Inverse Problems in Transient Elastography

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Abstract

Transient elastography is a reliable non-invasive method for characterizing the elasticity of soft tissue. In this method, a short pulsed boundary force creates interior displacement which is measured as a function of time and space. In this thesis, we study the problem of determining parameters from interior dynamical measurements. We focus mainly on the recent research works by RPI’s group, led by Prof. Joyce McLaughlin. We start by giving the motivation of the problem in medical science where these techniques could be of help. Indeed, since the elastic properties of abnormal tissue can be significantly different from that of the normal tissue, determining the shear wave speed gives a good way to decide if any part of the tissue is damaged or not. We start by exposing the theoretical results on the uniqueness character of the problem. We then describe an algorithm to reconstruct the shear wave speed from the interior displacement data. The algorithm is based on the arrival time of the shear wave. The arrival time, under Lipschitz continuous assumption, satisfies the Eikonal equation which gives relation to the shear wave speed. The first step of the algorithm is to determine the arrival time from the interior displacement data by using cross correlation. The resulting arrival time is used to solve the Eikonal equation by means of a reconstruction method. This yields the shear wave speed. Here, we implemented two reconstruction methods: the level set and the distance method. Numerical tests show that they have good stability in the presence of noise and that they give excellent identification of high speed region.
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Chapter 1

Introduction

It has been well known that many diseases are associated with alteration in tissue stiffness. One way to examine tissue stiffness for disease assessment is by using palpation, where a doctor presses his hand against the skin to detect abnormal tissue that is stiffer than normal tissue. This approach, however, is a qualitative method with low sensitivity. The abnormal tissue might be undetectable if it is located deep in the body or if it is too small. Therefore, elastography techniques have recently developed which is expected to detect stiffness alterations accurately. The aim of these techniques is to create a high resolution image of human tissue based on their stiffness variation. The resulting image is then used for diagnosis.

To reach this goal in elastography, several experiments have been proposed, depending on the nature of boundary forces. These include:

- Static experiment
  In this experiment, the tissue is compressed by a static or a slowly varying boundary force. The inversion technique are then applied to obtain images of displacement. Large displacement occurs in normal tissue and small displacement indicates regions where stiff or abnormal tissue occurs.

- Dynamic experiment
  Principally, static and dynamic experiment are almost the same, but in this
case a time harmonic excitation applied on the boundary creates a time harmonic displacement in the interior. Moreover, since the excitation force carries a frequency, using different excitation forces (frequencies) give different displacements. Data are recorded after the tissue has reached steady state. As in the static experiment, small displacement region indicates where the stiff inclusions occur.

- Transient elastography

In this experiment, tissue initially at rest is excited with a short pulsed force on the boundary. This pulse force creates a propagating wave inside the body. Stiff region can be identified by using the information from the shear wave speed inside the body. This is due the fact that shear wave speed in stiff region is 2-4 times faster than that in normal region.

The discussion about elastography with static and dynamic experiment has been proposed widely, see [17, 6, 22]. In this thesis, we focus on transient elastography. The aim is to reconstruct the shear wave speed from the interior displacement data.

In the three experiments mentioned above, the excitation force should be non-destructive since we are dealing with human tissue. Moreover, the resulting displacement should be measured in such a way that the measurement does not take place inside the tissue. We call this non-invasive measurement. For this reason, ultrasound and Magnetic Resonance Imaging (MRI) are chosen as measurement methods. The displacement are measured on a fine grid of points and are finally used to reconstruct tissue stiffness using inversion technique. Here we will consider several inversion techniques based on propagating wave front to reconstruct wave speed in order to identify tissue stiffness in isotropic medium. These approaches have been proposed by RPI’s group led by Prof. Joyce Mclaughlin.
This report is organized as follow. In chapter 2, we describe the forward and the inverse problems. The forward problems consist of linear elastic equation and wave equation. We study the forward problems by using the concept of weak solution, which is constructed by employing the Galerkin’s method. The existence, uniqueness and the smoothness of a solution of the forward problem are given. We describe the inverse problem from interior displacement data, which is the main concern in this thesis. To get more perspective, some related results from the same inverse problem are also described. The uniqueness results for the inverse problem are presented in Chapter 3. We discuss the uniqueness of the wave speeds and the elastic parameters from interior displacement data in isotropic medium. For the case of linear elastic equation, we state the uniqueness results while for the case of wave equation, we discuss the uniqueness results in details. The main tool to prove uniqueness is the shrink and spread argument. Furthermore, counter-example for non-uniqueness in anisotropic medium is given as well. Using arrival time information, which is a subset of interior displacement data, we show that the shear wave is identified uniquely. Furthermore, the arrival time information is not enough to identify the parameters \((\rho, \mu)\) simultaneously. Chapter 4 presents the reconstruction methods: the level set method and the distance method. We first describe the technique to find the arrival time from the interior displacement data. Then the reconstruction methods are used to recover the wave speed from the arrival time information. Chapter 5 focuses on the numerical results of the reconstruction methods. The stability and the accuracy of the methods are observed. Moreover, synthetic data, which are generated by solving the wave equation, are also used to test the methods. Finally, Chapter 6 summarizes the results of this study. Some possible extensions of this study are also described.
Chapter 2
Mathematical Models

2.1 The Forward Problems

Motivation. We start by analyzing the properties of the transient elastography experiments. The impulse force applied on the boundary of the medium creates two kinds of elastic waves that propagate into the medium. The first wave, called p-wave or compression wave, displaces the tissue in the direction in which the wave propagates. The second wave, called s-wave or shear wave, displaces the tissue orthogonally to the direction of propagation of the wave. Experiments show that propagating elastic waves have low variation of the amplitude, of order of microns. This suggests that we can use linear elastic model. A mathematical model that describes wave motion is based on the strain-stress curve that describes how the tissue deform when force is applied. In [20], the strain-stress curve for transient elastography experiment is shown to be linear. For these reasons, we use linear elastic model to describe the propagation of the elastic wave in the medium. We assume the medium is isotropic, meaning that the medium has the same material parameters in all directions. In this case, the relevant elastic parameters are density, \( \rho \), the Lamé parameters, \( \lambda \) and \( \mu \), or the compression and the shear wave speed, \( \sqrt{\frac{\lambda+2\mu}{\rho}} \) and \( \sqrt{\frac{\mu}{\rho}} \), respectively.

Let \( \Omega \subset \mathbb{R}^n (n = 2, 3) \) be an open connected \( C^2 \) domain and \( T > 0 \) fixed. In the next subsection, we consider different cases and state the forward problems described
by the initial-boundary value problems which model propagation of waves in elastic bodies. To be consistent with the experiments, in all cases, we assume that the medium begins at rest and that a displacement or a traction force on the boundary of the tissue initiates the propagation of waves into the tissue.

### 2.1.1 Vector displacement case

The vector elastic displacement in an isotropic medium is derived from the equation of motion in combination with Hooke’s law, which relates the stress and strain tensor, see [4]. The result is stated in the following theorem.

**Theorem 2.1.1.** Assume that \( \rho \in C^1(\bar{\Omega}) \) and \( \mu, \lambda \in C^2(\bar{\Omega}) \) satisfy \( \mu(x), \rho(x), \lambda(x) \geq \alpha_0 > 0 \). There exists a unique solution \( \vec{u} \in [H^2(\Omega \times (0, T))]^n \) that satisfies the following hyperbolic system

\[
\nabla(\lambda \nabla \cdot \vec{u}) + \nabla \cdot (\mu(\nabla \vec{u} + (\nabla \vec{u})^T)) = \rho \ddot{\vec{u}} \quad \text{in } \Omega \times (0, T) \tag{2.1.1}
\]

with homogenous initial condition

\[
\vec{u}(x, 0) = \vec{u}_i(x, 0) = 0 \quad \text{in } \Omega \tag{2.1.2}
\]

and one of the following boundary conditions

\[
\vec{u}(x, t) = \vec{f}(x, t) \quad \text{on } \partial\Omega \times (0, T) \tag{2.1.3}
\]

with \( \vec{f}(x, t) \in [H^{5/2}(\partial\Omega \times (0, T))]^n \) or

\[
[(\lambda(x)\nabla \cdot \vec{u})I + \mu(x)(\nabla \vec{u} + (\nabla \vec{u})^T)] \nu(x) = \vec{g}(x, t) \quad \text{on } \partial\Omega \times (0, T) \tag{2.1.4}
\]

with \( \vec{g}(x, t) \in [H^{3/2}(\partial\Omega \times (0, T))]^n \), where \( \vec{f}(x, t) \) and \( \vec{g}(x, t) \) satisfy some compatibility conditions. Here, \( \nu(x) \) is the outward normal to \( \partial\Omega \), \( I \) is the identity matrix, \( (\cdot)^T \) denotes the transpose of a matrix and \( x \in \Omega \).
Tissue in human body is mostly composed of water. For this reason, the compression wave travels at approximately the speed of sound in water, 1500 m/s. In contrast, the shear wave is very slow compared to compression wave, it travels at 1 m/s − 3 m/s. This large difference in speed means that in low frequency boundary excitation, the compression wave has a very long wavelength compared to that of the shear wave. This means that in a small or finite area of observation, while the shear wave is still propagating the compression wave has propagated out of the region of interest. Thus, the contribution of the compression wave in the displacement data can be considered as noise.

Since human tissue is mainly composed of water, which is incompressible liquid, we may assume that \( \nabla \cdot \vec{u} \approx 0 \). Then the contribution of the terms \( \nabla(\lambda \nabla \cdot \vec{u}) \) and \( \nabla \cdot (\mu(\nabla \vec{u})^T) \) in the linear elastic model is very small and can be treated as noise. Note that the term \( \nabla \cdot (\mu(\nabla \vec{u})^T) \) can be ignored if we know that \( \mu \) is slowly varying. Thus, for any single component, the linear elastic model can be approximated by the wave equation. In this case, for isotropic medium, the relevant elastic parameters are density, \( \rho \), the Lamé parameter, \( \mu \), or the shear wave speed, \( \sqrt{\frac{\mu}{\rho}} \).

### 2.1.2 Scalar displacement case

In this section, we study the model for the scalar shear displacement in an isotropic medium. We start by considering the following wave equation

\[
\begin{align*}
\frac{\partial^2 w}{\partial t^2}(x,t) - \nabla \cdot (\mu(x)\nabla w(x,t)) &= h(x,t) \quad \text{in } \Omega \times (0,T) \\
w(x,0) &= \xi(x), \quad w_t(x,0) = \eta(x) \quad \text{in } \Omega \\
w(x,t) &= 0 \quad \text{on } \partial \Omega \times (0,T)
\end{align*}
\]  

where \( h : \Omega \times (0,T) \to \mathbb{R}, \xi, \eta : \Omega \to \mathbb{R} \) are given and \( w : \Omega \times (0,T) \to \mathbb{R} \) is the unknown, \( w := w(x,t) \).

We first define and then construct an appropriate weak solution \( w \) of problem
(2.1.5). The next step is to show the existence, uniqueness and the smoothness of a weak solution to initial-boundary problem (2.1.5). We suppose initially that

\[ h \in L^2(\Omega \times (0, T)) \]
\[ \xi \in H^1_0(\Omega) \]
\[ \eta \in L^2(\Omega). \]  

(2.1.6)

Let us introduce the time dependent bilinear form,

\[ B[w, v; t] := \int_{\Omega} (\mu \nabla w) \cdot \nabla v \quad \forall w, v \in H^1_0(\Omega) \text{ and a.e. } t, \ 0 \leq t \leq T. \]  

(2.1.7)

**Motivation for the definition of weak solution.** Suppose \( w = w(x, t) \) is a smooth solution of (2.1.5). We are going to consider \( w \) not as a function of \( x \) and \( t \) together, but rather as a mapping \( w \) of \( t \) into the space \( H^1_0(\Omega) \) of \( x \), that is

\[ w : [0, T] \rightarrow H^1_0(\Omega) \]

defined by

\[ [w(t)](x) = w(x, t), \quad x \in \Omega, 0 \leq t \leq T. \]

Let us similarly define

\[ h : [0, T] \rightarrow L^2(\Omega) \]

by

\[ [h(t)](x) = h(x, t), \quad x \in \Omega, 0 \leq t \leq T. \]

Then if we fix a function \( v \in H^1_0(\Omega) \), multiply the equation \( w_{tt} - \nabla \cdot (\mu \nabla w) = h \) with \( v \) and integrate by parts in \( x \), we get

\[ (w'', v) + B[w, v; t] = (h, v) \quad \text{for a.e. } t, \ 0 \leq t \leq T; \]  

(2.1.8)

where \((\cdot, \cdot)\) denotes \( L^2(\Omega) \) inner product and \( ' := \frac{d}{dt} \).

Now, let us observe the equation

\[ w'' = h + \nabla \cdot (\mu \nabla w). \]  

(2.1.9)
Since \( h \in L^2(\Omega) \) and \( \mu \nabla w \in L^2(\Omega) \), for a.e. \( t, \ 0 \leq t \leq T \), the characterization of \( H^{-1}(\Omega) \) space implies that the right hand side of (2.1.9) lies in the Sobolev space \( H^{-1}(\Omega) \) for a.e. \( t, \ 0 \leq t \leq T \). This suggest that we should look for a weak solution \( w \) with \( w'' \in H^{-1}(\Omega) \) for a.e. \( t, \ 0 \leq t \leq T \). Then we can express (2.1.8) as

\[
\langle w'', v \rangle + B[w, v; t] = (h, v) \quad \text{for a.e. } t, \ 0 \leq t \leq T,
\]

(2.1.10)

where \( \langle \cdot, \cdot \rangle \) denotes duality pair of \( H^{-1}(\Omega) \) and \( H^1(\Omega) \).

This motivates the following

**Definition 2.1.1.** A function \( w \in L^2(0, T; H^1_0(\Omega)) \) with \( w' \in L^2(0, T; L^2(\Omega)) \) and \( w'' \in L^2(0, T; H^{-1}(\Omega)) \) is a weak solution of the hyperbolic initial-boundary value problem (2.1.5) if

(a) \( \langle w'', v \rangle + B[w, v; t] = (h, v) \quad \forall v \in H^1_0(\Omega) \) and a.e \( t, \ 0 \leq t \leq T \)

(b) \( w(0) = \xi \) and \( w'(0) = \eta \).

From Definition 2.1.1, we have \( w \in H^1_0(\Omega) \) with \( w' \in L^2(\Omega) \) and \( w'' \in H^{-1}(\Omega) \) for a.e. \( t, \ 0 \leq t \leq T \). We need to give sense to equality (b), that is for \( w \) and \( w' \) at \( t = 0 \). This is accomplished as follows. From the regularity of \( w \)

- \( w \in L^2(0, T; H^1_0(\Omega)) \) and \( w' \in L^2(0, T; L^2(\Omega)) \) imply

\[
w \in H^1(0, T; L^2(\Omega)) \hookrightarrow C([0, T]; L^2(\Omega))
\]

- \( w' \in L^2(0, T; L^2(\Omega)) \) and \( w'' \in L^2(0, T; H^{-1}(\Omega)) \) imply

\[
w' \in H^1(0, T; H^{-1}(\Omega)) \hookrightarrow C([0, T]; H^{-1}(\Omega))
\]

and as a consequence equality (b) makes sense.

To show the existence and uniqueness of solution for (2.1.5), we proceed in two stages. In the first part, we use Galerkin’s method together with energy inequality
(Theorem 2.1.3) to obtain existence. In this method, we construct the weak solution of (2.1.5) by first solving a finite dimension approximation and then pass to the limit.

More precisely, assume the functions \( y_k = y_k(x), k = 1, 2, \ldots \) are smooth,

\[
\{y_k\}_{k=1}^{\infty} \text{ is an orthogonal basis of } H^1_0(\Omega)
\]

and

\[
\{y_k\}_{k=1}^{\infty} \text{ is an orthonormal basis of } L^2(\Omega). \tag{2.1.11}
\]

For instance, we could take \( \{y_k\}_{k=1}^{\infty} \) to be the complete set of normalized eigen functions for \(-\Delta\) in \( H^1_0(\Omega)\). Fix a positive integer \( m \). We define the approximate solution \( w_m(t) \) of order \( m \) of the problem in the following way,

\[
w_m : [0, T] \to H^1_0(\Omega)
\]

with

\[
w_m(t) = \sum_{k=1}^{m} d_k^m(t)y_k, \tag{2.1.12}
\]

where \( d_k^m(t) \) are being determined such that

\[
(w_m'', y_k) + B[w_m, y_k, t] = (h, y_k) \quad k = 1, 2, \ldots, m \text{ and } 0 \leq t \leq T \tag{2.1.13}
\]

with

\[
w_m(0) = \sum_{k=1}^{m} d_k^m(0)y_k = \xi_m \to \xi \text{ in } H^1_0(\Omega) \text{ as } m \to \infty \tag{2.1.14}
\]

and

\[
w_m'(0) = \sum_{k=1}^{m} d_k^m(0)y_k = \eta_m \to \eta \text{ in } L^2(\Omega) \text{ as } m \to \infty. \tag{2.1.15}
\]

We can interpret (2.1.14)-(2.1.15) as projection of \( \xi \) and \( \eta \) to \( V_m = \text{span}\{y_k\}_{k=1}^{m} \), i.e.

\[
(\xi, y_p) = (\xi_m, y_p) \quad \text{and} \quad (\eta, y_p) = (\eta_m, y_p), \quad \text{for } p = 1, \ldots, m.
\]

This imply,

\[
(\xi_m, y_p) = \left( \sum_{k=1}^{m} d_k^m(0)y_k, y_p \right) = d_p^m(0)
\]
and
\[
(\eta_m, y_p) = \left( \sum_{k=1}^{m} d_{m}^{k}(0)y_k, y_p \right) = d^p_{m}(0).
\]
Then, (2.1.14)-(2.1.15) can be rewritten as
\[
d^k_m(0) = (\xi, y_k) \text{ and } d^k_m(0) = (\eta, y_k).
\]

Thus, we seek a function \( w_m \) of the form (2.1.12) that satisfies the projection (2.1.13) of the problem (2.1.5) onto a finite dimensional subspace \( V_m \). The following theorem provides sense for \( w_m(t) \) of the form (2.1.12).

**Theorem 2.1.2.** For each integer \( m = 1, 2, \ldots \), there exists a unique function \( w_m \) of the form (2.1.12) satisfying (2.1.13), (2.1.16).

**Proof.** Assume that \( w_m(t) \) is given by (2.1.12). Then,
\[
w''_m(t) = \sum_{p=1}^{m} d''_m(t)y_p,
\]
and using (2.1.11) we obtain
\[
(w''_m, y_k) = \left( \sum_{p=1}^{m} d''_m(t)y_p, y_k \right) = d''_m(t).
\]
Furthermore,
\[
B[w_m, y_k; t] = \int_{\Omega} \mu(x) \nabla w_m \cdot \nabla y_k \, dx
\]
\[
= \int_{\Omega} \mu(x) \sum_{p=1}^{m} d'_m(t) \nabla y_p \cdot \nabla y_k \, dx
\]
\[
= \sum_{p=1}^{m} d'_m(t) \int_{\Omega} \mu(x) \nabla y_p \cdot \nabla y_k \, dx
\]
\[
= \sum_{p=1}^{m} d'_m(t) B[y_p, y_k; t]
\]
\[
= \sum_{p=1}^{m} e^{pk}(t)d'_m(t),
\]
for $e^{pk}(t) := B[y_p, y_k; t]$, $p, k = 1, \ldots, m$.

We also write $h^k(t) := (h(t), y_k)$, $k = 1, \ldots, m$.

Consequently, (2.1.13) becomes the linear system of ODE

$$d_m^{k''}(t) + \sum_{p=1}^{m} e^{pk}(t)d_m^p(t) = h^k(t), \quad 0 \leq t \leq T, \quad k = 1, \ldots, m,$$

with initial conditions (2.1.16).

According to the standard theory of ODE, there exists a unique $H^2(0, T)$ function

$$D_m(t) := [d_m^1(t) \cdots d_m^m(t)]^T$$

satisfying (2.1.16) and solving (2.1.17) for $0 \leq t \leq T$.

From Theorem 2.1.2, it follows that $w_m$ defined by (2.1.12) solves (2.1.13) for a.e $t$, $0 \leq t \leq T$. We need to let $m$ tend to infinity and show that the solution $w_m$ of the approximate problem (2.1.13),(2.1.16) converges to a weak solution of (2.1.5). For this, we need some uniform a priori estimates.

**Theorem 2.1.3.** There exists a constant $C$, depending only on $\Omega, T$ and $\mu$ such that

$$\max_{0 \leq t \leq T} \left( ||w_m(t)||_{H^2_0(\Omega)} + ||w_m^\prime(t)||_{L^2(\Omega)} + ||w_m''(t)||_{L^2(0, T; H^{-1}(\Omega))} \right) \leq C \left( ||h||_{L^2(0, T; L^2(\Omega))} + ||\xi||_{H^1_0(\Omega)} + ||\eta||_{L^2(\Omega)} \right)$$

(2.1.18)

**Proof.** Multiply (2.1.13) by $d_m^k(t)$, take the summation over $k = 1, \ldots, m$ and recall (2.1.12), then

$$(w_m'', w_m^\prime) + B[w_m, w_m^\prime; t] = (h, w_m^\prime) \quad \text{for a.e. } t, \quad 0 \leq t \leq T.$$  

Since $(w_m'', w_m^\prime) = \frac{1}{2} \frac{d}{dt} ||w_m^\prime||^2_{L^2(\Omega)}$ and $B[w_m, w_m^\prime; t] = \frac{1}{2} \frac{d}{dt} B[w_m, w_m^\prime; t]$, we have

$$\frac{1}{2} \frac{d}{dt} \left( ||w_m^\prime||^2_{L^2(\Omega)} + B[w_m, w_m; t] \right) = (h, w_m^\prime) \leq ||h||_{L^2(\Omega)} ||w_m^\prime||_{L^2(\Omega)} \leq \frac{1}{2} \left( ||h||^2_{L^2(\Omega)} + ||w_m^\prime||^2_{L^2(\Omega)} \right).$$  

Using the fact that the bilinear form is coercive:

$$B[w, w; t] \geq \theta ||w||^2_{H^1_0(\Omega)}, \quad \forall \ w \in H^1_0(\Omega) \quad (2.1.19)$$

where $\theta$ is a positive constant, we obtain

$$\frac{d}{dt} \left( ||w'_m||^2_{L^2(\Omega)} + B[w_m, w_m; t] \right) \leq ||h||^2_{L^2(\Omega)} + ||w'_m||^2_{L^2(\Omega)} + B[w_m, w_m; t]. \quad (2.1.20)$$

Now write

$$\alpha(t) := ||w'_m||^2_{L^2(\Omega)} + B[w_m, w_m; t]$$

and

$$\beta(t) := ||h||^2_{L^2(\Omega)}.$$

Then inequality (2.1.20) reads,

$$\alpha'(t) \leq \alpha(t) + \beta(t)$$

for $0 \leq t \leq T$.

Using the Gronwall’s inequality, see [5], yields the estimate

$$\alpha(t) \leq e^t \left( \alpha(0) + \int_0^t \beta(s) \ ds \right) \leq C \left( \alpha(0) + \int_0^T \beta(s) \ ds \right) \quad \text{for } 0 \leq t \leq T. \quad (2.1.21)$$

But we have

$$\alpha(0) = ||w'_m(0)||^2_{L^2(\Omega)} + B[w_m(0), w_m(0); t]$$

with

$$||w'_m(0)||^2_{L^2(\Omega)} = \sum_{k=1}^m d_{m_k}^{k'}(0) y_k ||^2_{L^2(\Omega)} \leq || \sum_{k=1}^\infty d_{m_k}^{k'}(0) y_k ||^2_{L^2(\Omega)} = ||g||^2_{L^2(\Omega)}$$
and

\[ B[w_m(0), w_m(0); t] \leq \max_{x \in \Omega} |\mu(x)| \| \nabla w_m(0) \|_{L^2(\Omega)}^2 \leq C \|w_m(0)\|_{H^1_0(\Omega)}^2 \]

\[ = C \| \sum_{k=1}^m d_m^k(0) y_k \|_{L^2(\Omega)}^2 \leq C \| \sum_{k=1}^\infty d_m^k(0) y_k \|_{H^1_0(\Omega)}^2 \]

\[ = C \| \xi \|_{H^1_0(\Omega)}^2. \]

Thus, formula (2.1.21) provides the bound

\[ \| w_m' \|_{L^2(\Omega)}^2 + B[w_m, w_m; t] \leq C \left( \| \xi \|_{H^1_0(\Omega)}^2 + \| \eta \|_{L^2(\Omega)}^2 + \| h \|_{L^2(0,T;L^2(\Omega))}^2 \right). \]

From this estimate and using (2.1.19), we get

\[ \| w_m' \|_{L^2(\Omega)}^2 + \| w_m \|_{H^1_0(\Omega)}^2 \leq C \left( \| \xi \|_{H^1_0(\Omega)}^2 + \| \eta \|_{L^2(\Omega)}^2 + \| h \|_{L^2(0,T;L^2(\Omega))}^2 \right). \]

Since \( 0 \leq t \leq T \) was arbitrary, we see

\[ \max_{0 \leq t \leq T} \left( \| w_m' \|_{L^2(\Omega)}^2 + \| w_m \|_{H^1_0(\Omega)}^2 \right) \leq C \left( \| \xi \|_{H^1_0(\Omega)}^2 + \| \eta \|_{L^2(\Omega)}^2 + \| h \|_{L^2(0,T;L^2(\Omega))}^2 \right). \]

(2.1.22)

Fix any \( v \in H^1_0(\Omega) \) with \( \| v \|_{H^1_0(\Omega)} \leq 1 \). Write \( v = v^1 + v^2 \) with \( v^1 \in \text{span}\{y_k\}_{k=1}^m \) and \( (v^2, y_k) = 0 \) for \( k = 1, \ldots, m \). Then (2.1.12) and (2.1.13) imply

\[ \langle w_m'', v \rangle = (w_m'', v) = (w_m'', v^1 + v^2) = (w_m'', v^1) \]

\[ = (h, v^1) - B[w_m, v^1; t]. \]

Thus,

\[ |\langle w_m'', v \rangle| \leq C \left( \| h \|_{L^2(\Omega)} \| v^1 \|_{H^1_0(\Omega)} + \| w_m \|_{H^1_0(\Omega)} \| v^1 \|_{H^1_0(\Omega)} \right) \]

\[ = C \left( \| h \|_{L^2(\Omega)} + \| w_m \|_{H^1_0(\Omega)} \right) \| v^1 \|_{H^1_0(\Omega)} \]

\[ = C \left( \| h \|_{L^2(\Omega)} + \| w_m \|_{H^1_0(\Omega)} \right) \| v \|_{H^1_0(\Omega)}. \]
and this implies
\[ ||w''_m||_{H^{-1}(\Omega)} = \sup_{v \in H^1_0(\Omega), v \neq 0} \frac{||w''_m, v||}{||v||_{H^1_0(\Omega)}} \leq C \left( ||h||_{L^2(\Omega)} + ||w_m||_{H^1_0(\Omega)} \right). \]

Consequently,
\[
\int_0^T ||w''_m||^2_{H^{-1}(\Omega)} \, dt \leq C \int_0^T \left( ||h||^2_{L^2(\Omega)} + ||w_m||^2_{H^1_0(\Omega)} \right) \, dt \\
\leq C \left( ||\xi||^2_{H^1_0(\Omega)} + ||\eta||^2_{L^2(\Omega)} + ||h||^2_{L^2(0,T;L^2(\Omega))} \right). \quad (2.1.23)
\]

Combining (2.1.22) and (2.1.23) completes the proof. \(\square\)

Using Theorem 2.1.3, we will show that the function \(w_m\) constructed in this way is the weak solution of the problem (2.1.5). We proceed by passing \(m\) to infinity.

**Existence of solution.** Theorem 2.1.3 provides us that the sequence

\[
\{w_m\}_{m=1}^\infty \quad \text{is bounded in } C([0,T];H^1_0(\Omega)) \subset L^2(0,T;H^1_0(\Omega)), \\
\{w'_m\}_{m=1}^\infty \quad \text{is bounded in } C([0,T];L^2(\Omega)) \subset L^2(0,T;L^2(\Omega)) \quad \text{and} \\
\{w''_m\}_{m=1}^\infty \quad \text{is bounded in } L^2(0,T;H^{-1}(\Omega)).
\]

Since the sequence is bounded in a reflexive space, by the weak compactness theorem, see [5], there exists a subsequence \(\{w_{m_i}\}_{i=1}^\infty \subset \{w_m\}_{m=1}^\infty\) and \(w \in L^2(0,T;H^1_0(\Omega))\) with \(w' \in L^2(0,T;L^2(\Omega))\) and \(w'' \in L^2(0,T;H^{-1}(\Omega))\) such that

\[
w_{m_i} \rightharpoonup w \quad \text{in } L^2(0,T;H^1_0(\Omega)), \quad (2.1.24) \\
w'_m \rightharpoonup w' \quad \text{in } L^2(0,T;L^2(\Omega)), \quad (2.1.25) \\
w''_m \rightharpoonup w'' \quad \text{in } L^2(0,T;H^{-1}(\Omega)), \quad (2.1.26)
\]

where \(\rightharpoonup\) means weak convergence.

Fix an integer \(N\) and choose a function \(v \in C^1([0,T];H^1_0(\Omega))\) of the form

\[
v(t) = \sum_{k=1}^N d_k(t)y_k, \quad (2.1.27)
\]
where \( \{d^k\}_{k=1}^N \) are smooth functions.

Select \( m \geq N \), multiply (2.1.13) by \( d^k(t) \), take the summation over \( k = 1, \ldots, N \) and integrate with respect to time, we get

\[
\int_0^T \langle w_m'', v \rangle + B[w_m, v; t]dt = \int_0^T (h, v)dt. \tag{2.1.28}
\]

We set \( m = m_l \). By using (2.1.26), we obtain

\[
\int_0^T \langle w_m'', v \rangle dt \longrightarrow \int_0^T \langle w'', v \rangle dt \quad \text{in } H^{-1}(\Omega) \quad \text{as } m_l \to \infty.
\]

Using (2.1.24), we have

\[
\int_0^T B[w_m, v; t]dt \longrightarrow \int_0^T B[w, v; t]dt \quad \text{in } L^2(\Omega) \quad \text{as } m_l \to \infty.
\]

Combining these two results, from (2.1.28) we find, after taking the limit that

\[
\int_0^T \langle w'', v \rangle + B[w, v; t]dt = \int_0^T (h, v)dt \quad \forall v \text{ of the form (2.1.27)}.
\]

Since the set of the functions of the form (2.1.27) is dense in \( L^2(0, T; H^1_0(\Omega)) \), then

\[
\int_0^T \langle w'', v \rangle + B[w, v; t]dt = \int_0^T (h, v)dt \quad \forall v \in L^2(0, T; H^1_0(\Omega)). \tag{2.1.29}
\]

From (2.1.29), it follows that

\[
\langle w'', v \rangle + B[w, v; t] = (h, v) \quad \forall v \in H^1_0(\Omega) \text{ and a.e. } t, \ 0 \leq t \leq T. \tag{2.1.30}
\]

Let us now check that \( w(0) = \xi \) and \( w'(0) = \eta \). Choose any function \( v \in C^2([0, T]; H^1_0(\Omega)) \) with \( v(T) = v'(T) = 0 \). Integrating by parts twice in (2.1.29) with respect to time, we get

\[
\int_0^T \langle v'', w \rangle + B[w, v; t]dt = \int_0^T (h, v)dt - \langle w(0), v'(0) \rangle + \langle w'(0), v(0) \rangle. \tag{2.1.31}
\]

Similarly from (2.1.28), we get

\[
\int_0^T \langle v'', w_m \rangle + B[w_m, v; t]dt = \int_0^T (h, v)dt - \langle w_m(0), v'(0) \rangle + \langle w'_m(0), v(0) \rangle.
\]
We set \( m = m_l \) and let \( m_l \) tend to infinity. Using (2.1.24)-(2.1.25) we get

\[
w_{m_l} \rightharpoonup w \quad \text{in } H^1(0, T; L^2(\Omega)).
\]

Since \( H^1(0, T; L^2(\Omega)) \) is compactly embedded in \( C([0, T]; L^2(\Omega)) \), then

\[
w_{m_l}(0) \to w(0) \quad \text{in } L^2(\Omega). \tag{2.1.32}
\]

Using this, we have

\[
\langle w_{m_l}(0), v'(0) \rangle \to \langle \xi, v'(0) \rangle \quad \text{in } L^2(\Omega) \text{ as } m_l \to \infty.
\]

In the similar fashion, we can show that

\[
\langle w_{m_l}'(0), v(0) \rangle \to \langle \eta, v(0) \rangle \quad \text{in } H^{-1}(\Omega) \text{ as } m_l \to \infty.
\]

Summarizing, we find in the limit that

\[
\int_0^T \langle v''(t), w(t) \rangle + B[w(t), v(t)] dt = \int_0^T \langle h, v'(t) \rangle dt - \langle \xi, v'(0) \rangle + \langle \eta, v(0) \rangle. \tag{2.1.33}
\]

Comparing (2.1.31) and (2.1.33), we conclude \( w(0) = \xi \) and \( w'(0) = \eta \) since \( v \) is arbitrary. Hence, \( w \) is a weak solution of (2.1.5).

**Uniqueness of solution.** It is sufficient to show that the only weak solution of (2.1.5) with \( h = \xi = \eta = 0 \) is \( w = 0 \). Fix \( 0 \leq s \leq T \) and set

\[
v(t) = \begin{cases} \int_t^s w(\tau) d\tau, & 0 \leq t \leq s; \\ 0, & s \leq t \leq T. \end{cases} \tag{2.1.34}
\]

Then \( v(t) \in H^1_0(\Omega) \) for each \( 0 \leq t \leq T \) and we have

\[
\int_0^s (w'', v) + B[w, v; t] dt = 0.
\]

Integrate by parts in time,

\[
\int_0^s (w'', v) dt = (w', v)_{0}^{s} - \int_0^s (w', v') = -\int_0^s (w', v'),
\]
since \( w'(0) = v(s) = 0 \).

Set \( v' = -w \) in \( 0 \leq t \leq s \), we get
\[
\int_0^s (w', w) - B[v', v; t]dt = 0,
\]
(2.1.35)
and since \( \frac{1}{2} \frac{d}{dt} ||w||^2_{L^2(\Omega)} = (w', w) \) and \( \frac{1}{2} \frac{d}{dt} B[v, v; t] = B[v', v; t] \), (2.1.35) can be rewritten as
\[
\int_0^s \frac{d}{dt} \left( ||w||^2_{L^2(\Omega)} - B[v, v; t] \right) dt = 0 \quad \forall w, v \in H^1_0(\Omega).
\]
Consequently,
\[
||w(s)||^2_{L^2(\Omega)} - ||w(0)||^2_{L^2(\Omega)} - B[v(s), v(s); s] + B[v(0), v(0); 0] = 0.
\]
Since \( w(0) = v(s) = 0 \), we have
\[
||w(s)||^2_{L^2(\Omega)} + B[v(0), v(0); 0] = 0.
\]
By using coercivity property of \( B[v(0), v(0); 0] \), we obtain
\[
||w(s)||^2_{L^2(\Omega)} \leq ||w(s)||^2_{L^2(\Omega)} + C ||v(0)||^2_{L^2(\Omega)} \leq 0.
\]
We conclude that \( w(s) = 0 \) and since \( s \) is arbitrary, we have \( w = 0 \) in \( \Omega \) and a.e. \( t, 0 \leq t \leq T \). Thus, \( w \), as stated in Definition 2.1.1, is a unique weak solution of (2.1.5).

Now, we discuss the regularity of the weak solution \( w \). Suppose we have more regularity on the data in (2.1.6), then the regularity of \( w \) can be improved as well. To be precise, the result is stated in the following theorem.

**Theorem 2.1.4.** Assume \( h \in H^1(\Omega \times (0, T)) \), \( \xi \in H^2(\Omega) \) and \( \eta \in H^1_0(\Omega) \), then
\[
\begin{align*}
    w & \in L^2(0, T; H^2(\Omega)) \quad (2.1.36) \\
    w' & \in L^2(0, T; H^1_0(\Omega)) \quad (2.1.37) \\
    w'' & \in L^2(0, T; L^2(\Omega)). \quad (2.1.38)
\end{align*}
\]
Furthermore, we have \( w \in H^2(\Omega \times (0, T)) \).
Proof. The proof of the first part of the theorem can be found in [5]. We will show that \( w \in H^2(\Omega \times (0, T)) \). In other words, we need to show that

\[
D^{\alpha,\beta}w \in L^2(\Omega \times (0, T)), \quad |\alpha + \beta| \leq 2,
\]

(2.1.39)

where \( \alpha \) and \( \beta \) denote the order of spatial and time derivative, respectively. From Tonelli’s and Fubini’s theorem, see [2], we know that \( L^2(\Omega \times (0, T)) = L^2(0, T; L^2(\Omega)) \).

Thus, (2.1.39) is reduced to

\[
D^{\alpha,\beta}w \in L^2(0, T; L^2(\Omega)), \quad |\alpha + \beta| \leq 2.
\]

(2.1.40)

• From (2.1.36) − (2.1.38), we have \( w \in H^2(0, T; L^2(\Omega)) \). This implies,

\[
||w||_{H^2(0,T;L^2(\Omega))} < \infty,
\]

but

\[
||w||^2_{H^2(0,T;L^2(\Omega))} = \sum_{\beta \leq 2} ||D^{0,\beta}w||^2_{L^2(0,T;L^2(\Omega))} < \infty.
\]

Thus, \( D^{0,\beta}w \in L^2(0, T; L^2(\Omega)) \) for \( |\beta| \leq 2 \).

• From (2.1.36), we have \( w \in L^2(0, T; H^2(\Omega)) \). This implies,

\[
||w||_{L^2(0,T;H^2(\Omega))} < \infty,
\]

but

\[
||w||^2_{L^2(0,T;H^2(\Omega))} = \sum_{\alpha \leq 2} ||D^{\alpha,0}w||^2_{L^2(0,T;L^2(\Omega))} < \infty.
\]

Thus, \( D^{\alpha,0}w \in L^2(0, T; L^2(\Omega)) \) for \( |\alpha| \leq 2 \).

• From (2.1.37), we have \( w' \in L^2(0, T; H^1_0(\Omega)) \). This implies,

\[
||w'||_{L^2(0,T;H^1_0(\Omega))} < \infty,
\]

but

\[
||w'||^2_{L^2(0,T;H^1_0(\Omega))} = \sum_{\alpha \leq 1} ||D^{\alpha,1}w'||^2_{L^2(0,T;L^2(\Omega))} < \infty.
\]

Thus, \( D^{\alpha,1}w \in L^2(0, T; L^2(\Omega)) \) for \( |\alpha| \leq 1 \) and \( \beta = 1 \).
Hence, \( w \in H^2(\Omega \times (0, T)) \).

So far, we focussed on the wave equation with the case of homogenous Dirichlet boundary condition. However, in transient elastography, we need generalization of this case, i.e. non-homogenous boundary condition. Fortunately, non-homogenous boundary values can be easily transformed into homogenous setting, as we will see later.

The scalar shear displacement in an isotropic medium is governed by the initial-boundary value problem, which is stated in the following theorem.

**Theorem 2.1.5.** Assume that \( \rho \in C^0(\overline{\Omega}) \) and \( \mu \in C^1(\overline{\Omega}) \) satisfy \( \mu(x), \rho(x) \geq \alpha_0 > 0 \). There exists a unique (weak) solution \( u \), as stated in Definition 2.1.1, that satisfies the following hyperbolic equation

\[
\nabla \cdot (\mu(x) \nabla u(x,t)) = \rho(x)u_{tt}(x,t) \quad \text{in } \Omega \times (0,T) \tag{2.1.41}
\]

with homogenous initial condition

\[
u(x,0) = u_t(x,0) = 0 \quad \text{in } \Omega \tag{2.1.42}
\]

and one of the following boundary conditions

\[
u(x,t) = f(x,t) \quad \text{on } \partial \Omega \times (0,T) \tag{2.1.43}
\]

with \( f(x,t) \in H^{3/2}(\partial \Omega \times (0,T)) \) or

\[
u(x,t) = g(x, t) \quad \text{on } \partial \Omega \times (0,T) \tag{2.1.44}
\]

with \( g(x,t) \in H^{1/2}(\partial \Omega \times (0,T)) \), where \( f(x,t) \) and \( g(x,t) \) satisfy some compatibility conditions. Here, \( \nu(x) \) is the outward normal to \( \partial \Omega \), and \( x \in \Omega \).

**Proof.** For simplicity, we assume that \( \rho \equiv 1 \) and focus on the Dirichlet case. In this case, we need the compatibility condition : \( f(x,0) = u(x,0) = 0 \) on \( \partial \Omega \). We proceed
as follows. By the Trace theorem, see [3], there exists a function \( z \in H^2(\Omega \times (0, T)) \) such that \( \gamma_0 z = f \in H^{3/2}(\partial \Omega \times (0, T)) \), where \( \gamma_0 \) is the trace operator defined by

\[
\gamma_0 : H^2(\Omega \times (0, T)) \to H^{3/2}(\partial \Omega \times (0, T)).
\]

We make the change of the unknown variable

\[
u = w + z,
\]
then \( w \) satisfies the following homogenous Dirichlet problem

\[
w_{tt}(x, t) - \nabla \cdot (\mu(x)\nabla w(x, t)) = h(x, t) \quad \text{in } \Omega \times (0, T)
\]

\[
w(x, 0) = \xi(x), \quad w_t(x, 0) = \eta(x) \quad \text{in } \Omega
\]

\[
w(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T)
\]

with

\[
h(x, t) = \nabla \cdot (\mu(x)\nabla z(x, t)) - z_{tt}(x, t) \in L^2(\Omega \times (0, T))
\]

\[
\xi(x) = -z(x, 0) \in H^{3/2}(\Omega) \subset H^1(\Omega)
\]

\[
\eta(x) = -z_t(x, 0) \in H^{1/2}(\Omega) \subset L^2(\Omega).
\]

In addition, from the compatibility condition, we have \( \xi \in H^1_0(\Omega) \). Thus, using transformation (2.1.45) leads us again to the problem (2.1.5) with the data (2.1.6). Hence, a weak solution \( w \) to the problem (2.1.46) is as stated in Definition 2.1.1 and its existence and uniqueness are guaranteed. Thus, from transformation (2.1.45) again, the existence and uniqueness of a weak solution \( u \) follows from that of \( w \).

In chapter 3, we need the solution \( u \) of the wave equation to be continuous, in order to define arrival time. To obtain this high regularity of \( u \), we need to assume that the given data is smooth enough. This motivates the following theorem.

**Corollary 2.1.6.** Under the same hypotheses in Theorem 2.1.5, if we assume the boundary data (2.1.43) or (2.1.44) as \( f(x, t) \in H^{5/2}(\partial \Omega \times (0, T)) \) or \( g(x, t) \in H^{3/2}(\partial \Omega \times (0, T)) \), together with compatibility conditions, then we have \( u \in H^2(\Omega \times (0, T)) \).
\textit{Proof.} For simplicity, we assume that $\rho \equiv 1$ and focus on the Dirichlet case. In this case, we need the compatibility conditions: $f(x, 0) = u(x, 0) = 0$ and $f_t(x, 0) = u_t(x, 0) = 0$ on $\partial \Omega$. We proceed as follows. By the Trace theorem, see [3], there exists a function $z \in H^3(\Omega \times (0, T))$ such that $\gamma_0 z = f \in H^{5/2}(\partial \Omega \times (0, T))$, where $\gamma_0$ is the trace operator defined by

$$\gamma_0 : H^3(\Omega \times (0, T)) \rightarrow H^{5/2}(\partial \Omega \times (0, T)).$$

Using transformation (2.1.45), we find that $w$ satisfies the homogenous Dirichlet problem (2.1.46) with

\begin{align*}
h & \in H^1(\Omega \times (0, T)) \\
\xi & \in H^{5/2}(\Omega) \subset H^2(\Omega) \\
\eta & \in H^{3/2}(\Omega) \subset H^1(\Omega).
\end{align*}

The compatibility conditions lead us to $\eta \in H^1_0(\Omega)$. Then, by using Theorem 2.1.4, we conclude that $w \in H^2(\Omega \times (0, T))$. Finally, using (2.1.45) again with $w \in H^2(\Omega \times (0, T))$ and $z \in H^3(\Omega \times (0, T))$, we conclude that $u \in H^2(\Omega \times (0, T))$. \hfill $\Box$

Hereafter, we assume that the given data is smooth enough such that $u \in H^2(\Omega \times (0, T))$ for the case of wave equation, or $\vec{u} \in [H^2(\Omega \times (0, T))]^n$ for the case of linear elastic equation.
2.2 The Inverse Problems

The inverse problems related to isotropic elasticity model are to reconstruct elastic parameters $\rho$, $\mu$, $\lambda$ or some combination of them such as shear wave, $\sqrt{\frac{\mu}{\rho}}$, and compression wave, $\sqrt{\frac{\mu+2\lambda}{\rho}}$, from appropriate given data. These parameters appear in the linear elastic equation (2.1.1) and the wave equation (2.1.41). Restricting ourself to the wave equation (2.1.41), then the appropriate parameters to be considered are the density $\rho$ and the Lamé parameter $\mu$, or the shear wave speed $\sqrt{\frac{\mu}{\rho}}$. Depending on the given data, several aspects of some inverse problems have been discussed to identify the parameters. Some of them are described briefly in the next three subsections. In subsection 2.2.1, we describe the inverse problem from one or few dynamical measurements, which are taken in the subregion of the domain of interest. The inverse problem from boundary measurement using Dirichlet-to-Neumann map is presented in subsection 2.2.2. Finally, in subsection 2.2.3, we describe the inverse problem from a single interior measurement which will be our main focus in the coming chapters.

2.2.1 The inverse problem from one or few dynamical measurement

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^2$ boundary $\partial \Omega$ and assume that $\rho \equiv 1$. Consider the following acoustic equation

$$\nabla \cdot (\mu(x)\nabla u(x, t)) = u_{tt}(x, t) \quad \text{in } \Omega \times (0, T). \quad (2.2.1)$$

The solution $u(x, t)$ satisfies initial condition

$$u(x, 0) = a(x) \quad \text{and} \quad u_t(x, 0) = 0 \quad \text{in } \Omega \quad (2.2.2)$$

and the boundary condition

$$u(x, t) = b(x, t) \quad \text{in } \partial \Omega \times (0, T). \quad (2.2.3)$$
The unknown coefficient \( \mu = \mu(x) \in \mathcal{C}^1(\bar{\Omega}) \) is assumed to be positive, that is \( \mu(x) > 0 \) for all \( x \in \bar{\Omega} \). Let \( \omega \subset \Omega \) be a suitable subdomain of \( \Omega \) and \( T > 0 \) given.

**Definition.** The inverse problem from one or few dynamical measurement is to determine the coefficient \( \mu = \mu(x) \) that appears in acoustic equation (2.2.1) from a single interior measurement,

\[
u(x, t) \quad (x, t) \in \omega \times (0, T),
\]

assuming that the functions \( a(x), b(x, t) \) are suitably given. To be precise, let \( u = u(\mu) \) be a weak solution of (2.2.1) associated to \( \mu \). Do the values of \( u(\mu) \) in \( \omega \times (0, T) \) determine \( \mu \) uniquely? In other words,

\[
\text{Does } u(\mu_1)|_{\omega \times (0, T)} = u(\mu_2)|_{\omega \times (0, T)} \text{ imply } \mu_1(x) = \mu_2(x), x \in \Omega? 
\]

This problem has been addressed by Imanuvilov and Yamamoto firstly in [9]. They established the uniqueness and the stability for the inverse problem (2.2.1) with a single interior measurement for general spatial dimensions. In their paper, [9], they state that \( \omega \subset \Omega \) can not be an arbitrary subdomain. It has to satisfy the following condition

\[
\{ x \in \partial \Omega : (x - x_0) \cdot \nu(x) \geq 0 \} \subset \partial \omega,
\]

where \( \nu \) is the outward unit normal to \( \partial \omega \) at \( x \) and for some \( x_0 \notin \bar{\Omega} \). For example, in the case of disk \( \Omega = \{ x : x < R \} \) with \( R > 0 \), the restriction requires \( \omega \) to be a neighborhood of a subboundary which is larger than half of \( \partial \Omega \). With this condition, together with a suitable observation time \( T > 0 \) and conditions on \( a(x) \), they proved the stability of the inverse problem, i.e.

\[
||\mu_1 - \mu_2||_{L^2(\Omega)} \leq C \left( \sum_{j=2}^{3} ||\partial^j_t(u(\mu_1) - u(\mu_2))||_{L^2(\omega \times (0, T))} \right) \kappa
\]

for some \( \kappa \in (0, 1), C > 0 \) and \( \mu_1, \mu_2 \) belong to the admissible space of unknown coefficient. In the admissible space, the unknown coefficients satisfy a priori uniform
boundedness, compatible conditions and some positivity of the coefficient. Imanuvilov and Yamamoto proved the stability result by using Carleman estimates for a hyperbolic equation in $H^{-1}$ space.

Furthermore, if $u(\mu_1)$ and $u(\mu_2)$ are sufficiently smooth and $\mu_1 = \mu_2$ on $\partial \Omega$, then $u(\mu_1) = u(\mu_2)$ in $\omega \times (0, T)$ implies $\mu_1(x) = \mu_2(x)$ for $x \in \Omega$, here we have uniqueness of the inverse problem. Similar results can be obtained if the Dirichlet boundary (2.2.3) is replaced with the Neumann boundary case.

These results have been extended to other models, like elasticity, see [10, 8].

2.2.2 The inverse problem from the dynamical Dirichlet-to-Neumann map

Consider the dynamical problem (2.1.41)-(2.1.44). If the Dirichlet boundary data is specified with $f(x, t) \in H^1(\partial \Omega \times (0, T))$, we have $u(x, t) \in H^{3/2}(\Omega \times (0, T))$ and then $\frac{\partial u}{\partial \nu} \in L^2(\partial \Omega \times (0, T))$ with the estimate

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial \Omega \times (0, T))} \leq C \|f\|_{H^1(\partial \Omega \times (0, T))}.$$  

In section 2.1.2, we justified the forward problem for $f(x, t) \in H^{3/2}(\partial \Omega \times (0, T))$, but we can justify the problem for $f(x, t) \in H^1(\partial \Omega \times (0, T))$ as well.

From this estimate, we define the linear operator

$$\Lambda : H^1(\partial \Omega \times (0, T)) \rightarrow L^2(\partial \Omega \times (0, T))$$

which is called the dynamical Dirichlet-to-Neumann (DtN) map.

**Definition.** Let $(\mu_1, \rho_1)$ and $(\mu_2, \rho_2)$ be two families of parameters. We set $\Lambda_1$ and $\Lambda_2$ be the corresponding DtN map. The inverse problem for DtN map is as follows:

Does $\Lambda_1 = \Lambda_2$ imply $(\mu_1, \rho_1) = (\mu_2, \rho_2)$?
This problem has been established for long time, see [1]. In particular, Belishev [1] and Kurylev-Lassas [13] developed the so-called Boundary Control method to reconstruct the parameter \( \mu \) or \( \rho \) from the DtN map.

2.2.3 The inverse problem from interior measurement

In transient elastography, we are given interior displacement data. This data is measured by using ultrasound or MRI. The question related to the inverse problem is stated in the following definition.

**Definition.** The inverse problem from interior measurement is as follows: Does a single interior displacement data identify the wave speed or the elastic parameters uniquely? To be precise, let \( u(x,t) \) and \( \bar{u}(x,t) \) be the solution of the forward problems (2.1.41) and (2.1.1), respectively:

\[
\text{Does } u(x,t) \text{ (resp } \bar{u}(x,t)\text{)}, (x,t) \in \Omega \times (0,T) \text{ uniquely determine } \mu, \rho \text{ or/and } \lambda \text{ (resp } \sqrt{\frac{\mu}{\rho}} \text{ or/and } \sqrt{\frac{\lambda+2\mu}{\rho}}\text{)?}
\]

This question has been considered in [16]. The uniqueness results are available for the isotropic medium case. In the anisotropic case, a single interior displacement data is not enough to establish uniqueness. Recently, some reconstruction algorithms for transient elastography have been proposed. To reconstruct the Lamé parameter \( \mu \), we refer to [11], in which the algorithm is based on the asymptotic expansion of geometrical optics. The reconstruction of the wave speed can be achieved by using propagating front, see [14].

In the remaining part of this thesis, we are going to consider the inverse problem from a single interior measurement data. This includes the uniqueness of the wave speed and the elastic parameters, and the inversion techniques to recover the shear wave speed based on propagating front.
Chapter 3

Uniqueness of the Inverse Problem using Interior Measurements

In this chapter, we discuss the uniqueness result for the inverse problem related to the transient elastography. Particularly, the identification of the shear wave speed and elastic parameters \((\rho, \mu)\) for the scalar displacement case or wave equation. The uniqueness results for the vector displacement case or linear elastic equation are also available and will be discussed briefly in this thesis. Detail discussion for the vector displacement case can be found in [16].

The main tool to obtain uniqueness is the shrink and spread argument. This argument utilizes the properties of hyperbolic and elliptic equations. We are going to discuss the shrink and spread argument first before we discuss the uniqueness result. We start by recalling the notion of the finite propagation speed for dynamical problems and the unique continuation principle for elliptic equations.

**Definition.** Let \(B_\epsilon(x_0) := \{ x \in \mathbb{R}^n : |x - x_0| < \epsilon \} \subset \Omega \) be an open ball in \(\Omega\).

(a) \(\vec{U} = (U_1, \ldots, U_m) \in [H^2_{\text{loc}}(\Omega \times (0, T))]^m\) is said to have a finite propagation speed in \(B_\epsilon(x_0) \times (0, T)\) with the maximum speed \(c > 0\) if for any \(t_0 \in [0, T)\), \(\vec{U}(\cdot, t_0) = \vec{U}_I(\cdot, t_0) = 0\) in \(B_\epsilon(x_0)\) implies that \(\vec{U} = 0\) a.e. in a space time cone \(\bigcup_{0<s<\epsilon/c} C_s\), where \(C_s = C_s(x_0, t_0, \epsilon, c) := B_{c-\epsilon s}(x_0) \times \{t = t_0 + s\}\).
(b) $\vec{V} = (V_1, \ldots, V_m) \in [H^1_{\text{loc}}(\Omega)]^m$ is said to have a unique continuation property in $\Omega$ if $\vec{V} = 0$ in an open subset of $\Omega$ implies that $\vec{V} = 0$ in $\Omega$.

In the next section, we show that the solution of the wave equation and the linear elastic equation satisfy the finite propagation speed.

### 3.1 Finite Propagation Speed

For the wave equation, the solution $u$ of the hyperbolic equation (2.1.41) has a finite propagation speed as shown in the following theorem.

**Theorem 3.1.1.** Assume that $\rho \in C^0(\bar{\Omega})$ and $\mu \in C^1(\bar{\Omega})$ satisfy $\mu(x), \rho(x) \geq \alpha_0 > 0$. Let $u \in H^2(\Omega \times (0, T))$ be a solution to the hyperbolic equation

$$\nabla \cdot (\mu(x)\nabla u(x, t)) = \rho(x)u_{tt}(x, t) \quad \text{in} \quad \Omega \times (0, T),$$

Then for any open ball $B_\epsilon(x_0) \subset \Omega$, $u$ has a finite propagation speed in $B_\epsilon(x_0) \times (0, T)$ with the maximum speed $c = \sup_{x \in B_\epsilon(x_0)} \sqrt{\frac{\mu(x)}{\rho(x)}}$.

**Proof.** Fix any $t_0 \in [0, T)$ and assume that

$$u(\cdot, t_0) = u_t(\cdot, t_0) = 0 \quad \text{in} \quad B_\epsilon(x_0). \quad (3.1.1)$$

We need to show that $u = 0$ a.e. in $\bigcup_{0<s<\epsilon/c} C_s$, where $C_s = B_{c-s}(x_0) \times \{t = t_0 + s\}$.

Let $e(s)$ represent the elastic energy contained in $C_s$ given by

$$e(s) = \frac{1}{2} \int_{C_s} (\rho|u_t|^2 + \mu|\nabla u|^2) \, dx. \quad (3.1.2)$$

We will show that $e(s) = 0$ for all $s \in (0, \epsilon/c)$.

For a fixed $s \in (0, \epsilon/c)$, define $\Lambda(s) := \bigcup_{0<s<s} C_s$. We can interpret $\Lambda(s)$ as time-space dependent semi-cone with height $s$. Taking the inner product of (2.1.41) with $u_t$ and
integrating in $\Lambda(s)$, we get

$$0 = \int_{\Lambda(s)} u_t(\rho u_{tt} - \nabla \cdot (\mu \nabla u)) \, dx \, dt$$

Using identity $u_t \nabla \cdot (\mu \nabla u) = \nabla \cdot (u_t \mu \nabla u) - \nabla u_t \cdot (\mu \nabla u)$ and $\frac{1}{2}(\mu |\nabla u|^2)_t = \nabla u_t \cdot (\mu \nabla u)$, we obtain

$$0 = \frac{1}{2} \int_{\Lambda(s)} \left[(\rho |u_t|^2 + \mu |\nabla u|^2) - 2(\mu u_t \nabla u)\right] \, dx \, dt.$$

Applying the space-time divergence theorem, we get

$$0 = \frac{1}{2} \int_{\partial \Lambda(s)} \left[(\rho |u_t|^2 + \mu |\nabla u|^2) - 2(\mu u_t \nabla u)\right] \nu \, dS_{x,t} \quad \text{(3.1.3)}$$

where $\partial \Lambda(s) = C_0 \cup C_s \cup (\cup_{0<\tau<s} \partial C_\tau)$ is the space-time boundary of $\Lambda(s)$, $dS_{x,t}$ is the space-time boundary element. The outward unit normal vector of $\partial \Lambda(s)$ in the direction of $t$ and $x$ respectively, is given by

$$(\nu_x, \nu_t) = \begin{cases} 
(0, 1), & \text{on } C_s; \\
(0, -1), & \text{on } C_0; \\
\frac{1}{\sqrt{1+c^2}} \left(\frac{x-x_0}{|x-x_0|}, c\right), & \text{on } L := \cup_{0<\tau<s} \partial C_\tau.
\end{cases} \quad \text{(3.1.4)}$$

By using (3.1.4), (3.1.3) can be rewritten as

$$0 = -\frac{1}{2} \int_{C_0} (\rho |u_t|^2 + \mu |\nabla u|^2) \big|_{t=t_0} \, dx + \frac{1}{2} \int_{C_s} (\rho |u_t|^2 + \mu |\nabla u|^2) \big|_{t=t_0+s} \, dx$$

$$+ \frac{c}{2} \frac{1}{\sqrt{1+c^2}} \int_L \left((\rho |u_t|^2 + \mu |\nabla u|^2) - 2\frac{\mu}{c} u_t \nabla u \frac{x-x_0}{|x-x_0|}\right) \, dS_{x,t}$$

from which we get

$$e(s) - e(0) = \frac{1}{2} \frac{c}{\sqrt{1+c^2}} \int_L \left((\rho |u_t|^2 + \mu |\nabla u|^2) - 2\frac{\mu}{c} u_t \nabla u \frac{x-x_0}{|x-x_0|}\right) \, dS_{x,t}. \quad \text{(3.1.5)}$$

By using Cauchy-Schwarz inequality and the fact that $\sqrt{\frac{\mu}{\rho}} \leq c$,

$$2\frac{\mu}{c} u_t \nabla u \frac{x-x_0}{|x-x_0|} \leq 2\sqrt{\mu |u_t| \nabla u} \frac{x-x_0}{|x-x_0|}$$

$$\leq 2\sqrt{\mu |u_t||\nabla u|}.$$
Then we can estimate the integrand in (3.1.5) as follow

\[ \rho |u_t|^2 + \mu |\nabla u|^2 - 2 \frac{\mu}{c} u_t \nabla u \frac{x - x_0}{|x - x_0|} \geq \rho |u_t|^2 + \mu |\nabla u|^2 - 2 \sqrt{\rho \mu} |u_t||\nabla u| \]

\[ \geq (\sqrt{\rho} |u_t| - \sqrt{\mu} |\nabla u|)^2 \]

\[ \geq 0. \]

As a consequence, from (3.1.5) we have

\[ e(s) - e(0) \leq 0. \]

By using (3.1.1) and (3.1.2) we have \( e(0) = 0 \) and \( e(s) \geq 0 \), respectively.

From these two results, we obtain \( 0 \leq e(s) \leq e(0) = 0 \) and then finally \( e(s) = 0 \) \( \forall s \in (0, \epsilon/c) \).

To show that \( u = 0 \) almost everywhere, we use \( L^2 \) estimate for \( u_t \) in the cone \( \Lambda(\epsilon/c) = \bigcup_{0 < s < \epsilon/c} C_s \):

\[ ||u_t||^2_{L^2(\Lambda(\epsilon/c))} = \int_{\Lambda(\epsilon/c)} |u_t|^2 \, dx dt, \text{ since } \frac{\rho}{\alpha_0} \geq 1 \]

\[ \leq \int_{\Lambda(\epsilon/c)} \frac{\rho}{\alpha_0} |u_t|^2 \, dx dt \]

\[ \leq \frac{1}{\alpha_0} \int_{\Lambda(\epsilon/c)} (\rho |u_t|^2 + \mu |\nabla u|^2) \, dx dt \]

\[ \leq \frac{2}{\alpha_0} \int_0^{\epsilon/c} \left( \frac{1}{2} \int_{C_s} (\rho |u_t|^2 + \mu |\nabla u|^2) \, dx \right) ds \]

\[ \leq \frac{2}{\alpha_0} \int_0^{\epsilon/c} e(s) \, ds \]

We proved that \( e(s) = 0 \) \( \forall s \in (0, \epsilon/c) \) and this implies

\[ ||u_t||^2_{L^2(\Lambda(\epsilon/c))} \leq 0. \]

This concludes that \( u_t(x, t) = 0 \) a.e. in \( \Lambda(\epsilon/c) \), but by assumption on homogenous initial condition (3.1.1) we have \( u(x, t) = 0 \) a.e. in \( \bigcup_{0 < s < \epsilon/c} C_s \) which completes the proof. \( \square \)
For the vector displacement case, the displacement \( \vec{u} \) also has a finite propagation speed as shown in the following theorem.

**Theorem 3.1.2.** Assume that \( \rho \in C^0(\bar{\Omega}) \) and \( \mu, \lambda \in C^1(\bar{\Omega}) \) satisfy \( \rho(x), \mu(x), \lambda(x) \geq \alpha_0 > 0 \). Let \( \vec{u} \in [H^2(\Omega \times (0, T))]^n \) be a solution of the hyperbolic system

\[
\nabla (\lambda \nabla \cdot \vec{u}) + \nabla \cdot (\mu (\nabla \vec{u} + (\nabla \vec{u})^T)) = \rho \vec{u}_{tt} \quad \text{in } \Omega \times (0, T)
\]

Then for any open ball \( B_{\epsilon}(x_0) \subset \Omega \), \( \vec{u} \) has a finite propagation speed in \( B_{\epsilon}(x_0) \times (0, T) \) with the maximum speed \( c = \sup_{x \in B_{\epsilon}(x_0)} \sqrt{\frac{\lambda(x) + 2\mu(x)}{\rho(x)}} \).

Thus, the solution \( u \) of the scalar wave equation and the solution \( \vec{u} \) of the linear elastic equation both satisfy the finite propagation speed.

### 3.2 Unique Continuation Principle

It is well known that an elliptic system with the Laplacian, \( \Delta \), as the principle part has a unique continuation principle. The following lemma will help us to show that the difference of two solutions of the wave equation, as well as that of the linear elastic equation, satisfy the unique continuation principle.

**Lemma 3.2.1.** Let \( \Omega \subset \mathbb{R}^n \) and \( \vec{U} = (U_1, \ldots, U_m) \in [H^1_{loc}(\Omega)]^m \) be a solution of

\[
\Delta \vec{U} + B(\nabla \vec{U}) + V(\vec{U}) = 0
\]

in the distributional sense, where the lower-order operators \( B \) and \( V \) are given by

\[
[B(\nabla \vec{U})]_k = \sum_{i=1}^{n} \sum_{j=1}^{m} a^k_{ij}(x) \frac{\partial U_j}{\partial x_i} \quad \text{and} \quad V(\vec{U})_k = \sum_{j=1}^{m} b^k_j(x) U_j
\]

with coefficients \( a^k_{ij}, b^k_j \in L^\infty(\Omega) \) for \( k = 1, \ldots, m \). Then \( \vec{U} \) has a unique continuation principle in \( \Omega \).
The proof of Lemma 3.2.1 can be found in [19, 7]. It is based on the Carleman estimate.

In order to show unique identifiability in the inverse problem for the wave equation, it is natural to begin with a common solution that solves two wave equations (2.1.41) with different elastic parameters. Under this assumption, by using subtraction, we can see that the common solution \( u \) satisfies a differential equation without \( u_{tt} \) term.

The following theorem shows that \( u \) satisfies the unique continuation principle. This principle, however, is only valid in the region where the wave speeds are not identical.

**Theorem 3.2.2.** Assume that \( \rho_j \in C^0(\overline{\Omega}) \) and \( \mu_j \in C^1(\overline{\Omega}) \) for \( j = 1, 2 \) satisfies \( \mu_j(x), \rho_j(x) \geq \alpha_0 > 0 \). Let \( u \in H^2(\Omega \times (0, T)) \) be a common solution for \( j = 1, 2 \) to the hyperbolic equation

\[
\nabla \cdot (\mu_j(x)\nabla u(x, t)) = \rho_j(x)u_{tt}(x, t) \quad \text{in } \Omega \times (0, T)
\]

Then in any open subset \( D \subset \Omega \) satisfying

\[
\left| \frac{\mu_1}{\rho_1} - \frac{\mu_2}{\rho_2} \right| \geq \beta_0 > 0,
\]

\( u(\cdot, t_0) \) has a unique continuation principle in \( D \) for any \( t_0 \in (0, T) \).

**Proof.** Fix any point \( x_0 \in D \). Since \( D \) is an open subset of \( \Omega \), there exists an open ball \( B_\epsilon(x_0) \) in which we have \( \left( \frac{\mu_1}{\rho_1} - \frac{\mu_2}{\rho_2} \right) \neq 0 \). Multiply (3.2.1) with \( \frac{1}{\rho_j} \),

\[
\begin{align*}
\frac{1}{\rho_j}u_{tt} &= \frac{1}{\rho_j}(\nabla \cdot (\mu_j \nabla u)) \\
&= \frac{1}{\rho_j}(\mu_j \Delta u + \nabla \mu_j \cdot \nabla u)
\end{align*}
\]

and subtract one from the other for \( j = 1, 2 \), we get that the trace \( u(\cdot, t_0) \) for any \( t_0 \in (0, T) \) solves the following elliptic equation

\[
\left( \frac{\mu_1}{\rho_1} - \frac{\mu_2}{\rho_2} \right) \Delta u + \left( \frac{\nabla \mu_1}{\rho_1} - \frac{\nabla \mu_2}{\rho_2} \right) \cdot \nabla u = 0 \quad \text{in } B_\epsilon(x_0).
\]
Finally,
\[ \Delta u + \left( \frac{\mu_1}{\rho_1} - \frac{\mu_2}{\rho_2} \right)^{-1} \left( \frac{\nabla \mu_1}{\rho_1} - \frac{\nabla \mu_2}{\rho_2} \right) \cdot \nabla u = 0 \quad \text{in } B_\epsilon(x_0). \]
From Sobolev theory, we get that \( u(x,t_0) \in H^{3/2}(B_\epsilon(x_0)) \subset H^1(B_\epsilon(x_0)) \) for any fixed \( t_0 \in (0,T) \). In addition, due to the smoothness assumption on \( \rho_j \) and \( \mu_j \), we conclude that the coefficients of the above equation are in \( L^\infty(B_\epsilon(x_0)) \). By Lemma 3.2.1, \( u(x,t_0) \) has a unique continuation principle in \( B_\epsilon(x_0) \) for any fixed \( t_0 \in (0,T) \).

Not only the solution of the wave equation but the solution of the linear elastic equation also satisfies the unique continuation principle, stated in the following theorem.

**Theorem 3.2.3.** Assume that \( \rho_j \in C^1(\overline{\Omega}) \) and \( \mu_j, \lambda_j \in C^2(\overline{\Omega}) \) satisfy \( \rho_j(x), \mu_j(x), \lambda_j(x) \geq \alpha_0 > 0 \) for \( j = 1, 2 \). Let \( \vec{u} \in [H^2(\Omega \times (0,T))]^n \) be a common solution for \( j = 1, 2 \) of the hyperbolic systems
\[
\nabla (\lambda_j \nabla \cdot \vec{u}) + \nabla \cdot (\mu_j (\nabla \vec{u} + (\nabla \vec{u})^T)) = \rho_j \vec{u}_{tt} \quad \text{in } \Omega \times (0,T)
\]
Then in any open subset \( D \subset \Omega \) satisfying
\[
\min_D \left( \left| \frac{\mu_1}{\rho_1} - \frac{\mu_2}{\rho_2} \right|, \left| \frac{\lambda_1 + 2\mu_1}{\rho_1} - \frac{\lambda_2 + 2\mu_2}{\rho_2} \right| \right) \geq \beta_0 > 0,
\]
\( \vec{u}(\cdot,t_0) \) has a unique continuation principle in \( D \) for any \( t_0 \in (0,T) \).

Thus, the solution \( u \) and \( \vec{u} \) satisfy the unique continuation principle.

### 3.3 Shrink and Spread Argument

The shrink and spread argument states that in any region where the solution satisfies the finite propagation speed and the unique continuation principle, together with homogenous initial condition, it vanishes for all time.

**Theorem 3.3.1.** Let \( \vec{U} = (U_1, \ldots, U_m) \in [H^2_{\text{loc}}(\Omega \times (0,T))]^m \) and \( B_\epsilon(x_0) \subset \Omega \). Assume that \( \vec{U} \) satisfies the following assumptions:
(a) $\vec{U}$ has a homogenous initial condition:

$$\vec{U}(\cdot, 0) = \vec{U}_t(\cdot, 0) = 0 \quad \text{in } B_\epsilon(x_0)$$

(b) $\vec{U}$ has a finite propagation speed in $B_\epsilon(x_0) \times (0, T)$ with the maximum speed $c > 0$

(c) For any $t_0 \in [0, T)$, the trace $\vec{U}(\cdot, t_0)$ on $B_\epsilon(x_0) \times \{t = t_0\}$ has a unique continuation principle in $B_\epsilon(x_0)$.

Then $\vec{U} \equiv 0$ in $B_\epsilon(x_0) \times (0, T)$.

Proof. By (a) and (b), we get $\vec{U} = 0$ in $\bigcup_{0 < s < \epsilon/c} C_s$ where $C_s = B_{\epsilon-cs}(x_0) \times \{t = s\}$.

Using (c), for any $t_0 \in (0, \epsilon/c)$ we have $\vec{U}(\cdot, t_0) = 0$ implies $\vec{U} = 0$ in $B_\epsilon(x_0) \times \{0, \epsilon/c\}$.

Applying (b) with

$$\vec{U}(x, \frac{\epsilon}{2c}) = \vec{U}_t(x, \frac{\epsilon}{2c}) = 0 \quad \text{in } B_\epsilon(x_0) \times \{t = \frac{\epsilon}{2c}\}$$

as an initial condition and by (c) again we get $\vec{U} = 0$ in $B_\epsilon(x_0) \times \{0, \frac{3\epsilon}{2c}\}$. Iterating such procedures, we finally obtain $\vec{U} = 0$ in $B_\epsilon(x_0) \times (0, T)$. \hfill $\square$

Figure 3.1: The illustration of the shrink and spread argument.
The illustration of the shrink and spread argument is shown in Fig.3.1. At $t = 0$, apply (a) Shrink : $\vec{U} = 0$ in the space time cone by the finite propagation speed. (b) Spread : $\vec{U} = 0$ on each $t \in (0, \epsilon/c)$ by the unique continuation principle. (c) repeat (a) and (b), we will get that $\vec{U} = 0$ in the whole cylinder.

3.4 Uniqueness of Wave Speed in Isotropic Media

In this section, we discuss the uniqueness for the inverse problem, particularly for the wave speed. The shear wave speed, $\sqrt{\mu/\rho}$, and the compression wave speed, $\sqrt{\lambda+2\mu/\rho}$, are uniquely identified from a single interior displacement in any subregion where the wave has propagated, that is where the displacement is not zero for some $t \in (0, T)$.

First, we discuss the case of scalar wave equation.

**Theorem 3.4.1.** Assume that $\rho_j \in \mathcal{C}^0(\bar{\Omega})$ and $\mu_j \in \mathcal{C}^1(\bar{\Omega})$ for $j = 1, 2$ satisfies $\mu_j(x), \rho_j(x) \geq \alpha_0 > 0$. Let $u \in H^2(\Omega \times (0, T))$ be a common solution for $j = 1, 2$ to the hyperbolic equation

$$\nabla \cdot (\mu_j(x)\nabla u(x, t)) = \rho_j(x)u_{tt}(x, t) \quad \text{in } \Omega \times (0, T)$$

with the homogenous initial condition

$$u(x, 0) = u_t(x, 0) = 0 \quad \text{in } \Omega$$

and satisfying either the same Dirichlet boundary condition

$$u(x, t) = f(x, t) \quad \text{on } \partial\Omega \times (0, T),$$

or the same Neumann boundary condition

$$\mu_j(x)\nabla u(x, t) \cdot \nu = g(x, t) \quad \text{on } \partial\Omega \times (0, T),$$

where $\nu$ is the outward normal to $\partial\Omega$. Then we have

$$\frac{\mu_1}{\rho_1} = \frac{\mu_2}{\rho_2} \quad \text{in } \Omega \setminus \Omega_E$$

where $\Omega_E := \bigcup \{ V \subset \Omega \text{ is an open set satisfying } \|u\|_{L^2(V \times (0,T))} = 0 \}$.
Remark 3.4.1. Intuitively, $\Omega_E$ is the subset of $\Omega$ where the wave has not yet traveled during the time $(0, T)$. This means no wave has reached $\Omega_E$ during the time $(0, T)$. Of course, we can not conclude uniqueness in $\Omega_E$ if there is no information in that subregion.

Proof. Let $\Omega$ be expressed as the union of disjoint subset, $\Omega = \Omega^0 \cup \Omega^+ \cup \Omega^-$ where

$$\Omega^0 = \{ x \in \Omega : \frac{\mu_1}{\rho_1} = \frac{\mu_2}{\rho_2} \}$$

$$\Omega^\pm = \{ x \in \Omega : \frac{\mu_1}{\rho_1} \preceq \frac{\mu_2}{\rho_2} \}.$$

We will show that $\Omega^+ \cup \Omega^- \subset \Omega_E$ implying $\Omega \setminus \Omega_E \subset \Omega^0$.

To do this, first we show that $\Omega^+ \subset \Omega_E$ by using the shrink and spread argument. Fix any point $x_0 \in \Omega^+$. Since $\Omega^+$ is an open subset of $\Omega$, there exists an open ball $B_r(x_0) \subset \Omega^+$ on which we have

$$\alpha_1 \leq \frac{\mu_1}{\rho_1} - \frac{\mu_2}{\rho_2} \leq \alpha_2 \text{ for some } \alpha_1, \alpha_2 > 0.$$

By Theorem 3.1.1, $u$ already has a finite propagation speed in $B_r(x_0) \times (0, T)$ with the maximum speed $c = \sup_{x \in B_r(x_0)} \sqrt{\frac{\mu_1(x)}{\rho_1(x)}}$.

By Theorem 3.2.2, $u(\cdot, t_0)$ has a unique continuation principle in $B_r(x_0)$ for any $t_0 \in (0, T)$ since $B_r(x_0) \subset D$, where $D$ is as in Theorem 3.2.2.

Now, by using shrink and spread argument in Theorem 3.3.1 along with the homogeneous initial condition (3.4.2), we have $u = 0$ in $B_r(x_0) \times (0, T)$. This implies that $x_0 \in B_r(x_0) \subset \Omega_E$ and thus $\Omega^+ \subset \Omega_E$. Similarly, by using the same procedure, it can be proved that $\Omega^- \subset \Omega_E$, implying that $\Omega \setminus \Omega_E \subset \Omega \setminus \Omega^+ \cup \Omega^- = \Omega^0$. Thus, $\frac{\mu_1}{\rho_1} = \frac{\mu_2}{\rho_2}$ in $\Omega \setminus \Omega_E$ which completes the proof.

Thus, the shear wave speed is uniquely identified for the case of the scalar wave equation. For the case of the linear elastic equation, the wave speeds are also uniquely identified under a certain assumption. To be precise, if $\frac{\lambda}{\rho}$ is given in the interior,
then the shear wave speed is uniquely identified from a single interior measurement. However, this is true in subregion where the shear wave has propagated, that is \( \mathbf{u} \neq 0 \) for some time \( t \in (0, T) \).

**Theorem 3.4.2.** Assume that \( \rho_j \in C^1(\overline{\Omega}) \) and \( \mu_j, \lambda_j \in C^2(\overline{\Omega}) \) satisfy \( \rho_j(x), \mu_j(x), \lambda_j(x) \geq \alpha_0 > 0 \) for \( j = 1, 2 \). Let \( \bar{u} \in [H^2(\Omega \times (0, T))]^n \) be a common solution for \( j = 1, 2 \) of the hyperbolic systems

\[
\nabla(\lambda_j \nabla \cdot \bar{u}) + \nabla \cdot (\mu_j(\nabla \bar{u} + (\nabla \bar{u})^T)) = \rho_j \bar{u}_{tt} \quad \text{in } \Omega \times (0, T)
\]

with the homogenous initial condition

\[
\bar{u}(x, 0) = \bar{u}_t(x, 0) = 0 \quad \text{in } \Omega
\]

and satisfying either the same Dirichlet boundary condition

\[
\bar{u}(x, t) = \bar{f}(x, t) \quad \text{on } \partial\Omega \times (0, T)
\]

or the same Neumann boundary condition

\[
[(\lambda_j(x) \nabla \cdot \bar{u})I + \mu_j(x)(\nabla \bar{u} + (\nabla \bar{u})^T)] \nu(x) = \bar{g}(x, t) \quad \text{on } \partial\Omega \times (0, T)
\]

with \( \nu(x) \) is the outward normal to \( \partial\Omega \). If \( \frac{\lambda_1}{\rho_1} = \frac{\lambda_2}{\rho_2} \) in \( \Omega \), then we have

\[
\frac{\mu_1}{\rho_1} = \frac{\mu_2}{\rho_2} \quad \text{in } \Omega \setminus \Omega_E
\]

where \( \Omega_E = \bigcup \{ V \subset \Omega \text{ is an open set satisfying } \|\bar{u}\|_{L^2(V \times (0, T))} = 0 \} \)

If the elastic parameter \( \mu \) is given in the interior and the Neumann boundary condition is specified, the compression wave speed is uniquely identified from a single interior measurement in any subregion where the compression wave has propagated, that is \( \nabla \cdot \bar{u} \neq 0 \) for some time \( t \in (0, T) \).

**Theorem 3.4.3.** Assume that \( \rho_j \in C^1(\overline{\Omega}) \) and \( \mu_j, \lambda_j \in C^2(\overline{\Omega}) \) satisfy \( \rho_j(x), \mu_j(x), \lambda_j(x) \geq \alpha_0 > 0 \) for \( j = 1, 2 \). Let \( \bar{u} \in [H^2(\Omega \times (0, T))]^n \) be a common solution for \( j = 1, 2 \) of
the Neumann-type initial-boundary problem (3.4.5), (3.4.6) and (3.4.8). If $\mu_1 = \mu_2$ in $\Omega$, then we have

$$\frac{\lambda_1 + 2\mu_1}{\rho_1} = \frac{\lambda_2 + 2\mu_2}{\rho_2} \quad \text{in } \Omega \setminus \Omega_D$$

where $\Omega_D := \bigcup \{ V \subset \Omega \text{ is an open set satisfying } \| \nabla \cdot \vec{u} \|_{L^2(V \times (0,T))} = 0 \}$

If we replace the Neumann boundary condition (3.4.8) in Theorem 3.4.3 with Dirichlet boundary condition (3.4.7), then an extra condition for elastic parameter $\lambda$ on the boundary is required to obtain the same result.

**Corollary 3.4.4.** Under the same hypothesis in Theorem 3.4.3, that is $\mu_1 = \mu_2$ in $\Omega$, if the Neumann boundary condition (3.4.8) is replaced by the Dirichlet boundary condition (3.4.7) and, in addition, $\lambda_1 = \lambda_2$ on $\partial\Omega$ is assumed, we have $(\lambda_1 + 2\mu_1)/\rho_1 = (\lambda_2 + 2\mu_2)/\rho_2$ in $\Omega \setminus \Omega_E$

### 3.5 Simultaneous Identification in Isotropic Media

In the previous section, it is shown that the wave speeds are uniquely identified from the interior displacement data. In fact, the elastic parameters are also uniquely identified assuming that we have Neumann boundary condition (2.1.44). In case we have Dirichlet boundary condition (2.1.43), an extra condition on the elastic parameter has to be imposed on the boundary. We start with the Neumann case.

#### 3.5.1 Neumann Case

For the wave equation, if the Neumann boundary condition is specified in theorem 3.4.1, not only the shear wave speed but also the elastic parameters $\rho$ and $\mu$ are uniquely identified.

**Theorem 3.5.1.** Under the same hypothesis on $\rho_j$ and $\mu_j$ in Theorem 3.4.1, let $u \in H^2(\Omega \times (0,T))$ be a common solution to the Neumann-type initial-boundary value
problem (3.4.1), (3.4.2) and (3.4.4) for \( j = 1, 2 \). Then we have \( (\rho_1, \mu_1) = (\rho_2, \mu_2) \) in \( \Omega \setminus \Omega_E \), where

\[
\Omega_E := \bigcup \{ V \subset \Omega \text{ is an open set satisfying } \| u \|_{L^2(V \times (0,T))} = 0 \}
\]

**Proof.** From Theorem 3.4.1, we know that the shear wave speed is unique in \( \Omega \setminus \Omega_E \), that is \( c_s^2 = \frac{\mu_1}{\rho_1} = \frac{\mu_2}{\rho_2} \) in \( \Omega \setminus \Omega_E \). It is sufficient to prove that \( \mu_1 = \mu_2 \) in \( \Omega \setminus \Omega_E \). Let \( \Omega \) be expressed as the union of disjoint subsets, \( \Omega = \Omega^0 \cup \Omega^+ \cup \Omega^- \), with

\[
\Omega^0 = \{ x \in \Omega : \mu_1(x) = \mu_2(x) \} \\
\Omega^\pm = \{ x \in \Omega : \mu_1(x) \gtrless \mu_2(x) \}.
\]

We need to show \( \Omega^+ \cup \Omega^- \subset \Omega_E \). To do so, first we show that \( \Omega^+ \subset \Omega_E \). Subtracting (2.1.41) one from the other for \( j = 1, 2 \), we have:

\[
\nabla \cdot \left( \langle \mu \rangle \nabla u \right) = \langle \rho \rangle u_{tt} \tag{3.5.1}
\]

where \( \langle F \rangle = F_1 - F_2 \). Doing the same to the Neumann boundary condition (2.1.44), we have

\[
\langle \mu \rangle \nabla u(x,t) \cdot \nu(x) = 0 \quad \text{on } \partial \Omega \times (0,T). \tag{3.5.2}
\]

Taking the inner product of (3.5.1) with \( u_t \) and integrating in \( \Omega^+ \times (0,s) \) with \( s \in (0,T) \), we have

\[
0 = \int_0^T \int_{\Omega^+} \int_0^s \left[ (\langle \rho \rangle |u_t|^2 + \langle \mu \rangle |\nabla u|^2) - 2 \nabla \cdot (\langle \mu \rangle u_t \nabla u) \right] \, dt \, dx \, ds.
\]

Using the homogenous initial condition (2.1.42), we get

\[
\int_0^T \int_{\Omega^+} (\langle \rho \rangle |u_t|^2 + \langle \mu \rangle |\nabla u|^2)_{t=s} \, dx \, ds = 2 \int_0^T \int_{\Omega^+} \int_0^s \nabla \cdot (\langle \mu \rangle u_t \nabla u) \, dt \, dx \, ds. \tag{3.5.3}
\]

Applying the divergence theorem on the right-hand-side of (3.5.3), we obtain

\[
\int_0^T \int_{\Omega^+} (\langle \rho \rangle |u_t|^2 + \langle \mu \rangle |\nabla u|^2) \, dx \, ds = 2 \int_0^T \int_{\partial \Omega^+} \int_0^s (\langle \mu \rangle u_t \nabla u) \cdot \nu \, dt \, dS_x \, ds \tag{3.5.4}
\]
The smoothness assumption on $\mu$ implies that $\langle \mu \rangle = 0$ on $\partial \Omega^+ \setminus \partial \Omega$. Together with the Neumann boundary condition (3.5.2), the right-hand-side of (3.5.4) vanishes:

$$
\int_0^T \int_{\partial \Omega^+} \int_0^s \langle \mu \rangle u_t \nabla u \cdot \nu \, dt \, dS_x \, ds = \int_0^T \int_{\partial \Omega^+ \setminus \partial \Omega} \int_0^s \langle \mu \rangle u_t \nabla u \cdot \nu \, dt \, dS_x \, ds \\
+ \int_0^T \int_{\partial \Omega^+ \cap \partial \Omega} \int_0^s \langle \mu \rangle u_t \nabla u \cdot \nu \, dt \, dS_x \, ds = 0.
$$

Then, we have

$$
\int_0^T \int_{\Omega^+} (\langle \rho \rangle |u_t|^2 + \langle \mu \rangle |\nabla u|^2) \, dx \, ds = 0. \tag{3.5.5}
$$

On the other hand, since

$$
\mu_1 - \mu_2 = c_s^2 (\rho_1 - \rho_2) \quad \text{in } \Omega \setminus \Omega_E \tag{3.5.6}
$$

and $u_t = 0$ a.e. in $\Omega_E \times (0,T)$, we have $\langle \rho \rangle |u_t|^2 = \frac{\mu_1}{c_s^2} |u_t|^2$ a.e. in $\Omega \times (0,T)$. Hence,

$$
\int_0^T \int_{\Omega^+} (\langle \rho \rangle |u_t|^2 + \langle \mu \rangle |\nabla u|^2) \, dx \, ds = \int_0^T \int_{\Omega^+} \langle \mu \rangle \left( \frac{|u_t|^2}{c_s^2} + |\nabla u|^2 \right) \, dx \, ds = 0. \tag{3.5.7}
$$

Since $\langle \mu \rangle > 0$ in $\Omega^+$, (3.5.7) requires us to have $\left( \frac{|u_t|^2}{c_s^2} + |\nabla u|^2 \right) = 0$ in $\Omega^+ \times (0,T)$. This implies $u_t = \nabla u = 0$ in $\Omega^+ \times (0,T)$. By using the homogenous initial condition (3.4.2), we obtain $u \equiv 0$ in $\Omega^+ \times (0,T)$. Thus, we have $\Omega^+ \subset \Omega_E$. Similarly, we can prove that $\Omega^- \subset \Omega_E$, implying that $\Omega \setminus \Omega_E \subset \Omega \setminus \Omega^+ \cup \Omega^- = \Omega^0$. Thus, $\mu_1 = \mu_2$ in $\Omega \setminus \Omega_E$. Using this and from (3.5.6), it follows that $\rho_1 = \rho_2$ in $\Omega \setminus \Omega_E$ which completes the proof.

For the linear elastic equation, if the Neumann boundary condition is specified in Theorem 3.4.2, the elastic parameters $\rho$ and $\mu$ are uniquely identified.

**Theorem 3.5.2.** Under the same hypothesis on $\rho_j, \mu_j$ and $\lambda_j$ in Theorem 3.4.2, let $\vec{u} \in [H^2(\Omega \times (0,T))]^n$ be a common solution to the Neumann-type initial-boundary value problem (3.4.5), (3.4.6) and (3.4.8) for $j = 1, 2$. If $\lambda_1/\rho_1 = \lambda_2/\rho_2$ in $\Omega$, then we have $(\rho_1, \mu_1) = (\rho_2, \mu_2)$ in $\Omega \setminus \Omega_E$, where

$$
\Omega_E := \bigcup \{ V \subset \Omega \text{ is an open set satisfying } \| \vec{u} \|_{L^2(V \times (0,T))} = 0 \}.
$$
The same result is also deduced from Theorem 3.4.3. If the Neumann boundary condition is specified, the elastic parameters $\rho$ and $\lambda$ are uniquely identified.

**Theorem 3.5.3.** Under the same hypothesis on $\rho_j, \mu_j$ and $\lambda_j$ in Theorem 3.4.3, let $\vec{u} \in [H^2(\Omega \times (0, T))]^n$ be a common solution to the Neumann-type initial-boundary value problem (3.4.5), (3.4.6) and (3.4.8) for $j = 1, 2$. If $\mu_1 = \mu_2$ in $\Omega$, then we have $(\rho_1, \lambda_1) = (\rho_2, \lambda_2)$ in $\Omega \setminus \Omega_D$, where

$$\Omega_D := \bigcup \{V \subset \Omega \text{ is an open set satisfying } \|\nabla \cdot \vec{u}\|_{L^2(V \times (0, T))} = 0\}$$

A detailed proof of Theorem 3.5.2 and Theorem 3.5.3 can be found in [16].

### 3.5.2 Dirichlet Case

If the Dirichlet boundary condition is specified instead of the Neumann boundary condition, extra conditions need to be imposed on the boundary to identify the elastic parameters simultaneously. In the proof of Theorem 3.5.1, the Neumann boundary condition (3.5.2) is used to derive (3.5.5). If we specify Dirichlet boundary condition instead of Neumann, the simultaneous unique identification still holds if we assume $\langle \mu \rangle = 0$ on $\partial \Omega$ such that (3.5.2) still holds.

**Corollary 3.5.4.** Under the same hypothesis on $\rho_j$ and $\mu_j$ in Theorem 3.5.1, if the Neumann boundary condition (3.4.4) is replaced by the Dirichlet boundary condition (3.4.3) and, in addition, either $\rho_1 = \rho_2$ or $\mu_1 = \mu_2$ on $\partial \Omega$, then $(\rho_1, \mu_1) = (\rho_2, \mu_2)$ in $\Omega \setminus \Omega_E$.

For linear elastic equation, a priori knowledge on the elastic parameters on the boundary is needed if we replace the Neumann boundary condition with Dirichlet boundary condition.

**Corollary 3.5.5.** Under the same hypothesis in Theorem 3.5.2, that is $\lambda_1 / \rho_1 = \lambda_2 / \rho_2$ in $\Omega$, if the Neumann boundary condition (3.4.8) is replaced by the Dirichlet boundary
condition (3.4.7) and, in addition, either \( \rho_1 = \rho_2, \mu_1 = \mu_2 \) or \( \lambda_1 = \lambda_2 \) on \( \partial \Omega \) is assumed, we have \((\rho_1, \mu_1) = (\rho_2, \mu_2)\) in \( \Omega \setminus \Omega_E \).

**Corollary 3.5.6.** Under the same hypothesis in Theorem 3.5.3, that is \( \mu_1 = \mu_2 \) in \( \Omega \), if the Neumann boundary condition (3.4.8) is replaced by the Dirichlet boundary condition (3.4.7) and, in addition, \( \lambda_1 = \lambda_2 \) on \( \partial \Omega \) is assumed, we have \((\rho_1, \lambda_1) = (\rho_2, \lambda_2)\) in \( \Omega \setminus \Omega_D \).

Is it possible to obtain simultaneous unique identification without any additional assumption on the Dirichlet boundary? This question will be answered in the next counterexample which is constructed based on traveling waves. The following lemma will be used to construct counterexample for the unique identifiability in isotropic medium.

**Lemma 3.5.7.** Let \( U \in C^2(\mathbb{R}) \) satisfy \( U(s) = 0 \) for \( s < 0 \), and let \( \varphi \in C^2(\mathbb{R}^n) \) satisfy \( |\nabla \varphi| > 0 \) in \( \mathbb{R}^n_{\varphi > 0} := \{ x \in \mathbb{R}^n : \varphi(x) > 0 \} \). Let \( \Omega \subset \mathbb{R}^n_{\varphi > 0} \) be an open connected \( C^2 \) domain. If \( M \in [C^1(\bar{\Omega})]^{n \times n} \) and \( \rho \in C^0(\bar{\Omega}) \) satisfy

\[
\nabla \varphi \cdot M \nabla \varphi = \rho \quad \text{and} \quad \nabla \cdot (M \nabla \varphi) = 0 \quad \text{in} \ \Omega, \quad (3.5.8)
\]

then the traveling wave \( u(x, t) = U(t - \varphi(x)) \in C^2(\bar{\Omega}) \times [0, T] \) satisfies

\[
\nabla \cdot (M(x) \nabla u(x, t)) = \rho u_{tt}(x, t) \quad \text{in} \ \Omega \times (0, T)
\]

(3.5.9)

with the homogenous initial condition

\[
u(x, 0) = u_t(x, 0) = 0 \quad \text{in} \ \Omega, \quad (3.5.10)\]

and the Dirichlet boundary condition

\[
u(x, t) = U(t - \varphi(x)) \quad \text{on} \ \partial \Omega \times (0, T). \quad (3.5.11)\]

Moreover, the Neumann boundary condition is given by

\[
M(x) \nabla u(x, t) \cdot \nu = -\dot{U}(t - \rho(x)) (M(x) \nabla \varphi) \cdot \nu \quad \text{on} \ \partial \Omega \times (0, T), \quad (3.5.12)
\]

where \( \nu \) is the outward normal to \( \partial \Omega \) and \( \dot{U} \) represents the derivative of \( U \).
Proof. The partial derivatives of \( u(x,t) = U(t - \varphi(x)) \) are given by

\[
\begin{align*}
\nabla u &= -\dot{U} \nabla \varphi \\
\dot{u}_t &= \dot{U}, \quad u_{tt} = \ddot{U}, \quad \nabla \ddot{U} = -\dddot{U} \nabla \varphi
\end{align*}
\]

where \( \dot{U} \) represents the derivative of \( \dot{U} \). Then it follows that,

\[
\nabla \cdot (M(x) \nabla u(x,t)) = -\nabla \cdot (\dot{U} M \nabla \varphi)
\]

\[
\begin{align*}
&= -\dot{U} \underbrace{\nabla \cdot (M \nabla \varphi)}_{\equiv 0} - \nabla \cdot (M \nabla \varphi) \\
&= \dddot{U} \underbrace{\nabla \varphi \cdot (M \nabla \varphi)}_{\equiv \rho} \\
&= \rho \dddot{U}
\end{align*}
\]

Therefore,

\[
\nabla \cdot (M(x) \nabla u(x,t)) = \rho u_{tt} \quad \text{in \( \Omega \times (0, T) \)},
\]

which proves (3.5.9). From the construction of \( u \), for the homogenous initial condition (3.5.10) we have,

\[
u(x,0) = U(-\varphi(x)) = 0 \quad \text{and} \quad u_t(x,0) = \dot{U}(-\varphi(x)) = 0,
\]

since \( \varphi(x) \in \Omega \subset \mathbb{R}^n_{\varphi > 0} \) and \( U(s) = 0 \) for \( s < 0 \). The Dirichlet and Neumann boundary conditions are trivially satisfied. Hence, (3.5.9)-(3.5.12) are all satisfied, which completes the proof. \(\square\)

For the isotropic medium case, we have \( M = \mu I \) and (3.5.8) is equivalent to

\[
\rho = \mu |\nabla \varphi|^2 \quad \text{and} \quad \nabla \cdot (\mu \nabla \varphi) = 0 \quad \text{in} \ \Omega \quad \text{(3.5.13)}
\]

From lemma 3.5.7, the traveling wave \( u(x,t) = U(t - \varphi(x)) \) is the common solution of (3.5.9) with homogenous initial condition (3.5.10) and Dirichlet boundary condition (3.5.11), for different \( \rho \) and \( \mu \) that satisfy (3.5.13). There are a lot of pairs \( (\rho, \mu) \) that satisfy (3.5.13), so it is not possible to obtain simultaneous unique identification.
under Dirichlet boundary condition unless we specify \( \rho \) or \( \mu \) on the boundary, as shown in the following example. For simplicity, we choose \( n = 2 \).

**Example 3.5.8.** Let \( \Omega = \mathbb{R}^2_+ = \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0 \} \) and \( \varphi(x) = x_2 \). Fix \( U \in C^2(\mathbb{R}) \) that satisfies \( U(s) = 0 \) for \( s < 0 \) and choose any \( \omega \in C^1(\Omega) \) with \( \omega(x) := \omega(x_1) > 0 \) in \( \Omega \). Then for \( M = \mu I, \mu = \omega \) and \( \rho = \omega \), \( u(x, t) = U(t - x_2) \) solves (3.5.9)-(3.5.11) since (3.5.13) is satisfied. Thus, \( (\rho, \mu) = (\omega(x_1), \omega(x_1)) \), can be elastic parameters that have the same shear displacement \( u(x, t) = U(t - x_2) \) in \( \mathbb{R}^2_+ \) satisfying the same Dirichlet boundary condition.

### 3.6 Non-uniqueness in Anisotropic Media

In the previous section, we discussed the sufficient conditions to obtain the uniqueness results for the elastic parameters identification or some combination of them in isotropic medium. In anisotropic medium, however, the shear tensor \( M(x) \) may not be uniquely identified regardless of the type of specified boundary condition, even though the density \( \rho \) is assumed to be known. The counterexample for anisotropic medium is analogous to Example 3.5.8, but here for simplicity we take the case where the wave speed in one direction is a variable and the speed in the orthogonal direction is a known constant.

**Example 3.6.1.** Let \( \Omega = \mathbb{R}^2_+ = \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0 \} \) and \( \varphi(x) = x_2 \). Fix \( \rho(x) = 1 \) and \( U \in C^2(\mathbb{R}) \) satisfying \( U(s) = 0 \) for \( s < 0 \). Note that this time the density \( \rho \) is assumed to be known. Choose any \( 0 < \omega(x) \in C^1(\Omega) \). Then for \( M = \begin{pmatrix} \omega(x) & 0 \\ 0 & 1 \end{pmatrix} \), \( u(x, t) = U(t - x_2) \) solves (3.5.9)-(3.5.11) since (3.5.8) is satisfied. With this particular choice of data, the Neumann boundary condition (3.5.12),

\[
M(x) \nabla u(x, t) \cdot \nu = -\dot{U}(t - x_2))(M(x) \nabla \varphi) \cdot \nu \\
= \dot{U}(t - x_2) \quad \text{on } \partial \Omega \times (0, T),
\]
is independent of $\omega(x)$ where $\nu(x) = (0, -1)$.

Thus, any $M = \text{diag}(\omega(x), 1)$ can be an elastic tensor that gives the same shear displacement $u(x, t) = U(t - x_n)$ in $\mathbb{R}^2_+$ satisfying the same Dirichlet and Neumann boundary condition.

In the above example, the wave propagation is always in the $x_2$-direction at all points in the domain. So, it is expected that the displacement data do not contain any $x_1$-direction information, $\omega(x)$, in the anisotropic tensor.

### 3.7 Uniqueness of Wave Speed from Arrival Time

In the previous section, we discussed the uniqueness results from interior displacement data. However, from this interior displacement data, we can extract the arrival time, which is also an important feature of the data in transient elastography. Therefore, in this section, we will discuss the uniqueness result of the shear wave speed from the arrival time information. Moreover, we will show that the arrival time information is not enough to obtain simultaneous unique identification for the elastic parameters, in contrast to the interior measurements.

Here, we restrict ourselves to the wave equation (2.1.41) and assume that $u$ is continuous, $\rho \in C^1(\Omega)$ and the domain $\Omega$ is $C^1$.

**Definition 3.7.1.** Assume that $\rho \in C^1(\bar{\Omega})$ and $\mu \in C^1(\bar{\Omega})$ satisfy $\mu(x), \rho(x) \geq \alpha_0 > 0$. Let $u \in H^2(\Omega \times (0, T))$ be a solution to the initial-boundary value problem (2.1.41), (2.1.42) with boundary condition (2.1.43) or (2.1.44). Suppose further that the solution $u$ is continuous. Define

$$\Omega_{u \neq 0} := \{x \in \Omega : u(x, t) \neq 0 \text{ for some } t \in (0, T)\}$$

and the arrival time

$$\bar{T} : \Omega_{u \neq 0} \rightarrow [0, T] \text{ with } \bar{T}(x) := \inf\{t \in (0, T) : |u(x, t)| > 0\}. \quad (3.7.1)$$
Then the arrival time surface gives the position of the propagating wave front as the waves move through the medium.

Remark 3.7.1. For $\Omega \subset \mathbb{R}^2$, the solution $u \in H^2(\Omega \times (0, T))$ is continuous by Sobolev embedding theorem [3]. However, for $\Omega \subset \mathbb{R}^3$, we need more regularity on $u$ that is $u \in H^{2+\delta}(\Omega \times (0, T)), \delta > 0$ such that $H^{2+\delta}(\Omega \times (0, T)) \subset C^0(\bar{\Omega} \times (0, T))$ by Sobolev embedding theorem.

We would like to establish that $\hat{T}$ satisfies the Eikonal equation a.e. in $\Omega_{u \neq 0}$ under assumption that $\hat{T}$ is Lipschitz continuous. To accomplish this, we use the following two lemmas.

**Lemma 3.7.1.** Let $\hat{\Omega} \subset \mathbb{R}^n$ be open and $v \in C^0(\hat{\Omega})$ be differentiable at $x_0 \in \hat{\Omega}$. Then there exists $w \in C^1(\mathbb{R})$ with $w(x_0) = v(x_0)$ and $w < v$ in a punctured neighborhood of $x_0$, which implies $\nabla w(x_0) = \nabla v(x_0)$.

**Lemma 3.7.2.** Let $\hat{\Omega} \subset \Omega \subset \mathbb{R}^n$ be open and $(t_1, t_2) \subset (0, T)$. Let $\mu, \rho \in C^1(\bar{\Omega})$ and $u \in H^2(\Omega \times (0, T))$. Let $S$ be an $n$-dimensional $C^1$ surface in $\hat{\Omega} \times (t_1, t_2)$, $S = \{(x, t) \in \hat{\Omega} \times (t_1, t_2) : \phi(x, t) = 0\}$, which is the zero level set of a $C^1$ function $\phi(x, t)$ with $\nabla \phi \neq 0$ on $S$. Assume that $S$ is non-characteristic with respect to the operator $\nabla \cdot \mu \nabla - \rho \partial^2_t$. Then $u$ satisfies the unique continuation principle, in the sense that if $u(x, t) = 0$ for $(x, t)$ near $S$ satisfying $\phi(x, t) > 0$ then also $u(x, t) = 0$ for $(x, t)$ near $S$ satisfying $\phi(x, t) < 0$.

Then from Lemma 3.7.1 and 3.7.2, we establish the following.

**Theorem 3.7.3.** Assume that $\rho \in C^1(\bar{\Omega})$ and $\mu \in C^1(\bar{\Omega})$ satisfy $\mu(x), \rho(x) \geq \alpha_0 > 0$. Let $u \in H^2(\Omega \times (0, T)) \cap C^0(\Omega \times (0, T))$ be a solution to the initial-boundary value problem (2.1.41), (2.1.42) with boundary condition (2.1.43) or (2.1.44). Suppose further that the arrival time $\hat{T} : \Omega_{u \neq 0} \rightarrow [0, T]$ is Lipschitz continuous. Then $\hat{T}$ satisfies the Eikonal equation

$$\sqrt{\frac{\rho}{\mu}} = |\nabla \hat{T}| \quad \text{a.e. in } \Omega_{u \neq 0}. \tag{3.7.2}$$
Proof. We proceed by contradiction. Since $\hat{T}$ is Lipschitz continuous, $\nabla \hat{T}$ exists a.e. in $\Omega_{u \neq 0}$. Suppose $x_0 \in \Omega_{u \neq 0}$ be a point where $\nabla \hat{T}$ exists with

$$\sqrt{\frac{\rho(x_0)}{\mu(x_0)}} \neq |\nabla \hat{T}(x_0)|.$$ 

By lemma 3.7.1, we can construct $w \in C^1(\Omega_{u \neq 0})$ satisfying $w(x_0) = T(x_0), \nabla w(x_0) = \nabla T(x_0)$ and $w < \hat{T}$ in a punctured neighborhood of $x_0$. By continuity of $|\nabla \hat{T}|$, we have $\sqrt{\frac{\rho(x)}{\mu(x)}} \neq |\nabla w(x)|$ in a neighborhood of $x_0$ and therefore $A_w = \{(x, t) \in \Omega_{u \neq 0} \times [0, T] : t = w(x)\}$ is a non-characteristic $C^1$ surface in a neighborhood of $(x_0, w(x_0))$. Since $w < \hat{T}$ in the neighborhood of $x_0$ and by definition of $\hat{T}$, we have $u(x, t) = 0$ for $(x, t)$ near $(x_0, w(x))$ satisfying $t < w(x)$. From Lemma 3.7.2, we have $u(x, t) = 0$ for $(x, t)$ near $(x_0, w(x))$ satisfying $t > w(x)$. These two results imply, at $x = x_0$, we have $u(x_0, \hat{T}(x_0) + \epsilon) = 0$ for small $\epsilon > 0$. This contradicts the definition of $\hat{T}(x_0)$. Hence, $\sqrt{\frac{\rho(x_0)}{\mu(x_0)}} = |\nabla \hat{T}(x_0)|$.

From Theorem 3.7.3, it follows immediately that the wave speed is uniquely identified from the arrival time information and it depends continuously on $|\nabla \hat{T}|$.

**Corollary 3.7.4.** Assume that $\rho_j \in C^1(\Omega)$ and $\mu_j \in C^1(\Omega)$ satisfy $\mu_j(x), \rho_j(x) \geq \alpha_0 > 0$. Let $u \in H^2(\Omega \times (0, T)) \cap C^0(\Omega \times (0, T))$ be a solution to the initial-boundary value problem (2.1.41)-(2.1.44) with coefficients $\rho_j, \mu_j$ and the boundary input $f_j$ or $g_j$ for $j = 1, 2$. Suppose further that each arrival time $\hat{T}_j$ corresponding to the solution $u_j$ is Lipschitz continuous. Let $\hat{\Omega} := \Omega_{u_1 \neq 0} \cap \Omega_{u_2 \neq 0}$. Then we have,

(a) If $\hat{T}_1 = \hat{T}_2$ in $\hat{\Omega}$, then $\mu_1/\rho_1 = \mu_2/\rho_2$ in $\hat{\Omega}$

(b) Let $\alpha_0 := \min_{\Omega} \{\min\{\rho_1, \rho_2, \mu_1, \mu_2\}\}$ and $\alpha_1 := \max_{\Omega} \{\max\{\rho_1, \rho_2, \mu_1, \mu_2\}\}$. Then

$$\int_{\hat{\Omega}} \left| \sqrt{\frac{\mu_1}{\rho_1}} - \sqrt{\frac{\mu_2}{\rho_2}} \right| \, dx \leq \frac{\alpha_1}{\alpha_0} \int_{\hat{\Omega}} |\nabla (\hat{T}_1 - \hat{T}_2)| \, dx.$$
3.8 Non-uniqueness of Simultaneous Identification from Arrival Time

The arrival time $\hat{T}$ is a part of the information extracted from the interior shear wave displacement $u(x,t)$. Corollary 3.7.4 shows that the shear wave speed is uniquely identified from the arrival time information. In this section we will show that the arrival time information is not enough to obtain simultaneous unique identification for the elastic parameters.

For the counterexample, we need to find $(\rho_1, \mu_1) \neq (\rho_2, \mu_2)$ with $\hat{T}_1(x) = \hat{T}_2(x)$, $\hat{T}_j$ is the corresponding arrival time of the solution $u_j$ of (2.1.41)-(2.1.44) with elastic parameters $(\rho_j, \mu_j)$ for $j = 1, 2$, satisfying the same Dirichlet or the same Neumann boundary condition. From the uniqueness of the forward problem, if $u_1 \neq u_2$ then clearly $(\rho_1, \mu_1) \neq (\rho_2, \mu_2)$.

As in example 3.5.8, we will consider traveling waves. Let $\Omega = \mathbb{R}^2_+ = \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0 \}$ and $\varphi(x) = x_2$. Fix $U \in C^2(\mathbb{R})$ satisfying $U(s) = 0$ for $s < 0$. For $(\rho_1, \mu_1) = (1, 1)$, $u_1(x,t) = U(t - x_2)$ solves (2.1.41)-(2.1.44) and satisfies boundary conditions

$$u_1(x,t) = U(t) \quad \text{on } \partial \Omega \times (0,T) \quad (3.8.1)$$
$$\nabla u_1(x,t) \cdot \nu = \dot{U}(t) \quad \text{on } \partial \Omega \times (0,T) \quad (3.8.2)$$

with $\nu = (0, -1)$ on $\partial \Omega$.

By construction of $U(s)$, for a fixed point $x$, $u_1(x,t) = 0$ for $t < x_2$ and $|u_1(x,t)| > 0$ for $t \geq x_2$. This implies, by definition of arrival time (3.7.1), we have $\hat{T}_1(x) = x_2$.

For $(\rho_2, \mu_2) = (2, 2)$, $u_2(x,t) := \frac{1}{2}U(t - x_2)$ is the solution of (2.1.41),(2.1.42) and (2.1.44) satisfying the Neumann boundary condition $\nabla u_2 \cdot \nu = \dot{U}(t) = \nabla u_1 \cdot \nu$. Clearly, $\hat{T}_1(x) = \hat{T}_2(x)$ and $u_1(x,t) \neq u_2(x,t)$ with $(\rho_1, \mu_1) \neq (\rho_2, \mu_2)$.

For $(\rho_2, \mu_2) = ((x_2+1)^2, (x_2+1)^2)$, $u_2(x,t) := \frac{U(t-x_2)}{x_2+1}$ is the solution of (2.1.41),(2.1.42) and (2.1.43) satisfying the Dirichlet boundary condition $u_2(x,t) = U(t) = u_1(x,t)$. 

Clearly, $\hat{T}_1(x) = \hat{T}_2(x)$ and $u_1(x, t) \neq u_2(x, t)$ with $(\rho_1, \mu_1) \neq (\rho_2, \mu_2)$.

These two counterexamples show that it is not possible to obtain simultaneous unique identification from the arrival time information.

Comparing the uniqueness result from interior measurement and that from arrival time information, the shear wave speed is identified uniquely. Unfortunately, this is not the case for the simultaneous identification of parameters $(\rho, \mu)$. We have seen that the identification of parameters $(\rho, \mu)$ is not possible using the arrival time information, in contrast to that of the interior measurement. This is as expected, since the arrival time is a subset of the interior displacement data. So, the interior displacement data contains more information, and as a consequence, it gives more uniqueness results.
Chapter 4

Wave Speed Reconstruction Methods from Arrival Time Information

In this chapter, we will discuss some algorithms to recover the shear wave speed from the interior displacement data. The Eikonal equation (3.7.2) relates the arrival time with the shear wave speed. This implies that the shear wave speed can be determined from the arrival time information. Actually, we get an advantage by using (3.7.2) in recovering the shear wave speed. The Eikonal equation (3.7.2) contains one less derivative compared to that of the wave equation (2.1.41). Differentiation is an ill-posed operation especially in the case of noisy data, so it is preferable to have less order of differentiation.

To recover the wave speed from the arrival time information, we use an algorithm which is composed of two sub algorithms. First, to find the arrival time from the interior displacement data. Secondly, to find the shear wave speed from the arrival time using the Eikonal equation (3.7.2). For this, we need to assume that the shear wave displacement $u$ satisfies wave equation (2.1.41) and the condition in Theorem 3.7.3 is satisfied. To solve Eikonal equation (3.7.2), we describe two reconstruction methods to recover the shear wave speed from the arrival time information, that is
the distance method and the inverse level set method, both are discussed in section 4.2.

4.1 Finding the Arrival Time

This subsection will be devoted to describe a method for finding the arrival time from the shear wave displacement. We start by using pattern matching. The reason behind this idea is due to the fact that the overall shape of the time trace displacement (wave pulse) $u(x,t)$ is preserved as the wave propagated through the medium. To show this, we generate a synthetic displacement data by solving the wave equation with an appropriate given data. A detailed description on the data can be found in chapter 5, section 5.2, for the case of intermediate contrast. This synthetic displacement data is used to produce all illustrations in this chapter.

![Wave pulse](image)

Figure 4.1: (a) wave pulse at a point near the boundary. (b) wave pulse at a point in the interior

From the resulting displacement data, we fix two points, a point near to the source
boundary and a point in the interior, and then record the wave pulse at those points. The result is shown in Fig. 4.1 (a)-(b), that some parts of the wave pulse match the pattern of one to the other.

One way to do pattern matching is by identifying the feature of the wave pulse, e.g. the first non-zero displacement, the peak of the amplitude; and track this feature as the wave propagates through the medium. The simplest idea to do this is by using the fact that the shear displacement is zero before the shear wave arrives. So, detecting a feature of the wave with the first non-zero displacement could be a good estimate for the arrival time. Using this idea and from Definition (3.7.1), we can estimate the arrival time by

$$\hat{T}(x) \approx \inf\{t \in (0, T) : |u(x, t)| > \delta\}$$

where $\delta$ is a fixed threshold above noise level. We refer to this approach as thresholding.

This approach has some drawbacks though it is simple. Using the first non-zero displacement above a certain threshold will lead to an arrival time estimate that is too large. Moreover, in the region where the wave speed is not constant, for example in an increasing speed region, the wave pulse is scattered and its amplitude is decreasing. In a region where little scattering occurs, thresholding will give an arrival time estimate very close to the first non-zero displacement of the wave. In the region where scattering has degraded the amplitude significantly, for example close to noise level in the extreme case, thresholding will give an arrival time estimate very close to the peak amplitude of the wave pulse. So, thresholding detects the feature of the wave inconsistently. This will lead to large error in estimating the arrival time $\hat{T}$, especially in the region where the wave speed is not constant. Since we are mainly interested in the region with rapid change in speed, thresholding would not be an appropriate choice to estimate the arrival time.

We seek for a method which detects the feature of the wave pulse consistently.
With such a method we can hope that the error is roughly constant and this can be ignored since we are using $\nabla \hat{T}$ in (3.7.2). Furthermore, the method should work on a wide variety of wave pulses and uses as many detecting features of the wave pulses as possible. A possible choice for such method is to use cross correlation. This method measures the similarity of two wave pulses by comparing $u(x_{\text{ref}}, t)$, the wave pulse at a fixed reference point $x_{\text{ref}}$ with $u(x, t)$, the wave pulse at point $x$ in the domain of interest. This means that cross correlation uses the entire wave pulse as a detecting feature. The biased cross correlation of $u(x, t)$ and $u(x_{\text{ref}}, t)$ is computed using the following formula:

$$C(x, \delta t) := \frac{1}{T} \int_0^T u(x_{\text{ref}}, t) \hat{u}(x, t - \delta t) dt,$$

(4.1.1)

with

$$\hat{u}(x, t) = \begin{cases} 
  u(x, t), & 0 \leq t \leq T; \\
  u(x, t - T), & t > T; \\
  u(x, t + T), & t < 0.
\end{cases}$$

The formula (4.1.1) essentially shifts $u(x, t)$ along $t$-axis and evaluates the integral for all possible value of shifting in $[0, T]$. When $u(x, t - \delta t)$ and $u(x_{\text{ref}}, t)$ are nearly aligned, or in other words highly correlated, the value of $C(x, \delta t)$ is maximized. The arrival time $\hat{T}(x)$ is estimated by the time delay $\delta t$ that maximize the correlation between $u(x, t)$ and $u(x_{\text{ref}}, t)$, that is $\hat{T}(x) \approx \delta t_{\text{max}}$ where

$$\delta t_{\text{max}} := \arg\max_{\delta t \in [0, T]} C(x, \delta t).$$

The cross correlation between two wave pulses is illustrated in Fig.4.2 and Fig.4.3. The cross correlation at a reference point $x_{\text{ref}}$ near the boundary, $u(x_{\text{ref}}, t)$ is shown in Fig.4.2 (a). After passing through high speed region, the amplitude of the wave pulse has decreased significantly, as we can see in Fig.4.2 (b).

In Fig.4.3 (a), we see the cross correlation as a function of $\delta t$. In Fig.4.3 (b), time delayed displacement $u(x, t - \delta t)$ plotted together with the wave pulse at a reference
Figure 4.2: (a) wave pulse at a point near the boundary. (b) wave pulse at a point in the interior near high speed region

point, $u(x_{\text{ref}}, t)$. At this state, the wave pulses are highly correlated.

## 4.2 Reconstruction Methods

Having computed the arrival time using the method described previously, we use Eikonal equation (3.7.2) to compute the shear wave speed. Since $\hat{T}$ is now known, we could approximate $\nabla \hat{T}$, for example by using finite difference, to compute the wave speed $\sqrt{\frac{\mu}{\rho}}$. Unfortunately, this approach will not work in the case of noisy arrival time. We know that differentiation is an ill-posed operator. In the presence of noise, $\nabla \hat{T}$ could be very small even close to zero and this leads to speed blow up. To avoid this problem, we proceed as follows.

The basic idea is to represent, for each fixed time $t$, the contour $\{x : \hat{T}(x) = t\}$ as the zero level set of a higher dimensional Lipschitz continuous function $\phi(x, t)$, that is $\phi(x, \hat{T}(x)) = 0$. 
Figure 4.3: (a) Cross correlation between two wave pulses. (b) reference wave pulse (solid line) plotted together with wave pulse (dashed line) that is time delayed by an amount that maximizes the cross correlation

In standard level set notation, see [18], the zero level set of the level set function is normally a closed curve that separates the domain into two parts, interior and exterior part of the closed curve. The sign of the level set function is chosen positive in the exterior and negative in the interior domain. In our case, the zero level set of the level set function $\phi(x, t)$, that is the contour $\{x : \hat{T}(x) = t\}$, is not necessarily a closed curve but it does separate the domain into two parts, namely the part that is still at rest (in front of the wavefront) and the part that has experienced propagation (behind the wavefront). The sign convention has been changed to accommodate the domain segmentation. For the part behind the wavefront, we choose positive sign for $\phi(x, t)$ and conversely, negative for the part in front of the wavefront. To represent the level set function, we choose signed distance function:

$$
\phi(x, t) = \begin{cases} 
+d(x, t), & \text{for } \hat{T}(x) < t; \\
0, & \text{for } \hat{T}(x) = t; \\
-d(x, t), & \text{for } \hat{T}(x) > t.
\end{cases} \quad (4.2.1)
$$
where \( d(x, t) \) is a distance function defined as

\[
d(x, t) = \inf\{|x - \hat{x}| : \hat{x} \text{ satisfies } \hat{T}(\hat{x}) = t\}.
\]

A number of simplifications can be made when \( \phi(x, t) \) is a signed distance function since it has a special property that others do not have, that is \( |\nabla_x \phi| = 1 \). The signed distance function \( \phi(x, t) \) is essentially the distance from point \( x \) in the domain to the contour \( \{\hat{x} : \hat{T}(\hat{x}) = t\} \).

Now, we derive the level set equation. Since we assume \( \hat{T}(x) \) and \( \phi(x, t) \) are Lipschitz continuous, \( \phi(x, \hat{T}(x)) \) is Lipschitz continuous as well. This implies \( \nabla_x \phi \) and \( \phi_t \) exists almost everywhere on the zero level set of \( \phi(x, t) \), i.e. on \( \{x : \hat{T}(x) = t\} \). Then we can apply chain rule to \( \phi(x, \hat{T}(x)) = 0 \) to get

\[
\nabla_x \phi = -\nabla \hat{T}(x) \phi_t \quad \text{a.e. on } \{x : \hat{T}(x) = t\}. \tag{4.2.2}
\]

By the construction of \( \phi(x, t) \), we have \( \phi_t \geq 0 \) on the zero level set of \( \phi(x, t) \). This can be shown by considering the following. Fix \( t_1, t_2 \in [0, T] \) with \( t_2 \geq t_1 \). From (4.2.1) we have for \( \{x : \hat{T}(x) = t\} \)

\[
\phi(x, t_2) \geq \phi(x, t_1) \quad \text{for } \begin{cases} \hat{T}(x) \leq t_1 \leq t_2, \\ t_1 \leq \hat{T}(x) \leq t_2, \\ t_1 \leq t_2 \leq \hat{T}(x). \end{cases}
\]

and this implies \( \phi_t \) exists.

By taking the norm of both sides of (4.2.2),

\[
|\nabla_x \phi| = |\nabla \hat{T}(x)| \phi_t \quad \text{a.e. on } \{x : \hat{T}(x) = t\},
\]

and using Eikonal equation (3.7.2) to eliminate \( |\nabla \hat{T}| \) leads to the level set equation

\[
\sqrt{\frac{\mu}{\rho}} |\nabla_x \phi| = \phi_t \quad \text{a.e. on } \{x : \hat{T}(x) = t\}. \tag{4.2.3}
\]

Equation (4.2.3) is defined only on the zero level set of \( \phi(x, t) \). However, \( \phi \) is defined in a narrow band about the surface \( S_{\hat{T}} = \{x : \phi(x, \hat{T}(x)) = 0\} \) so that \( \phi_t \) can be
approximated on $S_{T}$. Now we take an advantage from the property of the signed distance function, that is $|\nabla_{x} \phi| = 1$ and (4.2.3) can be written as

$$\phi_t = \sqrt{\frac{\mu}{\rho}}.$$  \hfill (4.2.4)

Since $\phi$ is Lipschitz continuous, $\phi_t$ exists a.e. on the zero level set of $\phi(x, t)$. Then we can approximate $\phi_t$ in (4.2.4) using forward difference scheme and obtain the following equation,

$$\sqrt{\frac{\mu(x)}{\rho(x)}} \approx \frac{\phi(x, \hat{T}(x) + \Delta t) - \phi(x, \hat{T}(x))}{\Delta t}.$$  \hfill (4.2.5)

If $x$ is on the zero level set of $\phi(x, t)$, we have $\phi(x, \hat{T}(x)) = 0$ and the previous equation is equivalent to

$$\sqrt{\frac{\mu(x)}{\rho(x)}} \approx \frac{\phi(x, \hat{T}(x) + \Delta t)}{\Delta t}.$$  \hfill (4.2.6)

Using (4.2.2) and since $\Delta t > 0$,

$$\phi(x, \hat{T}(x) + \Delta t) = +d(x, \hat{T}(x) + \Delta t) \quad \text{for } \hat{T}(x) < \hat{T}(x) + \Delta t$$

$$= \inf\{|x - \hat{x}| : \hat{x} \text{ satisfies } \hat{T}(\hat{x}) = \hat{T}(x) + \Delta t\}. \hfill (4.2.7)$$

Then (4.2.5) can be rewritten as

$$\sqrt{\frac{\mu(x)}{\rho(x)}} \approx \frac{1}{\Delta t} \inf\{|x - \hat{x}| : \hat{x} \text{ satisfies } \hat{T}(\hat{x}) = \hat{T}(x) + \Delta t\}.$$  \hfill (4.2.7)

Thus, the shear wave speed on $\{x : \hat{T}(x) = t\}$, the zero level set of $\phi(x, t)$, can be approximated using (4.2.5) and (4.2.7). They are referred to as the first-order level set and the first-order distance method, respectively. The formulas essentially compute the distance from contour $\{x : \hat{T}(x) = t\}$ to the contour $\{\hat{x} : \hat{T}(\hat{x}) = \hat{T}(x) + \Delta t\}$, and divide by $\Delta t$ which yield the shear wave speed at the contour $\{x : \hat{T}(x) = t\}$. The accuracy of (4.2.5) and (4.2.7) are $O(\Delta t)$ since we use forward euler scheme to approximate $\phi_t$. If $\phi_t$ is Lipschitz continuous on the zero level set of $\phi(x, t)$, $\phi_{tt}$ exists
a.e. on the zero level set of $\phi(x,t)$ and higher order accuracy can be obtained by using center difference scheme to approximate $\phi_t$. Approximating $\phi_t$ in (4.2.4) using center difference scheme, we then obtain the following equation:

$$
\sqrt{\frac{\mu(x)}{\rho(x)}} \approx \frac{\phi(x, \hat{T}(x) + \Delta t) - \phi(x, \hat{T}(x) - \Delta t)}{2\Delta t}. \tag{4.2.8}
$$

Using (4.2.2) and since $\Delta t > 0$,

$$
\phi(x, \hat{T}(x) - \Delta t) = -d(x, \hat{T}(x) + \Delta t) \quad \text{for } \hat{T}(x) > \hat{T}(x) - \Delta t
$$

$$
= -\inf \{|x - \hat{x}| : \hat{x} \text{ satisfies } \hat{T}(\hat{x}) = \hat{T}(x) - \Delta t\}. \tag{4.2.9}
$$

Combining (4.2.6) and (4.2.9), we can rewrite (4.2.8) as

$$
\sqrt{\frac{\mu(x)}{\rho(x)}} \approx \frac{1}{2\Delta t} \left(\inf \{|x - \hat{x}^+| : \hat{x}^+ \text{ satisfies } \hat{T}(\hat{x}^+) = \hat{T}(x) + \Delta t\}
$$

$$
+ \inf \{|x - \hat{x}^-| : \hat{x}^- \text{ satisfies } \hat{T}(\hat{x}^-) = \hat{T}(x) - \Delta t\}\right),
$$

or in short,

$$
\sqrt{\frac{\mu(x)}{\rho(x)}} \approx \frac{1}{2\Delta t} \inf \{|x - \hat{x}^+| + |x - \hat{x}^-| : \hat{x}^\pm \text{ satisfies } \hat{T}(\hat{x}^\pm) = \hat{T}(x) \pm \Delta t\}. \tag{4.2.10}
$$

Equation (4.2.8) and (4.2.10) are referred to as the second-order level set and the second-order distance method, respectively. Their accuracy is $O(\Delta t^2)$. Note that using (4.2.5), (4.2.8), (4.2.7) and (4.2.10) yield the shear wave speed on the zero level set of $\phi(x,t)$, i.e. on $\{x : \hat{T}(x) = t\}$. While the level set method, (4.2.5) and (4.2.8), is identical with the distance method, (4.2.7) and (4.2.10), they suggest two different algorithms. We describe both methods in the next two subsections.

### 4.2.1 The Level Set Method

To obtain the shear wave speed using the level set method (4.2.8) or (4.2.10), we first build the signed distance function $\phi(x,t)$ for each fixed time $t$ with the contour
\{x : \hat{T}(x) = t\} as its zero level set. For this, we need to solve the Eikonal equation

$$|\nabla_x \phi(x, t)| = 1 \quad \text{with } \phi(x, t) = 0 \quad \text{on } \{x : \hat{T}(x) = t\}$$

(4.2.11)

for each fixed time \(t\). In level set terminology, solving (4.2.11) is referred to as reinitialization, see [18]. Solving (4.2.11) to find \(\phi(x, t)\) numerically is equivalent to solve

$$\phi_r(x, t, \tau) + |\nabla \phi(x, t, \tau)| = 1, \quad \phi_0(x, t) = 0 \quad \text{on } \{x : \hat{T}(x) = t\}$$

(4.2.12)

which is evolve in time (\(\tau\)) until a steady state is reached. At steady state, the value of \(\phi\) becomes constant with respect to time, implying that \(\phi_r = 0\) and (4.2.12) reduces to \(|\nabla \phi| = 1\) as desired. However, equation (4.2.12) propagates information in the normal direction of the zero level set with speed 1. This means that the information flows from smaller values of \(\phi\) to larger values of \(\phi\), implying that the zero level set is not guaranteed to stay fixed. This is because the zero level set is influenced by the information flowing from the negative values of \(\phi\). Since we want the zero level set to stay fixed at \(\{x : \hat{T}(x) = t\}\), equation (4.2.12) is modified as follows,

$$\phi_r + S(\phi_0)(|\nabla \phi| - 1) = 0$$

(4.2.13)

with

$$S(\phi_0) = \begin{cases} 
1, & \text{for } \hat{T}(x) < t; \\
0, & \text{for } \hat{T}(x) = t; \\
-1, & \text{for } \hat{T}(x) > t.
\end{cases}$$

(4.2.14)

This means that in \(\hat{T}(x) < t\) we have propagation in the normal direction while in \(\hat{T}(x) > t\) we have propagation in the opposite of normal direction. On \(\hat{T}(x) = t\), the zero level set will stay fixed since \(\phi_r = 0\).

Numerical tests indicate that better results are obtained when \(S(\phi_0)\) given by

$$S(\phi_0) = \frac{\phi_0}{\sqrt{\phi_0^2 + \Delta x^2}}$$

(4.2.15)
are numerically smoothed out near the zero level set of $\phi$. This numerical smoothing of the sign function decreases its magnitude, slowing down the propagation speed of information near the zero level set. Note that from (4.2.15), for $\Delta x \to 0$ we recover (4.2.14). Equation (4.2.13) with (4.2.15) can be solved numerically by combining forward euler discretization for time variable and Godunov’s method for the spatial variable, see [18].

Once we obtain $\phi(x, t)$ for each fixed time $t$, use (4.2.8) or (4.2.10) to approximate the shear wave speed on $\{x : \hat{T}(x) = t\}$. Fig.4.4 illustrates the second-order level set method to approximate the shear wave speed at point $x$ on the contour $\{x : \hat{T}(x) = t\}$. This illustration is generated by using synthetic data. We first find the signed distance function with the contour $\{\hat{x}^+ : \hat{T}(\hat{x}^+) = \hat{T}(x) + \Delta t\}$ as its zero level set. The distance from $x$ to the contour $\{\hat{x}^+ : \hat{T}(\hat{x}^+) = \hat{T}(x) + \Delta t\}$ is obtained by evaluating the signed distance function at $x$. This gives $\phi(x, \hat{T}(x) + \Delta t)$. The same procedure is applied to obtain $\phi(x, \hat{T}(x) - \Delta t)$. These values are then used to obtain the shear wave speed at point $x$ according to (4.2.8).

Figure 4.4: The second-order level set method.
4.2.2 The Distance Method

Instead of constructing the signed distance function \( \phi(x,t) \), the distance method (4.2.7) and (4.2.10) compute the (minimum) distance directly from the contour \( \{ x : \hat{T}(x) = t \} \) to the neighboring contours depending on the accuracy chosen. The calculation of the shear wave speed at point \( x \) on the contour \( \{ x : \hat{T}(x) = t \} \) using the second-order distance method is illustrated in Fig.4.5. To compare how the two different methods work, the data used in the following illustration is the same as the data used in the illustration of the level set method. First, find the distance from \( x \) to all points on the contours \( \{ \hat{x}^+ : \hat{T}(\hat{x}^+) = \hat{T}(x) + \Delta t \} \) and \( \{ \hat{x}^- : \hat{T}(\hat{x}^-) = \hat{T}(x) + \Delta t \} \). The closest distances from \( x \) to those two contours are then used to obtain the speed at point \( x \) according to (4.2.10).

Figure 4.5: The second-order distance method.

In the next chapter, the methods described previously will be compared and tested using a test problem and synthetic data as well.
Chapter 5

Results and Simulation

This chapter is devoted to discuss the numerical result of the method described in the previous chapter. In the first section, we will recover the shear wave speed for a given arrival time. Here, we will test the accuracy and stability of the reconstruction methods. In the second section, we will use the complete algorithm, that is cross correlation and reconstruction methods, to recover the shear wave speed from interior measurement.

5.1 Numerical Test of the Reconstruction Methods

In this section, we will test the accuracy of the distance and the level set method. Stability tests will also be performed on both methods by using noisy arrival data.

We start by considering the following test problem.

Let \( \Omega := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 10, 0 \leq y \leq 10\} \). We assume that the wave speed is explicitly given by

\[
\sqrt{\frac{\mu(x, y)}{\rho(x, y)}} = 2 + \sin(x),
\]

(5.1.1)

with \((x, y) \in \Omega\), \(\hat{T}(0, y) = 0\) and \(\hat{T}\) satisfies Eikonal equation. Using (3.7.2), the
exact arrival time is given by

\[ \hat{T}(x, y) = \int_0^x \frac{1}{2 + \sin(x_0)} \, dx_0. \]  

(5.1.2)

From the arrival time \( \hat{T}(x) \) in (5.1.2), we then reconstruct the wave speed given in (5.1.1), which is shown in Fig. 5.1 (a). The arrival time surface and its contour are shown in Fig. 5.1 (b)-(c).

![Graphs](image)

(a) Exact wave speed  
(b) Arrival time surface  
(c) Arrival time contour

Figure 5.1: Exact wave speed and Arrival time.

We start by testing the accuracy of the methods. We use Matlab in the implementation of the methods as follows.
First, we discretize the domain with an uniform grid of size $h$ and compute the arrival time $\hat{T}(x)$ at each grid point. Secondly, we find all contours $\{x : \hat{T}(x) = j\Delta t\}, j = 1, 2, \ldots$ of the arrival time. This is illustrated in Fig.5.1(c). The high speed region is indicated by far distance between the contours. Note that the points on the contours are not necessarily the grid point.

For the distance method, we calculate the minimum distance from each point on each contour to the neighboring contour according to (4.2.7) and (4.2.10). The wave speed at points on the contours is then obtained by dividing the resulting distance by $\Delta t$. Finally, we use Matlab cubic interpolation to get the wave speed at grid points.

For the level set method, for each contour we construct the signed distance function $\phi(x, j\Delta t)$ with the contour $\{x : \hat{T}(x) = j\Delta t\}, j = 1, 2, \ldots$ as its zero level set. We accomplished this by using Reinitialization routine implemented by Baris Sumengen, see [21]. The wave speed at points on the contours is then obtained by interpolating the signed distance function at the corresponding points according to (4.2.5) and (4.2.8). Finally, we use Matlab cubic interpolation to get the wave speed at grid points.

The reconstructed wave speed using the distance method and the level set method for $\Delta x = h = 0.1$ are shown in Fig.5.2.

In comparison with Fig.5.1 (a), we see that the first- and second-order of the methods give excellent recovery of the wave speed. We could barely distinguish the recovered and the exact wave speed. Comparing the result from the first-order methods, Fig.5.2 (a)-(b) with the second-order methods, Fig.5.2 (c)-(d), we hardly differentiate them visually, but they do give different results. Principally, the second-order methods should give better approximation compared to the first-order methods. To verify this, we redo the reconstruction with several different values of $\Delta t$ and $h$. We then calculate the $L^\infty$-error between the exact and the reconstructed wave speed. The resulting $L^\infty$-error is tabulated in table 5.1.

From table 5.1, for the first order distance and level set method we see that when
we halve the time step the $L^\infty$-error is halved accordingly. This verifies that the accuracy of the first order distance and level set method is $O(\Delta t)$. We observed the accuracy of the second-order distance and level set method as well. From the table, we see that when we halve the time step the $L^\infty$-error is reduced to a quarter of the previous one. This verifies that the accuracy of the second-order method is $O(\Delta t^2)$, as expected.

As stated previously, both methods need to find the contour of the arrival time. In the implementation, we use Matlab 2D contour plotter. The way Matlab contour
Table 5.1: $L^\infty$-error for the reconstruction methods

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>Distance method</th>
<th>Level set method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st order</td>
<td>2nd order</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2481</td>
<td>0.0698</td>
</tr>
<tr>
<td>0.1</td>
<td>0.01131</td>
<td>0.0185</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0561</td>
<td>0.0045</td>
</tr>
</tbody>
</table>

The plotter finds the contour is by interpolating the grid point. For this reason, every time we halve the time step we also need to halve the grid spacing. So, to gain accuracy in $t$ we need to balance with the accuracy in $x$.

To test the stability of the method, we add random Gaussian noise with mean zero and variance one to the exact arrival time, 

$$\hat{T}_\gamma(x, y) = \hat{T}(x, y) + \gamma \text{rand}(x, y),$$

with $\gamma$ is the noise level. The Gaussian random number is generated in Matlab.

The recovered wave speed with $\Delta x = h = 0.1$ and $\gamma = 0.01$ are shown in Fig.5.3. The recovery from the level set method are visually almost indistinguishable from that of the distance method. The second-order methods, indeed, give better recovery compared to the first-order methods, in the sense that they are smoother.

We also try for a higher noise level, $\gamma = 0.05$. The result is shown in Fig.5.4. We can now distinguish the recovery from both methods, however they give similar results. The first-order methods exhibit oscillation, but no large outliers in the recovered wave speed. This oscillation, however, is reduced in the second-order methods. Overall, they give reasonable recovery of the wave speed. We see that even in the presence of noise, the distance and the level curve method give good recovery for the wave speed. This observation shows that the distance and the inverse level curve method are robust with respect to noise.

We want to make comment on the computational time. With our implementation, the distance method is much faster than the level set method. What makes the level
set method very slow is because we construct each signed distance function in the whole domain. Actually, we only need to calculate $\phi$ at points needed in the scheme of the method and we know exactly which points are needed. This means that it is sufficient to construct the signed distance function in a narrow band where we need it, not in the whole domain. In [15], McLaughlin and Renzi show that using this narrow banding idea, the level set is faster than the distance method.
In this section, we test the full algorithm to recover the shear wave speed from interior displacement data. This is accomplished by combining the cross correlation and the reconstruction method described in the last chapter.

Let \( \Omega \) be the domain of interest given by \( \Omega := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 7, -2 \leq y \leq 2\} \). For the numerical simulation, we use synthetic data, i.e. the displacement.

Figure 5.4: Reconstructed wave speed with noise level \( \gamma = 0.05 \).

### 5.2 Reconstruction of the Shear Wave Speed

In this section, we test the full algorithm to recover the shear wave speed from interior displacement data. This is accomplished by combining the cross correlation and the reconstruction method described in the last chapter.

Let \( \Omega \) be the domain of interest given by \( \Omega := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 7, -2 \leq y \leq 2\} \). For the numerical simulation, we use synthetic data, i.e. the displacement.
data in $\Omega$ is generated by solving the wave equation
\[ \nabla \cdot (\mu(x, y)\nabla u(x, y, t)) = u_{tt}(x, y, t) \quad \text{in } \Omega \times (0, T), \]
with initial condition
\[ u(x, y, 0) = u_t(x, y, 0) = 0 \quad \text{in } \Omega, \]
and the Dirichlet boundary condition
\[ u(0, y, t) = \exp\left(-\frac{(t - 0.5)^2}{0.04}\right), \]
\[ u(x, -2, t) = u(x, 2, t) = u(7, y, t) = 0 \quad \text{on } \partial \Omega \times (0, T). \]

The simulation time $T$ is chosen such that the shear wave does not hit any of the boundary. This is because we want to prevent the reflected wave to influence the calculation of the arrival time. The wave equation is solved numerically using Finite element method in Matlab PDE ToolBox. We are mainly interested in the recovery of the wave speed especially in the region with rapid change in speed. To simulate this, we assume $\rho = 1$ and try for several different stiffness parameters $\mu(x, y)$. We use two scenarios for $\mu(x, y)$, the continuous case represented by a gaussian function and discontinuous case represented by a step function.

For all cases, we first need to find the arrival time. To calculate the arrival time $\hat{T}(x)$ at point $x$ from the displacement data, we first compute the cross correlation between a reference wave pulse $u(x_{\text{ref}}, t)$ and wave pulse $u(x, t)$ using Matlab discrete cross correlation, $\text{xcorr}$. The arrival time at point $x$ is estimated by the time delay $\delta t_{\text{max}}$ that maximizes the cross correlation, i.e. $\hat{T}(x) = \delta t_{\text{max}}$.

### 5.2.1 Continuous case

In this scenario, for the stiffness parameter $\mu(x, y)$ we use
\[ \mu(x, y) = \left(2 + \mu^* \exp\left(-\frac{(x - 3)^2}{0.5^2} - \frac{y^2}{0.5^2}\right)\right)^2 \]
with $\mu^* = 2, 6, 14$. Note that with $\rho = 1$, the wave speed is $\sqrt{\mu(x,y)}$. With our particular choice of parameter, the background speed is 2 and the maximum wave speed is $(2 + \mu^*)$, located at point $(3, 0)$ in the domain. We refer to $\mu^* = 2, 6, 14$ as the low, intermediate and high contrast case, respectively.

All the images created in the simulation are obtained using the second-order methods.

The recovery for the low contrast case is shown in Fig.5.5. The location of the high speed region is identified correctly. This region can also be identified from the contours of the arrival time, they bend when passing through the high speed region. However, the maximum wave speed is slightly undershot. The background wave speed is somehow overshot, there are some ripples trailing diagonally away from the high speed region. This can be explained as follows. When the wavefront reaches the high speed region, most of it propagates passing through this region, while some part of it is scattered and reflected. The ripples are caused by the scattered wave. The influence of the reflected wave, in this case - low contrast case, does not affect the calculation of the arrival time significantly. We can observe this phenomena clearly in high contrast case. The recovery from the level set method exhibit less oscillation compared to that from the distance method.

The recovery for the intermediate contrast case is shown in Fig.5.6. In this case, the wave speed in the high speed region is up to four times greater than the background speed. The recovery of the high speed region is excellent, for both the location and the value of the wave speed. As in the low contrast case, here we also have some ripples trailing diagonally away from the high speed region. However, the visibility of these ripples are not so clear due to the significant contrast between the background speed and the maximum wave speed. In front of the high speed region, specifically at $(2.5; 0)$, there is a small dip (artefact). If we recall Fig.4.2, where the wave pulses were generated from this case, we see that there is a small wave at $t = 3$. This is the wave pulse of the reflected wave. The dip is caused by the reflected wave, which
influences the calculation of the arrival time. Cross correlation uses the phase content of the wave pulse more than the amplitude. In the region where we have artefact, the reflected wave has the same phase as the forward wave. For this reason, the cross correlation has difficulty to separate out the forward wave from the reflected wave.

The visibility of this artefact is better in high contrast case, as shown in Fig.5.7. The artefact is indicated by dark blue color, located at (2.5, 0). The location of the high speed region is identified correctly, but the maximum wave speed is overshot up to approximately 19, quite far from the maximum of the target wave speed. The


(a) Exact wave speed  

(b) Arrival time contour  

(c) Recovery using the distance method  

(d) Recovery using the level set method

Figure 5.6: Intermediate contrast case.

Comparing Fig.5.5(b), Fig.5.6(b) and Fig.5.7(b), we see that the contours of the arrival time bend more in the region with higher speed.

5.2.2 Discontinuous case

In this scenario, for the stiffness parameter \( \mu(x,y) \) we use

\[
\mu(x,y) = \begin{cases} 
2^2, & \text{for } (x,y) \in \Omega \setminus \Omega_H; \\
4^2, & \text{for } (x,y) \in \Omega_H.
\end{cases}
\]
where \( \Omega_H := \{(x, y) \in \mathbb{R}^2 : 2.5 \leq x \leq 3.5, -0.5 \leq y \leq 0.5\} \).

With this choice of parameter, the background speed is 2 and the maximum wave speed is 4 located in \( \Omega_H \).

In this case, we can see clearly the effect of the reflected and scattered wave, a small dip in front of the high speed region and some ripples trailing diagonally away from the high speed region. The location of the high speed region is identified correctly, but not with the correct value of the wave speed. It overshot in the region about \( x = 2.5 \) and \( x = 3.5 \). This is caused by the jump in speed.
The algorithm gives good identification of the location of the high speed region for all cases. However, it works better in recovering the value of the wave speed if the maximum speed is up to 4 times of the background speed. The weakness of the cross correlation in calculating the arrival time is that it is affected by the reflected wave from the high speed region. Overall, the algorithm works well.
Chapter 6

Conclusion and Future Work

In this chapter, we summarize what we have done so far. The aim of this thesis was to reconstruct the wave speed from a single interior displacement data. We modeled the propagation of the elastic waves inside the body using the wave equation. We started by proving the existence and the uniqueness of a solution of the forward problem. Under assumption that the medium is isotropic, we proved that the shear wave speed is uniquely identified from a single interior measurement. Moreover, there exists at most one pair elastic parameters \((\rho, \mu)\) corresponding to an interior displacement data if \(\mu\) is either given on the boundary for the Dirichlet case or is determined from the boundary traction force for the Neumann case. The main ingredient to prove these uniqueness results was the shrink and spread argument, which utilizes the property of hyperbolic and elliptic problem.

In transient elastography, arrival time \(\hat{T}\), which is the subset of the interior displacement data, is an important feature of the data. We showed that under assumption that \(\hat{T}\) is Lipschitz continuous, it satisfies Eikonal equation which gives connection to the shear wave speed. From the arrival time information, the shear wave speed is uniquely identified. However, this is not the case for the identification of the parameters \((\rho, \mu)\).

We implemented an algorithm to reconstruct the shear wave speed from the arrival time information. The algorithm was composed of two sub-algorithms. First, we find
the arrival time from the interior displacement data using cross correlation method. Secondly, the resulting arrival time is used to recover the shear wave speed by using the level set method and the distance method. Numerical tests showed that both methods are robust with respect to noise. We also tested both methods by solving synthetic data, which was generated from the wave equation. The results showed that both methods gave excellent identification of the high speed region.

Though the simulations using synthetic data gave positive results, there are still some free space for improvement. Some extra work is needed to validate the algorithm using experimental displacement (real) data. The computational speed can be improved, especially for the level set method by using narrow banding idea. As we have seen that the cross correlation experienced some distortion due to backscattering from the high speed region. Therefore, finding a better method to estimate the arrival time could be one possible future work. Moreover, we assumed that the propagation of the elastic waves satisfy the wave equation and that the contribution of the compression wave is treated as noise. To get better result under more realistic model, we could use the linear elastic equation. For this, an extension of the arrival time algorithm is needed since both compression and shear wave give contribution in the displacement data from the linear elastic equation.

In this thesis, although in the model we assumed that the medium is isotropic, in reality elastic properties of damaged tissue could be anisotropic. So, another possible extension of this thesis is to consider parameter identification in anisotropic medium. All these considerations show that the inverse problem in transient elastography still remains a rich area for scientific research.
Bibliography


Eidesstattliche Erklärung

Ich, Kho Sinatra Canggih, erkläre an Eides statt, dass ich die vorliegende Masterarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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