The Flow of a Conducting Fluid past a Uniformly Magnetised Cylinder

Bachelor Thesis

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Abstract

This report deals with an introductory application of magnetohydrodynamics (MHD): the flow of a conducting fluid past a uniformly magnetised cylinder. Its mathematical formulation consists of the Maxwell equations, governing the electrodynamics of the problem, and conservation of mass and the Navier-Stokes equations, that describe the motion of the fluid. A dimensional analysis of the model brings out three dimensionless quantities by which the problem can be described: the Reynolds number $Re$, the magnetic Reynolds number $R_m$ and the Hartmann number $Ha$. The magnetic field was calculated analytically for the stationary case in which $R_m \ll 1$ and $Ha \ll 1$, both for a potential flow ($Re \gg 1$) and an Oseen flow ($Re \ll 1$). Afterwards, numerical simulations were done by using the software package COMSOL, covering a large regime of the three dimensionless quantities. The results show that the analytical calculations done in advance are consistent with the numerical model. Both the analytical and the numerical model appear to give physical results. Qualitatively, the field behaviour can be described as follows. The magnetic field is advected with the fluid flow, an effect that becomes stronger for increasing values of $dim R_m$. For $R_m \geq 1$ the initial field is completely dragged along by the fluid, leaving a smaller residual magnetic field in the vicinity of the cylinder. In the case of a Von Kármán vortex street in the velocity field, the magnetic field oscillates with the vortex shedding frequency and increasing $R_m$ to 10 causes the magnetic field lines to break and rejoin, creating closed curves that move upstream. Increasing $Ha^2/Re$ for $R_m \ll 1$ produces different flow phenomena for different values of $Re$. In general, for $Ha^2/Re \gg 1$ a region in the velocity field originates in which the velocity appears to be much smaller than in the rest of the field, but there seems to be a flow to and away from the magnetic poles of the cylinder. For $Re = 25$ the two vortices that appear in the wake of the cylinder increase in dimensions for larger values of $Ha^2/Re$. Also, these vortices are advected with the main flow. For $Re \geq 1000$ the Strouhal number of the Von Kármán vortex street was measured as a function of both $R_m$ and $Ha^2/Re$. The results of these measurements appear to be independent of the value of $Re$. 
## Contents

1 Introduction .................................................. 5

2 Formulation .................................................. 7
   2.1 Description of the model .................................. 7
   2.2 Basic equations of magnetohydrodynamics ............... 8
       2.2.1 Maxwell’s equations and Ohm’s law .................. 9
       2.2.2 Navier-Stokes equations and conservation of mass 9
       2.2.3 Dimensionless equations and basic approximations of MHD 10
   2.3 The magnetic field for $R_m \ll 1$ ......................... 12
   2.4 Two-dimensional flow .................................... 14
       2.4.1 Vorticity and vector potentials .................... 15
   2.5 Boundary conditions ..................................... 16

3 Analysis of Stationary Flows ................................. 19
   3.1 The magnetic field in a potential flow .................. 19
   3.2 Viscous flow ............................................. 24
       3.2.1 Stokes’ equations .................................. 24
       3.2.2 Oseen’s improvement ............................... 25
       3.2.3 The magnetic field in an Oseen flow ............... 27

4 Numerical Results ............................................ 31
   4.1 Model description in numerical software ............... 31
       4.1.1 Software and geometry ................................ 31
       4.1.2 Description of application modes .................. 34
       4.1.3 Boundary and initial conditions .................... 35
   4.2 Qualitative description of field behaviour ............. 37
       4.2.1 Uncoupled situation ............................... 37
       4.2.2 Behaviour of the magnetic field .................... 40
       4.2.3 Behaviour of the velocity field .................... 42
   4.3 Influence of the magnetic field on the vortex street .. 46

5 Conclusion .................................................. 49
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Bibliography</td>
<td>51</td>
</tr>
<tr>
<td>A  The Unperturbed Magnetic Field</td>
<td>53</td>
</tr>
<tr>
<td>B  Derivation of Equations for $\omega_z$, $A_z$, $\psi$ and $j_z$</td>
<td>55</td>
</tr>
<tr>
<td>C  Behaviour of Magnetic Field for $Re = 0.1$</td>
<td>57</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

Magnetohydrodynamics (MHD) is the field of physics concerned with the mutual interactions of the flow of an electrically conducting fluid and a magnetic field. The theory has applications in plasma physics, as well as other fields in which plasmas play an important role, such as geophysics and astrophysics. The latter, for example, studies the deflection of the solar wind, the flow of charged particles ejected from the sun, by the earth’s magnetic field. The magnetic field of the earth, in turn, is highly influenced by the solar wind.

This report concerns an introductory application of the theory of MHD as a final project for obtaining a Bachelor’s degree in Applied Physics and Applied Mathematics. Inspired by the situation of the solar wind, the flow of an arbitrary conducting fluid past a magnetised cylinder is examined. The cylinder has a uniform magnetisation perpendicular to its surface and it is extended to infinity in both directions. Therefore, the analysis will be essentially two-dimensional. Far from the cylinder, the fluid has a uniform velocity in the positive $x$-direction.

The main question of this report is how the velocity field is influenced by the presence of both the cylinder and its magnetic field. The formulation of the problem includes, among others, the Navier-Stokes equations and the entire problem cannot be solved analytically. Therefore, the problem will be ultimately solved by numerical methods. First however, some approximations will be made which make the problem analytically solvable. Some simpler cases will be analysed to get more mathematical insight in the problem.

Finally, the problem is solved numerically for a large parameter range with software based on the finite element method. The results of these calculations will be used to get qualitative insight in the physics for the different parameter regimes. Furthermore, some quantitative results will be presented for the influence of the magnetic interaction on the Strouhal number of the Von Kármán vortex street that appears at high Reynolds numbers.
Chapter 2

Formulation

2.1 Description of the model

First, the model will be described in more detail. An infinitely extended cylinder of radius \( a \) possesses a uniform magnetisation \( M^1 \) in its interior. The vector \( M \) is directed perpendicular to the surface of the cylinder. A Cartesian coordinate system is set up with the \( z \)-axis along the axis of the cylinder and the \( y \)-axis along the direction of \( M \), so that the magnetisation can be written as:

\[
M := M^0 e_y. \tag{2.1}
\]

Finding the magnetic field generated by the cylinder is now essentially a 2-dimensional problem. One can show (Appendix A) that the magnetisation above produces a magnetic field

\[
B^0 = \begin{cases} 
\frac{1}{2}\mu_0 M^0 e_y, & \text{if } x^2 + y^2 < a^2, \\
\frac{1}{2}\mu_0 M^0 a^2 \frac{2xy e_x - (x^2 - y^2) e_y}{(x^2 + y^2)^2}, & \text{if } x^2 + y^2 > a^2. 
\end{cases} \tag{2.2}
\]

in which \( \mu_0 \) is the magnetic permeability of free space. The magnetic field outside the cylinder can be expressed more conveniently in the usual polar coordinates \((r, \theta)\):

\[
B^0 = \frac{1}{2}\mu_0 M^0 \left( \frac{a}{r} \right)^2 (\sin \theta \, e_r - \cos \theta \, e_\theta), \quad r > 1. \tag{2.3}
\]

A schematic image of the situation including the magnetic field lines of (2.2) is shown in Fig. 2.1. Now, a conducting fluid flows into the region of the cylinder and its magnetic field \( B^0 \) from the left. The main flow of the fluid far from the cylinder is given by:

\[
v = V_\infty e_x, \quad x \to -\infty, \quad y \to \pm \infty. \]

\(^1\)The quantity \( M \) is the total magnetic dipole moment per unit volume of an object, see [3].
Figure 2.1: A schematic image of the model. The circle in the center represents the cylinder with a uniform magnetisation $\mathbf{M}$ in the positive $y$-direction. The blue curves are the field lines of the unperturbed magnetic field $\mathbf{B}^0$ generated by the cylinder. From the left, a uniform parallel flow in the positive $x$-direction with magnitude $V_\infty$ encounters the cylinder and its magnetic field.

The velocity field $\mathbf{v}$ will be distorted by the presence of the cylinder. Because the fluid is a conductor, charged particles can move freely inside the material, so there can be a distribution of electric current inside the fluid. The magnetic field $\mathbf{B}^0$ interacts with the currents inside the fluid. This interaction causes a body force that further influences the velocity field. The magnetic field, in turn, is distorted by the flow. From now on, the field $\mathbf{B}^0$ will be called the unperturbed field. In the following, the equations that govern the interactions between the velocity field and the magnetic field are derived.

### 2.2 Basic equations of MHD

In this section, the basic equations of magnetohydrodynamics are derived in the dimensionless form that is most convenient for the problem. Seeing that MHD is a combination of electromagnetism and fluid mechanics, the basic equations are derived from the Maxwell equations and the Navier-Stokes equations. Also, Ohm’s law and the continuity equation are needed to complete the formulation of MHD.
2.2 Basic equations of magnetohydrodynamics

2.2.1 Maxwell’s equations and Ohm’s law

Maxwell’s equations are as follows:

\[
\nabla' \cdot \mathbf{E}' = \frac{q'}{\varepsilon}, \quad \nabla' \times \mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t'},
\]

\[
\nabla' \cdot \mathbf{B}' = 0, \quad \nabla' \times \mathbf{B}' = \mu j' + \varepsilon \mu \frac{\partial \mathbf{E}'}{\partial t'}.
\]

(2.4)

Here, \( \mathbf{E}' \) [V/m] is the electric field, \( \mathbf{B}' \) [T] is the magnetic field, \( q' \) [C/m\(^3\)] the charge density, \( j' \) [A/m\(^2\)] is the current density, and \( \varepsilon \) [F/m] and \( \mu \) [H/m] are the electric permittivity and the magnetic permeability, respectively. Frequently in MHD, the fluid is assumed to be non-magnetisable, so that the value of \( \mu \) equals that of the permeability of free space \( \mu_0 \).

Ohm’s law can be derived by considering small volume elements from the conducting material (see, for example, [8]). It is most convenient in the following form:

\[
\mathbf{j}_c = \sigma (\mathbf{E}' + \mathbf{v}' \times \mathbf{B}').
\]

(2.5)

In this equation, \( \mathbf{j}_c \) is the so-called conduction current density, \( \mathbf{v}' \) is the velocity field of the conducting fluid and \( \sigma \) [S/m] is the electric conductivity of the fluid. The conduction current density is related to the total current density \( j' \) by

\[
\mathbf{j}' = \mathbf{j}_c + q' \mathbf{v}' + \frac{\partial \mathbf{P}'}{\partial t'},
\]

(2.6)

in which the second and third terms on the right side represent the current density due to conduction by the velocity field and polarisation of the conducting material, respectively. [8] The quantity \( \mathbf{P}' \) [C/m\(^2\)] in (2.6) is called the polarisation, which equals the total dipole moment of the conducting material per unit volume. For more information on electromagnetism, see [3].

2.2.2 Navier-Stokes equations and conservation of mass

The fluid-mechanical aspect of MHD is based on two principles: conservation of mass and conservation of momentum [7]. In general, conservation of mass is described by

\[
\frac{\partial \rho'}{\partial t'} + \nabla' \cdot (\rho' \mathbf{v}') = 0,
\]

\footnote{Primes are used to distinguish between the following quantities and their dimensionless analogues, which will be used frequently later on in this report.}
in which $\rho \ [\text{kg/m}^3]$ is the mass density of the fluid. In the model $\rho$ will be considered constant and uniform, so that the last equation is reduced to the continuity equation for incompressible flow:

$$\nabla' \cdot \mathbf{v}' = 0.$$  \hspace{1cm} (2.7)

Conservation of momentum is represented by the Navier-Stokes equations, the general form of which is:

$$\rho \left[ \frac{\partial \mathbf{v}'}{\partial t'} + (\mathbf{v}' \cdot \nabla')\mathbf{v}' \right] = -\nabla' p' + \nabla' \cdot \mathbf{T} + \mathbf{f}.$$  \hspace{1cm} (2.8)

Here, $p' \ [\text{N/m}^2]$ is the pressure in the fluid, $\mathbf{T} \ [\text{N/m}^2]$ is a tensor describing viscous surface forces and $\mathbf{f} \ [\text{N/m}^3]$ is an additional body force. For an incompressible, Newtonian fluid the viscosity term in (2.8) can be expressed by \cite{7}:

$$\nabla' \cdot \mathbf{T} = \eta \Delta' \mathbf{v}'.$$  \hspace{1cm} (2.9)

The coefficient $\eta \ [\text{Pa-s}]$ is called the dynamic viscosity. In MHD, the additional body force in (2.8) is caused by an electromagnetic field that acts on any distribution of charge or electric current present in the conducting fluid. It can be shown (see \cite{8}) to equal:

$$\mathbf{f} = q' \mathbf{E}' + j' \times \mathbf{B}'.$$  \hspace{1cm} (2.10)

Thus, from (2.8)–(2.10) one obtains:

$$\rho \left[ \frac{\partial \mathbf{v}'}{\partial t'} + (\mathbf{v}' \cdot \nabla')\mathbf{v}' \right] = -\nabla' p' + \eta \Delta' \mathbf{v}' + q' \mathbf{E}' + j' \times \mathbf{B}'.$$  \hspace{1cm} (2.11)

### 2.2.3 Dimensionless equations and basic approximations of MHD

The next step is to convert the equations presented above to a dimensionless form and to simplify them using some more or less obvious approximations. The following dimensionless quantities are introduced:

$$\nabla := a \nabla', \quad \Delta := a^2 \Delta', \quad \mathbf{v} := \frac{1}{V_\infty} \mathbf{v}', \quad \mathbf{B} := \frac{1}{B} \mathbf{B}', \quad \mathbf{E} := \frac{1}{BV_\infty} \mathbf{E}', \quad q := \frac{a}{\varepsilon BV_\infty} q', \quad j := \frac{1}{\sigma V_\infty} j', \quad j_c := \frac{1}{\sigma V_\infty} j'_c, \quad p := \frac{1}{\rho V_\infty^2} p'.$$

\hspace{1cm} (2.12)
2.2 Basic equations of magnetohydrodynamics

Here, \( a \) [m] is the radius of the cylinder and \( V_\infty \) [m/s] is the magnitude of the parallel flow far from the cylinder, as described in Section 2.1. The quantity \( B \) [T] is a characteristic magnitude of the magnetic field \( B' \), to be defined later. Applying these transformations to Maxwell’s equations (2.4) yields:

\[
\nabla \cdot E = q, \quad \nabla \times E = -\frac{\partial B}{\partial t},
\]
\[
\nabla \cdot B = 0, \quad \nabla \times B = R_m \dot{j} + Lo^2 \frac{\partial E}{\partial t}.
\] (2.13)

Two dimensionless numbers appear in (2.13), namely the magnetic Reynolds number:

\[
R_m := \frac{\mu_0 \sigma a V_\infty}{\nu} \quad (2.14)
\]

and the Lorentz number:

\[
Lo := \frac{V_\infty}{c}, \quad (2.15)
\]

in which \( c \) [m/s] is the speed of light in the fluid. This quantity has appeared through the relation \( c = 1/\sqrt{\varepsilon \mu} \), used in the derivation of (2.13). By applying the transformations of (2.12) to (2.5) and (2.6), one obtains:

\[
j_c = E + v \times B, \quad (2.16)
\]
\[
j = j_c + \frac{Lo^2}{R_m} \left( qv + \frac{\partial P}{\partial t} \right). \quad (2.17)
\]

Finally, the dimensionless forms of the continuity equation (2.7) and the Navier-Stokes equations (2.11) are found to be:

\[
\nabla \cdot v = 0, \quad (2.18)
\]
\[
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + \frac{1}{Re} \Delta v + \frac{Ha^2}{Re} \left( Lo^2 qE + j \times B \right). \quad (2.19)
\]

Equation (2.19) shows the Reynolds number

\[
Re := \frac{a V_\infty}{\nu}, \quad (2.20)
\]

in which \( \nu := \eta/\rho \) [m²/s] is the so-called kinematic viscosity, and the Hartmann number

\[
Ha := aB \sqrt{\frac{\sigma}{\eta}}. \quad (2.21)
\]
The Lorentz number appears in the dimensionless equations because, according to (2.12), the ratio between the magnitudes of $E'$ and $B'$ is assumed to be of the order of $V_{\infty}$. This assumption is valid if the frequencies of the electromagnetic waves in the problem are small, so that the time-dependence of the electromagnetic field due to changing charge distributions is negligible. [8] In the model the reasonable assumption $V_{\infty} \ll c$ is made, so $Lo \ll 1$. In fact, $Lo^2$ will be so small that one can safely assume $Lo^2 \ll R_m$. Therefore, all terms in equations (2.13), (2.16)–(2.19) that include $Lo$ can be neglected. Equation (2.17) then shows that $j = j_c$: only conduction contributes to the total current, whereas convection and polarisation are neglected. From now on the current will be represented by the single quantity $j$. Furthermore, the influence of the charge density $q$ is limited to Gauss’s Law, $\nabla \cdot E = q$. This equation can be discarded without jeopardising the solvability of the problem. Indeed, this shows that the electromagnetic fields do not essentially depend on changes in $q$.

The low-Lorentz number approximation now yields:

\begin{align}
\nabla \cdot B &= 0, \\
\nabla \times E &= -\frac{\partial B}{\partial t}, \\
\nabla \times B &= R_m j, \\
\n\nabla \cdot E &= \nabla \times (v \times B), \\
\n\nabla \cdot v &= 0,
\end{align}

(2.22a) (2.22b) (2.22c) (2.22d) (2.22e)

These are the basic equations of incompressible magnetohydrodynamics in dimensionless form for the unknowns $E$, $B$, $j$, $v$ and $p$.

### 2.3 The magnetic field for $R_m \ll 1$

Equation (2.22c) suggests that the magnetic Reynolds number $R_m$ can be seen as a parameter that determines the intensity of the source $j$ with respect to $\nabla \times B$. One can show how this affects the magnetic field $B$ by combining (2.22a)–(2.22d). Taking the curl of (2.22c) and (2.22d) and using the identity $\nabla \times (\nabla \times B) = \nabla (\nabla \cdot B) - \Delta B$, it follows that

\[
\frac{\partial B}{\partial t} = \frac{1}{R_m} \Delta B + \nabla \times (v \times B).
\]

(2.23)

This equation resembles the diffusion equation. The first term on the right can be regarded as a diffusive term and the second as a convective term. Equation (2.23) shows that $R_m$ represents the ratio of the magnitudes of the convective term and the diffusive term.
If \( R_m \) is large, the convective term dominates and the magnetic field is advected with the flow of the fluid. It can be shown that in a perfectly conducting fluid \((\sigma \to \infty)\) the magnetic flux through any loop moving with the fluid is constant. \([8]\) This means that magnetic field lines are essentially "frozen" inside the fluid and are dragged along with it.

For \( R_m \ll 1 \), the situation is entirely different. In this regime diffusion is dominant and (2.23) essentially becomes a Laplace equation,

\[
\Delta B = 0,
\]  

which is the equation of a stationary magnetic field in a region without any current distribution (a field that is both solenoidal and irrotational, according to (2.22a) and (2.22c)). The magnetic field that satisfies (2.24) is not generated inside the fluid and therefore it must be an externally imposed field. This means the magnetic field is hardly influenced by the fluid flow if the magnetic Reynolds number is low. Therefore, the total magnetic field \( B \) can be regarded as the sum of an externally imposed field \( B^0 \) and a small perturbation of order \( R_m \). This can be written as

\[
B = B^0 + R_m b,
\]  

in which \( b \) represents the perturbation\(^3\). By combining (2.23) and (2.25), keeping in mind that \( B^0 \) satisfies (2.24), one finds the following equation:

\[
R_m \frac{\partial b}{\partial t} = \Delta b + \nabla \times (v \times B^0) + R_m \nabla \times (v \times b).
\]  

Furthermore, (2.22c) and (2.22f) become:

\[
\nabla \times b = j,
\]  

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + \frac{1}{Re} \Delta v + \frac{Ha^2}{Re} j \times B^0 + R_m \frac{Ha^2}{Re} j \times b.
\]  

Together with \( \nabla \cdot v = 0 \) and \( \nabla \cdot b = 0 \), equations (2.26)-(2.28) are equivalent to (2.22) for finding the unknowns \( b, v, j \) and \( p \). After applying the approximation \( R_m \ll 1 \), these equations become:

\(^3\)The quantity \( b \) is regarded as being of the same order of magnitude as \( B^0 \), which is more convenient for notational purposes.
\[
\Delta b = -\nabla \times (v \times B^0), \quad (2.29a)
\]
\[
\nabla \times b = j, \quad (2.29b)
\]
\[
\nabla \cdot b = 0, \quad (2.29c)
\]
\[
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + \frac{1}{\text{Re}} \Delta v + \frac{Ha^2}{\text{Re}} j \times B^0, \quad (2.29d)
\]
\[
\nabla \cdot v = 0. \quad (2.29e)
\]

**2.4 Two-dimensional flow**

The model is essentially a two-dimensional problem, so the velocity field and the magnetic field have the following form:

\[
v = u e_x + v e_y, \quad B = (B_x^0 + R_m b_x)e_x + (B_y^0 + R_m b_y)e_y.
\]

Equations (2.26)–(2.28), (2.29c) and (2.29e) then become

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{1}{\text{Re}} \frac{\partial^2 u}{\partial x^2} + \frac{1}{\text{Re}} \frac{\partial^2 u}{\partial y^2} - \frac{Ha^2}{\text{Re}} j_z (B_x^0 + R_m b_x), \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \frac{1}{\text{Re}} \frac{\partial^2 v}{\partial x^2} + \frac{1}{\text{Re}} \frac{\partial^2 v}{\partial y^2} + \frac{Ha^2}{\text{Re}} j_z (B_y^0 + R_m b_y), \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial b_x}{\partial t} + \frac{\partial}{\partial y} (v b_x - u b_y) &= \frac{1}{R_m} \left( \frac{\partial^2 b_x}{\partial x^2} + \frac{\partial^2 b_x}{\partial y^2} \right) - \frac{1}{R_m} \frac{\partial}{\partial y} (v B_x^0 - u B_y^0), \\
\frac{\partial b_y}{\partial t} - \frac{\partial}{\partial x} (v b_x - u b_y) &= \frac{1}{R_m} \left( \frac{\partial^2 b_y}{\partial x^2} + \frac{\partial^2 b_y}{\partial y^2} \right) + \frac{1}{R_m} \frac{\partial}{\partial x} (v B_x^0 - u B_y^0), \\
\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y} &= j_z, \\
\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} &= 0.
\end{align*}
\]

Here, \( j_z \) is the \( z \)-component of the current density \( j \), which, according to (2.27) is the only component of \( j \) in a two-dimensional magnetic field.
2.4 Two-dimensional flow

2.4.1 Vorticity and vector potentials

Equations (2.30) can be reformulated by introducing three new quantities. The first of these is the vorticity $\omega$, which is defined as the curl of the velocity field $v$. In a two-dimensional flow, the vorticity only has a $z$-component, equal to:

$$\omega_z := \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \quad (2.31)$$

Furthermore, because the vector fields $v$ and $b$ are solenoidal, they can be described by vector potentials. Like the vorticity, these vector potentials only have $z$-components for two-dimensional fields. The magnetic vector potential $A$ is then defined by:

$$\frac{\partial A_z}{\partial x} := -b_y, \quad \frac{\partial A_z}{\partial y} := b_x. \quad (2.32)$$

The corresponding quantity for the velocity field is often called the stream function $\psi$. Like the magnetic vector potential, it is defined by:

$$\frac{\partial \psi}{\partial x} := -v, \quad \frac{\partial \psi}{\partial y} := u. \quad (2.33)$$

The quantities $\omega_z$, $A_z$, $\psi$ and $j_z$ satisfy the following equations (which are derived in Appendix B):

\[
\begin{align*}
\frac{\partial \omega_z}{\partial t} + [\omega_z, \psi] &= \frac{1}{Re} \left( \frac{\partial^2 \omega_z}{\partial x^2} + \frac{\partial^2 \omega_z}{\partial y^2} \right) + \frac{Ha^2}{Re} \left( B_x \frac{\partial j_z}{\partial x} + B_y \frac{\partial j_z}{\partial y} + R_m [j_z, A_z] \right), \quad (2.34a) \\
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} &= -\omega_z, \quad (2.34b) \\
\frac{\partial A_z}{\partial t} + [A_z, \psi] &= \frac{1}{R_m} \left( \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} \right) + \frac{1}{R_m} \left( B_x \frac{\partial \psi}{\partial x} + B_y \frac{\partial \psi}{\partial y} \right), \quad (2.34c) \\
\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} &= -j_z. \quad (2.34d)
\end{align*}
\]

The operator $[\cdot, \cdot]$ is defined by:

$$[A, B] := \frac{\partial A}{\partial x} \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y} \frac{\partial B}{\partial x}. $$
2.5 Boundary conditions

To solve the problem we need to consider some boundary conditions for the velocity field \( \mathbf{v} \) and the magnetic field \( \mathbf{B} \). The velocity field exists only in the region outside the cylinder, so for this vector field we need boundary conditions at the fixed surface of the cylinder and at infinity. For a realistic fluid, viscous effects at the surface of the cylinder will cause the tangential component of \( \mathbf{v} \) to vanish at the boundary. This is called the no-slip condition. Furthermore, the fluid cannot enter the region inside the cylinder, nor can there be a flow from inside the cylinder to the outer region. This means that the normal component of \( \mathbf{v} \) at the boundary must also be zero, the so-called impermeability condition. In terms of polar coordinates, these conditions can be expressed as:

\[
v_r = 0 \quad \text{and} \quad v_\theta = 0, \quad r = 1. \tag{2.35}\]

For \( x \to -\infty \) and \( y \to \pm \infty \), the flow is just the imposed parallel flow and \( \mathbf{v} \) needs to be constant, i.e.:

\[
\mathbf{v} \to \mathbf{e}_x = \cos \theta \, \mathbf{e}_r - \sin \theta \, \mathbf{e}_\theta , \quad r \to \infty, \quad \frac{1}{2} \pi \leq \theta \leq \frac{3}{2} \pi. \tag{2.36}\]

To the right of the cylinder, however, vortex shedding or turbulence can occur, so the condition that \( \mathbf{v} \) converges to a uniform vector field far away from the cylinder does not necessarily apply in this region.

Other boundary conditions apply for the magnetic field. In contrast to the velocity field, the magnetic field does penetrate the cylinder. There is a magnetic field inside the cylinder (denoted \( \mathbf{B}_I \)) and a magnetic field in the outer region (denoted \( \mathbf{B}_{II} \)). A general boundary condition for a magnetic field at a surface is [3]:

\[
\mathbf{B}_{II} - \mathbf{B}_I = \mathbf{K} \times \mathbf{n}, \tag{2.37}\]

in which \( \mathbf{K} \) is a (dimensionless) surface current and \( \mathbf{n} \) is the unit normal vector on the surface in the direction of the outer region (II). Equation (2.37) says that the normal component of \( \mathbf{B} \) is continuous at the boundary, whereas the tangential component is not. In the case of the magnetised cylinder, a surface current equal to \( \mathbf{M}^0 \times \mathbf{n} \) flows on the cylinder’s surface [3], so the tangential component of \( \mathbf{B}^0 \) is discontinuous at the boundary (this phenomenon can be seen in Fig. 2.1). Now, for the sake of simplicity, the magnetisation of the cylinder and the surface current produced by it are assumed to be unaffected by the flow of the fluid outside the cylinder. The total magnetic field \( \mathbf{B} = \mathbf{B}^0 + \mathbf{M}_b \) will still have a continuous normal component and a discontinuous tangential component. However, the discontinuity in the tangential component of both \( \mathbf{B} \) and \( \mathbf{B}^0 \) is the same. Therefore, both
components of the perturbation $\mathbf{b}$ must be continuous. In terms of the vector potential of $\mathbf{b}$ these conditions become\footnote{Because $\mathbf{b} = \nabla \times A_z \mathbf{e}_z$, the tangential component of $\mathbf{b}$ is $-\frac{\partial A_z}{\partial r}$. By assuming $\nabla \cdot A_z \mathbf{e}_z = 0$ (which can be accomplished by a gauge transformation that does not change $\mathbf{b}$ [3]), the continuity of $A_z$ follows from the divergence theorem.}:

$$A_{z,I} = A_{z,II} \quad \text{and} \quad \frac{\partial A_{z,I}}{\partial r} = \frac{\partial A_{z,II}}{\partial r}, \quad r = 1. \quad (2.38)$$

Finally, boundary conditions for the magnetic field at $r = 0$ and at infinity are needed. The magnetic field must remain finite in the origin, otherwise the solution would not be physical. Far from the cylinder the unperturbed magnetic field $\mathbf{B}^0$ converges to zero. Therefore, we expect the interactions between the fluid and the magnetic field to vanish for $r \to \infty$, at least in the region left from the cylinder, where there is no vortex shedding or turbulence. This leads to an \textit{a priori} condition that the perturbation $\mathbf{b}$ should also converge to zero at infinity. In terms of the vector potential these conditions can be expressed as:

$$\frac{1}{r} \frac{\partial A_z}{\partial \theta} < \infty \quad \text{and} \quad \frac{\partial A_z}{\partial r} < \infty, \quad r = 0. \quad (2.39)$$

$$\frac{1}{r} \frac{\partial A_z}{\partial \theta} \to 0 \quad \text{and} \quad \frac{\partial A_z}{\partial r} \to 0, \quad r \to \infty, \quad \frac{1}{2}\pi \leq \theta \leq \frac{3}{2}\pi. \quad (2.40)$$
Chapter 3

Analysis of Stationary Flows

In this chapter the stationary case of the problem is treated. Furthermore, the Hartmann number and the magnetic Reynolds number are assumed to be small to further simplify the problem. Basically this means that the interaction between the velocity field and the magnetic field is considered to be negligibly small. The magnetic term in the Navier-Stokes equations will indeed be neglected in this case, so that the velocity field can be found independently of the magnetic field. This particular case is still useful when the perturbation $b$ in the magnetic field is determined from its vector potential $A_z$ using (2.34c).

3.1 The magnetic field in a potential flow

First the case $Re \gg 1$ will be considered, which means that the fluid is assumed to be inviscid. Together with the assumptions $Ha^2/Re \ll 1$ and $R_m \ll 1$ this yields the following equations for the velocity field:

\[
\begin{align*}
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y}, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0.
\end{align*}
\]  

(3.1)

These equations are known to have an irrotational solution for the flow around a cylinder that satisfies the condition $v \to e_x$ for $x, y \to \pm \infty$. Because in this case $v$ can be described by a scalar potential, this type of flow is often called a potential flow. This solution of (3.1) in terms of the stream function $\psi$ and polar coordinates $(r, \theta)$ is [7]:

\[
\psi(r, \theta) = \left(1 - \frac{1}{r^2}\right) r \sin \theta.
\]  

(3.2)
Because \( \psi e_z = \nabla \times \mathbf{v} \), the radial and azimuthal components of the velocity are given by:

\[
v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \left( 1 - \frac{1}{r^2} \right) \cos \theta, \quad v_\theta = -\frac{\partial \psi}{\partial r} = -\left( 1 + \frac{1}{r^2} \right) \sin \theta.
\]

(3.3)

It is clear from (3.3) that on the surface of the cylinder \( r = 1 \) the radial component \( v_r \) vanishes, but the azimuthal component \( v_\theta \) does not. It follows that \( \mathbf{v} \neq \mathbf{0} \) on the surface of the cylinder, which is incompatible with no-slip boundary conditions. This is the main drawback of the potential flow model: it does not represent a realistic fluid flow, because it does not take into account any viscous effects at the boundary. It does, however, give a good description of the flow for \( r > 1 \) and it can still be useful for calculating the perturbation in the magnetic field. The field lines of the potential flow is shown in Fig. 3.1, along with the polar coordinate system used in the model.

In the stationary case and for \( R_m \ll 1 \), (2.34c) becomes:

\[
\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + B_0^r \frac{\partial \psi}{\partial x} + B_0^\theta \frac{\partial \psi}{\partial y} = 0.
\]

(3.4)

This is a Poisson equation of the form \( \Delta A_z = -f \), in which \( f \) is a function of the spatial coordinates. Outside the cylinder this source term can be expressed in polar coordinates as follows:

\[
f(r, \theta) := \mathbf{B}^0 \cdot \nabla \psi = B_0^r \frac{\partial \psi}{\partial r} + B_0^\theta \frac{1}{r} \frac{\partial \psi}{\partial \theta}.
\]

(3.5)

Figure 3.1: The field lines of the velocity field of the potential flow given by (3.2).
Inside the cylinder there is no fluid flow and the source term in (3.4) will vanish. The unperturbed field $B^0$ is taken from (2.3). This quantity still needs to be made dimensionless according to (2.12). The characteristic magnitude $B$ is now defined as $\mu_0 M^0/2$, and the dimensionless unperturbed field becomes:

$$B^0 = \frac{1}{r^2} \left( \sin \theta \mathbf{e}_r - \cos \theta \mathbf{e}_\theta \right).$$

Together with the boundary conditions from Section 2.5, equations (3.2), (3.4)–(3.6) form the following problem:

$$\begin{align*}
- \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \theta^2} &= 0, & 0 < r < 1, \quad 0 < \theta < 2\pi, \\
- \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \theta^2} &= -\frac{1}{r^4} + \frac{1}{r^2} \cos 2\theta, & 1 < r < \infty, \quad 0 < \theta < 2\pi, \\
\frac{1}{r} \frac{\partial A_z}{\partial r}, \frac{\partial A_z}{\partial \theta} &\rightarrow 0 \text{ for } r \rightarrow \infty, \\
A_z, \frac{\partial A_z}{\partial r} &\text{ continuous in } r = 1, \\
\frac{1}{r} \frac{\partial A_z}{\partial r}, \frac{\partial A_z}{\partial \theta} &\text{ bounded in } r = 0.
\end{align*}$$

To solve this problem, the following series expansion of $A_z$ is used:

$$A_z(r, \theta) = \sum_{n=-\infty}^{\infty} A_n(r) e^{in\theta}.$$  

Substituting this series expansion in the first equation of (3.7) yields the following set of ODEs for the $A_n$ for $0 < r < 1$, with corresponding general solutions:

$$\begin{align*}
r^2 A''_0 + r A'_0 &= 0 \quad \Rightarrow \quad A_0 = \alpha_0 + \beta_0 \log r, \\
r^2 A''_n + r A'_n - n^2 A_n &= 0 \quad \Rightarrow \quad A_n = \alpha_n r^{|n|} + \beta_n r^{-|n|} \quad \text{for } |n| \geq 1.
\end{align*}$$

Because of the boundedness conditions in $r = 0$, all $\beta_n$ must be zero. Therefore, the inner solution can be written as:

$$A_z(r, \theta) = \sum_{n=-\infty}^{\infty} \alpha_n r^{|n|} e^{in\theta}, \quad 0 \leq r \leq 1.$$ 

Because the velocity field is symmetric about the $y$-axis (there is no vortex shedding or turbulence in the potential flow), the boundary conditions of (2.40) are now valid in all directions.
The same technique can be used to find the outer solution. Substituting the series expansion of $A_z$ in the second equation of (3.7) gives the following set of ODEs for $r > 1$:

\[
\begin{align*}
  r^2 A_0'' + r A_0' &= -r^{-2} \quad \Rightarrow \quad A_0 = -\frac{1}{4} r^{-2} + a_0 \log r + b_0, \\
  r^2 A_{\pm 2}'' + r A_{\pm 2}' - 4 A_{\pm 2} &= \frac{1}{2} \quad \Rightarrow \quad A_{\pm 2} = -\frac{1}{8} + a_{\pm 2} r^2 + b_{\pm 2} r^{-2}, \\
  r^2 A_n'' + r A_n' - n^2 A_n &= 0 \quad \Rightarrow \quad A_n = a_n r^n + b_n r^{-n} \quad \text{for } |n| = 1, |n| \geq 3.
\end{align*}
\]

The conditions at infinity now require that the $a_n$ be zero for $|n| \geq 1$ (the $a_0$ term is kept). We now find the following expression for the outer solution:

\[
A_z(r, \theta) = -\frac{1}{4} r^{-2} - \frac{1}{8} (e^{2i\theta} + e^{-2i\theta}) + a_0 \log r + \sum_{n=-\infty}^{\infty} b_n r^{-|n|} e^{in\theta}, \quad r \geq 1. \quad (3.10)
\]

Equations (3.9) and (3.10) are now connected using the boundary conditions in $r = 1$. One degree of freedom then remains: the constant term $\alpha_0^2$. Choosing this term to be zero then leads to the following values of the coefficients:

\[
\begin{align*}
  \alpha_0 &= 0, \quad a_0 = -\frac{1}{2}, \quad b_0 = \frac{1}{4}, \\
  \alpha_{\pm 2} &= -\frac{1}{16}, \quad b_{\pm 2} = \frac{1}{16}, \\
  \alpha_n = b_n &= 0 \quad \text{for } |n| = 1, |n| \geq 3.
\end{align*}
\]

Thus, we find the following solution of (3.7):

\[
A_z(r, \theta) = \begin{cases} 
-\frac{1}{8} r^2 \cos 2\theta, & \text{if } 0 \leq r \leq 1, \\
-\frac{1}{2} \log r + \frac{1}{4} (1 - r^{-2}) + \frac{1}{8} (r^{-2} - 2) \cos 2\theta, & \text{if } 1 < r < \infty.
\end{cases} \quad (3.11)
\]

To visualise this solution, a contour plot is shown in Fig. 3.2. This contour plot represents the field lines of the magnetic perturbation $B$. To see the effect of this perturbation on the total magnetic field, we add $R_m A_z$ to the vector potential of the unperturbed field $B^0$ (which appears in Appendix A). A corresponding contour plot is shown in Fig. 3.3 and this plot represents the field lines of the total magnetic field. The value of the magnetic Reynolds number is chosen to be 0.1, the upper limit of the approximation $R_m \ll 1$. It appears that the magnetic field is slightly advected in the direction of the main flow (compare with Fig. 2.1). This is the expected behaviour that also occurs in the flow of the solar wind past the earth. [6] Thus, this simple model with many approximations still gives a physical result.

\[\text{2Because } A_z e_z \text{ is a vector potential, it is determined up to the gradient of a scalar field. Some degrees of freedom are already used by assuming that } \nabla \cdot A_z e_z = 0 \text{ in Section 2.5, but then } A_z \text{ is still determined up to a constant term.}\]
3.1 The magnetic field in a potential flow

Figure 3.2: The magnetic field lines of the perturbation $b$, given by the vector potential of (3.11). The contour of the cylinder is shown in the center.

Figure 3.3: The magnetic field lines of the total field $B = B^0 + R_m b$, for $R_m = 0.1$. 
3.2 Viscous flow

3.2.1 Stokes’ equations

Now we consider the case \( Re \ll 1 \), in which the fluid has a high viscosity. Again, the assumptions \( Ha^2/Re \ll 1 \) and \( R_m \ll 1 \) are made, so that the equations for the velocity field become:

\[
\begin{align*}
-\frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) &= 0, \\
-\frac{\partial p}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0.
\end{align*}
\]  
(3.12)

These are the well-known Stokes equations for viscous flow. They can be written in terms of the vorticity and stream function as follows (cf. (2.34a), (2.34b)):

\[
\begin{align*}
\frac{\partial^2 \omega_z}{\partial x^2} + \frac{\partial^2 \omega_z}{\partial y^2} &= 0, \\
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} &= -\omega_z.
\end{align*}
\]

Combining these two equations yields a biharmonic equation for the stream function \( \psi \):

\[\Delta^2 \psi = 0.\]

One can show that this equation has no solution satisfying both conditions (2.35) and (2.36). This amounts to solving the following problem:

\[
\begin{align*}
\Delta^2 \psi &= 0, & r > 1, & 0 < \theta < 2\pi, \\
\psi(1, \theta) &= \frac{\partial \psi}{\partial r}(1, \theta) = 0, & 0 < \theta < 2\pi, \\
\psi(r, \theta) &\to r \sin \theta & \text{for } r \to \infty.
\end{align*}
\]  
(3.13)

The last condition in (3.13) suggests a solution of the form \( \psi(r, \theta) = g(r) \sin \theta \), in which \( g \) is an unknown function of the radial coordinate. Using this ansatz to solve (3.13) will lead to the following ODE for \( g \):

\[r^4 g''' + 2r^3 g'' - 3r^2 g' + 3rg - 3g = 0,\]

\(^3\text{The pressure gradient terms must be retained to balance the viscous forces. This fact indicates that the pressure scale used in (2.12) is incorrect in this case [7].}\)
which has the general solution:

\[ g(r) = Ar^3 + Br \log r + Cr + Dr^{-1}. \]

The third condition in (3.13) implies that \( A = B = 0 \). However, if this were true, the solution could not satisfy the second condition (which is equivalent to \( g(1) = g'(1) = 0 \)). So Stokes’ problem for the viscous flow past a cylinder does not have a solution satisfying both the conditions on the cylinder’s surface and at infinity.

### 3.2.2 Oseen’s improvement

In the approximation described above, the non-linear terms of the Navier-Stokes equations representing inertia forces (denoted by \((v \cdot \nabla)v\)) are neglected completely. The failure of the Stokes’ approximation is due to the fact that these inertia forces become increasingly dominant for large values of \( r \). Therefore, they cannot be neglected completely if one wants to describe the fluid flow at a large distance from the cylinder. [1]

Oseen improved Stokes’ model by making a linear approximation of the inertia terms. This approximation is as follows. We assume that the velocity field deviates only slightly from the uniform parallel flow at infinity. Then, the velocity components can be written as:

\[ u = 1 + u', \quad v = v', \]

in which \( u' \) and \( v' \) are considered small. Inserting these expressions in the non-linear terms of the Navier-Stokes equation and neglecting products of the small terms yields:

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \approx \frac{\partial u'}{\partial x}, \quad u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \approx \frac{\partial v'}{\partial x}. \quad (3.14) \]

Substituting the approximations from (3.14) in the Navier-Stokes equations gives the following improvement of Stokes’ original problem ((3.12)):

\[
\begin{align*}
\frac{\partial u'}{\partial x} &= -\frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} \right), \\
\frac{\partial v'}{\partial x} &= -\frac{\partial p}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} \right), \\
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= 0, \\
u' = v' &= 0 \text{ at the cylinder’s surface,} \\
u', v' &\to 0 \text{ at infinity.}
\end{align*}
\quad (3.15)
\]
Batchelor [1] presents an approximate solution for the velocity field $u$ around a cylinder a radius $a$ moving with velocity $U$ in a fluid that is stationary at infinity, i.e. the following problem:

$$\begin{align*}
- \rho U \cdot \nabla u &= -\nabla p + \eta \Delta u, \\
\nabla \cdot u &= 0, \\
\text{u} &= U \text{ at the cylinder’s surface,} \\
\text{u} &\to 0 \text{ and } p - p_0 \to 0 \text{ at infinity.}
\end{align*}$$

The presented solution for (3.16) is:

$$u = U + C U \left( -\frac{1}{2} \log \frac{r}{a} - \frac{1}{4} + \frac{a^2}{4r^2} \right) + C e_x \frac{U \cdot e_x}{r^2} \left( \frac{1}{2} - \frac{a^2}{2r^2} \right).$$

This solution still has a logarithmic term that causes the velocity field to diverge at infinity. According to Batchelor there exists a solution of (3.16) that is self-consistent over the whole field and (3.17) is a reasonable approximation to this solution close to the cylinder, provided the constant $C$ is chosen as follows:

$$C = \frac{2}{\log(3.7/Re)}.$$  

(3.18)

Of course, this is only valid for small Reynolds numbers.

Problem (3.16) can be transformed into problem (3.15) by applying the substitutions $U \to -V_\infty e_x$ and $u \to v + V_\infty e_x$ and scaling correctly according to (2.12). The approximate solution (3.17) then becomes (expressed in polar coordinates):

$$v_r = C \left( \frac{1}{2} \log r - \frac{1}{4} + \frac{1}{4} r^{-2} \right) \cos \theta, \quad v_\theta = C \left( \frac{1}{4} r^{-2} - \frac{1}{4} - \frac{1}{2} \log r \right) \sin \theta,$$

(3.19)

in which $C$ is still according to (3.18). The stream function corresponding to (3.19) is:

$$\psi(r, \theta) = \frac{1}{4} C \left( 2 \log r + r^{-2} - 1 \right) r \sin \theta.$$  

(3.20)

The Oseen flow around the cylinder is visualised in Fig. 3.4. One can see that this flow does not converge to a parallel flow far away from the cylinder, unlike the potential flow shown in Fig. 3.1.

\footnote{Batchelor uses a Reynolds number which is twice the one defined in section 2.2.}
3.2.3 The magnetic field in an Oseen flow

To calculate the perturbation of the magnetic field in the Oseen flow, we again use \((3.4)\), this time with the stream function in \((3.20)\). Instead of \((3.7)\), the following problem is obtained:

\[
\begin{align*}
\frac{1}{r} & \frac{\partial}{\partial r} \left( r \frac{\partial A_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \theta^2} = 0, \quad 0 < r < 1, \quad 0 < \theta < 2\pi, \\
\frac{1}{r} & \frac{\partial}{\partial r} \left( r \frac{\partial A_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \theta^2} = -f(r, \theta), \quad 1 < r < \infty, \quad 0 < \theta < 2\pi, \\
f(r, \theta) &= \frac{1}{4} C\left( r^{-2} - r^{-4} \right) - \frac{1}{2} C r^{-2} \log r \cos 2\theta,
\end{align*}
\]

\((3.21)\)

This problem will now be solved with the method of section 3.1. Inside the cylinder, the equation for \(A_z\) is the same as in \((3.4)\). Therefore, the inner solution is just equal to \((3.9)\):

\[
A_z(r, \theta) = \sum_{n=-\infty}^{\infty} \alpha_n r^{|n|} e^{i n \theta}, \quad 0 \leq r \leq 1.
\]

\((3.22)\)
The outer solution is again found by substituting the series expansion (3.8) in the second equation of (3.21). We then obtain the following set of ODEs for \( r > 1 \), with corresponding general solutions:

\[
\begin{align*}
    r^2 A''_0 + r A'_0 &= \frac{1}{2} C (r^{-2} - 1) \quad \Rightarrow \quad A_0 = \frac{1}{16} \left( r^{-2} - 2(\log r)^2 \right) + a_0 \log r + b_0, \\
    r^2 A''_{\pm 2} + r A'_{\pm 2} - 4 A_{\pm 2} &= \frac{1}{4} C \log r \quad \Rightarrow \quad A_{\pm 2} = -\frac{1}{16} C \log r + a_{\pm 2} r^2 + b_{\pm 2} r^{-2}, \\
    r^2 A''_n + r A'_n - n^2 A_n &= 0 \quad \Rightarrow \quad A_n = a_n r^{|n|} + b_n r^{-|n|} \quad \text{for } |n| = 1, |n| \geq 3.
\end{align*}
\]

Applying the boundary conditions at infinity in (3.21) yields \( a_n = 0 \) for \( |n| \geq 1 \). We then obtain the following solution for the region outside the cylinder:

\[
A_z(r, \theta) = \frac{1}{16} \left( r^{-2} - 2(\log r)^2 \right) - \frac{1}{16} C \log r \left( e^{2i\theta} + e^{-2i\theta} \right) + a_0 \log r + \sum_{n=-\infty}^{\infty} b_n r^{|n|} e^{i\theta}, \quad r \geq 1.
\]

(3.23)

Again, we connect (3.22) and (3.23) with the boundary conditions at \( r = 1 \), choosing \( \alpha_0 \) to be zero. The following values of the coefficients are then obtained:

\[
\begin{align*}
    \alpha_0 &= 0, \quad a_0 = \frac{1}{8} C, \quad b_0 = -\frac{1}{16} C, \\
    \alpha_{\pm 2} &= b_{\pm 2} = \frac{1}{64} C, \\
    \alpha_n &= b_n = 0 \quad \text{for } |n| = 1, |n| \geq 3.
\end{align*}
\]

The solution of (3.21) thus found is:

\[
A_z(r, \theta) = \begin{cases} 
-\frac{1}{32} C r^2 \cos 2\theta, & \text{if } 0 \leq r \leq 1, \\
\frac{1}{16} C \left( r^{-2} - 1 + 2 \log r - 2(\log r)^2 \right) - \frac{1}{32} C (r^{-2} + 4 \log r) \cos 2\theta, & \text{if } 1 < r < \infty.
\end{cases}
\]

(3.24)

A contour plot of (3.24) is shown in Fig. 3.5, representing the perturbation \( b \) of the magnetic field in the Oseen flow. To calculate this perturbation, a Reynolds number of 0.1 is used. In the same way as in section 3.1 we can visualise the total magnetic field, as shown in Fig. 3.6. Like the magnetic field in a potential flow, this field seems to be advected in the direction of the main flow. Again, the simple model gives a physical result. There are some differences with the potential flow case in Fig. 3.3, mainly in the region far from the cylinder. This could be due to the fact that the approximate model used for the viscous flow around cylinder still fails at large distances.
3.2 Viscous flow

Figure 3.5: The magnetic field lines of the perturbation $b$, given by the vector potential of (3.24) for $Re = 0.1$. The contour of the cylinder in shown in the center.

Figure 3.6: The magnetic field lines of the total field $B = B^0 + R_m b$, for $Re = R_m = 0.1$. 
Chapter 4

Numerical Results

This chapter deals with solving the entire problem by numerical methods. The behaviour of both the velocity field and the magnetic field will be investigated for a large regime of the three dimensionless parameters: the Reynolds number $Re$, the magnetic Reynolds number $R_m$ and the Hartmann number $Ha$.

4.1 Model description in numerical software

4.1.1 Software and geometry

The numerical calculations are done with the software package COMSOL Multiphysics\textsuperscript{1}. This package has a solver routine based on the finite element method. In COMSOL one can choose various predefined model descriptions, called application modes, to formulate a problem that covers multiple areas of physics and other exact sciences. In our case, two application modes are used, one covering the electromagnetic aspect of the problem and the other describing incompressible fluid dynamics. A description of these application modes and the way in which they are used to solve the problem, are presented below.

First, the geometry created for this problem is described. In the model presented in section 2.1 the relevant equations are defined on an unbounded domain. This is not possible if one wants to do numerical calculations: the domain on which the solver operates must be bounded. Therefore, we create a geometry in which the outer region is bounded by a rectangular box. It is assumed that the error caused by the boundedness of the domain is small, if its dimensions are chosen sufficiently larger than the dimensions of the cylinder and the distance between the cylinder and the outer boundary is sufficiently large (provided we apply the appropriate boundary conditions, see below). Furthermore, the cylinder should not be precisely in the center of the domain, otherwise the symmetry could prevent a vortex street from appearing.

\textsuperscript{1}For more information, visit the website www.comsol.com
Experiments show that the geometry presented in Fig. 4.1 satisfies the conditions mentioned above well enough for our purposes. In this geometry, the outer domain is a square of 16 cm × 16 cm, centered at the origin. The cylinder is represented by a circle with a radius of 0.2 m, centered at the point (−4, 1). The next step is to define a mesh on this geometry. COMSOL can automatically generate or refine a mesh on a given geometry. The standard mesh is shown in Fig. 4.2 and this mesh proves to be of sufficient quality for our calculations. It consists of 1828 triangular elements bringing 11649 degrees of freedom into the problem. The minimum element quality is 0.6869.
**Figure 4.2:** The standard mesh generated by COMSOL on the geometry shown in Fig. 4.1.
4.1.2 Description of application modes

In the outer region of the geometry the application mode representing the fluid dynamics part of the problem is active. Within the boundaries of the cylinder, this application mode is disabled (there is not fluid flow inside the cylinder). The Incompressible Navier-Stokes application mode is based on the following set of equations\(^2\):

\[
\begin{aligned}
\rho \frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot [\eta (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{F}, \\
\nabla \cdot \mathbf{u} &= 0.
\end{aligned}
\]  

(4.1)

These equations are solved for the velocity vector \(\mathbf{u}\) and the pressure \(p\). Equation (4.1) has 4 parameters which can be defined by the user: the density \(\rho\), the dynamic viscosity \(\eta\) and the \(x\)- and \(y\)-components of the external body force \(\mathbf{F}\).

The other application mode is a predefined electromagnetic model for perpendicular induction currents, using the magnetic vector potential of the total field. It is part of COMSOL’s AC/DC Module and is fully described by the following equation:

\[
\sigma \frac{\partial A_z}{\partial t} - \nabla \cdot (\mu_0^{-1} \nabla A_z - \mathbf{M}) = \sigma \frac{\Delta V}{L} + J^e_z.
\]  

(4.2)

This equation is solved for \(A_z\), which is now the \(z\)-component of the vector potential of the total magnetic field. The user can define the following parameters: the conductivity \(\sigma\), the \(x\)- and \(y\)-components of an external magnetisation \(\mathbf{M}\), an external electric potential difference \(\Delta V\) over a length \(L\) in the \(z\)-direction and an external current density \(J^e_z\). The magnetic permeability of free space \(\mu_0\) is, of course, a constant.

The two application modes are combined in the outer region by defining the external body force \(\mathbf{F}\) and the external current density \(J^e_z\). The body force is chosen as follows:

\[
F_x = -J_z B_y, \quad F_y = J_z B_x,
\]  

(4.3)

in which \(J_z\) is a predefined quantity denoting the \textit{total} current density and \(B_x\) and \(B_y\) are the components of the total magnetic field, also predefined. Equation (4.3) corresponds to the \(\mathbf{j} \times \mathbf{B}\) body force used in previous chapters. The external current density is set to be equal to:

\[
J^e_z = -\sigma \left( u \frac{\partial A_z}{\partial x} + v \frac{\partial A_z}{\partial y} \right),
\]  

(4.4)

in which \(u\) and \(v\) are the two components of the velocity vector \(\mathbf{u}\).

\(^2\)COMSOL has no problem dealing with quantities that have dimensions. From now on, the model will be formulated with such quantities.
4.1 Model description in numerical software

The rest of the parameters are chosen in such a way that the user only has to vary the three dimensionless parameters of the problem to go to a different regime. In the outer region, we have the following:

\[ \rho = 1 \text{ kg/m}^3, \quad \eta = \frac{\rho a U_0}{Re}, \quad \sigma = \frac{R_m}{\mu_0 \mu_r a U_0}. \]

Here, \( a = 0.2 \text{m} \) is the radius of the cylinder and \( U_0 = 1 \text{ m/s} \) is the speed of the parallel inflow (see below). The quantity \( \mu_r \) appears when the user chooses the appropriate constitutive relation between the magnetic field \( B \) and the quantity \( H \), the latter being an auxiliary magnetic field which takes into account magnetisation effects [3]. This option effectively turns the second term of (4.2) into \( \nabla \times (\mu_0^{-1} \mu_r^{-1} \nabla \times A_z) \). We choose \( \mu_r \) to be 1, which means that outside the cylinder there is no magnetisation. Because there are no external electrical effects, the potential difference \( \Delta V \) is chosen to be zero (\( L \) can then have any value except zero, but we choose it to be 1 m).

Inside the cylinder the Navier-Stokes application mode is inactive, so the parameters of (4.1) need not to be considered. There is, however, a magnetic field inside the cylinder, so we still need the other application mode. In the inner region, \( \sigma, J_z \) and \( \Delta V \) are chosen to be zero and \( L \) is again equal to 1 m. This time a different constitutive relation between \( B \) and \( H \) is chosen and the second term in (4.2) does not change. This way the magnetisation inside the cylinder can be specified directly. As described in section 2.1, this magnetisation only has a \( y \)-component, which we choose to be:

\[ M_0 := 2HaU_0 \sqrt{\frac{\mu_r \rho}{\mu_0 R_m Re}}. \]

This is the magnetisation generating the initial unperturbed field, as will be clear from below. The dimensionless parameters used in the expressions above are defined as follows:

\[ Re = \frac{\rho a U_0}{\eta}, \quad R_m = \mu_0 \mu_r \sigma a U_0, \quad Ha = \frac{1}{2} \mu_0 M_0 a \sqrt{\frac{\sigma}{\eta}}. \quad (4.5) \]

From (2.22f) it is clear that the Hartmann number affects the fluid flow only through the dimensionless number \( Ha^2/Re \). Therefore, this number will be changed during the numerical experiments instead of \( Ha \). One can check that the dimensionless expressions in (4.5) are consistent with the values chosen for the parameters above.

4.1.3 Boundary and initial conditions

There are essentially five boundaries in the geometry of Fig. 4.1 on which boundary conditions can be specified: the cylinder’s surface and the four edges of the outer domain. Each application includes a number of predefined boundary conditions. For the Incompressible Navier-Stokes application mode the following boundary conditions are used:
**Left outer boundary:** Inlet, with normal inflow velocity \( U_0 = 1 \text{ m/s} \). This means the velocity on this boundary is specified by \( \mathbf{u} = -U_0 \mathbf{n} \), where \( \mathbf{n} \) is the unit normal vector pointing outwards.

**Upper and lower outer boundary:** Wall with slip, which means the boundary is impermeable (\( \mathbf{n} \cdot \mathbf{u} = 0 \)) and it does not exert any surface stress on the fluid:

\[
\mathbf{t} \cdot \left[ -p I + \eta \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \right] \mathbf{n} = 0,
\]

in which \( \mathbf{t} \) is a unit vector tangent to the boundary.

**Right outer boundary:** Outlet with a specified pressure \( p_0 = 0 \) and no viscous stress:

\[
\eta \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \mathbf{n} = 0.
\]

**Cylinder surface:** Wall with no-slip, meaning \( \mathbf{u} = 0 \) on the entire surface.

Because the electromagnetic application mode is active in both the outer and the inner region of the domain, the cylinder’s surface functions as an **interior boundary** for this application mode. COMSOL automatically applies continuity of flux across interior boundaries, unless specified otherwise. For the electromagnetic application mode, this means that \( \mathbf{n} \times \mathbf{H} \) is continuous across the cylinder’s surface.\(^3\) Thus, we only need to specify boundaries conditions on the outer boundary. The magnetic vector potential \( A_z \) on these boundaries is chosen to be equal to the unperturbed vector potential shown in Appendix A:

\[
A_z = -\frac{1}{2} \mu_0 M_0 \left( \frac{a}{r} \right)^2 \cos \theta.
\]

Initial conditions must be specified for all dependent variables, i.e. \( u, v, p \) and \( A_z \). The initial velocity field is set to be the field of a potential flow, as described in section 3.1. We define a function \( \psi \) in the outer region equal to (3.2) and specify the following values for \( u \) and \( v \):

\[
u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.
\]

The initial pressure is set to be zero. For the initial magnetic vector potential we choose the value of the unperturbed field in Appendix A:

\[
A_z = \begin{cases} 
-\frac{1}{2} \mu_0 M_0 r \cos \theta, & \text{inner region}, \\
-\frac{1}{2} \mu_0 M_0 a^2 \cos \theta / r^2, & \text{outer region}.
\end{cases}
\]

\(^3\)The continuity of \( \mathbf{n} \times \mathbf{H} \) implies that there are no free surface currents on the cylinder’s surface. There is, however, a bound surface current, generated by the magnetisation of the cylinder, which causes a discontinuity in \( \mathbf{B} \). [3]
4.2 Qualitative description of field behaviour.

4.2.1 Uncoupled situation

Now, the behaviour of the velocity field and the magnetic field will be described on the basis of the numerical results. First, the case in which both $R_m$ and $Ha^2/Re$ are small will be considered. In this case, the equations for $v$ and $B$ are uncoupled and we should obtain a situation in which the magnetic field is equal to its unperturbed version (2.2) and the flow is just the classical flow past a unmagnetised cylinder. We will use the values $R_m = Ha^2/Re = 0.001$.

Fig. 4.3 shows the magnetic field lines around the cylinder in this situation. Comparing this with Fig. 2.1 shows that it is indeed equal to the unperturbed field. The shape of this magnetic field proves to be independent of $Re$, as expected. The velocity field, of course, does depend on the value of the Reynolds number. It is calculated for $Re = 0.1$, 25 and 1000. Streamline plots of these three cases are shown in Fig. 4.4-4.6. The last two figures also include a colour plot of the vorticity. It is clear that in the case $Re = 0.1$ the velocity field resembles the Oseen flow shown in Fig. 3.4. In the case $Re = 25$ we obtain two vortices in the wake behind the cylinder and for $Re = 1000$ the famous Von Kármán vortex street appears. [7]
Figure 4.4: Streamline plot of velocity field for $Re = 0.1$ and $R_m = Ha^2/Re = 0.001$.

Figure 4.5: Streamline plot of velocity field for $Re = 25$ and $R_m = Ha^2/Re = 0.001$, including colour plot of the vorticity.
4.2 Qualitative description of field behaviour.

Figure 4.6: Streamline plot of velocity field for $Re = 1000$ and $Rm = Ha^2/Re = 0.001$, including colour plot of the vorticity.
4.2.2 Behaviour of the magnetic field

Next, the magnetic field will be investigated qualitatively for different values of $Re$ and $Rm$. The number $Ha^2/Re$ is chosen equal to 0.001, so that the velocity field is not influenced by the magnetic field. Fig. 4.7 shows the magnetic field lines for $Re = 25$ and $Rm = 0.1$. This figure has many similarities with the analytical solution in Fig. 3.3 (which was also calculated for $Re \gg 1$ and $Rm = 0.1$). In both cases the magnetic field is advected slightly in the direction of the fluid flow.

When we increase $Rm$, the effect of convection becomes stronger, as shown in Fig. 4.8. The numerical simulations show that there is a fundamental difference between the realisation of these fields and that of the field in Fig. 4.7. For $Rm = 0.1$, the magnetic field is only distorted slightly from the initial shape, whereas for $Rm \geq 1$, the initial field is being dragged along with the fluid, leaving a smaller residual magnetic field in the vicinity of the cylinder. This effect is quite strong for $Rm = 100$, in which case the magnetic field is concentrated very close to the cylinder’s surface.

For $Re = 0.1$ the behaviour of the magnetic field for different values of $Rm$ is similar to the ones presented here for $Re = 25$. Contour plots of $A_z$ in this case are shown in Appendix C. The only difference with the case $Re = 25$ is the shape of the residual magnetic field in the wake behind the cylinder for large values of $Rm$. 

**Figure 4.7**: Contour plot of $A_z$ for $Re = 25$, $Rm = 0.1$ and $Ha^2/Re = 0.001$. 

![Contour plot of $A_z$ for $Re = 25$, $Rm = 0.1$ and $Ha^2/Re = 0.001$.](image-url)
4.2 Qualitative description of field behaviour.

Figure 4.8: Contour plots of $A_z$ for $Re = 25$ and $Ha^2/Re = 0.001$. From top to bottom: $R_m = 1, 10, 100$. 
The cases $Re = 0.1$ and $Re = 25$ are similar with respect to the magnetic field, because the velocity field is stationary in these cases. For $Re = 1000$, however, a Von Kármán vortex street appears, as shown in Fig. 4.6, and this is not a stationary situation. The shed vortices move downstream, away from the cylinder. Therefore, we also expect instationary behaviour for the magnetic field.

The magnetic field lines in this case are shown in Fig. 4.9, for three values of $R_m$. In the case $R_m = 0.1$, the magnetic field is similar to the one shown in Fig. 4.7, for $Re = 25$ and the same value of $R_m$. However, numerical simulations show that the field is not stationary in the case $Re = 1000$. The field lines oscillate slightly along the vertical direction, with the same frequency as the Von Kármán vortex street. The influence of the vortex street becomes stronger for larger values of $R_m$. For $R_m = 1$, the second plot of Fig. 4.9 shows that the shape of the magnetic field lines is distorted slightly. The same oscillating behaviour as for $R_m = 0.1$ is seen in this case. For $R_m = 10$, we can see that the effect of the vortex street has become so significant that magnetic field lines break and rejoin, forming closed curves that move downstream. Apart from the oscillations, the magnetic field for $Re = 1000$ also exhibits the behaviour seen previously for $Re = 25$: the overall field is advected slightly by the fluid for $R_m = 0.1$, and dragged along completely for $R_m \geq 1$, leaving behind a smaller residual field.

### 4.2.3 Behaviour of the velocity field

The next step is to examine the effects of the unperturbed magnetic field on the velocity field when the number $Ha^2/Re$ is increased. We choose $R_m$ to be 0.001, so that the magnetic field is not in turn influenced by the fluid flow. It appears that for $Re = 0.1$ the magnetic field does not influence the main characteristics of the flow pattern until $Ha^2/Re$ reaches large values.

Fig. 4.10 shows the streamlines, as well as the vorticity, for $Ha^2/Re = 10^4$. Apparently, the streamlines are pushed outwards in the vicinity of the cylinder. Around the cylinder we see an atypical pattern in which the vorticity is nearly zero. The significance of this pattern becomes clearer when zooming in on the cylinder and plotting the $x$- and $y$-components of the velocity field, as shown in Fig. 4.11. It appears that most of the fluid flows around the region with the atypical vorticity pattern, but there is a small flow to and away from the magnetic poles of the cylinder.

The atypical flow in the vicinity of the cylinder was found to appear for $Ha^2/Re \geq 1000$ in the case $Re = 0.1$, covering a larger part of the domain for increasing values of $Ha^2/Re$. It also appears for $Re = 25$, but already at $Ha^2/Re = 100$. In the case $Re = 25$ we also encounter the phenomenon that the two vortices in the wake behind the cylinder (cf. Fig. 4.5) become larger in both directions for increasing $Ha^2/Re$. These vortices then appear to move downstream. An impression of this effect is shown in Fig. 4.12.
4.2 Qualitative description of field behaviour.

Figure 4.9: Contour plots of $A_z$ for $Re = 1000$ and $Ha^2/Re = 0.001$. From top to bottom: $R_m = 0.1, 1, 10$. 
Figure 4.10: Streamline plot of velocity field for $Re = 0.1$, $R_m = 0.001$ and $Ha^2/Re = 10^4$, including colour plot of the vorticity.

Figure 4.11: Colour plots of the $x$- and $y$-components of the velocity field for $Re = 0.1$, $R_m = 0.001$ and $Ha^2/Re = 10^4$. 
4.2 Qualitative description of field behaviour.

Figure 4.12: Streamline plot of velocity field for $Re = 25$, $R_m = 0.001$ and $Ha^2/Re = 100$, including colour plot of the vorticity.
4.3 Influence of the magnetic field on the vortex street

In the previous section, the influence of the magnetic field on the fluid flow was discussed qualitatively for $Re = 0.1$ and $Re = 25$. The cases $Re \geq 1000$, in which the Von Kármán vortex street appears, will now be dealt with quantitatively in the following manner. The vortex shedding occurs with a certain characteristic frequency $f$. One can define a characteristic time for the problem as $t = 1/f$. If this characteristic time is used to make the Navier-Stokes equation dimensionless, the following dimensionless number, called the Strouhal number, appears in front of the time-derivative of $v$ [7]:

$$Sr = \frac{Df}{U},$$

(4.6)

in which $D$ is a characteristic length (i.e. the diameter of the cylinder) and $U$ is a characteristic measure of speed (i.e. the magnitude of the uniform parallel flow). The quantity $U/f$ is the so-called hydrodynamic wavelength, which is essentially the wavelength we see in the vortex street. Therefore, the Strouhal number can be seen as a dimensionless measure of this wavelength, which is scaled by the cylinder’s diameter. It is known that $Sr \approx 0.2$ for the frequency in the Von Kármán vortex street for a large regime of the Reynolds number, starting with $Re = 300$ [2] [4] [5]. To investigate the influence of the magnetic field on the vortex street, we search for deviations from this value.

The Strouhal number is determined by letting COMSOL measure the vorticity as a function of time, over a period of 10 s, in the point $(−2, 1)$ behind the cylinder, once a steady vortex street has been established. The obtained signal is exported to MATLAB to calculate the corresponding frequency spectrum, using the Fast Fourier transform routine. The most intense frequency is used in (4.6), together with the values $D = 0.4$ m and $U = 1$ m/s, to calculate the corresponding Strouhal number.

For $Re = 1000$ and $R_m = Ha^2/Re = 0.001$ the Strouhal number was found to be equal to $0.22 \pm 0.02$. The same value was found for the following Reynolds numbers: $Re = 5000$, $10000$, $50000$, $100000$. These results confirm that the approximate value of 0.2 mentioned above also holds for our numerical model, and will be used as a reference point for the results below. Further simulations demonstrate that this value for the Strouhal number is also independent of $R_m$, provided $Ha^2/Re$ remains 0.001.

Fig. 4.13 shows a complete overview of the results obtained in these simulations for $Re = 1000$. It is clear that the Strouhal number is a function of both $R_m$ and $Ha^2/Re$. In general, the Strouhal number decreases when $Ha^2/Re$ increases, but the rate of this decrease depends on the value of $R_m$. The cases $R_m = Ha^2/Re = 1$ and $R_m = 10, Ha^2/Re = 100$ were also investigated for $Re = 5000$, $10000$, $50000$ and $100000$. For all these values of the Reynolds number, the same Strouhal number was found as mentioned in Fig. 4.13, i.e. $0.19 \pm 0.02$ and $0.09 \pm 0.02$ respectively.

\[4\]For more information, visit www.mathworks.com.
4.3 Influence of the magnetic field on the vortex street

Figure 4.13: The Strouhal number in the vortex street as a function of the number $Ha^2/Re$, for different values of $R_m$. These results were obtained for $Re = 1000$. $Sr = 0$ means that no vortex street has appeared in that particular situation. The maximum numerical error due to data processing is estimated to be 0.02.
Chapter 5

Conclusion

The previous chapter showed numerous phenomena associated with the magnetic field and the velocity field for different values of the number $Re$, $R_m$ and $Ha^2/Re$. First of all, both fields prove to behave as expected when the mutual interactions are negligibly small, i.e. when $R_m = Ha^2/Re = 0.001$. Fig. 4.3 shows that the magnetic field does not deviate from the unperturbed field in this case, and Fig. 4.4-4.6 give the right fluid flow for the right Reynolds number [7]. This indicates that the numerical model gives physical results, at least in these cases.

When $R_m$ is increased to 0.1, the magnetic field is advected slightly in the direction of the flow, as shown in Fig. 4.7. The same phenomenon was seen in chapter 3, where the magnetic field was calculated analytically for $R_m \ll 1$ and $Ha^2/Re \ll 1$. Thus, the analytical model, with all its approximations, is consistent with the numerical results. It was already mentioned in chapter 3 that the analytical model gives physical results, so we can conclude that the numerical model does so as well.

The behaviour of the magnetic field seen in Fig. 4.8 and 4.9 for other values of $R_m$ is consistent with the theory presented in chapter 2. Increasing $R_m$ increases the influence of convection on the magnetic field, according to (2.23). This effect is also visible in the numerical results. The simulations show that the magnetic field is advected more strongly by the fluid for higher values of $R_m$. For $R_m \geq 1$, the initial field is dragged along completely. This is consistent with Alfvén’s theorem, which says that in a perfectly conducting fluid the magnetic flux through any loop moving with the fluid is constant. [8]

It was already mentioned in section 2.3 that this means that the magnetic field is essentially ‘frozen’ inside the fluid, i.e. field lines are dragged along with the flow. This is precisely the effect we see in the numerical results for high values of $R_m$: the magnetic field tends to follow the flow pattern, but due to a finite conductivity there remains a smaller residual field.

A more dramatic example of a physical phenomenon satisfying Alfvén’s theorem is shown in Fig. 4.9. This time, the Reynolds number is equal to 1000 and a Von Kármán vortex street appears. Again, the magnetic field tends to follow the fluid motion. Initially, for $R_m = 0.1$, 

the magnetic field is just slightly advected by the fluid, as before, but it also oscillates with the vortex shedding frequency. When $R_m$ increases, these oscillations become fiercer, up to point at which the magnetic field lines break. Then, the magnetic Reynolds number is so large that a kind of ‘magnetic vortex shedding’ occurs. This is also consistent with Alfvén’s theorem: the magnetic field tends to follow the flow pattern and the field lines are constantly broken and rejoined by the passing vortices.

As for the behaviour of the velocity field, some examples were shown of phenomena occurring for $R_m = 0.001$ and large values of $Ha^2/Re$ (Fig. 4.10-4.12). In general, we can conclude that the magnetic field forces the greater part of the fluid to flow in a greater path past the cylinder, creating a region in the vicinity of the cylinder in which the velocity is much smaller than elsewhere. It is not really clear what is happening close to the cylinder, but it appears that some fluid flows from and to its magnetic poles. Furthermore, the strength of this effect appears to depend on the Reynolds number. Further research in this regime is needed to account for the physics of this situation. It is not even clear if there exist real situations in which $Ha^2/Re$ has such high values.

The number $Ha^2/Re$ also influences the vortices appearing for $Re \gg 1$. For $Re = 25$, the two vortices in the wake behind the cylinder increase in dimensions when $Ha^2/Re$ increases. It appears that these vortices are also advected with the main flow when $Ha^2/Re \geq 1$. For $Re = 1000$, increasing $Ha^2/Re$ will prevent a vortex street from appearing, provided $R_m$ is kept small. The cause of these phenomena should become clear after further research.

The situation becomes more complicated when varying both $R_m$ and $Ha^2/Re$. First of all, we found $Sr = 0.22 \pm 0.02$ for $R_m = Ha^2/Re = 0.001$, independent of the Reynolds number. This is consistent with the results found by Strouhal and his successors [2] [4] [5]. Fig. 4.13 shows the Strouhal number of the Von Kármán vortex street for different values of $R_m$ and $Ha^2/Re$. Apparently, the vortex street is influenced by both numbers. This is to be expected, because there is a truly mutual interaction between magnetic field and velocity field when $R_m$ and $Ha^2/Re$ have significant values. For two further cases, mentioned in the previous section, the Strouhal number was again found to be independent of $Re$. This indicates that Fig. 4.13 as a whole might be independent of the Reynolds number.

The graph shows that the Strouhal number decreases for increasing values of $Ha^2/Re$, up to point at which no vortex street appears at all. However, the point at which the vortex street fails to appear shifts to the right when $R_m$ is increased. A rough explanation of this phenomenon might be the following. Increasing $Ha^2/Re$ strengthens the influence of the magnetic field on the flow. Because the magnetic field is initially stationary, it restricts the mobility of the fluid, thereby reducing the tendency of creating a vortex street. When $R_m$ is increased, however, the magnetic field tends to follow the fluid flow and becomes less ‘rigid’, which in turn imposes a lesser restriction on the velocity field. Thus, switching on both sides of the interaction causes the system to regain its former dynamics, and the vortex street reappears. Again, further research is needed to truly understand this phenomenon, mainly in the significance of the interaction terms in the governing equations.
Bibliography


Appendix A

The Unperturbed Magnetic Field

To determine the unperturbed magnetic field $B^0$ described in section 2.1 we first introduce the magnetic vector potential $A^0$ of this field. It is defined by:

$$B^0 = \nabla \times A^0.$$  \hspace{1cm} (A.1)

From Maxwell’s equation one can derive a Poisson equation for $A^0$ [3]:

$$\Delta A^0 = -\mu_0 j,$$  \hspace{1cm} (A.2)

in which $j$ is the total (volume) current density. In the unperturbed case there is only an electric current on the surface of the cylinder, which generates the magnetisation. Therefore, the volume current density $j$ is zero everywhere, except on the cylinder’s surface. Equation (A.2) then reduces to a Laplace equation:

$$\Delta A^0 = 0.$$  \hspace{1cm} (A.3)

The magnetic field lines are assumed to lie in the $xy$-plane, so that $A^0$ only has a $z$-component. Furthermore, we introduce the usual polar coordinates $(r, \theta)$ with $r = 0$ at the axis of the cylinder and $\theta = 0$ along the positive $x$-axis. Equation (A.3) then becomes:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A^0_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A^0_z}{\partial \theta^2} = 0,$$  \hspace{1cm} (A.4)

in which $A^0_z$ is the $z$-component of $A^0$. Separation of variables, together with boundedness conditions for $B^0$ for $r = 0$ and $r \to \infty$, yields the following solutions for (A.4):

$$A^0_z = \sum_{n=1}^{\infty} a_n r^n \cos n \theta, \hspace{0.5cm} r < a,$$

$$A^0_z = \sum_{n=1}^{\infty} b_n r^{-n} \cos n \theta, \hspace{0.5cm} r > a.$$  \hspace{1cm} (A.5)
The magnetic vector potential must be continuous at the surface of the cylinder \([3]\). Applying this condition to \((A.5)\) yields \(b_n = a^{2n}b_n\). It follows from \((A.1)\) that the \(r\)- and \(\theta\)-components of \(B^0\) are equal to:

\[
B^0_r = -\frac{\partial A^0_z}{\partial r}, \quad B^0_\theta = \frac{1}{r} \frac{\partial A^0_z}{\partial \theta}.
\]  
(A.6)

Finally, the following boundary condition must be applied to \(B^0\) \([3]\):

\[
B^0_\| - B^0_I = K \times n,
\]  
(A.7)

in which \(B^0_I\) is the solution for \(r < a\), \(B^0_\|\) is the solution for \(r > a\), \(K\) is the surface current density on the surface of the cylinder and \(n\) is a unit vector normal to the cylinder’s surface. The surface current density in terms of the magnetisation \(M\) of the cylinder is as follows:

\[
K = M \times n = M^0 \left( \sin \theta \, e_r + \cos \theta \, e_\theta \right) \times e_r = -M^0 \cos \theta \, e_z.
\]  
(A.8)

Inserting this expression in \((A.7)\) and applying this boundary condition to the magnetic field found from \((A.6)\), the coefficients in the general solution are found to be:

\[
a_1 = -\frac{1}{2} \mu_0 M^0, \quad b_1 = -\frac{1}{2} \mu_0 M^0 a^2, \quad a_n = b_n = 0 \quad \text{for} \quad n > 1.
\]

Thus, the unperturbed magnetic field generated by the magnetised cylinder is:

\[
B^0 = \begin{cases} 
\frac{1}{2} \mu_0 M^0 (\sin \theta \, e_r + \cos \theta \, e_\theta), & \text{if } r < a, \\
\frac{1}{2} \mu_0 M^0 a^2 (\sin \theta \, e_r - \cos \theta \, e_\theta)/r^2, & \text{if } r > a.
\end{cases}
\]  
(A.9)

The magnetic vector potential of this field is equal to:

\[
A^0_z = \begin{cases} 
-\frac{1}{2} \mu_0 M^0 r \cos \theta, & \text{if } r < a, \\
-\frac{1}{2} \mu_0 M^0 a^2 \cos \theta/r^2, & \text{if } r > a.
\end{cases}
\]  
(A.10)
Appendix B

Derivation of Equations (2.34)

To derive the equation for the vorticity, (2.34a), consider the first two equations of (2.30) (representing conservation of momentum). Taking the \( x \)-derivative of the equation for \( v \) and the \( y \)-derivative of the equation for \( u \) yields:

\[
\frac{\partial^2 v}{\partial x \partial t} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} + u \frac{\partial^2 v}{\partial x^2} + v \frac{\partial^2 v}{\partial x \partial y} =
\]

\[
- \frac{\partial p}{\partial x \partial y} + \frac{1}{Re} \left( \frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 v}{\partial x \partial^2 y} \right) + \frac{Ha^2}{Re} \left( B_x^0 + R_m b_x \right) \frac{\partial j_z}{\partial x} \tag{B.1a}
\]

\[
+ \frac{Ha^2}{Re} j_z \frac{\partial}{\partial x} \left( B_x^0 + R_m b_x \right),
\]

\[
\frac{\partial^2 u}{\partial y \partial t} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} + u \frac{\partial^2 u}{\partial y \partial^2 x} + v \frac{\partial^2 u}{\partial y^2} =
\]

\[
- \frac{\partial p}{\partial x \partial y} + \frac{1}{Re} \left( \frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial y \partial^2 x} \right) - \frac{Ha^2}{Re} \left( B_y^0 + R_m b_y \right) \frac{\partial j_z}{\partial y} \tag{B.1b}
\]

\[- \frac{Ha^2}{Re} j_z \frac{\partial}{\partial y} \left( B_y^0 + R_m b_y \right).\]

The next step is to subtract (B.1b) from (B.1a) (which is equivalent to taking the curl of the Navier-Stokes equation), inserting the definitions of \( \omega_z \), \( A_z \) and \( \psi \) and applying the conditions \( \nabla \cdot v = \nabla \cdot B_0 = \nabla \cdot b = 0 \). When done correctly, one indeed finds (2.34a).

Equation (2.34b) is easily obtained by inserting (2.33) in (2.31). Likewise, one finds (2.34d) by inserting (2.32) in the 2-dimensional version of (2.27).

Now consider Ohm’s law:

\[
j = E + v \times \left( B^0 + R_m b \right). \tag{B.2}\]

In general, the electric field can be expressed by [3]:

\[
E = -\nabla V - \frac{\partial A_{tot}}{\partial t}, \tag{B.3}
\]
in which $V$ is the electric potential and $A_{\text{tot}}$ is the total magnetic vector potential. One can show that the electric field $E$ only has a $z$-component (cf. (2.22b)). Because there is no variation in the $z$-direction, the $z$-component of $\nabla V$ is zero. Furthermore, the total vector potential can be written as:

$$A_{\text{tot}} = A^0 + R_m A,$$

(B.4)

in which $A^0$ is the vector potential of the unperturbed magnetic field, which is constant in time. Therefore, it follows from (B.3) that the $z$-component of $E$, $E_z$ say, and the vector potential $A_z$ of the perturbation $b$ are related by:

$$E_z = -R_m \frac{\partial A_z}{\partial t}.$$

(B.5)

One obtains (2.34c) by inserting (2.34d), (B.5) and the definitions of $A_z$ and $\psi$ in (B.2).
Appendix C

Behaviour of Magnetic Field for $Re = 0.1$

In this appendix some contour plots are presented that show the behaviour of the magnetic field for different values of $R_m$ in the case of a viscous flow, in which $Re = 0.1$. Comparing these figures with Fig. 4.7 and Fig. 4.8 shows that the effects are very similar to the case $Re = 25$. The magnetic field is again being advected by the fluid flow and this effect becomes stronger for larger values of $R_m$. In the cases for which $R_m \geq 1$ the initial field is completely dragged along by the fluid, leaving a residual magnetic field close to the cylinder. There is one clear difference between the plots shown below and the ones for $Re = 25$. For $Re = 0.1$ the field lines in the wake behind the cylinder retain their oval shape, whereas for $Re = 25$ the two vortices in the wake create a dent in this shape, in the upstream direction.
Figure C.1: Contour plot of $A_z$ for $Re = 0.1$, $R_m = 0.1$ and $Ha^2/Re = 0.001$.

Figure C.2: Contour plot of $A_z$ for $Re = 0.1$, $R_m = 1$ and $Ha^2/Re = 0.001$. 
Figure C.3: Contour plot of $A_z$ for $Re = 0.1$, $R_m = 10$ and $Ha^2/Re = 0.001$.

Figure C.4: Contour plot of $A_z$ for $Re = 0.1$, $R_m = 100$ and $Ha^2/Re = 0.001$. 