \) and \( C^n \). By Proposition 3.4, we have that

\[
\forall i \in \{1, 2\} : h^{\theta_i} = (C^0, C^n)_{\theta_i/n} \hookrightarrow (C^0, C^n)_{\theta_i/n, \infty}.
\]

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which implies that we can apply Theorem 3.6.(i) (set \(E_i = h^{\theta_i}, X = C^0\) and \(Y = C^n\)).

The condition in Theorem 3.6.(ii) is also satisfied, because of the following. (3.3) shows that \(h^{\theta_i}\) is an intermediate space between \(C^0\) and \(C^n\). Proposition 3.3 implies that \(C^{\theta_i} \in J_{\theta_i/n}(C^0, C^n)\), i.e.

\[
\forall i \in \{1, 2\} \exists c_i > 0 \forall f \in C^n : \|f\|_{C^{\theta_i}} \leq c_i \|f\|_{C^0}^{1 - \sigma} \|f\|_{C^n}^\sigma.
\]

Since \(\|\cdot\|_{h^{\theta_i}} = \|\cdot\|_{C^{\theta_i}}\), it is immediate that also \(h^{\theta_i} \in J_{\theta_i/n}(C^0, C^n)\), which implies that we can apply Theorem 3.6.(ii) too. Therefore,

\[
(h^{\theta_1}, h^{\theta_2})_\sigma = (C^0, C^n)_{((1 - \sigma)\theta_1 + \sigma\theta_2)/n} = h^{(1 - \sigma)\theta_1 + \sigma\theta_2},
\]

\*: see Theorem 3.9.

In [18], it is also not shown that one can generalize the interpolation properties of (little) Hölder spaces on subsets of \(\mathbb{R}^n\) to (little) Hölder spaces on bounded, closed, smooth hypersurfaces. This is true. We will illustrate the techniques to prove this in the proof of the next Corollary.

**Corollary 3.11.** Let \(\theta_1, \theta_2 \in \mathbb{R}_+ \setminus \mathbb{N}\) such that \(\theta_1 < \theta_2\), let \(\sigma \in (0, 1)\), \(n \in \mathbb{N}\) such that \(n \geq 2\), and \(\Gamma\) a bounded, closed, smooth hypersurfaces of \(\mathbb{R}^n\). If \(\tau := \theta_1 + \sigma(\theta_2 - \theta_1) \notin \mathbb{N}\), then

\[
(h^{\theta_1}(\Gamma), h^{\theta_2}(\Gamma))_\sigma = h^{\tau}(\Gamma).
\]

**Proof.** The proof consists of showing that the two concerning spaces are continuously embedded into each other.

Consider the operators \(P\) and \(Q\) as given in Proposition A.10. These operators give a translation between little Hölder spaces on hypersurfaces and products of little Hölder spaces on domains of \(\mathbb{R}^n\). This allows us to use the previous interpolation results.

Let \(f \in (h^{\theta_1}(\Gamma), h^{\theta_2}(\Gamma))_\sigma\). Proposition A.10 implies that

\[
P \in \mathcal{L}(h^{\theta_1}(\Gamma), \prod_{i=1}^N h^{\theta_1}(U_i)) \cap \mathcal{L}(h^{\theta_2}(\Gamma), \prod_{i=1}^N h^{\theta_2}(U_i)).
\]

Then it follows from Theorem 3.5 that

\[
P f \in \left(\prod_{i=1}^N h^{\theta_1}(U_i), \prod_{i=1}^N h^{\theta_2}(U_i)\right)_\sigma.
\]

(3.4)

Observe that

\[
\left(\prod_{i=1}^N h^{\theta_1}(U_i), \prod_{i=1}^N h^{\theta_2}(U_i)\right)_\sigma \overset{*_1}{=} \prod_{i=1}^N \left(h^{\theta_1}(U_i), h^{\theta_2}(U_i)\right)_\sigma \overset{*_2}{=} \prod_{i=1}^N h^{\tau}(U_i),
\]

(3.5)

\(*_1\): see Proposition 3.8,

\(*_2\): see Corollary 3.10.

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Hence, we have by Proposition A.10 that
\[ f = Q(Pf) \in h^\pi(\Gamma), \]
\( \ast : \text{see (3.4) and (3.5)}. \)

The inclusion of the spaces in Corollary 3.11 from right to left, can be shown by applying the same steps. The only main difference is that we have to apply Theorem 3.5 to \( Q \) instead of applying it to \( P \).

Although we have used the interpolation space as defined in (3.2), we can use similar arguments to generalize (other) interpolation properties related to (3.1) to (little) Hölder spaces defined on bounded, closed, smooth hypersurfaces.

We close this chapter with an interpolation property of Sobolev spaces. In [18], the author does not treat interpolation properties of Sobolev spaces. This is done, for example, in [2], Chapter 5.

**Theorem 3.12.** Let \( \Omega \in \mathbb{R}^n \) be a bounded domain with a smooth boundary. Then for \( p \in (n + 2, \infty) \), it holds that
\[
(L_p(\Omega), H^2_p(\Omega))_{1 - \frac{1}{p}, p} \hookrightarrow C^{2-(n+2)/p}(\Omega).
\]

**Proof.** The proof relies strongly on some of the equalities and embeddings stated in [2], Chapter 5. We will not introduce all the concerning spaces here.

In [2], for \( k \in \mathbb{N}, H^k_p(\Omega) \) is denoted by \( W^k_p \). We now switch to the notation used in [2], including the references to other equations. Be warned; by \( H^k_p \) we now denote a different space (called the Bessel potential space).

The proof is basically given in the following line:
\[
(W^0_p, W^2_p)_{1 - \frac{1}{p}, p} \overset{(5.2)}= (H^0_p, H^2_p)_{1 - \frac{1}{p}, p} \overset{(5.21)}= B^{2 - 2/p}_p \overset{(5.5)}= B^{2 - 2/p}_{p, \infty} \overset{(5.8)}= B^{2 - (n+2)/p}_{\infty, \infty} \overset{(5.1)}= C^{2-(n+2)/p},
\]
where \( C^{2-(n+2)/p} \) denotes the Hölder space which we denote similarly by \( C^{2-(n+2)/p}(\Omega) \).

**Remark 3.13.** The extension of Theorem 3.12 to bounded, closed, smooth hypersurfaces \( \Gamma \), can be proven to hold analogously to the proof of Corollary 3.11. Since for the chart \( U \) of any local representation of \( \Gamma \), it holds that \( \dim U = \dim \Gamma = n \), the condition \( p \in (n + 2, \infty) \) is not altered.
4 Semigroup theory

This chapter gives a selective summary of the first three chapters of [22]. It is our goal to describe the main results and ideas of this theory, rather than deriving and proving them. This theory will not be directly applicable to our problem, but it is included in this thesis because it lays the foundation on which the theory discussed in chapters 5 and 6 is built upon.

Before diving into the abstract theory, we start by giving a motivation to consider semigroup theory. This is not done in [22]. The first section will encompass this, while the second section will be about the abstract framework.

4.1 Motivation

Consider the evolution equation
\[
\begin{aligned}
  u'(t) - Au(t) &= 0 & t \in (0, \infty) \\
  u(0) &= u_0,
\end{aligned}
\]

where \( A, u_0 \in \mathbb{R} \). It is well-known that the solution to this initial value problem is given by \( u(t) = e^{At}u_0 \).

It is slightly more interesting if we consider \( u_0 \in \mathbb{R}^n \) and \( A \in \mathbb{R}^{n \times n} \). Still, the solution is given by \( u(t) = e^{At}u_0 \), where
\[
e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.
\]

This expression makes sense, since we know what powers of matrices are, and because \( A \) is bounded in any matrix norm. So
\[
\|e^{At}\| \leq \sum_{k=0}^{\infty} \frac{t^k \|A\|^k}{k!} \leq \sum_{k=0}^{\infty} \frac{t^k \|A\|^k}{k!} = e^{\|A\|t}.
\]

The following step is to consider a Banach space \( X \), and take \( u_0 \in X, A \in \mathcal{L}(X) \). It can be shown that again \( u(t) = e^{At}u_0 \), but now \( e^{At} \in \mathcal{L}(X) \). Still, (4.2) makes sense, because we know what powers of operators are, and the boundedness of \( A \) implies that \( e^{At} \) is bounded.

The most interesting case that we consider, occurs when we generalize this setting to unbounded operators \( A \). For example, a vast class of linear partial differential equations can...
be written as in (4.1). For such \( A \), \( e^{At} \) is not well-defined by (4.2). In this chapter, we present results which state for which operators \( A \), (4.1) has a unique solution \( u(t) = e^{At}u_0 \), where \( e^{At} \in \mathcal{L}(X) \) satisfies a number of properties (which we make precise in the following section).

If such operator exists, then \( \{ e^{At} \mid t \in (0, \infty) \} \) is going to be the semigroup coupled to \( A \).

### 4.2 Theory

Unlike the previous section suggests, we can also start with a semigroup \( \{ T(t) \mid t \in (0, \infty) \} \), and see which operator \( A \) is coupled to this semigroup. First, we need to define what a semigroup is.

Throughout this section, \( X \) denotes a Banach space.

**Definition 4.1.** (Semigroup). \( \hat{T} := \{ T(t) \mid t \in (0, \infty) \} \subset \mathcal{L}(X) \) is called a semigroup of bounded linear operators on \( X \) if

(i) \( T(0) = I \), the identity map on \( X \),
(ii) \( \forall t, s \geq 0 : T(t+s) = T(t)T(s) \).

(ii) is called the semigroup property. We will call \( \hat{T} \) simply a semigroup.

**Remark 4.2.** In view of the previous section, we need (i) to ensure that \( u(0) = u_0 \). (ii) states that if we would solve (4.1) up to time \( s \), and then solve (4.1) again with initial condition \( T(s)u_0 \) up to time \( t \), that we find the same solution as in the case where we would start at \( u_0 \), and solve (4.1) up to \( t+s \).

One can regard \( \hat{T} \) as the range of \( T : [0, \infty) \rightarrow \mathcal{L}(X) \).

**Definition 4.3.** (Infinitesimal generator). Let \( \hat{T} \) be a semigroup. The linear operator \( A \) defined by

\[
D(A) = \left\{ x \in X \left| \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\} \right.,
\]

\[
Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t}, \quad x \in D(A),
\]

is called the infinitesimal generator of \( \hat{T} \).

This definition might look odd at first sight. To get an idea about why it is defined this way, one can substitute \( T(t) \) by \( e^{At} \) and assume that all usual properties of the exponential function still hold. This is of course a very formal way of looking at this theory, but at least it gives an idea where this definition and the definitions still to come, come from.

**Definition 4.4.** (Uniformly continuous semigroup). A semigroup \( \hat{T} \) is called uniformly continuous if \( T \) is continuous at 0, i.e.

\[
\lim_{t \downarrow 0} \| T(t) - I \| = 0.
\]
Remark 4.5. By \(\|\cdot\|\) we either mean \(\|\cdot\|_X\) or \(\|\cdot\|_{\mathcal{L}(X)}\), depending on its argument. Furthermore, by the semigroup property, we have that
\[
\|T(t + h) - T(t)\| \leq \|T(h)(T(t) - I)\|, \quad h > 0, \\
\|T(t - h) - T(t)\| \leq \|T(t - h)(I - T(t))\|, \quad h \in (0, t),
\]
which implies by the boundedness of \(T(t)\), that \(T\) is continuous everywhere if it is continuous at 0.

It can be shown (see [22], Section 1.1) that for any uniformly continuous semigroup \(\hat{T}\), the infinitesimal generator \(A\) is bounded, and \(T(t) = e^{At}\). Conversely, every \(A \in \mathcal{L}(X)\) generates a unique \(\hat{T}\), which happens to be uniformly continuous.

More interesting are those semigroups which are not uniformly continuous, but which are still strongly continuous:

Definition 4.6. (Strongly continuous semigroup). A semigroup \(\hat{T}\) is called strongly continuous (or a \(C_0\)-semigroup) if \(T(\cdot)x\) is continuous at 0 for any \(x \in X\), i.e.
\[
\forall x \in X : \lim_{t \downarrow 0} T(t)x = x.
\]
The set of all \(C_0\)-semigroups is denoted by \(C_0\).

The following Theorem lists basic properties of \(\hat{T} \in C_0\):

Theorem 4.7. Let \(\hat{T} \in C_0\) and \(A\) its infinitesimal generator. Then for any \(t \in [0, \infty)\), the following statements are true:

1. \(\exists M \geq 1, \omega \geq 0 : \|T(t)\| \leq Me^{\omega t}, \quad (M, \omega \text{ independent of } t),\)
2. \(\forall x \in X : T(\cdot)x \in C([0, \infty); X),\)
3. \(\forall x \in X : \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x \, ds = T(t)x, \quad (\text{for } t = 0, \text{ take } h \downarrow 0),\)
4. \(\bigcap_{n=1}^{\infty} D(A^n) \text{ is dense in } X,\)
5. \(\forall x \in X : A \int_0^t T(s)x \, ds = T(t)x - x,\)
6. \(\forall x \in D(A) : \frac{dT}{dt}(t)x = AT(t)x = T(t)Ax,\)
7. \(\forall x \in D(A), t \in [0, t) : T(t)x - T(s)x = \int_s^t AT(\tau)x \, d\tau = \int_s^t T(\tau)Ax \, d\tau,\)
8. if \(S \in C_0\) has infinitesimal generator \(B\), and \(A = B\), then \(S = \hat{T}\),
9. \(A\) is closed.
In the following, if we take \( \hat{T} \in C_0 \), we do not mention that we denote by \( A \) its infinitesimal generator, and that we denote by \( M \) and \( \omega \) some constants that satisfy statement (i) of Theorem 4.7.

Note that (ix) implies that \((D(A), \| \cdot \|_A)\) is a Banach space (\( \| \cdot \|_A \) denotes the graph norm of \( A \)). Therefore, one can regard \( A \) as a bounded linear operator from \( D(A) \) to \( X \), i.e. \( A \in \mathcal{L}(D(A), X) \). We say that

\[
A \in \mathcal{G}(D(A), X) \subset \mathcal{L}(D(A), X)
\]

iff it generates a \( C_0 \)-semigroup. Now, if we take an \( A \in \mathcal{G}(D(A), X) \), we immediately define the corresponding \( \hat{T} \in C_0 \) (Theorem 4.7.(viii) states that \( \hat{T} \) is unique), and denote by \( M \geq 1 \) and \( \omega \geq 0 \) some constants satisfying part (i) of Theorem 4.7, without mentioning it.

In practice (like (4.1)), a (unbounded) linear operator \( A \) is given, while it is not known whether \( A \in \mathcal{G}(D(A), X) \). If \( A \) is bounded, we know that this is true, and moreover, \( \hat{T} \) is uniformly continuous. If \( A \) is unbounded, the following Theorem says whether it is a generator of a \( C_0 \)-semigroup or not.

**Theorem 4.8.** (Hille-Yosida). \( A \in \mathcal{G}(D(A), X) \) with \( M \) and \( \omega \) satisfying part (i) of Theorem 4.7 iff

(i) \( A \) is closed and \( D(A) \) is dense in \( X \), and

(ii) \( (\omega, \infty) \subset \rho(A) \), and \( \forall \lambda > \omega, n \in \mathbb{N} : \| R(\lambda : A)^n \| \leq \frac{M}{(\lambda - \omega)^n} \).

**Remark 4.9.** Theorem 4.8 is even stronger. Whenever \( A \in \mathcal{G}(D(A), X) \), then not only (ii) is true, but also the following holds:

\[
\left\{ \lambda \in \mathbb{C} \mid \Re \lambda > \omega \right\} \subset \rho(A) \), and \( \forall \lambda > \omega, n \in \mathbb{N} : \| R(\lambda : A)^n \| \leq \frac{M}{(\Re \lambda - \omega)^n} \).
\]

In Theorem 4.8, \( \rho(A) \subset \mathbb{C} \) denotes the resolvent set of \( A \), and \( R(\lambda : A) = (\lambda I - A)^{-1} \in \mathcal{L}(X) \) its resolvent operator.

In practice, Theorem 4.8 is still not satisfactory. This is because it is usually very hard to prove estimates for powers of the resolvent operator. The Remark does not help either, because it is only useful when one already knows that \( A \in \mathcal{G}(D(A), X) \). However, from a theoretical point of view, the Remark is very useful.

Now we can show a theorem which tells us for a given \( \hat{T} \in C_0 \) how to interpret \( e^{At} \) (other than just equating it to \( T(t) \)).

**Theorem 4.10.** Let \( \hat{T} \in C_0 \), \( x \in X \). Then

\[
T(t)x = \lim_{n \to \infty} \left( I - \frac{t}{n} A \right)^{-n} x = \lim_{n \to \infty} \left( \frac{n}{t} R\left( \frac{n}{t} : A \right) \right)^n x.
\]

The limit is uniform in \( t \) on bounded time intervals.
Remark 4.11. Note that the second equality in Theorem 4.10 is straightforward. Theorem 4.8 tells us that the right hand side makes sense whenever \( n/t > \omega \), which for \( n \) large enough is the case when \( t > 0 \) is fixed. For \( t = 0 \), the other expression tells us that \( T(0) = I \).

Note that for \( a \in \mathbb{R} \), we have

\[
e^{at} = \left(e^{-at}\right)^{-1} = \lim_{n \to \infty} \left(1 - \frac{t}{n}a\right)^{-n},
\]

so that the following definition is consistent:

\[
e^{At}x := \lim_{n \to \infty} \left(1 - \frac{t}{n}A\right)^{-n}x, \quad x \in X.
\]

This tells us that it makes sense that for given \( A \in \mathcal{G}(D(A), X) \), we can write \( e^{At} \) instead of \( T(t) \). Note that this is also consistent with what one would expect from Theorem 4.7.(vi).

We are now going to define a special subclass of \( C_0 \)-semigroups, called analytic semigroups, and are interested in which \( A \in \mathcal{G}(D(A), X) \) generate such semigroups.

Definition 4.12. (Differentiability). \( \hat{T} \in C_0 \) is called \( t \)-differentiable if

\[
\forall_{x \in X} : T(\cdot)x \in C^1([t, \infty); X).
\]

If \( t = 0 \), we just call \( \hat{T} \) differentiable.

Note that for any \( \hat{T} \in C_0 \), Theorem 4.7.(vi) already states that \( T(\cdot)x \in C^1([0, \infty); X) \), but that this only holds for \( x \in D(A) \).

Definition 4.13. (Analytic semigroup). Let \(-\pi < \varphi_1 < 0 < \varphi_2 \leq \pi \) and \( \Sigma := \{z \in \mathbb{C} \mid \arg z \in (\varphi_1, \varphi_2)\} \). Then \( T_\Sigma := \{T(z) \in \mathcal{L}(X) \mid z \in \Sigma\} \) is an analytic semigroup in \( \Sigma \) if

(i) \( T \) analytic in \( \Sigma \), i.e. \( T \in C^\omega(\Sigma; \mathcal{L}(X)) \),

(ii) \( \forall_{x \in X} : \lim_{\substack{z \to 0 \\text{\scriptsize{\textbullet}} \\text{\scriptsize{\textbullet}}
\Sigma}} \, T(z)x = x \), and

(iii) \( \forall_{z_1, z_2 \in \Sigma} : T(z_1 + z_2) = T(z_1)T(z_2) \).

Remark 4.14. There are two reasons for this particular domain \( \Sigma \). First, we want that \( T\big|_{[0, \infty)} \) (called the real part of \( T_\Sigma \)) defines a differentiable \( \hat{T} \in C_0 \). This implies that we need \([0, \infty) \subset \Sigma \). Second, we need \( \Sigma \) to be an additive semigroup of complex numbers such that (iv) makes sense, i.e. \( z_1 + z_2 \in \Sigma \). In the following, if we say that \( \hat{T} \) is analytic, we mean that there exists a \( \Sigma \) with the properties described above.

From (ii), it makes sense to extend \( T \) to the set \( \Sigma \cup \{0\} \) by \( T(0) = I \). Then it is easy to see that \( T\big|_{[0, \infty)} \) describes a semigroup, which shows that analytic semigroups are a class of semigroups.

We will only consider domains \( \Sigma \) for which \( -\pi = \varphi_1 = \varphi_2 =: \varphi \in (0, \pi] \), and denote these by \( \Sigma_\varphi \).

Definition 4.15. (Bounded analytic semigroup). An analytic semigroup \( T_{\Sigma_\varphi} \) is bounded if

\[
\forall_{\varphi \in (0, \varphi)} \exists_{M_{\varphi} \geq 1} \forall_{z \in \Sigma_\varphi} : \|T(z)\| < M_{\varphi}.
\]
The following Theorem gives important characterizations of analytic semigroups.

**Theorem 4.16.** Let $\hat{T} \in C_0$ such that $\omega = 0$ and $0 \in \rho(A)$. Then the following statements are equivalent:

(i) There exists a $\delta \in (0, \pi)$ and a bounded analytic semigroup $S_{\Sigma_i}$ satisfying $S|_{[0, \infty)} = T$,

(ii) $\exists_{C>0} \forall_{\lambda \in \mathbb{C}, \text{Re}\lambda>0} : \|R(\lambda : A)\| \leq \frac{C}{|\text{Im} \lambda|},$

(iii) $A$ is sectorial, i.e. there exists $\delta \in (0, \pi/2), C > 0$ such that

$$\Sigma_{\delta} \subset \rho(A),$$

$$\forall_{\lambda \in \Sigma_{\delta} \setminus \{0\}} : \|R(\lambda : A)\| \leq \frac{C}{|\lambda|}.$$  

(iv) $\hat{T}$ is differentiable, and $\exists_{C>0} \forall_{t>0} : \|\hat{T}(t)\| \leq \frac{C}{t}$.

**Remark 4.17.** The extra demands on $\hat{T}$ at the beginning of Theorem 4.16, are no real constraints. To see this, let $\hat{T} \in C_0, \epsilon > 0$ arbitrary. Then $S(t) := e^{-\epsilon(t+\epsilon)^2}T(t)$ defines a $C_0$-semigroup which satisfies $\|S(t)\| \leq M$ and is generated by $\hat{A} := A - (\omega + \epsilon)I$. $\hat{A}$ is the operator that one gets if one moves the spectrum of $A$ along the real axis to the left by a distance of $\omega + \epsilon$. Since $(\omega, \infty) \subset \rho(A)$, we therefore have $(0, \infty) \subset \rho(\hat{A})$, which shows that $\hat{S}$ does satisfy the conditions of Theorem 4.16. So, if $S$ happens to be the real part of a bounded analytic semigroup, then Theorem 4.16 applies to $\hat{S}$, and all characterizations (i)-(iv) can easily be translated to properties of $\hat{T}$. \hfill $\blacksquare$

We say for $A \in \mathcal{H}(D(A), X)$ that $A \in \mathcal{H}(D(A), X)$ iff $\hat{T}$ is the real part of an analytic semigroup.

We close this chapter with an important perturbation result for analytic semigroups.

**Theorem 4.18.** Let $A \in \mathcal{H}(D(A), X)$. Then there exists a $\delta > 0$ such that for all closed linear operators $B$ satisfying $D(B) \supset D(A)$ and

$$\exists_{b \geq 0} \forall_{x \in D(A)} : \|Bx\| \leq \delta \|Ax\| + b\|x\|,$$

it holds that $A + B \in \mathcal{H}(D(A), X)$.

It is obvious that when $B$ is bounded, Theorem 4.18 applies (just take $b = \|B\|$, and the inequality holds regardless of $\delta$). The following Corollary is less obvious, but still not hard.

**Corollary 4.19.** $\mathcal{H}(D(A), X)$ is open in $\mathcal{L}(D(A), X)$.

**Proof.** Let $A \in \mathcal{H}(D(A), X)$ and $\delta$ given by Theorem 4.18. It is sufficient to show that for any $B \in B_{\mathcal{L}(D(A), X)}(0, \delta)$, it holds that $A + B \in \mathcal{H}(D(A), X)$.

Let $B \in B_{\mathcal{L}(D(A), X)}(0, \delta)$. Then

$$\delta > \|B\|_{\mathcal{L}(D(A), X)} = \sup_{x \in D(A)} \frac{\|Bx\|}{\|x\|} = \sup_{x \in D(A)} \frac{\|Bx\|}{\|Ax\| + \|x\|}.$$
So for any $x \in D(A)$, we have that

$$\|Bx\| < \delta \|Ax\| + \delta \|x\|.$$ 

By taking $b = \delta$, we can apply Theorem 4.18 to conclude that $A + B \in \mathcal{H}(D(A), X)$. □
5 Local existence and uniqueness

The theory of DaPrato and Grisvard is a tool for showing local existence and uniqueness for certain parabolic initial value problems. Our mathematical model of the cell wall can be cast into such a problem, so this theory will be very useful.

In this chapter, we present the framework of this theory and the main result (Theorem 5.7) which will give us local existence and uniqueness for our problem. [5] gives a clear description of this abstract theory. This chapter contains a selective summary of this article. For the proofs of various statements, we refer to this article.

Let
\[ E = (E_1, E_0) \]
be a Banach couple, i.e. \( E_0, E_1 \) are Banach spaces such that \( E_1 \) is densely and continuously embedded in \( E_0 \). We denote by \( \mathcal{L}(E) := \mathcal{L}(E_1, E_0) \) the space of bounded linear maps from \( E_1 \) to \( E_0 \), which is also a Banach space with respect to the operator norm induced by the norms on \( E_0 \) and \( E_1 \). To be more precise, let \( A \in \mathcal{L}(E) \), then
\[
\|A\|_{\mathcal{L}(E)} = \sup_{x \in E_1} \frac{\|Ax\|_{E_0}}{\|x\|_{E_1}} = \sup_{\|x\|_{E_1} = 1} \|Ax\|_{E_0}.
\]

Any \( A \in \mathcal{L}(E) \) can also be seen as a linear operator in \( E_0 \) with domain \( D(A) = E_1 \). This operator may be unbounded in \( E_0 \), i.e.
\[
\|A\|_{\mathcal{L}(E_0, E_0)} = \sup_{x \in E_0} \frac{\|Ax\|_{E_0}}{\|x\|_{E_0}}
\]
may not exist. In any case, if \( A \) generates a \( C_0 \)-semigroup (denoted by \( \{e^{At} \mid t \in [0, \infty)\} \), see also Chapter 4), we say \( A \in \mathcal{G}(E) \). Furthermore, we define
\[
\mathcal{H}(E) := \{ A \in \mathcal{G}(E) \mid A \text{ generates an analytic } C_0 \text{-semigroup} \},
\]
which is an open subset of \( \mathcal{L}(E) \) (see Corollary 4.19).

The following Proposition could also have been included in Chapter 4, but we have chosen to follow [5] step by step.

**Proposition 5.1.** For any \( A \in \mathcal{G}(E) \), \( \| \cdot \|_{E_1} \) is equivalent to \( \| \cdot \|_A \) defined by \( \|x\|_A = \|Ax\|_{E_0} + \|x\|_{E_0} \).

We continue by defining \( X_\theta(E) \) and \( Y_\theta(E) \) for \( \theta \in (0, 1] \) fixed and \( T > 0 \), which contain paths in \( E_0 \) parametrized on the interval \( (0, T] \). To economize our notation, we will not denote the
dependence of these spaces on \( T \). For \( \theta = 1 \)

\[
X_1(E) := C([0, T]; E_0),
Y_1(E) := C([0, T]; E_1) \cap C^1([0, T]; E_0),
\]

and for \( \theta \in (0, 1) \)

\[
X_\theta(E) := \left\{ u \in C((0, T]; E_0) \left| \lim_{t \downarrow 0} t^{1-\theta} \|u(t)\|_{E_0} = 0 \right. \right\},
Y_\theta(E) := \left\{ u \in C((0, T]; E_1) \cap C^1((0, T]; E_0) \left| \lim_{t \downarrow 0} t^{1-\theta} (\|u'(t)\|_{E_0} + \|u(t)\|_{E_1}) = 0 \right. \right\}.
\]

We equip these spaces with the following norms:

\[
\|u\|_{X_\theta(E)} := \sup_{t \in (0, T]} t^{1-\theta} \|u(t)\|_{E_0},
\|u\|_{Y_\theta(E)} := \sup_{t \in (0, T]} t^{1-\theta} (\|u'(t)\|_{E_0} + \|u(t)\|_{E_1}).
\]

Formally speaking, the role of \( \theta \) is that it controls the rate at which a path (or its derivative) may explode as \( t \to 0 \). From the definitions, it is immediate that for any \( 0 < \vartheta < \theta \leq 1 \),

\[
Y_\vartheta(E) \subset X_\theta(E), \quad X_\vartheta(E) \subset X_\theta(E), \quad Y_\theta(E) \subset Y_\vartheta(E).
\]

Since \( \theta \) is fixed, it is convenient to write \( X \) and \( Y \) instead of \( X_\theta(E) \) and \( Y_\theta(E) \).

For any \( u \in Y \), it holds that \( \exists C > 0 : \|u'(t)\|_{E_0} \leq Ct^{\theta-1} \). Therefore

\[
\left\| \int_0^T u'(t) \, dt \right\|_{E_0} \leq \int_0^T Ct^{\theta-1} \, dt = T^\theta \int_0^1 C \, \frac{1}{t} < \infty,
\]

so we can extend the domain of \( u \) to \([0, T]\), where

\[
u(0) = u(T) - \int_0^T u'(t) \, dt.
\]

We are now going to define the interpolation space \( E_\theta \). For this definition, we fix \( T = 1 \). Fixing another value of \( T \) yields the same space \( E_\theta \) with another (but still equivalent) norm. Since \( u(0) \) is well-defined, the following space is also well-defined:

\[
E_\theta := \{ u(0) \mid u \in Y \}. \tag{5.1}
\]

For \( E_\theta \) it can be shown that it is a Banach space when it is equipped with the following norm:

\[
\|x\|_{E_\theta} := \inf \left\{ \|u\|_Y \mid x = u(0), u \in Y \right\}.
\]

\( E_\theta \) is known to be the continuous interpolation space of exponent \( \theta \) for the couple \( E \). In view of (3.2), we can also write \( E_\theta = (E_0, E_1)_{\theta} \).
Now that we have defined the interpolation space $E_0$, we again consider general $T > 0$. For $\theta \in (0, 1)$, we are going to introduce $\mathcal{M}_\theta(E) \subset \mathcal{H}(E)$. First, we define the time differential operator:
\[ \partial_t : Y \to X, \quad (\partial_t u)(t) = u'(t). \]
In the following, we will omit the brackets around $\partial_t u$. From the norms on $X$ and $Y$, it immediately follows that $\partial_t \in \mathcal{L}(Y, X)$.

Second, for any $A \in \mathcal{L}(E)$, we define
\[ \hat{A} : Y \to X \times E_0, \quad \hat{A}u = (\partial_t u(t) - Au(t), u(0)), \]
where $X \times E_0$ is a Banach space when we equip it with (a norm equivalent to) the norm
\[ \mathcal{V}_{(x,e) \in X \times E_0} : \|(x, e)\|_{X \times E_0} := \|x\|_X + \|e\|_{E_0}. \]

A similar argument used for showing $\partial_t \in \mathcal{L}(Y, X)$, can be used to show that $\hat{A} \in \mathcal{L}(Y, X \times E_0)$. Moreover,
\[ (A \mapsto \hat{A}) \in C(\mathcal{L}(E); \mathcal{L}(Y, X \times E_0)), \quad (5.2) \]
which follows easily after writing the definition of continuity in terms of the norms on $E_0$ and $E_1$.

Third, for $Z_1$ and $Z_2$ Banach spaces, we define
\[ \mathcal{L}_{is}(Z_1, Z_2) = \{ B \in \mathcal{L}(Z_1, Z_2) \mid B \text{ invertible} \}. \]
The subscript ‘is’ stands for isomorphism. One of the central theorems of linear operators on Banach spaces states that $B \in \mathcal{L}_{is}(Z_1, Z_2) \implies B^{-1} \in \mathcal{L}(Z_2, Z_1)$.

**Definition 5.2.** Using the setting above, let
\[ \mathcal{M}_\theta(E) := \{ A \in \mathcal{H}(E) \mid \hat{A} \in \mathcal{L}_{is}(Y, X \times E_0) \}. \]

**Remark 5.3.** Perhaps it is more intuitive to say that $\mathcal{M}_\theta(E)$ contains precisely those $A \in \mathcal{H}(E)$ for which
\[ \begin{cases} x'(t) - Ax(t) = f(t) & t \in (0, T) \\ x(0) = x_0 \end{cases} \quad (5.3) \]
has a unique solution $x \in Y$ for any data $f \in X$ and $x_0 \in E_0$. Note that this is exactly what it means for $\hat{A}$ to be invertible. In this setting, $\hat{A}^{-1}$ is given by $\hat{A}^{-1}(f, x_0) = x$.

For $A \in \mathcal{M}_\theta(E)$, we consider the operator
\[ J_A := \hat{A}^{-1}|_{X \times \{0\}}, \quad (5.4) \]
which we will regard as an operator on $X$ mapping to
\[ Y^{(0)} := \{ u \in Y \mid u(0) = 0 \}. \quad (5.5) \]
It is easy to see that $J_A \in \mathcal{L}_{is}(X, Y^{(0)})$, and per definition, $J_A^{-1} = \partial_t - A$. ■
For $A \in \mathcal{M}_\theta(E)$, the solution to (5.3) is given by

$$x(t) = (\hat{A}^{-1}(f, x_0))(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} f(s) \, ds, \quad t \in (0, T],$$

(5.6)

which one can think of as the generalization of the variation of constants formula. Remember that $e^{tA}$ denotes the analytic semigroup generated by $A$.

To get an idea about why not all $A \in H(E)$ result in a unique solution to (5.3), we take a closer look at (5.6). We need $x \in Y$, which implies - among other estimates - that we need to be able to show that

$$\int_0^t A e^{(t-s)A} f(s) \, ds \in E_0,$$

where $A$ in the integrand comes from the derivative of $e^{(t-s)A}$. From Theorem 4.16.(iv) we have that $\|A e^{tA}\|_{L(E_0, E_0)} \leq C/t$, which is not good enough to ensure the convergence of this integral. Therefore, we expect $\mathcal{M}_\theta(E) \subsetneq H(E)$.

On the other hand, we want $\mathcal{M}_\theta(E)$ to be non-empty, which is not trivial. In fact, if $E_0$ is reflexive, then $\mathcal{M}_1(E) = \emptyset$. Therefore, we need some restrictions on $E$ in order to guarantee that $\mathcal{M}_\theta(E)$ is not empty.

The following Proposition will be useful later on. Since its proof is very short, we copy it from [5].

**Proposition 5.4.** $\mathcal{M}_\theta(E)$ is open in $H(E)$.

**Proof.** $\mathcal{L}_\text{ps}(Y, X \times E_0)$ is open in $\mathcal{L}(Y, X \times E_0)$. Note that

$$\mathcal{M}_\theta(E) \ni (A \mapsto \hat{A}^{-1}(\mathcal{L}_\text{ps}(Y, X \times E_0))),$$

(5.7)

$\ast$ : in this context, $f^{-1}(W)$ denotes the preimage of a function $f$ from a subset $W$ of its image.

Then because of (5.2), the right hand side of (5.7) is open in $\mathcal{L}(E)$. Together with Corollary 4.19, this completes the proof.

We are going to use the DaPrato-Grisvard construction, because it will be of great help to us. In the following, we will explain how it works. Let $(F_1, F_0)$ be a Banach couple, and let $A \in \mathcal{H}(F_1, F_0)$. Then define

$$F_2 := D(A^2) = \{x \in F_1 \mid Ax \in F_1\},$$

(5.8)

which is a Banach space when it is equipped with $\|x\|_{F_2} = \|Ax\|_{F_1} + \|x\|_{F_1}$, or with a norm equivalent to $\|\cdot\|_{F_2}$.

For any $\sigma \in (0, 1)$, introduce the continuous interpolation spaces

$$E_i := (F_{i+1}, F_i)_\sigma =: F_{i+\sigma}, \quad i \in \{0, 1\},$$

(5.9)

which are defined, together with their norms, similarly as $E_0$ in (5.1). Moreover, $E := (E_1, E_0)$ is a Banach couple. This is seen by using the following basic property of interpolation spaces:

$$F_2 \hookrightarrow E_1 \hookrightarrow F_1 \hookrightarrow E_0 \hookrightarrow F_0.$$
Furthermore, we need that $E_1$ is dense in $E_0$. This follows by an easy argument from Proposition 3.7 and $F_2$ being dense in $F_1$.

In [5], the authors continue by stating that $A \in \mathcal{H}(F_1, F_0)$ implies $A \in \mathcal{H}(E)$ by the interpolation property, without giving any proof. We will state this in the following Theorem, and prove it. The proof shows how one can combine the theory from Chapters 3 and 4.

**Theorem 5.5.** Let $\sigma \in (0, 1)$ arbitrary and $F_i$ and $E_j$ as above. Then $A \in \mathcal{H}(F_1, F_0)$ implies $A \in \mathcal{H}(E)$.

**Proof.** Let $A \in \mathcal{H}(F_1, F_0)$, and assume that it satisfies the conditions of Theorem 4.16. By the definition of $F_2$, we have $A_1 := A|_{F_2} \in \mathcal{L}(F_2, F_1)$. Then by Theorem 3.5, we also have $A_\sigma := A|_{E_1} \in \mathcal{L}(E_1, E_0)$. Now we want to show that $A_\sigma$ and $A_1$ satisfy part (iii) of Theorem 4.16.

We start by showing that $\rho(A_1), \rho(A_\sigma) \supset \rho(A)$. To this end, let $\lambda \in \rho(A)$. Then

$$\lambda I - A \in \mathcal{L}_{is}(F_1, F_0).$$

(5.10)

This directly implies that $\lambda I - A_1 = (\lambda I - A)|_{F_2}$ is injective. For showing surjectivity of $(\lambda I - A_1)$ onto $F_1$, let $x \in F_1$. Then also $x \in F_0$, for which (5.10) implies that there exists a $z \in F_1$ such that $(\lambda I - A)z = x$. Rewriting this yields $Az = \lambda z - x \in F_1$. Then we have by (5.8) that $z \in F_2$, so $x = (\lambda I - A_1)z$. Hence $\lambda I - A_1$ is surjective, and together with injectivity, we have $\lambda I - A_1 \in \mathcal{L}_{is}(F_2, F_1)$. Together with (5.10), we can use Theorem 3.5 twice (once for $\lambda I - A_1$, and once for $R(\lambda : A) := (\lambda I - A)^{-1}$) to show that

$$\lambda I - A_\sigma \in \mathcal{L}_{is}(E_1, E_0).$$

(5.11)

So $\lambda \in \rho(A_\sigma) \cap \rho(A_1)$. Since $\lambda$ was arbitrary, we have shown $\rho(A_1) \cap \rho(A_\sigma) \supset \rho(A)$.

In order to show that $A_\sigma$ and $A_1$ satisfy part (iii) of Theorem 4.16, we also need the resolvent estimate. Let $x \in F_1$, then

$$\|R(\lambda : A_1)x\|_{F_1} = \|R(\lambda : A)x\|_{F_0} \leq \hat{C} \left( \|R(\lambda : A)Ax\|_{F_0} + \|R(\lambda : A)x\|_{F_0} \right) \leq \hat{C} \left( \frac{C}{|\lambda|} \|Ax\|_{F_0} + \frac{C}{|\lambda|} \|x\|_{F_0} \right) = \hat{C} \frac{C}{|\lambda|} \|x\|_{F_1},$$

(5.12)

*1 : see Proposition 5.1, and note that $A$ and $R(\lambda : A)$ commute on $F_1$. $\hat{C}$ comes from the equivalence of norms,

*2 : since $A$ satisfies part (iii) of Theorem 4.16.

With (5.12) we not only have that $A_1$ satisfies Theorem 4.16.(iii) (from which we can conclude $A_1 \in \mathcal{H}(F_2, F_1)$), but by Theorem 3.5 we also have

$$\|R(\lambda : A_\sigma)\|_{\mathcal{L}(E_0)} \leq \|R(\lambda : A_\sigma)\|_{\mathcal{L}(F_0)}^{1-\sigma} \|R(\lambda : A_\sigma)\|_{\mathcal{L}(F_1)}^{\sigma} \leq \hat{C} \frac{C}{|\lambda|}. $$

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So also \( A_x \) satisfies Theorem 4.16.(iii), and therefore \( A_x \in \mathcal{H}(E) \), thus \( A \in \mathcal{H}(E) \).

To complete the proof, we have to show that we still have \( A \in \mathcal{H}(E) \) when we drop the condition that \( \mathcal{A} \) is chosen such that the conditions of Theorem 4.16 are satisfied. So, let \( A \in \mathcal{H}(F_1, F_0) \) arbitrary. For \( a > 0 \) (to be specified in a moment), let \( \tilde{A} := A - aI \). By Theorem 4.18, \( \tilde{A} \in \mathcal{H}(F_1, F_0) \) for any \( a > 0 \). Remark 4.17 implies that there exists an \( a > 0 \) such that \( \tilde{A} \) is such that the conditions of Theorem 4.16 are satisfied. For \( \tilde{A} \) we have already proven that \( \tilde{A} \in \mathcal{H}(E) \). Then again by Theorem 4.18, we have that \( A = \tilde{A} + aI \in \mathcal{H}(E) \).

\[ \square \]

The main theorem describing the DaPrato-Grisvard construction, uses Theorem 5.5, and reads:

**Theorem 5.6.** Let \((F_1, F_0)\) be a Banach couple, \(A \in \mathcal{H}(F_1, F_0)\) and \(F_2\) as defined in (5.8). For any \(\sigma \in (0, 1)\), let the Banach couple \(E := (E_1, E_0)\) be defined by (5.9). Then for any \(\theta \in (0, 1]\), it holds that \(A \in \mathcal{M}_\theta(E)\), and there exists a \(C > 0\) which only depends on \(A, T, \sigma\) and \(\theta\), such that

\[ \|J_A\|_{L^1(E, \mathcal{Y}^{\theta(0)})} \leq C. \]

For the definition of \(J_A\), see (5.4). The proof of this Theorem is quite technical, and we omit it here.

The use of Theorem 5.6 will become clear in Theorem 5.7, for which we need the following preliminaries. Let \(E = (E_1, E_0)\) be a Banach couple, \(\mathcal{Y} \in E_1\) an open subset. Then for any \(f \in C^1(\mathcal{Y}, E_0)\) and any \(x_0 \in \mathcal{Y}\), we consider the initial value problem

\[ \begin{cases} 
  x'(t) = f(x(t)) & t \in [0, T] \\
  x(0) = x_0. 
\end{cases} \tag{5.13} \]

We call \(x\) a strict solution to (5.13) if it is an element of \(C([0, T]; E_1) \cap C^1([0, T]; E_0)\).

**Theorem 5.7.** If \(\forall x \in \mathcal{Y}: f'(x) \in \mathcal{M}_1(E)\), then there exists a \(T > 0\) such that (5.13) has a unique strict solution.

The proof of Theorem 5.7 is relatively easy. We give a brief sketch here. By a simple argument, one can restrict oneself by considering (5.13) for \(x_0 = 0\). Since \(f \in C^1(\mathcal{Y}, E_0)\), we can write it in the following Taylor expansion

\[ f(x) = f_0 + Ax + r(x), \]

where \(A = f'(0) \in \mathcal{M}_1(E)\) and \(r \in C^1(\mathcal{Y}, E_0)\) such that \(r(0) = 0\) and \(r'(0) = 0\). Then it is easy to see that (5.13) can be rewritten as

\[ x = J_A(r(x) + f_0), \]

which can be solved by a contraction mapping argument. This argument requires \(\|x^{(n)}(t)\|_{E_0}\) to be small (\(x^{(n)}\) is the \(n\)-th iterate of the contraction), which can be realized by choosing \(T > 0\) small enough (but independent of \(n\)).
6 Stability

In this section we consider autonomous quasilinear evolution equations on Banach spaces, which have a set of equilibria $E$ that locally looks like an $m$-dimensional $C^1$-manifold $M$. We give sufficient conditions for which this manifold is attractive around a point $u_0 \in M$. This means that the evolution equation at hand has (for initial values close to $u_0$) a unique solution existing on an infinite time interval, which converges to some $u_\infty \in E$. In general, we do not know what $u_\infty$ is, which requires special treatment.

Theorem 6.3 gives a precise description of the statement mentioned in the previous paragraph. This theorem is posed and proven in [25] as Theorem 2.1. We introduce the setting needed to understand this theorem, and give a brief sketch of its proof. In Section 7.5 we are going to show that our evolution equation fits in this framework, so that Theorem 6.3 gives us the desired stability result.

Let $(X_1, X_0)$ be a Banach couple, i.e. $X_0$ and $X_1$ are Banach spaces such that $X_1$ is continuously and densely embedded in $X_0$. For any $p \in (1, \infty)$, let

$$X_\sigma := (X_0, X_1)_{1-1/p, p},$$

(6.1)

$*: $ see (3.1) for the definition of the interpolation space. $\sigma$ is used to abbreviate $1 - 1/p \in (0, 1)$.

We assume that the autonomous quasi-linear problem of interest, given by

$$\begin{cases}
\dot{u}(t) - A(u(t))u(t) = F(u(t)) & t > 0 \\
u(0) = u_0 \in X_\sigma,
\end{cases}$$

(6.2)

satisfies the following:

$$\exists V \subset X_\sigma \text{ open} : A \in C^1(V, \mathcal{L}(X_1, X_0)), F \in C^1(V, X_0).$$

(6.3)

This assumption looks quite technical. To get a feeling about what this assumption implies, one can take a look at the beginning of Section 7.5, where we give an example of a quasilinear problem which satisfies (6.3).

Let

$$E := \{u \in V \cap X_1 \mid -A(u)u = F(u)\}$$

(6.4)

be the set of all equilibrium solutions to (6.2). Fix $u_\ast \in E$. If there happens to be an $m \in \mathbb{N}_+$, a $\delta > 0$, a $U \subset \mathbb{R}^m$ open, and a $\varphi \in C^1(U; X_1)$ such that
(i) \( 0 \in U, \varphi(0) = u_s, \text{ and } \varphi(U) \subset \mathcal{E} \).
(ii) \( \mathcal{N}(\varphi'(0)) = \{0\}, (\varphi'(0) \in \mathcal{L}(\mathbb{R}^m, X_1) \) is the Fréchet derivative of \( \varphi \) at 0.
(iii) \( \mathcal{E} \cap B_{X_1}(u_s, \delta) \subset \varphi(U) \),

then we know by the inverse function Theorem that there is a neighbourhood \( \hat{U} \subset U \) of 0 and an \( M \subset \mathcal{E} \) such that \( \varphi|_{\hat{U}} : \hat{U} \to M \) is a homeomorphism. Therefore, \( M \) is an \( m \)-dimensional \( C^1 \)-manifold in \( X_1 \). We want to remark that we have only defined manifolds as subsets of \( \mathbb{R}^n \) (see Definition 2.1). The definition of manifolds in Banach spaces is quite similar. Then one easily sees that \( M \) is a 1-regular manifold. Since we can describe \( M \) by the single local representation \( (\hat{U}, \varphi|_{\hat{U}}, M) \), we have that \( M \) is a \( C^1 \)-manifold. The tangent space of \( M \) at \( u_s \) is given by

\[ T_{u_s}M := \{ \varphi'(0)x \mid x \in \mathbb{R}^m \}. \]

So, in order to show that there is a neighbourhood \( M \subset \mathcal{E} \) around \( u_s \) which is a \( C^1 \)-manifold, one needs to show that there are \( m \in \mathbb{N}_+, \delta > 0, U \subset \mathbb{R}^m \) open, and \( \varphi \in C^1(U; X_1) \) such that all three conditions in (6.5) are satisfied.

Given that we are close to an equilibrium, we cast (6.2) into a form that is more convenient to work with. For this purpose, we fix \( u_s \in \mathcal{E} \), and we are going to rewrite (6.2) in terms of \( v = u - u_s \). Note that

\[ \dot{u} = \frac{d}{dt}(v + u_s) \equiv \dot{v}, \]

\( *: u_s \in \mathcal{E} \), i.e. \( u_s \) is an equilibrium solution to (6.2).

Since we expect \( v \) to be small, it makes sense to look at the linearization of (6.2) around \( u_s \), i.e.

\[ A(u)u + F(u) = A(v + u_s)(v + u_s) + F(v + u_s) \]
\[ \equiv A(u_s) + A'(u_s)v + o(v)(v + u_s) + \left( F(u_s) + F'(u_s)v + o(v) \right) \]
\[ \equiv A(u_s)v + \left( A'(u_s)v \right)u_s + F'(u_s)v + o(v), \]

\( *_1 \) : the validity of this Taylor expansion follows from (6.3). For the definition of \( o(v) \), see Remark B.2. We would like to remark that \( A'(u_s) \) is the Fréchet derivative at \( u_s \) of a mapping that maps \( V \subset X_\sigma \) to \( \mathcal{L}(X_1, X_0) \). Therefore, \( A'(u_s) \in \mathcal{L}(X_\sigma, \mathcal{L}(X_1, X_0)) \).

\( *_2 \) : since \( u_s \in \mathcal{E} \), we have \( A(u_s)u_s + F(u_s) = 0 \). Furthermore, \( A'(u_s)v = O(v) \), so applying this operator to \( v \) yields a term that is \( o(v) \).

So

\[ A_0v := A(u_s)v + \left( A'(u_s)v \right)u_s + F'(u_s)v \]

(6.7)
is the linearization of \( A(u)u + F(u) \) around \( u_s \). Just by adding and removing terms in both sides of (6.2), one can rewrite this equation into

\[ \dot{v}(t) - A_0v(t) = G_1(v(t)) + G_2(v(t), v(t)), \]

(6.8)

\[ G_1(v) := \left( F(u_s + v) - F(u_s) - F'(u_s)v \right) + \left( A(u_s + v) - A(u_s) - A'(u_s)v \right)u_s, \]
\[ G_2(v, w) := (A(u_s + v)w - A(u_s))w. \]
Let $V := \{ v - u \mid v \in V \}$. Then one easily shows with the help of (6.3) that

$$G_1 \in C^1(V; X_0), \quad G_2 \in C^1(V \times X_1; X_0),$$

$$G_1(0) = G_2(0, 0) = 0, \quad G_1'(0) = 0, \quad G_2'(0, 0) = 0.$$

(6.9)

We do not derive this here. In [25], (6.9) is stated as a preparation for the proof of [25], Theorem 2.1. We present (6.9) just to show the reader that (6.8) describes our quasi-linear problem by a linear part, and a non-linear part that has a (very) small contribution (in the sense of $o(v)$) for small $v$. So if there exists a solution $v$ to (6.8) which moreover stays small, the linearization of (6.2) around $u_*$ will still give a good approximation to $u$ at any time $t$.

Although this is a very formal statement, it gives us an idea about how we can find stability results for (6.2). More details will be given in Remark 6.4.

We still need two more definition in order to state the main theorem of this chapter:

**Definition 6.1. (Maximal $L_p$-regularity.)** $A_0 \in \mathcal{L}(X_1, X_0)$ has maximal $L_p$-regularity if there exists a $T > 0$ such that for all $f \in L_p((0, T), X_0)$,

$$\begin{cases}
\dot{u} - A_0 u = f & t \in (0, T] \\
u(0) = 0
\end{cases}$$

has a unique solution $u \in H^1_p((0, T), X_0) \cap L_p((0, T), X_1)$.

Note that in [25], the authors use the definition of maximal $L_p$-regularity for operators with the opposite sign.

**Definition 6.2. (Normally stable.)** $u_* \in \mathcal{E}$ is called normally stable if the following four statements are satisfied:

(i) there exists a neighbourhood $M \subset \mathcal{E}$ around $u_*$ which is an $m$-dimensional $C^1$-manifold in $X_1$,

(ii) $T_{u_*} M = \mathcal{N}(A_0)$,

(iii) $0$ is a semisimple eigenvalue, i.e. $\mathcal{R}(A_0)$ is closed, and $\mathcal{N}(A_0) \oplus \mathcal{R}(A_0) = X_0$,

(iv) $\sigma(A_0) \setminus \{ 0 \} \subset \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda < 0 \}$.

The following theorem is the main theorem of this chapter. For the full proof (which takes a couple of pages), we refer to [25], the proof of Theorem 2.1. We give a brief sketch of the proof in Remark 6.4.

**Theorem 6.3.** Let $p \in (1, \infty)$ (note that this fixes $X_0$ by (6.1)) and consider (6.2) in which $A$ and $F$ satisfy (6.3). Let $u_* \in \mathcal{E}$. If $A(u_*)$ has maximal $L_p$-regularity and $u_*$ is normally stable, then there exists a $\delta > 0$ such that for all $u_0 \in B_{X_0}(u_*, \delta)$, problem (6.2) has a unique solution

$$u \in H^1_p((0, \infty); X_0) \cap L_p((0, \infty); X_1).$$

Moreover,

$$\exists u_\infty \in \mathcal{E} \exists C_{\omega, \delta} > 0 : \| u(t) - u_\infty \|_{X_0} < Ce^{-\omega t}.$$
Remark 6.4. Since Theorem 6.3 looks quite technical, we start with a formal description of it. Choose a proper equilibrium solution \( u_* \) to (6.2). Then Theorem 6.3 gives us a \( \delta \), which is a measure on how much we are allowed to perturb \( u_* \) to \( u_0 \). Theorem 6.3 then tells us that (6.2) with this \( u_0 \) as initial condition, has a unique solution \( u \) that exists on the whole of \( \mathbb{R}_+ \), and which converges exponentially fast to some \( u_\infty \in \mathcal{E} \) (\( u_\infty \) does not have to be equal to \( u_* \)).

We now give a brief sketch of the proof. First, the spectral projection \( P \) according to the spectral set \( \{0\} \subset \sigma(A_0) \) (see [18], Appendix A, for its definition and main properties), yields a direct composition of the spaces \( X_i \) \((i \in \{0,1\})\). This can be done because \( u_* \) is normally stable (see Definition 6.2). The idea of this splitting is to split the problem into two parts; one dealing with the shift of the equilibrium point (which is going to be \( u_\infty \) instead of \( u_* \)), and one dealing with the convergence of \( u \) to an equilibrium.

We obtain \( X_0^* := PX_0 = \mathcal{N}(A_0) = T_u M \) and \( X_0^* := (I - P)X_0 = \mathcal{R}(A_0) \) (\( c \) and \( s \) stand for ‘center’ and ‘stable’ part). Then also \( A_0 \) can be split into a center and a stable part, i.e. \( A_c := AP = A|_{\mathcal{N}(A_0)} \equiv 0 \) and \( A_s := A(I - P) = A|_{\mathcal{R}(A_0)} \).

It can be shown that \( A_s \) has maximal \( L_p \)-regularity, even for \( T = \infty \) (see Definition 6.1). This means that for all \( f \in L_p((0, \infty), X_0^*) \),

\[
\begin{align*}
\dot{w}_1 - A_s w_1 &= f & t \in (0, \infty) \\
\dot{w}_1(0) &= 0
\end{align*}
\]

(6.10)

has a unique solution \( w_1 \in H^1_p((0, \infty), X_0^*) \cap L_p((0, \infty), X_1) \), which is given by

\[
w_1(t) = \int_0^t e^{A_s(t-\tau)} f(\tau) \, d\tau, \quad t \in (0, \infty),
\]

where \( e^{A_s t} \) denotes the analytic semigroup generated by \( A_s \). One of the properties of an (analytic) semigroup, is that for all \( x \in X_0^* \),

\[
\begin{align*}
\dot{w}_2 - A_s w_2 &= 0 & t \in (0, \infty) \\
\dot{w}_2(0) &= x
\end{align*}
\]

(6.11)

is solved by \( w_2(t) = e^{A_s t} x \). Since problems (6.10) and (6.11) are linear, the following problem

\[
\begin{align*}
\dot{w} - A_s w &= f & t \in (0, \infty) \\
\dot{w}(0) &= x
\end{align*}
\]

(6.12)

is solved by

\[
w(t) = e^{A_s t} x + \int_0^t e^{A_s(t-\tau)} f(\tau) \, d\tau, \quad t \in (0, \infty)
\]

(6.13)

for any \( f \in L_p((0, \infty), X_0^*) \) and \( x \in X_0^* \). This yields various estimates of \( w_1, w_2 \) and \( w \) in terms of \( f \) and \( x \). Furthermore, since 0 is an isolated spectral point (i.e. there exists a neighbourhood \( \Lambda \) of 0 in \( \mathbb{C} \) such that \( \sigma(A_0) \cap \Lambda = \{0\} \)) and \( A_s \) generates a bounded analytic semigroup, we can shift the spectrum of \( A_s \) a little bit to the right (by considering \( A_s + \omega I \) for
\( \omega > 0 \) small) such that \( A_s + \omega I \) still generates a bounded analytic semigroup. This yields a term \( e^{-\omega t} \) in some of the estimates, which in the end is going to give us the exponential rate of convergence as stated in Theorem 6.3.

Up to this point, we have only considered linear problems, while we are interested in the solution of the quasi-linear problem given by (6.8). Also this problem can be split into a coupled system of quasi-linear problems, but there is some freedom in how one does this splitting. The authors of [25] use a splitting such that the coupled system on \( X_0' \times X_0' \) as given by

\[
\begin{align*}
    \dot{v}_1 &= r_1(v_1, v_2), & v_1(0) &= v_1^0 \\
    \dot{v}_2 &= A_s v_2 = r_2(v_1, v_2), & v_2(0) &= v_2^0
\end{align*}
\]  

(6.14)

has the following properties:

- for \( v \) close enough to 0, there is an isomorphism between \( v \) and \((v_1, v_2)\),
- \( v_2(t) = 0 \) \iff \( v(t) \in \{ w - u_s \mid w \in \mathcal{E} \} \).

We have already stated that \( A_s = 0 \). Hence, there is no additional term in the left hand side of the first equation in (6.14). From (6.9) it can be shown that

\[
\| r_1(v_1^0, v_2^0) \|_{X_0} + \| r_2(v_1^0, v_2^0) \|_{X_0} \leq C \| v_0^0 \|_{X_1},
\]

for \( v_0 \) close to 0. So, provided that we can show that \( v \) stays close enough to 0, the nonlinear terms \( r_i \) stay small, and even go to 0 as \( v_2 \) goes to 0 (which corresponds to \( u \) converging to \( \mathcal{E} \)).

It can be shown that there is an \( a > 0 \) such that for each \( v_0 \in B_{X_0}(0, a) \), (6.8) has a unique solution

\[
v \in H^1_p((0, t_1); X_0) \cap L^p((0, t_1); X_1) \subset C([0, t_1]; X_0)
\]

for some \( t_1 > 0 \). We can extend this solution to a maximal interval of existence \([0, t_s]\). If \( t_s \) is finite, then either \( v(t) \) leaves the ball \( B_{X_0}(0, a) \) at time \( t_s \), or the limit of \( v(t) \) for \( t \uparrow t_s \) does not exist in \( X_0 \). With the help of the estimates that we mentioned before, the authors of [25] show that both cases cannot happen. So \( t_s = \infty \), and the exponential rate of convergence of \( v(t) \) to \( v_\infty \) also follows from those estimates. The solution \( u \) to (6.2) is merely a translation of \( v \) by \( u_s \). This ends the sketch of the proof.

The authors of [25] are brief about the extension of the maximal interval of existence of the local solution \( v \) to (6.8). They merely state it, but we will prove it here in full detail.

**Proposition 6.5.** Let \( v \) be the unique solution to (6.8) on the time interval \([0, t_s]\) with initial condition \( v_0 \in B_{X_0}(0, r) \) (see the setting in Remark 6.4). If

\[
\exists v_n \in B_{X_0}(0, r) : \| v(t) - v_n \|_{X_0} \to 0 \quad \text{as} \quad t \uparrow t_s,
\]

then \( v \) can be extended to the unique solution of (6.8) on the time interval \([0, t_s + t_1]\) for some \( t_1 > 0 \).
It is easy to show that \( v \) satisfies properties of \( X \) even have this for \( X_1 \) instead of \( X_0 \) and that used \( v \) as an initial condition (we have used this uniqueness result in Remark 6.4). Let \( v_2 \) be a shift of \( v \) in time, i.e. \( v_2(t) := v(t - t_s) \) for \( t \in [t_s, t_s + t_1] \). Now we want to show that the extension of \( v_1 \), given by

\[
v(t) := \begin{cases} v_1(t) & t \in [0, t_s) \\ v_2(t) & t \in [t_s, t_s + t_1]. \end{cases}
\]

(6.15)

satisfies

\[
v \in H^1_p((0, t_s + t_1); X_0) \cap L_p((0, t_s + t_1); X_1).
\]

(6.16)

This follows directly from

\[
\|v\|_{L_p((0, t_s + t_1); X_1)}^p = \int_0^{t_s+t_1} \|v(t)\|_{X_1}^p \, dt
\]

(6.15)

\[
\leq \int_0^{t_s} \|v_1(t)\|_{X_1}^p \, dt + \int_{t_s}^{t_s+t_1} \|v_2(t)\|_{X_1}^p \, dt < \infty.
\]

For showing that \( v \in H^1_p((0, t_s + t_1); X_0) \), we need to show that \( v \in L_p((0, t_s + t_1); X_0) \) (we even have this for \( X_1 \) instead of \( X_0 \)) and that

\[
\exists w \in L_p((0, t_s + t_1); X_0) \forall \varphi \in C_0^\infty((0, t_s + t_1)): \int_0^{t_s+t_1} \varphi' v = -\int_0^{t_s+t_1} \varphi w.
\]

(6.17)

If such \( w \) exists, it is called the weak derivative of \( v \), which is often denoted by \( v' \). Let \( \varphi \in C_0^\infty((0, t_s + t_1)) \). Then

\[
\int_0^{t_s+t_1} \varphi' v = \int_0^{t_s} \varphi' v_1 + \int_{t_s}^{t_s+t_1} \varphi' v_2.
\]

(6.18)

Let us focus on the first integral of the sum in the right hand side. We are going to use basic properties of \( L_p \) and \( H^1_p \) functions which are Banach space valued. These properties can be found, for example, in [11], Section 5.9.2. We know that \( \varphi \in C^\infty([0, t_s]) \), \( v_1 \in L_p((0, t_s + t_1); X_0) \) and \( v_2' \in L_p((0, t_s + t_1); X_0) \). Then we also know that \( \varphi' \in C^\infty([0, t_s]), \varphi v_1 \in L_p((0, t_s + t_1); X_0) \) and \( \varphi v_2' \in L_p((0, t_s + t_1); X_0) \). Then by the weak product rule, we have that \( (\varphi v_1)' \in L_p((0, t_s + t_1); X_0) \) and \( \varphi v_2' \in L_p((0, t_s + t_1); X_0) \). Then we also have \( \varphi v_1 \in H^1_p((0, t_s + t_1); X_0) \), from which we conclude that

\[
\int_0^{t_s} \varphi' v_1 + \int_{t_s}^{t_s+t_1} \varphi' v_2' = \int_0^{t_s} (\varphi v_1)' = \varphi(t_s)v_1(t_s) - \varphi(0)v_1(0).
\]

(6.19)
So applying (6.19) on both integrals in the right hand side of (6.18) yields

\[ \int_0^{t_\ast} \varphi^\prime v = \varphi(t_\ast)v_1(t_\ast) - \varphi(0)v_1(0) - \int_0^{t_\ast} \varphi v_1^\prime \]

\[ + \varphi(t_\ast + t_1)v_2(t_\ast + t_1) - \varphi(t_\ast)v_2(t_\ast) - \int_t^{t_\ast + t_1} \varphi v_2^\prime \]

\[ = - \int_0^{t_\ast} \varphi v_1^\prime - \int_t^{t_\ast + t_1} \varphi v_2^\prime. \]

\[ \ast : 0 = \varphi(0) = \varphi(t_\ast + t_1), \] so the corresponding terms vanish. Furthermore, \( v_1(t_\ast) = v_2(t_\ast), \) so the corresponding terms cancel each other out.

Let

\[ w(t) := \begin{cases} v_1'(t) & t \in [0, t_\ast) \\ v_2'(t) & t \in [t_\ast, t_\ast + t_1] \end{cases}. \]

Note that \( w \in L_p((0, t_\ast + t_1); X_0), \) and that \( w \) is independent of \( \varphi. \) Since \( \varphi \) was chosen arbitrarily, (6.17) follows from (6.20). This completes the proof.
7 Osmotic cell swelling

7.1 Derivation of the model

In this section, we will model the evolution of a cell wall. This means that we are going to make a lot of assumptions for our cell wall, such that all irrelevant phenomena will not be taken into account, and such that we are left with a short description of its motion (see (7.6)). We conclude by giving a motivation why we are going to show local existence and uniqueness of a solution to (7.6), and stability of this solution around a configuration of the cell wall at which it will be in equilibrium.

Consider an $\Omega^+ \subset \mathbb{R}^n$ which is open, bounded and connected, such that $\Gamma := \partial \Omega^+$ is a connected $C^2$-boundary. Figure 7.1 shows a schematic overview. Note that this implies that $\Omega^- := \mathbb{R}^n \setminus (\Omega^+ \cup \Omega^-)$ is open, unbounded and connected, and $\partial \Omega^- = \Gamma$. For $n = 3$, $\Omega^+$ corresponds to the interior of a cell, and $\Gamma$ describes the cell wall.

Figure 7.1: Schematic view of the cell wall, together with the notation of certain objects related to it. Convexity of $\Omega^+$ is not demanded.

In our simplified model of this cell, the interior consist of a fluid (one can think of water) with a solute (e.g. salt, dissolved in the fluid), while the exterior only consist of the fluid. The cell wall has infinitesimal thickness, is semipermeable, has no mass and has the same permeability everywhere, even if it is stretched or bent. In our model, semipermeability means that the fluid can flow through the cell wall, while the solute can not. No mass implies that there is no inertia. Furthermore, we assume that the solute molecules do not precipitate on the cell wall.

Given an initial configuration, the goal is to describe the evolution in time of the cell wall. There are no external forces (e.g. gravity) acting on the cell or the fluid around it. Neither will there be barriers (e.g. other neighboring cells) preventing parts of the cell wall to move...
in a certain direction. The only effects that will be taken into account, are osmosis across \( \Gamma \) due to the difference in concentration of the solute, and the pressure difference of the fluid across \( \Gamma \). In the sequel, we will discuss these two effects separately. This is also done in [30], Chapter 1, where the model is slightly different.

For a certain quantity \( \psi \), we denote by \( \psi^+ \) its value in \( \Omega^+ \) (which may depend on the position), and by \( \psi^- \) its value in \( \Omega^- \). We denote \( [\psi] := \psi^+ - \psi^- \) for the jump of \( \psi \) across the boundary.

We consider a one phase model, which means in our case that all quantities with a minus as subscript are known constants.

We consider \( n_s > 0 \) mol of solute inside \( \Omega^+ \) and no solute in \( \Omega^- \). This gives a positive concentration \( u^+ \) (measured in mol m\(^{-n}\)), and yields \( u^- = 0 \). Because we consider a semipermeable membrane, \( n_s \) stays constant and \( u^- \) stays 0. Therefore, \( [u] = u^+ \).

Let \( P \) denote the hydrostatic pressure (measured in Pa = N m\(^{-n-1}\)) exerted on the boundary. \( P^+ \) is unknown. We assume \( P^- \) to be a given constant.

In our model, \( [u] \) and \( [P] \) are the two phenomena that cause the fluid to flow through the boundary. So, we also need to describe the tendency of the solvent molecules (molecules of the fluid) to flow across the boundary. Let \( [\mu] \) denote the magnitude of this tendency, measured in J mol\(^{-1}\). \( [\mu] \) is known as the chemical potential. If \( [\mu] > 0 \) at a point \( \xi \in \Gamma \), the fluid tends to flow out of the cell at \( \xi \).

The following formula describes how \( [u] \), \( [P] \) and \( [\mu] \) are related:

\[
[mu] = V[P] - VRT[u], \quad \text{on } \Gamma,
\]

\( V \): molar volume of the fluid, in m\(^n\) mol\(^{-1}\),
\( R \): gas constant, in J mol\(^{-1}\) K\(^{-1}\),
\( T \): temperature, in K.

(7.1) describes a thermodynamical equilibrium. A derivation of this formula can be found in [27].

We still need to specify how \( [u] \), \( [P] \) and \( [\mu] \) depend on \( \Omega^+ \) and \( \Gamma \). To do this, we need more assumptions. We will elaborate on this in the sequel.

\( u^+ \) is assumed to be constant in space. This is what makes our model different from the usual models in this field of research, which do incorporate the diffusion of the solute inside \( \Omega^+ \). Our assumption comes from considering fast diffusion. If the diffusion speed is much greater than the average speed at which the boundary moves, the solute will be almost uniformly distributed in \( \Omega^+ \). So it seems reasonable to assume that it really is uniformly distributed whenever the diffusion speed is expected to be fast.

As a result of this assumption, we obtain

\[
[u] = \frac{n_s}{|\Omega^+|},
\]

(7.2)

The jump in the hydrostatic pressure across \( \Gamma \) is caused by surface tension. One can compare this with a balloon filled with air, for which it is obvious that \( [P] > 0 \). It is less obvious, but still true, that \( [P] \) is linked with the curvature of the surface, denoted by \( H \), by the following relation:

\[
[P] = -2\gamma H.
\]

(7.3)
This equation is known as the Young-Laplace equation, and is derived, for example, in [26], Section 13.5, and [19], Section 2.1. \( \gamma \) is the surface tension, measured in \( \text{N m}^{-1} \), which is assumed to be constant in space and time. \( H \) is a measure for curvature, measured in \( \text{m}^{-1} \).

For \( n = 2 \), \( H \) is the well-known curvature of a curve. For \( n \geq 2 \), there is no unique extension of this definition of \( H \). However, the extension is unique when we impose that \( H = -1 \) for the unit sphere (in \( \mathbb{R}^n \)) at any point. This extension is given in Section 2.1.

We have already stated that \( [\mu] \neq 0 \) implies that the fluid will flow through \( \Gamma \). However, we have not yet spoken about the flow of the fluid. In fact, we assume that there is no flow of fluid at any time. Fluid flowing through \( \Gamma \) simply means that \( \Gamma \) moves in the opposite direction. So if \( [\mu] > 0 \) at some \( \xi \in \Gamma \) (fluid flows out of \( \Omega^+ \)), the normal velocity \( V \) (measured in \( \text{m s}^{-1} \)) of \( \Gamma \) at \( \xi \) will be negative (with respect to the outward pointing normal vector of \( \Gamma \)). The relation between \( V \) and \( [\mu] \) is derived in [30], Section 1.3:

\[
V = -\mathcal{P}[\mu],
\]

(7.4)

where \( \mathcal{P} \) is the permeability of the membrane, measured in \( \text{m mol s}^{-1} \text{J}^{-1} \), which we assume to be a given constant.

Finally, we assume \( T \) to be constant. Then, substitution of (7.2), (7.3) and (7.4) into (7.1), yields

\[
V = 2\mathcal{P}V\gamma H + \mathcal{P}V\mathcal{R}Tn_s\left|\frac{1}{\Omega}\right|,
\]

(7.5)

in which only \( V \), \( H \) and \( |\Omega| := |\Omega^+| \) may (in general) depend on time or space. All other quantities can mathematically be considered as given constants.

In practice, there are two types of constants. They are important when one wants to apply formula (7.5) to a cell in an experiment. The first type of constants, are ones that are fixed regardless of the type of membrane or environment in which the shape of the membrane evolves. \( V \) and \( \mathcal{R} \) are of this type. \( \mathcal{P} \), \( \gamma \), \( T \) and \( n_s \) are of the other type of constants, which do depend on the membrane and the environment. However, in a specific experiment, the environment will be known, and the quantities related to the membrane can be derived or estimated beforehand. So, these four constants can be considered as the parameters of our model.

Again, mathematically, the type of a constant considered above, is not important. Therefore, it is convenient to cast (7.5) into the form

\[
V = C_1H + C_2\frac{1}{|\Omega|},
\]

(7.6)

\[
C_1 = 2\mathcal{P}V\gamma, \quad [C_1] = \text{m}^2\text{s}^{-1},
\]

\[
C_2 = \mathcal{P}V\mathcal{R}Tn_s, \quad [C_2] = \text{m}^{n+1}\text{s}^{-1}.
\]

(7.6) describes the model we are going to consider. It describes how the cell wall is going to evolve in time, given an initial configuration \( \Gamma_0 \). Figure 7.2 shows an example. We still need to prescribe which choices for \( \Gamma_0 \) are allowed. This will be done later on.

Let us first look at equilibrium configurations of (7.6). These are defined to be solutions of (7.6) for \( V = 0 \), i.e. solutions of

\[
-H = \frac{C_2}{C_1}|\Omega|.
\]

(7.7)
For a $\Gamma^*$ satisfying (7.7), we know that $|\Omega|$ is constant. Then (7.7) implies that $H$ is constant. This implies, by a non-trivial theorem which requires $\Gamma^*$ to be bounded, that $\Gamma^*$ is a sphere of radius $R$. Then we know that $H = -1/R$ and $|\Omega| = \omega_n R^n$, where

$$\omega_n := \frac{\pi^{n/2}}{\Gamma(1 + n/2)} = \begin{cases} \frac{\pi^k}{k!} & n = 2k \\ \frac{2^k \pi^{k-1}}{\prod_{i=1}^k (2i - 1)} & n = 2k - 1 \end{cases}$$ \hspace{1cm} (7.8)$$

is the well-known formula for the $n$-dimensional unit ball, in which $\Gamma(\cdot)$ denotes the Gamma function. By using these expressions in (7.7), we obtain

$$R = \left( \frac{C_2}{C_1 \omega_n} \right)^{1/n}. \hspace{1cm} (7.9)$$

We have thus shown that $\Gamma^*$ is an equilibrium solution to (7.6) if and only if it is a sphere with radius $R$. So the shape of an equilibrium solution to (7.6) is unique, but this sphere can be situated anywhere in $\mathbb{R}^n$.

If we consider other shapes of $\Gamma$ than just spheres, (7.6) becomes hard to solve. However, with a hand-waving argument, we can give some physical insight in (7.6) and discuss its global behaviour. If we would remove the volume term $|\Omega|^{-1}$ from (7.6), the resulting equation would describe the well-known mean curvature flow, for which $\Gamma$ is known to shrink to a point in finite time. Our volume term prevents $\Gamma$ from shrinking to a point, which is shown by the following reasoning for sphere-like boundaries $\Gamma$ (i.e. $H$ has a constant order of magnitude). When $d$ is the typical width of $\Omega$ (for a ball, $d$ would be the diameter), then $|\Omega| \sim d^n$ while

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$H \sim -d^{-1}$. So for $d > 0$ small enough, the right hand side of (7.6) will be positive, causing $\Omega$ to expand. For the same reason, when $|\Omega|$ is very large (i.e. $d$ very large), $H$ will dominate, causing $\Gamma$ to behave like a curvature flow, which means that $\Omega$ will shrink.

To see what happens when we relax the condition $H \sim -d^{-1}$, we consider the following example. Let $\Omega$ be a stretched ellipsoid such that $|\Omega|$ is of order 1. Then, at the sharpest points of the ellipsoid, $-H$ is very large, causing $-V$ to be very large. Around the center of the ellipsoid, $|H|$ is very small, so $V$ is of the same order as $1/|\Omega|$. Hence, the ellipsoid will shrink fast in the direction in which it is stretched, and it will expand in the perpendicular plane to this direction, but at a much slower rate. So in general, we expect that $H$ smoothens out all peaks occurring at the surface $\Gamma$, i.e. making $\Gamma$ more sphere-like.

These hand-waving arguments motivate us to show that the equilibrium shape of $\Gamma$ is stable. This means that if one perturbs this equilibrium shape of $\Gamma$ a bit, that the cell wall will evolve under (7.6) to the same equilibrium shape, although it may happen that this shape is situated elsewhere.

Besides showing stability of the equilibrium shape of the solution to (7.6), we also want to show local existence and uniqueness. This property would show that (7.6) is well-posed, in the sense that it gives a unique description about how the cell wall is going to move in time, even if it is just for a small time interval. Besides, we will need local existence and uniqueness for showing stability. The reason for not showing global existence and uniqueness, will be given later on.

### 7.2 Dimensionless formulation

The idea of making (7.6) dimensionless, is that it reveals the typical length scales and time scales of our problem. If there is more than one length scale (or more than one time scale), it gives us a dimensionless number (depending on $C_1$ and $C_2$) which value is essential to the behaviour of the dimensionless version of the model. For our model, described by (7.6), there happens to be one time scale and one length scale (we are going to show this). This implies that none of the parameters essentially influences the nature of our problem. We can even state how the evolution of the cell wall depends on a certain parameter, without explicitly knowing how the cell wall evolves.

Since

$$[V] = m \text{ s}^{-1}, \quad [H] = m^{-1}, \quad [|\Omega|] = m^n,$$

it seems reasonable to scale in time and space by quantities $L$ and $T$, which respectively have units m and s, such that $\hat{V}$, $\hat{H}$ and $|\hat{\Omega}|$, defined by

$$V =: \frac{L}{T} \hat{V}, \quad H =: \frac{1}{L} \hat{H}, \quad |\Omega| =: L^n |\hat{\Omega}|,$$

are dimensionless. Note that we are free to choose $L$ and $T$. We will do this later on.

Substitution of (7.10) into (7.6) yields

$$\hat{V} = C_1 \frac{T}{L^2} \hat{H} + C_2 \frac{T}{L^{n+1}} \frac{1}{|\hat{\Omega}|}.$$

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Let
\[ \hat{C}_1 := C_1 \frac{T}{L^2}, \quad \hat{C}_2 := C_2 \frac{T}{L^{n+1}}, \]
(7.12)
and note that these are dimensionless constants. Now, it seems a straightforward choice to choose \( L \) and \( T \) such that \( \hat{C}_1 \) and \( \hat{C}_2 \) equal 1, by which we move the dependence of our problem on the parameters to the time and length scale. However, there is a hidden constant in our model, which is the radius \( \hat{R} \) of the \((n-1)\)-dimensional sphere \( S_{n-1}(\hat{R}) \) that is the shape of the equilibrium solutions to (7.6). Observe that
\[ \hat{R} := \frac{R}{L} = \left( \frac{C_2}{C_1 \omega_n} \right)^{\frac{1}{n-1}} \frac{1}{L} \]
(7.13)
shows that the three dimensionless numbers \( \hat{C}_1, \hat{C}_2 \) and \( \hat{R} \) are coupled independently of the parameters. So, by choosing \( L \) and \( T \) properly, we can make all these three numbers independent of the parameters. First, we want \( \hat{R} = 1 \). From (7.13), we then see that we have to choose
\[ L = \left( \frac{C_2}{C_1 \omega_n} \right)^{\frac{1}{n-1}} = \left( \frac{RT n_s}{2\gamma \omega_n} \right)^{\frac{1}{n-1}}. \]
(7.14)
Then, from (7.12), we obtain \( \hat{C}_1 = 1 \) if we choose
\[ T = \frac{L^2}{C_1} = \left( \frac{RT n_s}{\omega_n} \right)^{\frac{2}{n-1}} \left( 2\gamma \right)^{\frac{n+1}{n-1}} \frac{\omega_n}{P^2 \Omega}. \]
(7.15)
Then from (7.13) it easily follows that \( \hat{C}_2 = \omega_n \).

After our choice of scaling (see (7.14) and (7.15)), we obtain from (7.11) that
\[ \hat{V} = \hat{H} + \frac{\omega_n}{|\Omega|}. \]
(7.16)
The scaling also influences our initial configuration \( \Gamma_0 \). By our scaling, \( x \in \Gamma_0 \) is mapped to \( \hat{x} = \frac{1}{L} x \). Therefore, \( \Gamma_0 \) is mapped to
\[ \hat{\Gamma}_0 = \left\{ \frac{1}{L} x \mid x \in \Gamma_0 \right\}. \]
(7.17)
(7.16) with initial condition (7.17) is the problem of interest. If one would be able to find a solution to this problem (we have not even defined what a solution of this equation is. For the moment, we consider a solution to be a family of hypersurfaces \( \{ \hat{\Gamma}(t) \in \mathbb{R}^n \mid t \in [0, T] \} \)), then our scaling (see (7.14) and (7.15)) tells us how the cell wall \( \Gamma(t) \) will evolve in time. \( \Gamma(t) \) will be \( L \) times as big as \( \hat{\Gamma}(t) \) in any direction, and it will evolve \( T \) times as slow. So if, for example, the permeability \( \mathcal{P} \) in a next experiment will be twice as big, then \( T \) will be twice as small, causing the cell wall to move twice as fast. Since \( L \) does not depend on \( \mathcal{P} \), the intermediate configurations of the cell wall will not change. So it evolves exactly in the same way, but only with a faster rate.

This example shows that the dependence of the parameters on our problem, is completely described by (7.14) and (7.15). Therefore, we only have to solve (7.16) in order to estimate

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how a cell wall will evolve for a given experiment. The word ‘estimate’ is chosen because of all the assumptions that are made to arrive at (7.6).

So, in order to solve (7.16), it is sufficient to look solely at this equation, and forget about its derivation and nondimensionalizing. There is no need to conserve the hats anymore, so from now on, we will consider

\[
\begin{align*}
V &= H + \frac{\omega_n}{|\Omega|} \quad t \in (0, T] \\
\Gamma(0) &= \Gamma_0.
\end{align*}
\]  

(7.18)

7.3 Mathematical formulation

Problem (7.18) still does not have a convenient form to work with, because it is a free boundary problem, of which its solution will be a family of hypersurfaces \( \{ \Gamma(t) \in \mathbb{R}^{n+1} \mid t \in [0, T] \} \). We can get rid of this free boundary by describing \( \Gamma(t) \) by graphs of certain functions from a fixed function space. One way to do this, is described in Section 2.2. Here, we transform our problem as given by (7.18) into this functional setting, and see whether this imposes restrictions to solutions to our problem.

Consider the setting in Section 2.2. First, we need to fix a smooth hypersurface \( \Gamma \subset \mathbb{R}^{n+1} \) (the reason that we consider \( \mathbb{R}^{n+1} \) instead of \( \mathbb{R}^n \), is that \( \Gamma \) is an \( n \)-dimensional hypersurface in this setting). Then we describe \( \Gamma(t) \) by a mapping \( \rho(t) : \Gamma \rightarrow \mathbb{R} \) as is done in (2.19). See also Figure 2.3. We have to be careful on how to choose \( \Gamma \), because the existence of such a mapping \( \rho(t) \) is not guaranteed for any \( \Gamma \). In fact, it is not trivial at all that for any given \( \Gamma_0 \), there exist a \( \Gamma \) such that there exists a \( \rho(0) : \Gamma \rightarrow \mathbb{R} \) which describes \( \Gamma_0 \) by (2.19). Of course, in the case that \( \Gamma_0 \) is smooth, we can just take \( \Gamma = \Gamma_0 \), but we want to consider a more general class of initial conditions.

To avoid this difficulty, we restrict ourselves to those \( \Gamma_0 \) for which there exists a smooth hypersurface \( \Gamma \) and a \( \rho(0) : \Gamma \rightarrow \mathbb{R} \) regular enough (we will come back to this in a moment) which describes \( \Gamma_0 \). First, we need \( \rho(0) \in C^2(\Gamma) \). This implies that the volume, the normal vector and the curvature related to \( \Gamma_0 \) are well-defined. Second, we need

\[
\max_{\xi \in \Gamma} |\rho(0)(\xi)| < \delta,
\]

where \( \delta \) depends on \( \Gamma \) and is defined by (2.6) and (2.7). This condition ensures that \( \xi \mapsto X(\xi, \rho(0)(\xi)) \) (\( X \) is defined by (2.8)) is a homeomorphism from \( \Gamma \) onto \( \Gamma_0 \). Third, for technical reasons, we fix \( \alpha \in (0, 1) \) and we want \( \rho(0) \in h^{2+\alpha}(\Gamma) \) (\( h^{2+\alpha}(\Gamma) \) denotes the little Hölder space on \( \Gamma \). See Appendix A for its definition and its properties), by which \( \rho(0) \) is required to be slightly more regular than \( C^2 \).

To summarize these restriction, we demand

\[
\rho(0) \in \mathcal{U} := \left\{ f \in h^{2+\alpha}(\Gamma) \mid \max_{\xi \in \Gamma} |f(\xi)| < \delta \right\},
\]

which describes the restriction on the choice of \( \Gamma_0 \).
Now, if the hypersurface $\Gamma(0)$ is going to evolve to $\Gamma(t)$ according to (7.18), we want to prove that there exists a $T > 0$ such that there is a unique $\rho(t) \in \mathcal{U}$ describing $\Gamma(t)$ for $t \in (0, T]$. We are not going to look at general $T > 0$ for a unique solution to (7.18) (unless $\Gamma_0$ is close to an equilibrium configuration, which we will discuss in section 7.5). This has two reasons:

- For general $\Gamma$, we can expect $\Gamma(t)$ to leave the domain $\Omega_\delta$, which is the collar manifold surrounding $\Gamma$ (see (2.6)), at a certain time $t_1$. Then $\rho(t_1) \notin \mathcal{U}$, so it might happen that we cannot describe $\Gamma(t)$ for $t > t_1$.
- We do not know if it can happen that $\Gamma(t) \subseteq \Omega_\delta$ can not be described by any $\rho \in h^{2+\alpha} (\Gamma)$ (one can think of such $\Gamma(t)$ as a hypersurface that is folded inside $\Omega_\delta$).
- See Figure 7.3. In this case, one would expect that the cell wall will touch itself.

![Figure 7.3](image)

**Figure 7.3**: A choice for $\Gamma_0$ for which one would expect that $\Gamma(t)$ is going to intersect itself. At the horizontal part of $\Gamma_0$, $H$ is zero and $1/|\Omega|$ is positive. Then (7.18) implies that $V$ is positive, causing the horizontal parts to move towards each other. The magnitude of $V$ is independent of $\varepsilon$, so if we take $\varepsilon$ small enough, $\Gamma(t)$ is going to intersect itself at a certain time $t$.

The first two reasons are restrictions caused by our functional setting and not by the problem itself. This is the price we have to pay for using this functional setting. However, the third reason shows that our problem becomes significantly harder when we would consider general $T > 0$. If one wants to show global existence and uniqueness, one has to take into account that the cell wall might touch itself, which makes the evolution equation much harder. One might be able to avoid this difficulty by restricting oneself to those $\Gamma_0$ for which $\Gamma(t)$ will not intersect itself. Then the problem is to find such a class of initial conditions. We will not discuss this problem here, and move on with defining what we exactly mean by local existence and uniqueness of a solution to (7.18).

In Sections 2.2 and 2.3, it is shown how one can write $V$, $H$ and $1/|\Omega|$ as operators acting on $\rho$ (see (2.22), Theorem 2.36 and (2.35)). By substituting these formulas into (7.18), we obtain the autonomous evolution equation

\[
\begin{cases}
\frac{\partial \rho}{\partial t}(t) = L(\rho(t))(H(\rho(t)) + V(\rho(t))) =: F(\rho(t)) & t \in (0, T] \\
\rho(0) = \rho_0.
\end{cases}
\]  

(7.19)  

where $\rho_0 \in \mathcal{U}$. By autonomous, we mean that $F$ does not depend on time (but $F(\rho(t))$ does, of course).

The formulation of our evolution equation given by (7.19) suggests that we are dealing with an autonomous fully nonlinear problem. However, our problem has a nicer structure; it is
quasi-linear. To see this, use Theorem 2.36 to rewrite (7.19) into
\[ \frac{\partial \rho}{\partial t}(t) + L(\rho(t))P(\rho(t))\rho(t) = L(\rho(t)) \left( -Q(\rho(t)) + V(\rho(t)) \right) =: \hat{F}(\rho(t)). \] (7.20)

All operators appearing in (7.20) (i.e. \(L, P, Q\) and \(V\)) only involve first order derivatives of \(\rho\). The only place where second order derivatives of \(\rho\) appear, is in \(P(\rho(t))\rho(t)\), because \(P(\rho(t))\) is a linear second order differential operator. This implies that our problem is quasi-linear.

A more technical way to show that our problem is quasi-linear, is given in the following. Let
\[ A : \Omega \to \mathcal{L}(h^{2+\alpha}(\Gamma), h^\alpha(\Gamma)), \quad A(f) := -L(f)P(f). \] (7.21)

Then the regularity of our operators (given by Theorem 2.36 and its remark, Proposition B.6 and Appendix C) imply that
\[ A \in C^\infty(\Omega_1; \mathcal{L}(h^{2+\alpha}(\Gamma), h^\alpha(\Gamma))), \quad \hat{F} \in C^\infty(\Omega_1; h^\alpha(\Gamma)), \] (7.22)

where \(\Omega_1\) is given by (2.32). From (7.22) it directly follows that our problem is quasi-linear.

For showing local existence and uniqueness, we do not need this quasi-linear structure; it suffices to regard our problem as fully non linear. We do need to specify the function space in which we seek for a unique solution to (7.19). To determine this space, we have a closer look at \(F\). We want at least \(\rho(t) \in \Omega\), such that \(\Gamma_{\rho(t)}\) describes the configuration of the cell wall. Therefore, it suffices to take \(\Omega\) as the domain of \(F\). On the other hand, \(F\) is only defined on this set, but this reasoning shows that this is not a restriction.

Since \(h^\alpha(\Gamma)\) is a Banach algebra and \(L, H, V \in C^\infty(\Omega_1; h^\alpha(\Gamma))\), we also have
\[ F \in C^\infty(\Omega_1; h^\alpha(\Gamma)). \] (7.23)

An immediate consequence of this is that \(F(\rho(t)) \in h^\alpha(\Gamma)\) for all \(t \in [0, T]\), because we already imposed \(\rho(t) \in \Omega\). Then by (7.19), we also need
\[ \frac{\partial \rho}{\partial t}(t) \in h^\alpha(\Gamma). \]

This brings us to the next definition of a solution to (7.19):

**Definition 7.1.** (Strict solution). \(\rho\) is a **strict solution** to (7.19) if
\[ \rho \in C^1([0, T]; h^\alpha(\Gamma)) \cap C([0, T]; \Omega) \]
and \(\rho\) satisfies (7.19).

It is possible to make our setting slightly more general. The next remark makes this precise.

**Remark 7.2.** Instead of considering \(h^{2+\alpha}(\Gamma)\) as the underlying space, we also could have considered \(h^{2+k+\alpha}(\Gamma)\) for general \(k \in \mathbb{N}\). This would ensure more regularity of \(\rho(t)\). The only thing we have to do to gain more regularity, is to choose \(\rho_0 \in h^{2+k+\alpha}(\Gamma)\). Even in the following, one may as well consider \(h^{2+k+\alpha}(\Gamma)\).

Now, our mathematical formulation is complete. In the next section, we show that this formulation is well-posed in the sense that there exists a unique strict solution to (7.19) for some \(T > 0\) and ‘suitable’ chosen \(\rho_0\).
7.4 Local existence and uniqueness

In this section, we prove that there is a $T > 0$ such that there is precisely one strict solution to (7.19). A rigorous version of this statement is given by Theorem 7.3, which is the first of the two main theorems of this thesis. For its proof, we use the theory of Chapter 5. The main part of the proof consists of showing that our problem fits into that framework.

**Theorem 7.3.** Let $\Gamma \subset \mathbb{R}^{n+1}$ be a bounded, closed, smooth hypersurface, and let $\alpha \in (0, 1)$ be fixed. Then there exists a $T > 0$ and a non-empty $\Theta \subset h^{2+\alpha}(\Gamma)$ open such that for all $\rho_0 \in \Theta$, (7.19) has a unique strict solution.

See Definition 2.21 for the definition of a closed, smooth hypersurface. Strict solutions are defined in Definition 7.1.

**Proof.** The proof is divided in several parts. If any of these parts consists of several subparts, we will introduce these subparts at the beginning of the concerning part. For now, we start with introducing the main two parts.

Part (I) is the biggest part of the proof. It is about showing how we can apply Theorem 5.7 and why its conditions are satisfied. Part (II) is about showing how the existence and uniqueness result from Theorem 5.7 implies Theorem 7.3.

(I) In this part we show that we can apply Theorem 5.7 to our case. Table 7.1 shows how we have to choose the objects appearing in Theorem 5.7, in our case.

<table>
<thead>
<tr>
<th>Objects from Theorem 5.7</th>
<th>Objects from Theorem 7.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_0$</td>
<td>$h^\alpha(\Gamma)$</td>
</tr>
<tr>
<td>$E_1$</td>
<td>$h^{2+\alpha}(\Gamma)$</td>
</tr>
<tr>
<td>$f$</td>
<td>$F$</td>
</tr>
<tr>
<td>$x, x_0$</td>
<td>$\rho, \rho_0$</td>
</tr>
</tbody>
</table>

Table 7.1: The use of Theorem 5.7 in our setting. The left column refers to the objects in the general setting of Chapter 5. The right column refers to the related objects from our setting.

However, to apply Theorem 5.7 to our setting, we have to show that all needed conditions on the objects in the right column of Table 7.1 are satisfied. We do this in subparts (a), (b) and (c). In (a), we show that

$$(h^{2+\alpha}(\Gamma), h^\alpha(\Gamma)) = h^{2+\alpha}_\alpha$$

is a Banach couple. For subparts (b) and (c), we need to show that there exists a non-empty $\Theta \subset h^{2+\alpha}(\Gamma)$ open such that:

(b) $F \in C^1(\Theta, h^\alpha(\Gamma))$, \hspace{1cm} (7.24)

c) $\forall \rho \in \Theta : F'(\rho) \in M_1(h^{2+\alpha}_\alpha)$, \hspace{1cm} (7.25)

Subparts (a) and (b) are very short, while (c) will be the main challenge. (a), (b) and (c) together will imply that all conditions for applying Theorem 5.7 in our setting are satisfied. This will conclude part (I).

(a) $h^{2+\alpha}(\Gamma)$ being dense in $h^\alpha(\Gamma)$ follows from Theorem A.7. $h^{2+\alpha}(\Gamma) \hookrightarrow h^\alpha(\Gamma)$ follows directly from the definition of their norms (see (A.3)).
 expansion, we obtain

First, it follows from (I.a) that

where

Second, we set

so

and the real number

are in our setting. We have to show that for our choices, the conditions of Theorem 5.6 are satisfied. This will be the main part of this proof.

(i) In the following, we can fix any

but for convenience, we take

Then for

we set

appearing in the setting of Theorem 5.6, equal to

and the conditions

are satisfied. We have to show that for our choices, the conditions of Theorem 5.6 are satisfied. This will be the main part of this proof.

First, it follows from (I.a) that

is a Banach couple.

Second, we set

and we have to show that

To see this, we start with calculating

In view of (7.19), we are going to look at the linearization of

around 0 separately. Let

such that

is small. By using a Taylor expansion, we obtain

which implies that

(see Remark B.2 for the definition of

in this context).

We split

into

and

. The Fréchet derivative of

is calculated in the proof of [10], Lemma 3.1. This calculation is given by

so

is the Fréchet derivative of

. From (2.31) we easily obtain

(7.26)

* : see Remark 2.34.

is derived in [7], Lemma 5.1. There, the authors use another approach to linearize

based on the same ideas discussed in Section 2.3. The result is that

where

are the principal curvatures of

(see Definition 2.29). For clarity, we note that

, and that the linear operator

just multiplies its argument with

. In Appendix C, it is shown that

By gathering the previous results, we obtain for the Fréchet derivative of

around 0:

(7.29)
So, \( A = F'(0) \) consists of a sum of three operators. The first one, \(-P(0) = \Delta_t/n\), is known to be an element of \( \mathcal{H}(h_0^{2+\beta}) \) for any \( \beta \in (0, 1) \). Unfortunately, this seems to be a folk Theorem which everyone in this field of research knows, but which is (for as far as we know) nowhere explicitly stated. We give the interested reader three options. First, \([9]\) is a very technical paper of which one of the many possible corollaries would be \( \Delta_t \in \mathcal{H}(h_0^{2+\beta}) \). Second, [3], Corollary 5.5.5 gives a vast subclass of \( \mathcal{H}(h_0^{2+\beta}) \) of which \( \Delta_t \) is an element. This is a corollary of Theorem 5.4.2, and one needs to know the definition of \( \mathcal{E}_t \), which is given on page 56, and that Amann denotes \( L_p(\mathbb{R}^n) \) just by \( L_p \). Third, in [18], Theorem 3.1.14, it is stated that those elements from \( \mathcal{L}(C^{2+\beta}(\mathbb{R}^n), C^\beta(\mathbb{R}^n)) \) that are uniformly elliptic, are elements of \( \mathcal{H}(C^{2+\beta}(\mathbb{R}^n), C^\beta(\mathbb{R}^n)) \). \( \Delta \) is such an operator. To conclude from this that \( \Delta_t \in \mathcal{H}(h_0^{2+\beta}) \), one needs to use several techniques that are well-known, but which we will not discuss here.

The other two operators of which \( A \) is composed, are bounded operators on \( h^\beta(\Gamma) \). To see this, let \( h \in h^\beta(\Gamma) \) arbitrary, and denote \( \| \cdot \|_{h^\beta(\Gamma)} \) by \( \| \cdot \|_\beta \). Then

\[
\| Q'(0) h \|_\beta = \frac{1}{n} \| \Sigma h \|_\beta \leq \| \Sigma \|_\beta \| h \|_\beta,
\]

\[
\left\| \int_\Gamma h \right\|_\beta \leq \int_\Gamma \| h \|_\beta = |\Gamma| \| h \|_\beta.
\]

By Theorem 4.18, we then have that \( A \in \mathcal{H}(h_0^{2+\beta}) \), which completes the second part of (I.c.i).

Third, since \( \Delta_t \) is an elliptic operator (see Remark 2.34), it follows from regularity of elliptic operators (see for example [12], Lemma 6.16) that

\[
h^{2+\beta}(\Gamma) = \{ \rho \in h^\beta(\Gamma) \mid \Delta_t \rho \in h^\beta(\Gamma) \}.
\]

We have already seen that the other two operators appearing in the sum of (7.29) are bounded, so (7.30) also holds if we replace \( \Delta_t \) by \( A \). Together with Remark 7.2, we see that the condition on \( F_2 \) (see (5.8)) is satisfied for \( F_2 = h^{4+\beta}(\Gamma) \).

Fourth, we need to find \( \sigma \in (0, 1) \) such that

\[
h^{2+\sigma}(\Gamma) = \left( h^\beta(\Gamma), h^{2+\beta}(\Gamma) \right)_\sigma,
\]

\[
h^{4+\sigma}(\Gamma) = \left( h^{2+\beta}(\Gamma), h^{4+\beta}(\Gamma) \right)_\sigma.
\]

For \( \sigma = \alpha/4 \), one easily sees that this follows from Corollary 3.11 (note that we have chosen \( \beta = \alpha/2 \)).

Now we have shown that all conditions of Theorem 5.6 are satisfied in our setting. The only result that we need from this theorem, is that \( F'(0) \in \mathcal{M}_1(h_1^{2+\alpha}) \). This completes subpart (i).

(ii) Since \( \mathcal{M}_1(h_1^{2+\alpha}) \) is open in \( \mathcal{L}(h_1^{2+\alpha}) \) (see Proposition 5.4), we know that

\[
\exists \delta_1 > 0 : B_{\mathcal{L}(h_1^{2+\alpha})}(A_1, \delta_1) \subset \mathcal{M}_1(h_1^{2+\alpha}),
\]

where \( B_{\mathcal{L}(h_1^{2+\alpha})}(A_1, \delta_1) \) denotes the ball of radius \( \delta_1 \) with center \( A \) in the space \( \mathcal{L}(h_1^{2+\alpha}) \). On the other hand, from (I.b) we have that \( F' \in C(\Omega, \mathcal{L}(h_1^{2+\alpha})) \), which gives us a \( \delta_2 > 0 \) such that

\[
\forall \rho \in B_{\mathcal{L}(h_1^{2+\alpha})}(0, \delta_2) : \| F'(\rho) - F'(0) \|_{\mathcal{L}(h_1^{2+\alpha})} < \delta_1.
\]
Hence, for any $\rho \in B_{h^{2+\alpha}}(0, \delta_2)$, we know that $F'(\rho) \in B_{\mathcal{L}(h^{2+\alpha})}(A, \delta_1) \subset \mathcal{M}_1(h^{2+\alpha})$. So if we choose

$$\mathcal{O} = B_{h^{2+\alpha}}(0, \delta_2) \subset \mathcal{U},$$

we know that (7.25) is satisfied. This completes subpart (I.c), and also part (I).

(II) Part (I) allows us to use Theorem 5.7, which states that there exists a $T > 0$ such that for any $\rho_0 \in \mathcal{O}$ (given by (7.31)) there is exactly one element

$$\rho \in C([0, T]; h^{\alpha}(\Gamma)) \cap C([0, T]; \mathcal{O})$$

which satisfies (7.19). The difference with Theorem 7.3 is that that theorem requires the uniqueness of a strict solution $\rho$ (see Definition 7.1), which is a slightly larger class than the one given in (7.32). Since this class is slightly larger, we immediately have existence of a strict solution. For uniqueness, assume that there is another strict solution $\hat{\rho}$ satisfying (7.19) on $[0, T]$. Since $\hat{\rho}(0) = \rho_0 \in \mathcal{O}$, $\mathcal{O}$ open in $h^{2+\alpha}(\Gamma)$ and $\hat{\rho} \in C([0, T]; \mathcal{O})$, it holds that

$$\exists t_1 \in (0, t_1] \forall t \in [0, t_1] : \hat{\rho}(t) \in \mathcal{O},$$

which means that $\hat{\rho}|_{(0, t_1]} \in C([0, t_1]; \mathcal{O})$, so that we have by Theorem 5.7 that $\hat{\rho}|_{(0, t_1]} = \rho|_{(0, t_1]}$. Now we are going to extend this interval $[0, t_1]$ to $[0, t_2]$, where

$$t_2 := \sup \{ t \in [0, T] \mid \hat{\rho}(t) = \rho(t) \}.$$ (7.33)

Since $\hat{\rho}, \rho \in C([0, T]; h^{2+\alpha}(\Gamma))$, the limit $t \to t_2$ in $h^{2+\alpha}(\Gamma)$ of $\hat{\rho}(t)$ and $\rho(t)$ exists. By (7.33), we then have $\hat{\rho}(t_2) = \rho(t_2) \in \mathcal{O}$. If $t_2 = T$, we would be done. If $t_2 < T$, we can apply the same reasoning as above to conclude that there exists $t_3 > t_2$ such that $\hat{\rho}|_{[t_2, t_3]} = \rho|_{[t_2, t_3]}$. But then

$$t_3 \leq \sup \{ t \in [0, T] \mid \hat{\rho}(t) = \rho(t) \} = t_2,$$

which contradicts with $t_3 > t_2$. So $t_2 < T$ can not happen, which implies $\hat{\rho}|_{[0, T]} = \rho|_{[0, T]}$. Therefore, we also have uniqueness of strict solutions to (7.19). This completes part (II) (i.e. the final step) of the proof.

\[\Box\]

### 7.5 Stability

In this section, we prove the main result of this thesis. We have already shown that the only equilibrium solution to (7.18) is a sphere with radius 1. What we show in this section, formally speaking, is that if we perturb this sphere a little bit in any direction, that $\Gamma_{\rho(t)}$ will evolve to a sphere with radius 1, which might be located elsewhere. We also give a lower bound about how fast $\Gamma_{\rho(t)}$ converges to this sphere. The rigorous version of this statement is given by Theorem 7.4. To understand this theorem, we first need to introduce some important related objects.

Theorem 6.3 will be the key for proving Theorem 7.4. In order to use this theorem, we have to show that our setting fits into the corresponding framework of Theorem 6.3. Showing this
consists of two parts: relating our setting to the framework of Theorem 6.3, and proving that all the conditions of Theorem 6.3 are satisfied in our setting. For stating Theorem 7.4, we need to do the first part first. The latter part will be the proof of this theorem.

We start by fixing our reference hypersurface \( \Gamma \). We know that the equilibrium shape of (7.18) is the sphere of radius 1. Given such an equilibrium configuration, we choose our coordinate system such that \( S_n \subset \mathbb{R}^{n+1} \) (the unit sphere in \( \mathbb{R}^{n+1} \)) describes this equilibrium of (7.18). \( S_n \) is a bounded, closed, smooth hypersurface, so we can take it as our reference hypersurface \( \Gamma \). Its tubular neighbourhood can be chosen to be \( \Omega_1 \) (see (2.6) for \( \delta = 1 \)).

In the following, we consider the functional formulation of (7.18), given by (7.19) with \( T = \infty \) and \( \Gamma = S_n \). With (7.20) and \( A \) as given by (7.21), this functional formulation is given by

\[
\begin{align*}
\frac{\partial \rho}{\partial t}(t) - A(\rho(t))\rho(t) &= \hat{F}(\rho(t)) \quad t > 0 \\
\rho(0) &= \rho_0.
\end{align*}
\]

(7.34)

The set of all \( \rho_0 : S_n \rightarrow \mathbb{R} \) that we are going to consider, will be specified in Theorem 7.4. It gives an upper bound on how much we can perturb the unit sphere such that we know that the resulting hypersurface is going to evolve to a sphere of radius 1.

Just like in the proof of Theorem 7.3, we give a table like Table 7.1 to show the reader how our setting fits into the framework of Theorem 6.3. We have not yet defined all objects in the right column of Table 7.2. We will do so in the following paragraphs.

<table>
<thead>
<tr>
<th>Objects from Theorem 6.3</th>
<th>Objects from Theorem 7.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_0 )</td>
<td>( L_p(S_n) )</td>
</tr>
<tr>
<td>( X_1 )</td>
<td>( H^2_p(S_n) )</td>
</tr>
<tr>
<td>( F )</td>
<td>( \hat{F} )</td>
</tr>
<tr>
<td>( u )</td>
<td>( \rho )</td>
</tr>
<tr>
<td>( U )</td>
<td>( B_{\mathbb{R}^{n+1}}(0, 1) )</td>
</tr>
<tr>
<td>( u_\ast )</td>
<td>0</td>
</tr>
<tr>
<td>( A_0 )</td>
<td>( F'(0) )</td>
</tr>
</tbody>
</table>

Table 7.2: The use of Theorem 6.3 in our setting. The left column refers to the objects in the general setting of Chapter 6.

The reason that we do not take the little Hölder spaces as \( X_0 \) and \( X_1 \), is that we do not know whether the condition of Theorem 6.3 about maximal \( L_p \)-regularity holds for these spaces. \( H^2_p \) denote the (classical) Sobolev spaces of order 2. Similar to Definition A.6, we can define \( L_p(S_n) \) and \( H^2_p(S_n) \).

Let

\[
X_\sigma := (L_p(S_n), H^2_p(S_n))_{1-p,p},
\]

(7.35)

\[
V := \left\{ \rho \in X_\sigma \left| \frac{\text{ess sup}}{\xi \in S_n} |\rho(\xi)| < 1 \right. \right\}.
\]

(7.36)

We continue by examining how \( \mathcal{E} \) as defined by (6.4) looks like. In our case, \( \mathcal{E} \) contains those \( \rho \in V \) for which \( \Gamma_\rho \) is a sphere of radius 1. One easily sees that the center \( c \) of the sphere
$\Gamma_\rho$ must satisfy $|c| < 1$ ($|\cdot|$ denotes the 2-norm), because otherwise $\rho \not\in V$. For $|c| < 1$, it is stated in [10], a part of the proof of Proposition 6.4, that $\varphi(c)$ defined by

$$
\varphi \in C^\infty(B_{\mathbb{R}^{n+1}}(0, 1); C^\infty(S_n)),
\varphi(c) = (\xi \mapsto \xi \cdot c - 1 + \sqrt{1 + (\xi \cdot c)^2 - c \cdot c}),
$$

(7.37)
equals this $\rho$ for any $c \in B_{\mathbb{R}^{n+1}}(0, 1)$. There, it is also stated that

$$
\varphi'(0)h = (\xi \mapsto \xi \cdot h), \quad h \in \mathbb{R}^{n+1}.
$$

(7.38)

Summarizing the previous statements yield

$$
E = \varphi(B_{\mathbb{R}^{n+1}}(0, 1)).
$$

(7.39)

In view of Chapter 6, taking $U = B_{\mathbb{R}^{n+1}}(0, 1)$ and $u_* = 0 \in h^{2+\alpha}(S_n)$, we see that (7.37), (7.38) and (7.39) imply that all three conditions of (6.5) are satisfied. This implies that there is an $M \subset E$ with $0 \in M$ such that $M$ is a $C^1$-manifold. Although we do not need it, we would like to remark that (7.37) implies that $M = E$ is a smooth manifold.

From (6.6) we see that

$$
T_0M = \{\varphi'(0)h \mid h \in \mathbb{R}^{n+1}\}.
$$

(7.40)

Fix $h \in \mathbb{R}^{n+1}$, and let

$$
p_h : \mathbb{R}^{n+1} \rightarrow \mathbb{R}, \quad p_h(x) = \sum_{i=1}^{n+1} h_i x_i.
$$

From Definition D.1 one easily sees that $p_h|_{S_n} \in H^{n+1}_1$. On the other hand, (7.38) implies $p_h|_{S_n} = \varphi'(0)h$. This implies

$$
T_0M \subset H^{n+1}_1.
$$

From their definitions, it follows that $T_0M$ and $H^{n+1}_1$ are finite dimensional vector spaces. Then by Proposition D.3, we have

$$
\dim H^{n+1}_1 = n + 1 = \dim T_0M,
$$

and therefore

$$
T_0M = H^{n+1}_1.
$$

(7.41)

There is still one object left in Chapter 6 which we have not yet described in terms of our problem. This is $A_0$, the linearization of $A(u)u + F(u)$ around $u_*$. In our case, $A_0$ is given by the linearization of $F(\rho)$ (see (7.19)) around zero. Let $h \in H^2_\rho(S_n)$, then

$$
A_0h = F'(0)h = \frac{1}{n} \Delta_{S_n}h + h - \frac{1}{\omega_{n+1}} \int_{S_n} h,
$$

(7.42)

* see (7.29), in which we have taken $\Gamma = S_n$. On $S_n$, it is well-known that all principal curvatures $\kappa_i$ (see Definition 2.29) equal 1. Therefore, we see from (7.27) that $\Sigma = n$. Furthermore, by definition, $|\Omega| = \omega_{n+1}$, which explains the third term in the right hand side.
Now we are ready to state and prove the following theorem:

**Theorem 7.4.** Let $n \in \mathbb{N}_+$. For any $p \in (n + 2, \infty)$, there exists a $\delta > 0$ such that for all $\rho_0 \in B_{X_\sigma}(0, \delta)$, (7.34) has a unique solution

$$\rho \in H^1_p((0, \infty); L_p(S_\delta)) \cap L_p((0, \infty); H^2_p(S_\delta)).$$

Moreover,

$$\exists \rho_\infty \exists C, \omega > 0 : \|\rho(t) - \rho_\infty\|_{X_\sigma} < Ce^{-\omega t}.$$

$X_\sigma$ is defined in (7.35), and by $B_{X_\sigma}(0, \delta)$ we denote the ball in $X_\sigma$ with center 0 and radius $\delta$.

**Proof.** The proof is rather long, so just like the proof of Theorem 7.3, we divide it in several parts and subparts. We start with the global outline of the proof.

Theorem 6.3 states exactly what we need to prove, so we only have to show that all its conditions are satisfied in our setting. Each of the parts (I) - (III) states that one of these conditions is satisfied. To be more precise:

(I) our problem has a quasi-linear structure, i.e. (6.3) is satisfied,

(II) 0 $\in \mathcal{E}$ is normally stable,

(III) $A(0)$ has maximal $L_p$-regularity.

(I) We need to show that

$$A \in C^1(V; \mathcal{L}(H^2_p(S_\delta), L_p(S_\delta))), \quad \hat{F} \in C^1(V; L_p(S_\delta)).$$

(7.43)

Since $p \in (n + 2, \infty)$, we can use Theorem 3.12 (see also Remark 3.13) to conclude that

$$X_\sigma \hookrightarrow C^{2-n + 2 \frac{2}{p}}(S_\delta).$$

(7.44)

By a similar reasoning that we used to show (7.22), we can show that

$$\hat{F} \in C^\infty\left(\left\{ f \in C^{2-n + 2 \frac{2}{p}}(S_\delta) \mid \max_{\xi \in S_\delta} |f(\xi)| < 1 \right\} ; C^{1-n + 2 \frac{2}{p}}(S_\delta)\right),$$

which by (7.44) and (7.36) clearly implies the second statement of (7.43).

For showing the first statement of (7.43), observe that for $g \in X_\sigma$ we have that

$$A(g) \overset{(7.21)}{=} -L(g)P(g) \overset{\ast}{=} f_{ij}(g, \partial_k g)\partial_i \partial_j + f_i(g, \partial_k g)\partial_i,$$

(7.45)

$\ast$: see (2.29) and (2.31). By $\partial_k g$ in one of the arguments of $f_{ij}$ and $f_i$, we mean that these operators depend on all the first order derivatives of $g$.

A similar proof of Proposition B.6 can be used to show that the operators $f_{ij}$ and $f_i$ depend smoothly upon $g \in V$ and all its first order derivatives. We omit the details. Then it is easy to see that the first part of (7.43) follows from (7.45). This completes part (I).
(II) We show that $0 \in \mathcal{E}$ is normally stable directly from its definition (see Definition 6.2). Therefore, we need to show that all four conditions (i) - (iv) from Definition 6.2 are satisfied. We do this in the related four subparts (i) - (iv).

(i) While explaining the setting of Theorem 7.4, we have already shown (i).

Before showing that subparts (ii) - (iv) are satisfied too, we first derive some properties of $A_0$. For us, its most important property is the special form of its spectrum, which we are able to give explicitly.

Since $p > n + 2$, we have that $H^2_p(S_n), L_p(S_n) \subset L_2(S_n)$, so the inner product on $L_2^2(S_n)$, given by

$$(f, g) := \int_{S_n} f \overline{g}, \quad f, g \in L_2(S_n),$$

is well-defined on $H^2_p(S_n)$ and $L_p(S_n)$. With respect to this inner product, we will show that $A_0$ is symmetric, i.e. for all $f, g \in H^2_p(S_n)$ it holds that

$$(A_0 f, g) = (f, A_0 g). \quad (7.46)$$

(7.42) states that

$$A_0 = \frac{1}{n} \Delta_{S_n} + I - \left( f \mapsto \frac{1}{\omega_{n+1}} \int_{S_n} f \right). \quad (7.47)$$

where we regard $1/\omega_{n+1}$ as an element from $L_p(S_n)$. It is easy to see that the last two operators of the sum are symmetric. So in order to show (7.46), we only need to show that $\Delta_{S_n}$ is symmetric. Let $f, g \in H^2_p(S_n)$, then

$$\begin{align*}
(\Delta_{S_n} f, g) &= \int_{S_n} \Delta_{S_n} f \overline{g} \\
&\overset{*1}{=} \int_{S_n} f \Delta_{S_n} \overline{g} \\
&\overset{*2}{=} \int_{S_n} f \Delta_{S_n} g \\
&= (f, \Delta_{S_n} g),
\end{align*}$$

*$1$: apply (2.4) twice,
*$2$: $\Delta_{S_n}$ is a real operator; this is easily seen from Remark 2.34,

from which (7.46) follows.

We continue by showing that

$$\left\{ 1 - \frac{k}{n} (k + n - 1) \mid k \in \mathbb{N}_+ \right\} \cup \{-n\} \subset \sigma(A_0). \quad (7.48)$$

Proposition D.3 will be of great help to us. Part (iii) and (iv) of this proposition imply that the first and the third operator in the sum in (7.47) have the spherical harmonics as their eigenvectors. The eigenvalues of the latter operator corresponding to these eigenvectors are easily calculated; for $k \geq 1$, Proposition D.3.(iv) implies for any $f \in \mathcal{H}_k^{n+1}$ that

$$\frac{1}{\omega_{n+1}} \int_{S_n} f = 0,$$
which implies that the spherical harmonics of order greater equal 1 have eigenvalue 0. For \( f \in \mathcal{H}_n^{n+1} \) (i.e. \( f \) is constant), we have

\[
\frac{1}{\omega_{n+1}} \int_{S_n} f = f \left|_{S_n} \right| = (n+1)f.
\]

Combining these results with Proposition D.3.(iii), we obtain from (7.47) that for all \( k \in \mathbb{N} \), we have for \( f \in \mathcal{H}_k^{n+1} \) that

\[
A_0 f = -\frac{k}{n}(k+n-1)f + f - (n+1)\chi_{\{k=0\}}f =: \lambda_k f.
\]

which implies that (7.48) holds.

Actually, it holds that both sets in (7.48) are equal. We show this in two steps. First, we show that

\[
\{\lambda_k | k \in \mathbb{N} \} = \sigma_p(A_0), \tag{7.50}
\]

where \( \lambda_k \) is defined in (7.49), and \( \sigma_p(A_0) \) denotes the point spectrum of \( A_0 \). We reason by contradiction. Assume there exists an eigenvector \( u \) of \( A_0 \) with eigenvalue

\[
\lambda \notin \{\lambda_k | k \in \mathbb{N} \}. \tag{7.51}
\]

Let \( k \in \mathbb{N} \) and \( f \in \mathcal{H}_k^{n+1} \) arbitrary. Then

\[
\lambda(u, f) = (u, A_0 f) \overset{\#_1}{=} (u, A_0 f) \overset{\#_2}{=} \lambda_k (u, f), \tag{7.52}
\]

\*1: \( A_0 \) is symmetric,
\*2: \( \lambda_k \in \mathbb{R} \), so \( \lambda_k = \lambda_k \).

From (7.52) we have that either \( \lambda = \lambda_k \), or \( u \perp f \). The first cannot happen, because we have assumed that (7.51) holds. Therefore, we must have \( u \perp f \). Since \( k \) and \( f \) were arbitrary, we therefore must have

\[
u \in \left( \bigoplus_{k=0}^{\infty} \mathcal{H}_k^{n+1} \right) \perp. \tag{7.53}
\]

Together with Proposition D.3.(i), (7.53) implies that \( u = 0 \), which contradicts with \( u \) being an eigenvector. Therefore there are no other eigenvalues of \( A_0 \) than the \( \lambda_k \), so (7.50) holds.

Furthermore, we know what the eigenspace \( E_{\lambda_k} \) coupled to \( \lambda_k \) is:

\[
E_{\lambda_k} = \bigcup_{\ell \in \{\ell \in \mathbb{N} | \lambda_\ell = \lambda_k \}} \mathcal{H}_\ell^{n+1}. \tag{7.54}
\]

We are now going to show

\[
\sigma_p(A_0) = \sigma(A_0), \tag{7.55}
\]

for which we mimic [29], the proof of Lemma 2.17. It follows from coercivity estimates and standard \( L_p \)-elliptic regularity results that \( \rho(A_0) \neq \emptyset \). Then we also have that \( A_0 \) is closed.

Let \( \lambda \in \rho(A_0) \), then

\[
R(\lambda : A_0) \in \mathcal{L}(L_p(S_n), H^2_p(S_n)) \overset{\#}{\subset} K(L_p(S_n), L_p(S_n)), \tag{7.56}
\]

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\( H^2_p(S^n) \) is compactly embedded in \( L_p(S^n) \). The space on the right hand side contains all linear bounded operators that are compact.

(7.56) and \( A_0 \) being closed, imply that we can use Theorem 6.29 from Chapter 3 of [13] to show that (7.55) holds. Together with (7.50), this implies

\[
\{ \lambda_k \mid k \in \mathbb{N} \} = \sigma(A_0),
\]

\[
\lambda_k = -\frac{k}{n}(k + n - 1) + 1 - (n + 1)\chi(k=0).
\] (7.57)

Let us have a closer look at \( \sigma(A_0) \). Table 7.3 gives the first few eigenvalues \( \lambda_k \) (as defined in (7.49)) for different values of \( n \).

<table>
<thead>
<tr>
<th>( n ) ( \backslash k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-3</td>
<td>-8</td>
<td>-15</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>0</td>
<td>-2</td>
<td>-5</td>
<td>-9</td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
<td>0</td>
<td>-5/3</td>
<td>-4</td>
<td>-7</td>
</tr>
</tbody>
</table>

Table 7.3: Eigenvalues \( \lambda_k \) of \( A_0 \) for different values of \( n \).

With these results, it is easy to do parts (ii) and (iv). We show part (iii) with the help of part (ii).

(iv) From (7.57) it is easy to see that for all \( n \in \mathbb{N}_+ \)

\[ \sigma(A_0) \subset (-\infty, 0]. \]

(ii) From (7.57) it is also easy to see that \( \lambda_k = 0 \) if and only if \( k = 1 \). Then it follows from (7.54) that \( \mathcal{H}^{n+1}_1 \) is the eigenspace coupled to the eigenvalue 0. Hence

\[ \mathcal{N}(A_0) = \mathcal{H}^{n+1}_1. \]

Together with (7.41) this concludes part (ii).

(iii) Let \( \tilde{A}_0 \) be the extension of \( A_0 \) (i.e. \( \tilde{A}_0 \) is given by the expression in (7.42)) to \( H^2_p(S^n) \). We show first that

\[ \mathcal{R}(\tilde{A}_0) \oplus \mathcal{N}(\tilde{A}_0) = L_2(S^n). \] (7.58)

Analogously to the derivation of (7.57), we can derive that \( \tilde{A}_0 \) has the same spectrum as \( A_0 \), where the eigenspaces are again given by the spherical harmonics. A direct consequence is that \( \mathcal{N}(\tilde{A}_0) = \mathcal{N}(A_0) \).

Let \( P \) denote the orthogonal projection of \( L_2(S^n) \) onto \( \mathcal{H}^{n+1}_1 \). Then \( P(L_2(S^n)) \) and \( (I - P)(L_2(S^n)) \) are closed, and

\[ P(L_2(S^n)) \oplus (I - P)(L_2(S^n)) = L_2(S^n). \] (7.59)

Together with part (ii), it follows that \( \mathcal{N}(\tilde{A}_0) = P(L_2(S^n)) \). Therefore,

\[ \mathcal{R}(\tilde{A}_0) \subset (I - P)(L_2(S^n)) = \bigoplus_{k=0}^{\infty} \mathcal{H}^{n+1}_k, \]
Since \( \phi \) for showing that \( f \) see Remark D.4.

With these results, it suffices to show

\[
\mathcal{R}(\tilde{A}_0) \supset \bigoplus_{k=0}^{\infty} \mathcal{H}_k^{n+1}
\]

in order to prove (7.58). Let

\[
f \in \bigoplus_{k=0}^{\infty} \mathcal{H}_k^{n+1}, \quad f = \sum_{k=0}^{\infty} \sum_{\ell=1}^{N(k)} f_{k\ell} \phi_{k\ell},
\]

\( \ast: \) see Remark D.4.

For showing that \( f \in \mathcal{R}(\tilde{A}_0) \), it suffices to show that there is a \( u \in H^2_2(S_n) \) such that \( \tilde{A}_0 u = f \).

Since \( \phi_{k\ell} \) are the eigenvectors of \( \tilde{A}_0 \) with eigenvalue \( \lambda_k \), an obvious choice for \( u \) is

\[
u := \sum_{k=0}^{\infty} \sum_{\ell=1}^{N(k)} \frac{f_{k\ell}}{\lambda_k} \phi_{k\ell}.
\] (7.60)

Observe that

\[
\|u\|^2_{H^2_2(S_n)} \overset{(D.4)}{=} \sum_{k=0}^{\infty} \sum_{\ell=1}^{N(k)} \left(\frac{k^4 + 1}{\lambda_k} \right)^2 f_{k\ell}^2
\]

\[
\overset{(7.57)}{=} \sum_{k=0}^{\infty} \sum_{\ell=1}^{N(k)} \frac{(k^4 + 1)}{(k(k + n - 1)/n + 1 - (n + 1)\chi(k=0))^2} |f_{k\ell}|^2
\]

\[
\leq \sum_{k=0}^{\infty} \sum_{\ell=1}^{N(k)} C |f_{k\ell}|^2
\]

\[
\overset{(D.3)}{=} C \|f\|^2_{L_2(S_n)}.
\]

\( \ast: \) the rational polynomial with variable \( k \) converges to \( n^2 \) as \( k \to \infty \). Since \( \lambda_k = 0 \) if and only if \( k = 1 \), the denominator will not vanish for any other \( k \in \mathbb{N} \). Hence, the rational polynomial is bounded on \( \mathbb{N} \setminus \{1\} \) by a constant \( C \).

Therefore, \( u \in H^2_2(S_n) \). Then it easily follows from (7.60) and \( \tilde{A}_0 \in \mathcal{L}(H^2_2(S_n), L_2(S_n)) \) that \( \tilde{A}_0 u = f \).

This was the final step for showing (7.58). Let \( f \in L_p(S_n) \subset L_2(S_n) \). Then by (7.58) we can write \( f = f_R + f_N \) with \( f_R \in \mathcal{R}(\tilde{A}_0) \subset L_2(S_n) \) and \( f_N \in \mathcal{N}(\tilde{A}_0) \subset C^\infty(S_n) \). Therefore, \( f_R \in L_p(S_n) \) and there exists a \( g \in H^2_2(S_n) \) such that \( \tilde{A}_0 g = f_R \). By regularity of elliptic operators (see for example [12], Lemma 6.16), we then have that \( g \in H^2_2(S_n) \). Hence, \( \mathcal{R}(\tilde{A}_0) \cap L_p(S_n) \subset \mathcal{R}(A_0) \), from which it easily follows that

\[
\mathcal{R}(A_0) \oplus \mathcal{N}(A_0) = L_p(S_n),
\] (7.61)
which completes the proof of (II.iii). This was the final step of part (II), i.e. we have shown that $0 \in \mathcal{E}$ is a normally stable.

(III) Note that

\[ A(0) \stackrel{[7.21]}{=} -L(0)P(0) = \frac{1}{n} \Delta S_n, \]

*; use $L(0) = 1$ and (7.26),

so it is sufficient to show that $\Delta S_n \in \mathcal{L}(H^2(S_n), L^p(S_n))$ has maximal $L^p$-regularity. It is well-known that this is true, but its proof is rather technical. It is beyond the scope of this thesis to be very precise about this. We give the interested reader two different guidelines for proving maximal $L^p$-regularity of $\Delta S_n$.

Theorem 4.10.7 in [1] gives sufficient conditions for a class of operators to have maximal $L^p$-regularity. These conditions are (translated to our setting):

- $L^p(S_n)$ is a UMD space. This follows from [1], Theorem 4.5.2.(vi).

- Condition (4.10.1) in [1]. For this condition, we need $\Delta S_n \in \mathcal{H}(H^2(S_n), L^p(S_n))$, which follows from the same theory to which we referred in the proof of Theorem 7.3 to show that $\Delta \Gamma \in \mathcal{H}(h^{2+a}(\Gamma), h^a(\Gamma))$. We also need the estimate for the resolvent as given by part (iii) of Theorem 4.16. This estimate is satisfied if we consider $\Delta S_n - aI$ for some $a > 0$ (i.e. by moving the spectrum of $\Delta S_n$ by distance $a$ to the left).

- $\Delta S_n - aI$ has bounded imaginary powers (see p. 162 of [1] for the definition). This follows from [4], Corollary 10.4, for some $a > 0$.

Now, Theorem 4.10.7 states that $\Delta S_n - aI$ has maximal $L^p$-regularity. Then from Definition 6.1, it easily follows that $\Delta S_n$ has maximal $L^p$-regularity too.

Another way of proving that $\Delta S_n$ has maximal $L^p$-regularity, is by using (10.12) on p. 355 of [15]. It is not directly clear how the property of maximal $L^p$-regularity follows from this inequality. Furthermore, it only implies that $\Delta$ has maximal $L^p$-regularity in the case where domains of $\mathbb{R}^n$ are considered instead of bounded, closed, smooth hypersurfaces. The ideas to generalize this result to bounded, closed, smooth hypersurfaces, are known, but we will not present them here.

$\Delta S_n$ having maximal $L^p$-regularity, completes part (III), which is the final step of the proof.

\[ \square \]

7.6 An extension of the model

We can extend our model by relaxing the condition that $u_- = 0$ to the case $u_- > 0$ (but still constant in space and time). This corresponds to the case in which solute is present in the environment around the cell. This extended model is more realistic for the setting of the experiments discussed in the introduction. We still assume that the solute is uniformly distributed, and that its concentration does not change if the cell swells or shrinks (hence $u_-$ is constant).
Physically, the difference of this extended model with our previous model, is that the osmotic pressure will be smaller, causing the cell to swell less fast. It may even happen that the concentration of solute outside the cell is bigger than inside. Then both the surface tension and osmosis will cause the cell to shrink. This effect is not present in our previous model. It is therefore not surprising that in our extended model, a dimensionless number is going to appear.

It is relatively easy to show that Theorem 7.3 and Theorem 7.4 still hold for this extended model. To see this, we will state the differences with our previous model.

The only change in Section 7.1 caused by $u_+ > 0$, is that (7.2) changes into

$$[u] = \frac{n_s}{|\Omega^+|} - u_+.$$  (7.62)

Then (7.5) changes into

$$V = 2\mathcal{P}\mathcal{V}\gamma H + \mathcal{P}\mathcal{V}\mathcal{R}\mathcal{T}\left(\frac{n_s}{|\Omega|} - u_+\right),$$  (7.63)

which we can write as

$$V = C_1 H + C_2 \frac{1}{|\Omega|} - C_3,$$

where $C_i > 0$ for $i \in \{1, 2, 3\}$.

By similar arguments, one can show that the equilibrium will be a sphere again. However, the radius $R$ of this sphere now satisfies

$$0 = -C_1 \frac{1}{R} + C_2 \frac{1}{\omega_R R^n} - C_3.$$  (7.65)

It is not hard to see that $C_i > 0$ implies that there is exactly one positive real solution $R$ to (7.65).

The dimensionless formulation changes significantly. Again, we scale time by $T$ and scale space by $L$. Our dimensionless quantities are again given by (7.10). We can cast (7.64) into

$$\hat{V} = \hat{C}_1 \hat{H} + \hat{C}_2 \frac{1}{|\hat{\Omega}|} - \hat{C}_3,$$

where $\hat{C}_i$ are given by

$$\hat{C}_1 = C_1 \frac{T}{L^2}, \quad \hat{C}_2 = C_2 \frac{T}{L^{n+1}}, \quad \hat{C}_3 = C_3 \frac{T}{L}.$$  (7.64)

(7.65) transforms into

$$0 = -\hat{C}_1 \frac{1}{\hat{R}} + \hat{C}_2 \frac{1}{\omega_{\hat{R}} \hat{R}^n} - \hat{C}_3.$$  (7.66)

We choose $T$ and $L$ such that $\hat{R} = 1 = \hat{C}_1$. We obtain the first equality by imposing

$$\omega_{\hat{R}} (\hat{C}_3 + 1) = \hat{C}_2.$$
because for this choice, (7.66) has $\hat{R} = 1$ as a solution, which is the only positive solution as we have already stated. For this scaling, we have

$$\hat{V} = \hat{H} + (\hat{C}_3 + 1) \frac{\omega_n}{|\Omega|} - \hat{C}_3,$$

so our problem depends on the dimensionless number $\hat{C}_3 > 0$. In the following, we remove the hats.

In our functional setting, we then have

$$F(\rho) = L(\rho)(H(\rho) + (C_3 + 1)V(\rho) - C_3)$$

instead of $F$ as defined in (7.19). It is easily seen that this does not change the fact that

$$F \in C^\infty(\Omega; h^n(\Gamma)).$$

Since $L'(0) = 0$, the only thing that changes in the expression of $F'(0)$ (see (7.29)), is the constant in front of the integral. Therefore, the proof of Theorem 7.3 applies directly to our extended model, which gives us local existence and uniqueness. The same is true for stability of the equilibria, but this extension result is a little bit more involved. The constant $C_3 + 1$ in front of the integral in the linearization does change the spectrum. However, for all $C_3 > 0$, it only results in $\lambda_0$ being smaller than the $\lambda_0$ we got in Section 7.5, which does not change the conclusion of the concerning part of the proof.
8 Discussion

We succeeded in proving Theorem 7.3 and Theorem 7.4, but we have not shown that these theorems really state what we want. In fact, this is not even the case. We will elaborate on this in the following. We also speculate about how these results can be generalized to more difficult problems.

The problem with using Theorem 7.3 to show well-posedness of our model for any initial condition $\Gamma_0$ which is regular enough for the curvature to be defined, is that one needs to find a bounded, closed, smooth hypersurface $\Gamma$ such that $\Gamma_0$ is described by the graph of an element from $\mathcal{O}$. We have already addressed this issue in Section 7.3 for an even larger set of initial conditions. In the past, many have assumed that one can always find such $\Gamma$ for given $\Gamma_0$. However, recently, in [6], by a non trivial proof, conditions are given for which this is possible. Unfortunately, we can not guarantee that these conditions are satisfied for our model, because $\mathcal{O}$ depends on $\Gamma$ in an intricate way.

One way to fix this, is to require $\Gamma_0$ to be a bounded, closed, smooth hypersurface. Then we can just take $\Gamma = \Gamma_0$ to get the desired well-posedness of local classical solutions by Theorem 7.3. However, this is a waste in the sense that Theorem 7.3 can handle a much wider class of initial configurations $\Gamma_0$; it is just hard to tell whether a regular enough hypersurface is contained in this class.

Another way to fix this, is to regard $\Gamma_0$ as the reference hypersurface, even though it is not smooth. The problem with this is that there is little literature about function spaces defined on manifolds that are not smooth. If one wants to do this, one needs to extend all the concerning proofs about these function spaces on Riemannian manifolds (smooth manifolds with a notion of an inner product) to non smooth manifolds. This will be a time consuming task, and success is not guaranteed.

Extension of Theorem 7.3 to global existence and uniqueness will also be very hard. We have already discussed this in Section 7.3. Furthermore, the proof of Theorem 7.3 strongly relies on the linearization of $F$ around $0$, i.e. linearizing the problem around the reference hypersurface. This linearization only guarantees to give a good approximation to $F$ around $0$, while the membrane may evolve to be relatively far away from the reference hypersurface, rendering the linearization around $0$ useless.

One idea to extend the time interval, is to redefine $\Gamma$ at $t = T$ (as given by Theorem 7.3), and to use Theorem 7.3 again for this $\Gamma$ as the reference hypersurface, and the configuration of the cell at time $t = T$ as the initial condition. This works inductively. The question is how to choose $\Gamma$, and what one can expect about the maximal length for the time interval. Figure 7.3 shows an initial condition for which this time interval surely is finite, so one also has to think about suitable initial conditions.
Theorem 7.4 shows that we have global existence if the initial condition is close enough to an equilibrium configuration (a sphere with a fixed radius). However, it ensures not as much regularity of the solution in space and time as Theorem 7.3 does. It might be possible to combine Theorem 7.3 and Theorem 7.4 to show global existence and uniqueness of classical solutions for initial conditions close to the equilibrium configuration.

For more difficult problems, Theorem 5.7 might be useful. It allows $F$ to be fully nonlinear, and for the regularity of $F$ it only needs $F$ to be $C^1$. However, we still need to cast this problem into a formulation like (7.19), where there are no boundary conditions. This is already not possible (for the methods that we used) in the case where one considers diffusion of solute or the flow of water, because this requires functions defined on $\Omega$, the domain enclosed by the membrane, while our model deals with functions that are only defined on the reference hypersurface.

Theorem 7.4 is even harder to generalize to models that are more involved. In the setting of this theorem, our problem was described by a radially symmetric operator. Therefore, it is not surprising that the eigenfunctions of our linearization were the spherical harmonics. This allowed us to compute the spectrum of the linearization explicitly, which helped us while proving that our equilibria were normally stable. The chances are very small that the same technique applies to models that do not involve a radially symmetric operator.
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A (Little) Hölder spaces

(Little) Hölder spaces are well-known function spaces. It is our goal to introduce them here, and to extend their definition to functions defined on bounded, closed, smooth hypersurfaces. This is done, for example, in [14], Chapter 3. There, a self-contained overview is given of (little) Hölder spaces and their basic properties. We only list those definitions and properties that we need. Proposition A.10 is not stated in [14], so we prove it ourselves. This proposition proves to be useful while showing the regularity of a class of operators (see Appendix B), or while extending interpolation properties of little Hölder spaces on Euclidean domains to hypersurfaces (see Corollary 3.11).

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain. For $f \in C(\Omega)$, we define for any $\alpha \in (0, 1)$

$$[f]_\alpha := \sup_{x, y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha},$$

where $|\cdot|$ denotes a norm in $\mathbb{R}^k$ for any $k \in \mathbb{N}_+$. If $k = 1$, this norm is just the absolute value, and for $k \geq 2$, one can take any norm in $\mathbb{R}^k$, since all norms in $\mathbb{R}^k$ are equivalent.

Now we define the Hölder space on $\Omega$ of exponent $\alpha$ by

$$C^\alpha(\Omega) := \{ f \in C(\Omega) | [f]_\alpha < \infty \}.$$

The little Hölder space is defined as

$$h^\alpha(\Omega) := \left\{ f \in C^\alpha(\Omega) \left| \lim_{\delta \to 0} \sup_{x, y \in \Omega, 0 < |x - y| < \delta} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right. \right\}.$$

To get some insight in (little) Hölder spaces, consider the following example:

**Example A.1.** Let $\Omega = (0, 1)$, $f \in C([0, 1])$ given by $f(x) = \sqrt{x}$. Then $f \in C^\alpha([0, 1])$ iff $\alpha \leq 1/3$, and $f \in h^\alpha([0, 1])$ iff $\alpha < 1/3$.

For any $k \in \mathbb{N}$, we can extend our definitions of (little) Hölder spaces to more regular functions. In these definitions, we use the following notation: for any multi-index $j \in \mathbb{N}^m$, we define

$$\partial^j := \frac{\partial^{|j|}}{\partial x_1^{j_1} \cdots \partial x_m^{j_m}}.$$

Then the (little) Hölder spaces are given by
K^α := \{ f \in C^k(\Omega) \mid \forall j \in \mathbb{N}^m, |j| = k : \partial^j f \in C^\alpha(\Omega) \},

h^{k+\alpha}(\Omega) := \{ f \in C^k(\Omega) \mid \forall j \in \mathbb{N}^m, |j| = k : \partial^j f \in h^\alpha(\Omega) \}.

Note that for \( k = 0 \), this definition is equivalent to the previous one. If we equip the (little) Hölder spaces with the following norm:

\[ \| f \|_{k+\alpha} := \max_{|j| \leq k} \| \partial^j f \|_0 + \max_{|j| = k} [\partial^j f]_\alpha, \tag{A.1} \]

both \( C^{k+\alpha}(\Omega) \) and \( h^{k+\alpha}(\Omega) \) are Banach spaces (\( \| f \|_0 \) denotes the maximum of \( |f| \) over its domain). Moreover, these spaces are actually Banach algebras, which means that

\[ \exists c > 0 \forall f, g \in C^{k+\alpha}(\Omega), \text{ and } \|fg\|_{k+\alpha} \leq c \|f\|_{k+\alpha} \|g\|_{k+\alpha}. \]

The same holds true for \( h^{k+\alpha}(\Omega) \).

The following proposition follows directly from the definition of the (little) Hölder spaces:

**Proposition A.2.** Let \( k \in \mathbb{N}_+ \). If \( f, g \in C^{k+\alpha}(\Omega) \), then \( g \circ f \in C^{k+\alpha}(\Omega) \). The same is true for \( h^{k+\alpha}(\Omega) \).

This concludes the part about (little) Hölder spaces on the closure of bounded domains of \( \mathbb{R}^m \). The idea of defining Hölder continuous functions on a bounded, closed, smooth hypersurface \( \Gamma \), is by translating such a function to charts. This translation is done by push forwards and pull backs (which we define in the following) of functions with compact support. To be more precise, we define

\[ C^{k+\alpha}_c(\Omega) := \{ f \in C^{k+\alpha}(\Omega) \mid \text{supp } f \subset \Omega \}. \tag{A.2} \]

Analogously we define \( h^{k+\alpha}_c(\Omega) \).

**Definition A.3.** (Push forward, pull back). Let \( (U, \varphi, V) \) be a local representation of \( \Gamma \) (see Definition 2.1). For any \( g \in C_c(U) \) and \( f \in C(\Gamma) \) with \( \text{supp } f \subset V \), we define the **push forward** \( \varphi_* \) and **pull back** \( \varphi^* \) by

\[ \varphi_* g := g \circ \varphi^{-1} \in C(\Gamma), \]
\[ \varphi^* f := f \circ \varphi \in C_c(U), \]

\( \ast \) : \( \varphi_* g \) is only defined on \( V \subset \Gamma \), but we can extend it to \( \Gamma \) by defining \( \varphi_* g \equiv 0 \) on \( \Gamma \setminus V \). \( g \) having compact support in \( U \) ensures that this extension conserves continuity. A similar argument is used to extend \( \varphi^* f \) by 0 on \( \partial U \).

**Remark A.4.** In literature, push forwards and pull backs are much more general. They apply to any homeomorphism between two sets \( A \) and \( B \), and move any object related to \( A \) to a similar object related to \( B \) and vice versa. We, however, only use push forwards and pull backs in the setting of Definition A.3.
The following proposition is a direct consequence of Definition 2.3:

**Proposition A.5.** Consider the setting of Definition A.3, and let \( \ell \in \mathbb{N}_+ \cup \{\infty\} \). If we additionally have that \( g \in C^\ell(U) \) and \( f \in C^\ell(\Gamma) \), then \( \varphi_+ g \in C^\ell(\Gamma) \) and \( \varphi_+ f \in C^\ell(U) \).

With the help of pull backs, we can define the space of (little) Hölder continuous functions on \( \Gamma \):

**Definition A.6.** Let \( k \in \mathbb{N} \) and \( \alpha \in (0, 1) \). Let \( L := \{(U_i, \varphi_i, V_i, \psi_i) \mid i \in \{1, \ldots, N\}\} \) a localizing system on \( \Gamma \). We define

\[
C^{k+\alpha}_L(\Gamma) := \left\{ f \in C(\Gamma) \mid \forall i \in \{1, \ldots, N\}: \varphi_i^*(\psi_i f) \in C^{k+\alpha}_c(U_i) \right\},
\]

\[
h^{k+\alpha}_L(\Gamma) := \left\{ f \in C(\Gamma) \mid \forall i \in \{1, \ldots, N\}: \varphi_i^*(\psi_i f) \in h^{k+\alpha}_c(U_i) \right\}.
\]

Since we mostly work with little Hölder spaces on bounded, closed, smooth hypersurfaces, we focus our attention on \( h^{k+\alpha}_L(\Gamma) \). However, the following is also true for \( C^{k+\alpha}_L(\Gamma) \).

It is not hard to see that \( h^{k+\alpha}_L(\Gamma) \) is a vector space. Actually, it is a normed vector space, because one can show that

\[
\|f\|_{h^{k+\alpha}_L(\Gamma)} := \max_{i \in \{1, \ldots, N\}} \|\varphi_i^*(\psi_i f)\|_{k+\alpha}
\]

(A.3)

defines a norm on \( h^{k+\alpha}_L(\Gamma) \). In fact,

\[
\left( h^{k+\alpha}_L(\Gamma), \| \cdot \|_{h^{k+\alpha}_L(\Gamma)} \right)
\]

is a Banach space, and even a Banach algebra.

One can show that for any two localizing systems \( L_1 \) and \( L_2 \) on \( \Gamma \), it holds that the Banach spaces \( h^{k+\alpha}_{L_1}(\Gamma) \) and \( h^{k+\alpha}_{L_2}(\Gamma) \) are equal. Therefore, we just write \( h^{k+\alpha}(\Gamma) \) for this Banach space, and \( \| \cdot \|_{h^{k+\alpha}(\Gamma)} \) for its norm, knowing that changing our localizing system yields a different - but equivalent - norm.

An important characterization of \( h^{k+\alpha}(\Gamma) \) is given by the following theorem (which does not hold for \( C^{k+\alpha}(\Gamma) \)):

**Theorem A.7.**

\[
h^{k+\alpha}(\Gamma) = C^\infty(\Gamma)^{\| \cdot \|_{h^{k+\alpha}(\Gamma)}}.
\]

Properties of elements from \( h^{k+\alpha}(\Gamma) \) are proven by pulling them back to charts of a localizing system (this can be seen from the proofs that we omitted here). We have used this strategy in Section 2.1 to define differentiability of functions on \( C^k \)-manifolds, for which we needed compatibility of the mappings \( \psi_i \) from the charts to the manifold. We need a similar notion of compatibility for showing that various properties of (little) Hölder spaces on the closure of bounded domain in \( \mathbb{R}^n \) still hold for (little) Hölder spaces on bounded, closed, smooth hypersurfaces. Specifically, these properties are an interpolation property (see Corollary 3.11) and an easy way for showing regularity of operators between these spaces (see Corollary B.5). The compatibility condition is described in Theorem A.8, which states that this condition is satisfied:
Theorem A.8. Let \((U_1, \varphi_1, V_1)\) and \((U_2, \varphi_2, V_2)\) be two local representations of \(\Gamma\) around \(\xi \in \Gamma\). Then
\[
\varphi_2^* \varphi_1^* \in \mathcal{L}(H_1, H_2),
\]
where \(H_i := \{ f \in h^{k+\alpha}(\overline{U_i}) \mid \text{supp } f \subset \varphi_i^{-1}(V_1 \cap V_2) \} \) for \(i \in \{1, 2\}\).

Remark A.9. Formally speaking, \(\varphi_2^* \varphi_1^*\) translates a function \(f\) from one chart to another via \(\Gamma\). This only makes sense for those parts of the charts that correspond to the same part of \(\Gamma\) (in our case, this part is \(V_1 \cap V_2\)). Furthermore, we restrict ourselves to those \(f\) which have compact support in the corresponding parts of the charts.

We write \(\varphi_1^k\) instead of \((\varphi_1)_k\), although the latter would be more consistent with Definition A.3. Note that \(\varphi_2^k \varphi_1^*\) is the push forward of the transition map \(\varphi_2^{-1} \circ \varphi_1\). From this it is not hard to see that \((\varphi_2^k \varphi_1^*)^{-1} = \varphi_1^* \varphi_2^k\). ■

Let \(\{(U_i, \varphi_i, V_i, \psi_i) \mid i \in \{1, \ldots, N\}\} := L\) be a localizing system on \(\Gamma\). We are now going to show a translation of element from \(h^{k+\alpha}(\Gamma)\) to
\[
h^{k+\alpha}(U_1) \times \ldots \times h^{k+\alpha}(U_N) =: \prod_{i=1}^N h^{k+\alpha}(U_i) =: X(k + \alpha; L).
\]

This translations allows us to regard \(f \in h^{k+\alpha}(\Gamma)\) fully in terms of elements from \(h^{k+\alpha}(U_i)\) for every \(i \in \{1, \ldots, N\}\), for which several properties are known.

This translation is given by the operators \(P\) and \(Q\) defined below. The use of this translation is given by Proposition A.10.

For \(i \in \{1, \ldots, N\}\), let \(\Psi_i \in C^\infty(\Gamma)\) such that \(\Psi_i|_{\text{supp } \psi_i} \equiv 1\) and \(\text{supp } \Psi_i \subset V_i\). For \(f \in h^{k+\alpha}(\Gamma)\), we define
\[
P_i f := \varphi_i^* (\Psi_i f) =: \varphi_i^* (\Psi_i f) \in h^{k+\alpha}(U_i), \quad i \in \{1, \ldots, N\},
P f := (P_i f)_{i=1}^N \in X(k + \alpha; L), \quad \text{(A.4)}
\]

\(*\) : see Definition A.6.

For \(g = (g_1, \ldots, g_N) \in X(k + \alpha; L)\), we define
\[
Q_i g_i := \psi_i \varphi_i^* (g_i) =: C^k(\Gamma), \quad i \in \{1, \ldots, N\},
Q g := \sum_{i=1}^N Q_i g_i \in h^{k+\alpha}(\Gamma), \quad \text{(A.5)}
\]

\(*\) : see Proposition A.5. Actually, it holds that \(Q_i g_i \in h^{k+\alpha}(\Gamma)\), as can be seen from the following proposition.

Proposition A.10. Fix a localizing system \(L\) on \(\Gamma\). For \(P\) and \(Q\) defined by (A.4) and (A.5) (note that they depend on this localizing system), it holds that
\[
P \in \mathcal{L}(h^{k+\alpha}(\Gamma), \prod_{i=1}^N h^{k+\alpha}(U_i)),
\]
\[
Q \in \mathcal{L}\left(\prod_{i=1}^N h^{k+\alpha}(U_i), h^{k+\alpha}(\Gamma)\right).
\]

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Furthermore, $Q$ is the left inverse of $P$, i.e. $QP = I \in \mathcal{L}(h^{k+\alpha}(\Gamma), h^{k+\alpha}(\Gamma))$.

Proof. From their definitions, it is easy to see that $P$ and $Q$ are linear. We continue by showing that $Q$ is bounded. In the sequel, we will not be precise about the constant $C$. Let $g \in X(k+\alpha; L)$, then

$$
\|Qg\|_{h^{k+\alpha}(\Gamma)} \overset{1}{=} \max_{j \in \{1, \ldots, N\}} \left\| \varphi_j^* \left( \sum_{i=1}^{N} \psi_i^* \varphi_i^*(g_i) \right) \right\|_{h^{k+\alpha}(\Gamma)}
$$

$$
\leq \max_{j \in \{1, \ldots, N\}} \sum_{i=1}^{N} \left\| \varphi_j^* \left( \psi_j \psi_i \varphi_i^*(g_i) \right) \right\|_{h^{k+\alpha}(\Gamma)}
$$

$$
\overset{2}{=} \max_{j \in \{1, \ldots, N\}} \sum_{i=1}^{N} \left\| \varphi_j^* \varphi_i^* \left( \tilde{\psi}_{ij} g_i \right) \right\|_{h^{k+\alpha}(\Gamma)}
$$

$$
\overset{3}{=} \max_{j \in \{1, \ldots, N\}} \sum_{i=1}^{N} C \left\| \tilde{\psi}_{ij} g_i \right\|_{h^{k+\alpha}(\Gamma)}
$$

$$
\overset{4}{=} C \sum_{i=1}^{N} \left\| g_i \right\|_{h^{k+\alpha}(\Gamma)}
$$

*1: use (A.3) and (A.5),
*2: note that $\psi_j \psi_i \in C^\infty(\Gamma)$ with supp $\psi_j \psi_i \subset V_i \cap V_j$. Let $\tilde{\psi}_{ij} := \varphi_i^*(\psi_j \psi_i) \in C^\infty(\overline{U_i})$ (see Proposition A.5),
*3: since we have that $\tilde{\psi}_{ij} g_i \in H_i$ as defined in Theorem A.8, we can use this theorem,
*4: $h^{k+\alpha}(\overline{U_i})$ is a Banach algebra.

For the boundedness of $P$ we use similar arguments. Let $f \in h^{k+\alpha}(\Gamma)$, then

$$
\| Pf \|_{X(k+\alpha; L)} = \sum_{i=1}^{N} \left\| \varphi_i^*(\tilde{\psi}_i f) \right\|_{h^{k+\alpha}(\Gamma)}
$$

$$
\overset{1}{=} \sum_{i,j=1}^{N} \left\| \varphi_i^* \left( \psi_j \tilde{\psi}_i f \right) \right\|_{h^{k+\alpha}(\Gamma)}
$$

$$
\overset{2}{=} \sum_{i,j=1}^{N} \left\| \varphi_i^* \varphi_j^* \varphi_i^* \left( \psi_j \tilde{\psi}_i f \right) \right\|_{h^{k+\alpha}(\Gamma)}
$$

$$
\overset{3}{=} \sum_{i,j=1}^{N} C \left\| \varphi_j^* \left( \psi_j \tilde{\psi}_i f \right) \right\|_{h^{k+\alpha}(\Gamma)}
$$

$$
\overset{4}{=} \sum_{i,j=1}^{N} \left\| \varphi_j^* \left( \psi_j \tilde{\psi}_i f \right) \right\|_{h^{k+\alpha}(\Gamma)} \left\| \varphi_i^* \left( \tilde{\psi}_i f \right) \right\|_{h^{k+\alpha}(\Gamma)}
$$

$$
\overset{5}{=} C \| f \|_{h^{k+\alpha}(\Gamma)},
$$

*1: use $f = \sum_{i=1}^{N} \psi_i f$ and the triangle inequality,
\[ \text{Remark A.11.} \text{ Note that for any } f \in h^{k+\alpha}(\Gamma), \text{ it holds that } \text{supp } P_i f \subset \text{supp } \psi_i \subset U_i. \text{ Therefore, for any domain } W_i \text{ satisfying } \text{supp } \psi_i \subset W_i \subset U_i, \text{ we may as well consider } h^{k+\alpha}(W_i) \text{ as the space to which } P_i \text{ maps. This is convenient whenever certain theorems (like Theorem 3.9 and Theorem B.3) require certain properties about } U_i. \text{ If } U_i \text{ does not satisfy the requirement for any of these two theorems, we can always find such } W_i \text{ for which these requirements are fulfilled. We will not go into more details about this.} \]

\[ \text{Remark A.12.} \text{ The extension of function spaces from domains of } \mathbb{R}^n \text{ to bounded, closed, smooth manifolds (as discussed in Definition A.6) does not only apply to (little) Hölder spaces. Similarly, for } p \in [1, \infty] \text{ and } k \in \mathbb{N}, \text{ one can define } H^k_p(\Gamma) \text{ as the extension of the Sobolev space of order } k \text{ (} k = 0 \text{ corresponds to } L_p \text{) to bounded, closed, smooth hypersurfaces. We will not discuss the details here.} \]
B Regularity of Operators on (little) Hölder spaces

In Section 2.2 we introduce operators acting on little Hölder spaces defined on a bounded, closed, smooth hypersurface Γ'. In this chapter, we show that these operators are smooth. The reason that we do this in the Appendix, is that the proof for their smoothness is extensive, while one would intuitively guess that these operators are smooth because of their explicit representation in terms of ρ.

Let us first recall the definition of the Fréchet derivative:

**Definition B.1.** Let $X, Y$ be Banach spaces, $\mathcal{O} \subset X$ open, and let $F \in C(\mathcal{O}; Y)$. Then $F$ is said to be differentiable at $x \in \mathcal{O}$ if there exists an $A \in \mathcal{L}(X, Y)$ such that

$$\frac{\|F(x + h) - F(x) - Ah\|_Y}{\|h\|_X} \to 0 \quad \text{as} \quad h \to 0 \quad \text{in} \quad X.$$  

If such $A$ exists, it is called the Fréchet derivative of $F$ at $x$, denoted by $F'(x)$. Furthermore, we denote by $C^1(\mathcal{O}; Y)$ the class of all $F \in C(\mathcal{O}; Y)$ that are differentiable at any $x \in \mathcal{O}$, and for which $F' \in C(\mathcal{O}; \mathcal{L}(X, Y))$.

**Remark B.2.** For $F \in C^1(\mathcal{O}; Y)$ we know by definition that $F' \in C(\mathcal{O}; \mathcal{L}(X, Y))$. Since $\mathcal{L}(X, Y)$ is a Banach space, it can happen that $F'$ is differentiable (in the sense of Definition B.1) at any $x \in \mathcal{O}$ too. Let $C^2(\mathcal{O}; Y)$ be the class of those $F \in C^1(\mathcal{O}; Y)$ for which this happens. Then for $F \in C^2(\mathcal{O}; Y)$, we have $F' \in C^1(\mathcal{O}; \mathcal{L}(X, Y))$, and then from Definition B.1 we have

$$F'' := (F')' \in C\left(\mathcal{O}; \mathcal{L}(X, \mathcal{L}(X, Y))\right) \cong C(\mathcal{O}; \mathcal{L}^2(X, Y)),$$

where $\mathcal{L}^n(X, Y)$ denotes the space of multilinear operators which map $n$ elements of $X$ to a single element of $Y$. In general, for any $k \in \mathbb{N}$, we say that $F \in C^k(\mathcal{O}; Y)$ if $F^{(k)} \in C(\mathcal{O}; \mathcal{L}^k(X, Y))$. Furthermore, we set

$$C^\infty(\mathcal{O}; Y) = \bigcap_{k=0}^{\infty} C^k(\mathcal{O}; Y).$$

We call a mapping $F : \mathcal{O} \to Y$ smooth if $F \in C^\infty(\mathcal{O}; Y)$.

For calculating $F'$ at $x \in \mathcal{O}$ for given $F \in C^1(\mathcal{O}; Y)$, we will usually take an arbitrary $h \in X$, and try to write $F(x + h) - F(x)$ as $Ah + r(h)$, where $r : X \to Y$ such that $r(h)$ is small, in the sense that

$$\frac{\|r(h)\|_Y}{\|h\|_X} \to 0 \quad \text{as} \quad h \to 0 \quad \text{in} \quad X. \quad (\text{B.1})$$

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Then from Definition B.1 it is easy to see that \( A = F'(x) \). In the following, we say that any \( r : X \to Y \) satisfying (B.1) is \( o(h) \), and we write \( Ah + o(h) \) instead of \( Ah + r(h) \). Note that \( o(h) \) depends on the norms of the Banach spaces \( X \) and \( Y \).

Remark B.2 states what we mean by a smooth operator. However, to show that an operator is smooth, it is cumbersome to use the definition in Remark B.2. Theorem B.3, which is stated in [28], Theorem 4.4, will be of great help to us:

**Theorem B.3.** Let \( \Omega \in \mathbb{R}^n \) open and bounded such that

\[
\exists c_0 > 0, \forall v \in C^1(\Omega) : \sup_{x, y \in \Omega, x \neq y} \frac{|v(x) - v(y)|}{|x - y|} \leq c_0 \sum_{j=1}^n \sup_{x \in \Omega} |\frac{\partial v}{\partial x_j}(x)|.
\]

Furthermore, let \( k, N \in \mathbb{N}_+ \), \( m \in \mathbb{N} \) and \( O \in \mathbb{R}^N \) open. If \( f \in C^{1+k+\max\{1, m\}}(\overline{\Omega} \times O) \), then for any \( \alpha \in (0, 1] \), it holds for the mapping \( F(g) := (x \mapsto f(x, g(x)) \) that

\[
F \in C^k\left( \left\{ g \in C^{m+\alpha}(\overline{\Omega}; \mathbb{R}^N) \, | \, g(\overline{\Omega}) \subset O \right\} ; C^{m+\alpha}(\overline{\Omega}) \right).
\]

**Remark B.4.** Remark 1.1 of [28] implies that the condition on \( \Omega \) is satisfied if \( \Omega \) has a \( C^1 \) boundary. For our application of Theorem B.3, we are only going to consider such domains (see Remark A.11 for the details).

The use of Theorem B.3 is that it couples the regularity of \( f \) (which is often easy to determine) to the regularity of \( F \), which is much harder to determine.

We want to use Theorem B.3 on little Hölder spaces. This can be done because of the following. Consider the setting in Theorem B.3 in which the conditions are satisfied. Since \( h^{m+\alpha}(\overline{\Omega}; \mathbb{R}^N) \subset C^{m+\alpha}(\overline{\Omega}; \mathbb{R}^N) \) and the norms on both spaces are equal, we then also have

\[
F \in C^k\left( \left\{ g \in h^{m+\alpha}(\overline{\Omega}; \mathbb{R}^N) \, | \, g(\overline{\Omega}) \subset O \right\} ; C^{m+\alpha}(\overline{\Omega}) \right).
\]

So in order to use Theorem B.3 on little Hölder spaces, we only have to show that

\[
F\left( \left\{ g \in h^{m+\alpha}(\overline{\Omega}; \mathbb{R}^N) \, | \, g(\overline{\Omega}) \subset O \right\} \right) \subset h^{m+\alpha}(\overline{\Omega}).
\]

**Corollary B.5.** Let \( \Gamma \) a bounded, closed, smooth hypersurface of \( \mathbb{R}^{n+1} \), and let \( k, N \in \mathbb{N}_+ \), \( m \in \mathbb{N} \) and \( O \in \mathbb{R}^N \) open. If \( f \in C^{1+k+\max\{1, m\}}(\Gamma \times O) \), then for any \( \alpha \in (0, 1) \), it holds for the mapping \( \mathcal{F}(\rho) := (\xi \mapsto f(\xi, \rho(\xi)) \) that

\[
\mathcal{F} \in C^k\left( \left\{ \rho \in C^{m+\alpha}(\Gamma; \mathbb{R}^N) \, | \, \rho(\Gamma) \subset O \right\} ; C^{m+\alpha}(\Gamma) \right).
\]

**Proof.** We start by defining the translation of \( f \) and \( \mathcal{F} \) to functions defined on the charts of \( \Gamma \). Let \( \{(U_i, \psi_i, V_i, \psi_i) \, | \, i \in \{1, \ldots, N\}\} \) be a localizing system on \( \Gamma \), and let \( \Psi_i \in C^\infty(\Gamma) \) such that \( \psi_i \big|_{\text{supp} \psi_i} \equiv 1 \) and \( \text{supp} \Psi_i \subset V_i \). For convenience, we define

\[
Z(\Gamma) := \left\{ \rho \in C^{m+\alpha}(\Gamma; \mathbb{R}^N) \, | \, \rho(\Gamma) \subset O \right\},
\]

\[
Z(U_i) := \left\{ g \in C^{m+\alpha}(U_i; \mathbb{R}^N) \, | \, g(U_i) \subset O \right\}, \quad i \in \{1, \ldots, N\}.
\]

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Consider the operators $P$ and $Q$ from Proposition A.10. Let $i \in \{1, \ldots, N\}$, $u \in U_i$, $a \in O$ and $g \in Z(U_i)$. Then we define
\[
\begin{align*}
  f_i(u, a) &:= P f(u, a) = \Psi_i(\varphi_i(u)) f(\varphi_i(u), a), \\
  F_i(g) &:= \{ u \mapsto f_i(u, g(u)) \}.
\end{align*}
\] (B.2)

It holds that
\[
f \in C^{1+k+\max\{1,m\}}(\Gamma \times O) \Rightarrow f_i \in C^{1+k+\max\{1,m\}}(U_i \times O),
\] *: The open set $O$ prevents us from using Proposition A.10 directly. However, it is not hard to show that the implication still holds. We omit the details.

Now we can apply Theorem B.3. This yields
\[
F_i \in C^k(Z(U_i); C^{m+\alpha}(U_i)).
\]

Let $F := (F_1, \ldots, F_N)$. Then
\[
F \in \prod_{i=1}^{N} C^k(Z(U_i); C^{m+\alpha}(U_i)) \subset C^k(\prod_{i=1}^{N} Z(U_i); \prod_{i=1}^{N} C^{m+\alpha}(U_i)).
\]
and therefore
\[
QFP \in C^k(Z(\Gamma); C^{m+\alpha}(\Gamma)).
\] (B.3)

In the sequel, we show that $F = QFP$. Together with (B.3), this will complete the proof.

Observe that on $V_i \subset \Gamma$, it holds for $\rho \in Z(\Gamma)$ that
\[
\begin{align*}
  (\varphi_i^* F_i \varphi_i^*) \rho &= (\varphi_i^* F_i)(\rho \circ \varphi_i) \\
  &\overset{(B.2)}{=} \varphi_i^* f_i(u \mapsto f_i(u, (\rho \circ \varphi_i)(u))) \\
  &= (\xi \mapsto f_i(\varphi_i^{-1}(\xi), \rho(\xi))) \\
  &\overset{(B.2)}{=} (\xi \mapsto \Psi_i(\xi) f(\xi, \rho(\xi))) \\
  &= \Psi_i F(\rho).
\end{align*}
\] (B.4)

Again, take $\rho \in Z(\Gamma)$. Then
\[
\begin{align*}
  (QFP) \rho &= \sum_{i=1}^{N} \psi_i \varphi_i^*(F_i(\varphi_i^*(\Psi_i \rho))) \\
  &\overset{\ast_1}{=} \sum_{i=1}^{N} \psi_i (\varphi_i^* F_i \varphi_i^*)(\rho) \\
  &\overset{(B.4)}{=} \sum_{i=1}^{N} \psi_i \Psi_i F(\rho) \\
  &\overset{\ast_2}{=} F(\rho),
\end{align*}
\]

*$_1$ : $\Psi_i|_{\text{supp} \psi_i} \equiv 1$, 
*$_2$ : see *$_1$ and use part (iv) of Definition 2.17.
We are now going to apply Corollary B.5 to show that $L$, as defined by (2.29), is smooth. But first, we have to define $L$ globally (we have only defined it locally yet). Since (2.29) implies that $L(\rho)(\xi) = \| \nabla N(\bar{X}(\rho)(\xi)) \|$ for any $\xi \in \Gamma$, the value of

$$L(\rho)(\xi) = \sqrt{\tilde{g}^{ij}(\rho)(\xi) \partial_i \rho(\xi) \partial_j \rho(\xi) + 1}$$

can not depend on which local representation around $\xi$ we have chosen. Now, let \{(U_k, \varphi_k, V_k, \psi_k) \mid k \in \{1, \ldots, N\}\} be a localizing system of $\Gamma$, and let for any $k \in \{1, \ldots, N\}$

$$L_k : \Omega \rightarrow \{ f : V_k \rightarrow \mathbb{R} \}, \quad L_k(\rho)(\xi) := \sqrt{\tilde{g}^{ij}_k(\rho)(\xi) \partial_i \rho(\xi) \partial_j \rho(\xi) + 1},$$

(B.5)

where $\Omega$ is defined by (2.18). Then for any $\rho \in \Omega$ and $\xi \in \Gamma$, observe that

$$\sum_{k=1}^{N} \psi_k(\xi) L_k(\rho)(\xi) \geq L(\rho)(\xi) \sum_{k=1}^{N} \psi_k(\xi) = L(\rho)(\xi),$$

(B.6)

$\ast$: if $\xi \notin \text{supp} \psi_k \subset V_k$, then $\psi(\xi) = 0$. If $\xi \in V_k$, then $L_k(\rho)(\xi)$ is well-defined by (B.5), and by our previous reasoning, it equals $L(\rho)(\xi)$, which is independent of $k$.

(B.6) gives a global definition of $L$ as defined by (2.29). The following Proposition states its regularity.

**Proposition B.6.**

$$L \in C^\infty(\Omega; h^{1+\alpha}(\Gamma)).$$

**Proof.** We start by defining smoothness of the auxiliary function $\mathcal{L}$. Let $O = (-\delta, \delta) \times \mathbb{R}^n$, where $\delta$ is chosen such that (2.7) holds, and where $n$ is the dimension of $\Gamma$. Let \{(U_k, \varphi_k, V_k, \psi_k) \mid k \in \{1, \ldots, N\}\} be a localizing system on $\Gamma$, and let $\psi_k \in C^\infty(\Gamma)$ such that $\psi_k|_{\text{supp} \psi_k} \equiv 1$ and $\text{supp} \psi_k \subset V_k$. Let

$$\ell : \Gamma \times O \rightarrow \mathbb{R}, \quad \ell(\xi, (\eta, a)) := \sum_{k=1}^{N} \psi_k(\xi) \sqrt{\psi_k(\xi) \left( (g^*)_{ij}^k(\varphi_k^{-1}(\xi), \eta) \right) a_i a_j + 1},$$

where $(g^*)_{ij}^k(\varphi_k^{-1}(\xi), \eta)$ is defined by (2.15). Here, the subscript $k$ denotes that $(g^*)_{ij}^k$ is defined on the chart $(U_k, \varphi_k, V_k)$. Since $(g^*)_{ij}^k \in C^\infty(U_k^*)$ (see the line below (2.15)), we can use Proposition A.5 to conclude that

$$((\xi, \eta) \mapsto \psi_k(\xi) (g^*)_{ij}^k(\varphi_k^{-1}(\xi), \eta)) \in C^\infty(\Gamma \times (-\delta, \delta)).$$

Now it is easy to see that $\ell \in C^\infty(\Gamma \times O)$ (note that the matrix with entries $(g^*)_{ij}^k(\varphi_k^{-1}(\xi), \eta)$ is positive definite (see Remark 2.12), which implies that the argument of the square root is greater than 1). Therefore, we can use Corollary B.5 to conclude that for $\mathcal{L}(\varphi) := (\xi \mapsto \ell(\xi, \varphi(\xi)))$ it holds that

$$\mathcal{L} \in C^\infty\left( C^{1+\alpha}(\Gamma; O); C^{1+\alpha}(\Gamma) \right),$$

$h^{1+\alpha}(\Gamma)$ being a Banach algebra and Proposition A.2 are the main tools for showing that

$$\mathcal{L} \in C^\infty\left( h^{1+\alpha}(\Gamma; O); h^{1+\alpha}(\Gamma) \right).$$

(B.7)
We omit the details here. Note that the construction of $\mathcal{L}$ is such that for $\rho \in \mathcal{U}$, it holds that
\[ L(\rho) = \mathcal{L}(\rho, \partial_1 \rho, \ldots, \partial_n \rho), \tag{B.8} \]
where $\partial_i$ is defined by Definition 2.3. Let
\[ D : \mathcal{U} \rightarrow h^{1+\alpha}(\Gamma; O), \quad D\rho := (\rho, \partial_1 \rho, \ldots, \partial_n \rho). \]

Basically, the construction of the spaces $h^{1+\alpha}(\Gamma)$ is such that
\[ D \in \mathcal{L}(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma; \mathbb{R}^{n+1})). \tag{B.9} \]

Therefore, we omit the proof of (B.9). It follows from (B.7), (B.8) and (B.9) that
\[ L = \mathcal{L}D \in C^\infty(\mathcal{U}; h^{1+\alpha}(\Gamma)), \]
which is exactly what we needed to show.

In the proof of Theorem 2.36, the authors are very brief in the part of the proof which is about the regularity of $P$ and $Q$. One can show the regularity of these operators in a similar way as we did this for the operator $L$.

Unfortunately, we cannot apply this strategy to show the regularity of $\rho \mapsto |\rho|^{-1}$, because this operator cannot be written as $\xi \mapsto v(\xi, \rho(\xi))$. This disables us to use Corollary B.5.

Similar results also hold for Sobolev spaces instead of (little) Hölder spaces. The following theorem, given by Theorem 4.3 in [28], provides the details. In Remark B.8 we state its main difference from Theorem B.3. We do not derive the equivalence of Corollary B.5 for Sobolev spaces. Its proof is similar to the proof of Corollary B.5.

**Theorem B.7.** Let $\Omega \in \mathbb{R}^n$ open and bounded such that
\[ \exists_{\alpha > 0} \forall v \in C^1(\overline{\Omega}) : \sup_{x, y \in \Omega \atop x \neq y} \frac{|v(x) - v(y)|}{|x - y|} \leq c_\Omega \sum_{i=1}^n \sup_{x \in \Omega} \left| \frac{\partial v}{\partial x_i}(x) \right|. \]

Furthermore, let $m, k, N \in \mathbb{N}_+$ and $O \in \mathbb{R}^N$ open. If $f \in C^{k+m}(\overline{\Omega} \times \overline{O})$, then for any $r \in \mathbb{N}$ and any $p > 1$ that satisfy $(r + m)p > n$, it holds for the mapping $F(\rho) := (x \mapsto f(x, \rho(x)))$ that
\[ F \in C^k \left( \{ \rho \in H^{n+r}_p(\overline{\Omega}; \mathbb{R}^N) \mid \rho(\overline{\Omega}) \subseteq O \} ; H^{n+r}_p(\overline{\Omega}) \right). \]

**Remark B.8.** The main difference with Theorem B.3 is that $m = 0$ is not allowed anymore. Hence, we can only consider operators $F$ for which their domain and range are (at least) included in $H^{n+r}_p(\overline{\Omega})$ if we want to show their regularity with the help of Theorem B.7.

For our applications, it will be necessary to take $r = 0$. Then Theorem B.7 demands that $p > n/m$. This will be no important restriction for our application, because for other reasons we already need $p > n + 2$. 

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C Fréchet derivative and regularity of \( V \)

To determine the regularity of the operator \( V \) (defined by (2.22)), it turns out to be useful to determine the Fréchet derivative of \( V \). We do this step by step.

Let \((U, \varphi, V, \psi)\) be an element of a localizing system. Let \( \rho \in \mathcal{U} \) (see (2.18)), and let

\[
I(\rho) := \int_U \psi(\varphi(u)) \int_0^{\rho(\varphi(u))} \sqrt{g^*(u, u_{m+1})} \, du_{m+1} \, du.
\]

Note that \( I(\rho) \) is an element of the sum in (2.21). Observe that for \( h \in h^{2+\alpha}(\Gamma) \) small, we have that

\[
I(\rho + h) - I(\rho) = \int_U \psi(\varphi(u)) \int_0^{\rho+h(\varphi(u))} \sqrt{g^*(u, u_{m+1})} \, du_{m+1} \, du
= \int_U \psi(\varphi(u)) \int_0^{h(\varphi(u))} \sqrt{g^*(u, u_{m+1} + \rho(\varphi(u)))} \, du_{m+1} \, du
\]

\[
= \int_U \psi(\varphi(u)) \left( h(\varphi(u)) \sqrt{g^*(u, \rho(\varphi(u)))} + o(h(\varphi(u))) \right) \, du.
\]

* : Taylor expansion on the inner integral.

For the error term, we have that

\[
\left| \int_U \psi(\varphi(u)) \, o(h(\varphi(u))) \, du \right| \leq \int_U \psi(\varphi(u)) \, o(\|h\|_0) \, du
\]

\[
\leq o(\|h\|_0) |\varphi^{-1}(\text{supp } \psi)|
\leq o(\|h\|_{h^{2+\alpha}(\Gamma)}).
\]

* : for \( f \in C(\Gamma) \), we define \( \|f\|_0 := \max_{\xi \in \Gamma} |f(\xi)| \),

* : \( \text{supp } \psi \) is compact in \( V \), so \( \varphi^{-1}(\text{supp } \psi) \) is bounded in \( U \). Furthermore, \( \psi \leq 1 \).

Then we easily see from (C.1) and Definition B.1 that

\[
I'(\rho) h = \int_U \psi(\varphi(u)) \, h(\varphi(u)) \sqrt{g^*(u, \rho(\varphi(u)))} \, du.
\]

Now we are going to use all elements of the localizing system, so we introduce an index \( i \) with range \( \{1, \ldots, N\} \). Let

\[
\tilde{I}(\rho) := \sum_{i=1}^N I_i(\rho) = \sum_{i=1}^N \int_{U_i} \psi_i(\varphi_i(u)) \int_0^{\rho(\varphi_i(u))} \sqrt{g_i^*(u, u_{m+1})} \, du_{m+1} \, du.
\]

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Then by the linearity of the Fréchet derivative, we obtain

\[ \tilde{I}'(\rho)h = \sum_{i=1}^{N} \int_{U_i} \psi_i(\varphi_i(u)) h(\varphi_i(u)) \sqrt{g_i^*(u, \rho(\varphi_i(u)))} \, du. \]

Finally, we calculate

\[
\frac{1}{\omega_{m+1}} \mathcal{V}(\rho + h) \overset{(2.21)}{=} \frac{1}{|\Omega| + \tilde{I}(\rho + h)} = \frac{1}{|\Omega| + \tilde{I}(\rho) + \tilde{I}'(\rho)h + o(h)} = \frac{1}{|\Omega| + \tilde{I}(\rho)} \left( 1 - \frac{\tilde{I}'(\rho)h}{|\Omega| + \tilde{I}(\rho)} + o(h) \right) = \frac{1}{\omega_{m+1}} \mathcal{V}(\rho) - \frac{\tilde{I}'(\rho)h}{(|\Omega| + \tilde{I}(\rho))^2} + o(h),
\]

which shows that

\[
\frac{1}{\omega_{m+1}} \mathcal{V}'(\rho)h = -\frac{\mathcal{V}(\rho)h}{(|\Omega| + \tilde{I}(\rho))^2} = -\frac{\mathcal{V}(\rho)^2}{\omega_{m+1}^2} \tilde{I}'(\rho)h. \quad (C.2)
\]

Although the last formula looks awkward, it shows us the regularity of \( \mathcal{V} \). We know that \( \mathcal{V}^2 \) is Fréchet differentiable, and \( \tilde{I}(\rho)(h) \) is a sum of integrals, of which the integrands depend smoothly on \( \rho \). Therefore, \( \mathcal{V} \in C^\infty(\mathfrak{M}; \mathcal{L}(h^{2+\alpha}(\Gamma))) \), so

\[
\mathcal{V} \in C^\infty(\mathfrak{M}; h^{2+\alpha}(\Gamma)). \quad (C.3)
\]

Note that for any \( \rho \in \mathfrak{M} \) and \( h \in h^{2+\alpha}(\Gamma) \), we regard \( \mathcal{V}(\rho) \) and \( \mathcal{V}'(\rho)h \) as constant functions on \( \Gamma \).

For our application, \( \mathcal{V}'(0) \) has a particular interest. To calculate it, we have to plug \( \rho = 0 \) into (C.2). Note that \( \mathcal{V}(0) = \omega_{m+1}/|\Omega| \), and that

\[
\sqrt{g_i^*(u, 0)} \overset{(2.14)}{=} \sqrt{g_i(u)},
\]

from which we obtain

\[
\tilde{I}'(0)h = \sum_{i=1}^{N} \int_{U_i} \psi_i(\varphi_i(u)) h(\varphi_i(u)) \sqrt{g_i(u)} \, du = \int_{\Gamma} h,
\]

\( *: \) see Definition 2.19.

Now we see from (C.2) that

\[
\mathcal{V}'(0)h = -\frac{1}{|\Omega|^2} \int_{\Gamma} h.
\]
D Spherical harmonics

The spherical harmonics are smooth functions defined on $S_n$, i.e. the unit sphere in $\mathbb{R}^{n+1}$. They are important to us, because they happen to be the eigenfunctions of the linearization of our problem around equilibria (see Section 7.5).

**Definition D.1.** *(Spherical harmonics.)* Let $n, k \in \mathbb{N}$ and let $\mathcal{P}(\mathbb{R}^{n+1})$ be the space of all polynomials of $n+1$ variables. Then

$$\mathcal{H}^{n+1}_k := \{ p|_{S_n} \mid p \in \mathcal{P}(\mathbb{R}^{n+1}), \Delta p = 0, \forall \lambda, x \in \mathbb{R}^{n+1} : p(\lambda x) = \lambda^k p(x) \} \quad (D.1)$$

is called the set of spherical harmonics of order $k$ in $n+1$ variables.

**Remark D.2.** The word 'spherical' is used because we restrict the polynomials on $\mathbb{R}^{n+1}$ to $S_n$. The word 'harmonics' is used because we demand $\Delta p = 0$. The last condition for $p$ is equivalent to requiring that $p$ is a homogeneous polynomial of order $k$. ■

Definition (D.1) looks quite technical, so let us look at $\mathcal{H}^{n+1}_k$ for small $n$. $\mathcal{H}^{1+1}_1$ is either empty, or equivalent to a pairing of 2 real numbers, so $n = 0$ gives no feeling for the spherical harmonics. $\mathcal{H}^{2+1}_2$ is much more interesting. One can show that

$$\mathcal{H}^{2+1}_2 \cong \langle t \mapsto \cos kt, t \mapsto \sin kt \rangle, \quad k \geq 1. \quad (D.2)$$

We have used the symbol ‘$\cong$’ to indicate that both spaces contain different objects; the space in the left hand side contains real-valued functions defined on $S_1 \subset \mathbb{R}^2$, whereas the space in the right hand side contains $2\pi$-periodic, real-valued functions defined on $\mathbb{R}$. Of course, there is a natural isomorphism between these sets.

In the following Proposition, we list the properties of the spaces of spherical harmonics that we need in this thesis. Property (i) is stated in [21], Lemma 4 on page 5 (one needs to use (11), from which Property (i) follows by straightforward calculations). Properties (ii) - (iv) can be found in [20], Theorem 2 on page 26, Lemma 1 on page 82, and Theorem 5 on page 21.

To get a feeling about any of these properties, one is advised to take $n = 1$ and use (D.2) to see what the concerning property says.

**Proposition D.3.** Let $k, n \in \mathbb{N}$, then

(i) $\dim \mathcal{H}^{n+1}_k = \left( \begin{array}{c} n+k \\end{array} \right) - \left( \begin{array}{c} n+k-2 \\end{array} \right) =: N(k)$,

(ii) $\bigoplus_{k=0}^{\infty} \mathcal{H}^{n+1}_k$ is dense in $L_2(S_n)$,
(iii) $\mathcal{H}_k^{n+1}$ is the eigenspace of $\Delta_{S_n}$ (see Definition 2.33) belonging to the eigenvalue $-k(k + n - 1)$, i.e.
\[ f \in \mathcal{H}_k^{n+1} \iff \Delta_{S_n} f = -k(k + n - 1) f, \]

(iv) $\forall \ell \in \mathbb{N} \setminus \{k\} : \mathcal{H}_k^{n+1} \perp \mathcal{H}_\ell^{n+1}$ with respect to the inner product on $L_2(S_n)$, i.e.
\[ \forall \ell \in \mathbb{N} \setminus \{k\} \forall f \in \mathcal{H}_k^{n+1} \forall g \in \mathcal{H}_\ell^{n+1} : \int_{S_n} f \overline{g} = 0. \]

**Remark D.4.** For our application of spherical harmonics, $n$ is fixed. Therefore, we do not denote the dependence of $N(k)$ in part (i) on $n$ explicitly.

Parts (ii) and (iv) of Proposition D.3 gives us an orthonormal basis of $L_2(S_n)$ in terms of spherical harmonics. Since $\mathcal{H}_k^{n+1}$ is finite dimensional, it has an orthonormal basis
\[ \{ \varphi_{k1}, \ldots, \varphi_{kN(k)} \}. \]

From part (iv), we then have that
\[ \{ \varphi_{k\ell} \mid k \in \mathbb{N}, \ell \in \{1, \ldots, N(k)\} \} \]
is an orthonormal basis of $L_2(S_n)$ (this follows from part (ii)). With this orthonormal basis, we can write any $f \in L_2(S_n)$ as
\[ f = \sum_{k=0}^{\infty} \sum_{\ell=1}^{N(k)} f_{k\ell} \varphi_{k\ell}, \]
where $f_{k\ell}$ are real constants. Here, the infinite sum should be understood as a limit in $L_2(S_n)$.

It can be shown that
\[ \| f \|_{L_2(S_n)}^2 := \sum_{k=0}^{\infty} \sum_{\ell=1}^{N(k)} |f_{k\ell}|^2 \tag{D.3} \]
defines a norm on $L_2(S_n)$ which is equivalent to any norm on $L_2(S_n)$ defined with the help of a localizing system on $S_n$ (such norms are defined in Appendix A in the setting of (little) Hölder spaces).

This norm can be generalized to Sobolev spaces on $S_n$. For example, it can be shown that
\[ \| f \|_{H^2_2(S_n)}^2 := \sum_{k=0}^{\infty} \sum_{\ell=1}^{N(k)} (k^4 + 1) |f_{k\ell}|^2 \tag{D.4} \]
defines a norm on $H^2_2(S_n)$, which again is equivalent to norms on $H^2_2(S_n)$ that are defined with the help of a localizing system on $S_n$.
Bibliography


