Credit Portfolio Management
A Mark-to-Market Portfolio Loss Model

Master’s Thesis By
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Preface

This thesis is written to obtain my master’s title for my study Industrial and Applied Mathematics at Eindhoven University of Technology, the Netherlands. My graduation project is done during an internship at Rabobank Nederland, Utrecht, where I have been from December 2006 till July 2007.

During my internship I did two separate projects:
- Developing a Mark-to-Market credit portfolio loss model;
- Developing methodology for mapping external to internal ratings.

This thesis is about the first project. In the appendix, a project justification of the second project can be found.

My internship at Rabobank was my first real experience in the financial world. Although I did two financial projects at Eindhoven University of Technology, the banking world was completely new to me. In the beginning all kinds of banking terms were thrown at me. But after some time a lot of terms became clearer and clearer. I still have a long road to go, but in the past few months I think I made quite some progress.

I have learned a lot during this internship. A pleasant aspect of this internship was the translation of the used mathematics into a language everybody within Rabobank can understand. It was also very nice to see that I could bring some courses I followed into practice. For example, I used my knowledge of statistics, probability theory, stochastics and Matlab.

First of all, I would like to thank dr. Máčé Mesters for guiding me during the internship and many fruitful discussions. I would also like to thank Máčé for giving me the opportunity of working for Rabobank after my internship. Furthermore, I would like to thank Iryna Snihir, for bringing me in contact with Máčé. For helpful comments and discussions I thank ir. Xinzheng Huang and dr. Viktor Tchistiakov. From Eindhoven University of Technology, I was supervised by dr. Johan van Leeuwaarden, to whom I am vary grateful. Last, but certainly not least, I would like to thank my girlfriend Paulien, my parents and my girlfriend’s parents for supporting me throughout my study and all my fellow students, in particular Dennis Vos and Paul Frenken, for the extraordinary time at the university.

Martin van Jole
Eindhoven, August 2007
"All of life is the management of risk, not its elimination"
Walter Wriston, former chairman of Citicorp
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## Symbols

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<td>$A_i$</td>
<td>asset value of obligor $i$.</td>
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<td>$c_{i,j,k}$</td>
<td>$k$-year (after the horizon) cumulative default probability of obligor $i$ with rating $j$ at the horizon.</td>
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<tr>
<td>$\delta_{i,j}$</td>
<td>path for obligor $i$ with rating $j$ at the horizon.</td>
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<td>$EAD_i$</td>
<td>exposure at default of obligor $i$.</td>
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<tr>
<td>$EC$</td>
<td>economic capital.</td>
</tr>
<tr>
<td>$EL$</td>
<td>expected loss.</td>
</tr>
<tr>
<td>$EVA$</td>
<td>economic value added.</td>
</tr>
<tr>
<td>$F_{i,j}$</td>
<td>cash flows after the horizon of obligor $i$ with rating $j$.</td>
</tr>
<tr>
<td>$K$</td>
<td>the number of ratings, including the default status.</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>the market’s Sharpe ratio.</td>
</tr>
<tr>
<td>$L$</td>
<td>loss.</td>
</tr>
<tr>
<td>$L_i$</td>
<td>loss of obligor $i$.</td>
</tr>
<tr>
<td>$L^s$</td>
<td>loss in a certain state of the world.</td>
</tr>
<tr>
<td>$L_i^s$</td>
<td>loss of obligor $i$ in a certain state of the world.</td>
</tr>
<tr>
<td>$L^{(s)}$</td>
<td>loss in a certain scenario.</td>
</tr>
<tr>
<td>$L_i^{(s)}$</td>
<td>loss of obligor $i$ in a certain scenario.</td>
</tr>
<tr>
<td>$LGD_i$</td>
<td>loss given default of obligor $i$.</td>
</tr>
<tr>
<td>$\mu_i$</td>
<td>drift of asset value of obligor $i$.</td>
</tr>
<tr>
<td>$m_{i,j,k}$</td>
<td>marginal default probability of obligor $i$ in the $k$-th year after the horizon with rating $j$ at the horizon.</td>
</tr>
<tr>
<td>$\bar{m}_{i,j,k}$</td>
<td>risk-neutral marginal default probability of obligor $i$ in the $k$-th year after the horizon with rating $j$ at the horizon.</td>
</tr>
<tr>
<td>$M_i$</td>
<td>maturity of obligor $i$’s loan.</td>
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<td>$MtM_i^H$</td>
<td>Mark-to-market value of obligor $i$ at the horizon.</td>
</tr>
<tr>
<td>$MtM_{i,j}^H$</td>
<td>Mark-to-market value of obligor $i$ at the horizon given that the obligor becomes $j$-rated.</td>
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<tr>
<td>$MtM_{i,ND}^H$</td>
<td>Mark-to-market value of obligor $i$ at the horizon given the obligor does not default.</td>
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<tr>
<td>$N$</td>
<td>normal distribution.</td>
</tr>
<tr>
<td>$N_2$</td>
<td>bivariate normal distribution.</td>
</tr>
<tr>
<td>$n$</td>
<td>number of obligors.</td>
</tr>
<tr>
<td>$N_s$</td>
<td>number of different states of the world.</td>
</tr>
<tr>
<td>$N_{sc}$</td>
<td>number of scenarios.</td>
</tr>
<tr>
<td>$p_{i,j}$</td>
<td>probability that obligor $i$ becomes $j$-rated.</td>
</tr>
<tr>
<td>$\tilde{p}_{i,j}$</td>
<td>risk-neutral probability that obligor $i$ becomes $j$-rated.</td>
</tr>
<tr>
<td>$\tilde{p}_{i_1,i_2,j_1,j_2}$</td>
<td>joint probability that obligor $i_1$ becomes $j_1$-rated and obligor $i_1$ becomes $j_1$-rated.</td>
</tr>
</tbody>
</table>
\( q_{i,j} \) = probability that obligor \( i \) becomes at most \( j \)-rated.
\( \tilde{q}_{i,j} \) = risk-neutral probability that obligor \( i \) becomes at most \( j \)-rated.
\( \rho_{i_1,i_2} \) = correlation between obligor \( i_1 \) and \( i_2 \).
\( r^c \) = cost of capital.
\( r^f \) = risk-free rate.
\( r_i \) = interest rate of obligor \( i \).
\( R^2_i \) = correlation of obligor \( i \) with the market.
\( \sigma_i \) = volatility of asset value of obligor \( i \).
\( \Sigma \) = correlation matrix.
\( s_{i,j} \) = probability that obligor \( i \) becomes \( j \)-rated given that he does not default.
\( \tau \) = the number of years to default, starting at the horizon.
\( T_t \) = the \( t \)-year migration matrix.
\( VaR_{\alpha} \) = value-at-risk.
\( X_i \) = standardized asset value of obligor \( i \).
\( \tilde{X}_j \) = standardized asset value threshold.
\( Y \) = common factor.
\( Z_i \) = obligor specific factor (idiosyncratic factor).

**Functions**

\( E[\cdot] \) = expectation.
\( E^{RN}[\cdot] \) = expectation under risk-neutral probabilities.
\( P[\cdot] \) = probability.
\( 1_{\{\cdot\}} \) = indicator function
\( \phi(\cdot) \) = standard normal probability distribution function.
\( \Phi(\cdot) \) = standard normal cumulative distribution function.
\( \Phi^{-1}(\cdot) \) = standard normal inverse cumulative density.
\( V[\cdot] \) = value.
Chapter 1

Introduction

Credit risk has become one of the most challenging and important areas of risk management since the early 1990’s. Corporate loans have always constituted a large portion of the balance sheet for Rabobank. The financial crises facing large economies like the U.S. and Japan, emerging markets and banks in the past few decades have focused the attention of credit market participants on quantifying and managing risk.

Historically, credit risk has been managed qualitatively with limit enforcement and counter party monitoring. These approaches have proven to be inadequate as financial products have become more complex. Even institutions originating from standard corporate loans have suffered from the correlated nature of distress in the economy. Many banks have failed despite the presence of rules on controlling an institution’s overall credit exposure. Financial institutions looking to build portfolios with superior return-to-risk characteristics are increasingly seeking to quantify their credit risk. As a result, there is a strong need for quantitative and coherent credit models.

A few years ago, Rabobank developed an in-house credit portfolio loss model. The backbone of this model is the one-factor Merton Default model (see Chapter 2). Furthermore, only one year loans are considered and cash flows (such as interest and repayment) are ignored. The goal of the in-house credit portfolio loss model is to obtain a loss distribution, from which several risk measures can be derived. In the current model, Rabobank uses Monte Carlo simulation to obtain this distribution, which is used to determine the expected loss of a portfolio and the capital that Rabobank needs to reserve to cover for unexpected losses on portfolio level.

In this thesis, we will extend the in-house credit portfolio loss model, to a Mark-to-Market model, in which credit migration is allowed, loans can have a maturity of more than one year and cash flows are taken into account. These assumptions resemble reality much more than the assumptions in the current in-house model. The goal of this new credit portfolio loss model is again to obtain loss distributions, from which the same risk measures as in the current model can be derived. Furthermore, we extend the number of risk measures that can be determined, for example risk contributions (e.g. capital that Rabobank needs to reserve to cover for unexpected losses on client level) and return measures.

There exists some vendor models in which very detailed credit portfolio loss models are implemented. Two major vendor models are KMV’s Portfolio Manager (PM) and J.P. Morgan’s CreditMetrics. The loss model in this thesis is somewhat a mixture of both vendor models. It uses the view on migration from CreditMetrics, but the view on cash flows and valuation of PM.

We conclude this introduction with some main assumption we use throughout this thesis. Then, two chapters will follow about Merton’s theory on migration and default, which will form the backbone of this thesis. Subsequently, we give the methodology of calculating the loss distribution
analytically and by simulation. Thereupon, we conduct a comparative study, in which we investigate several sources of credit risk. Finally, a conclusion and some recommendations for future research are given.

1.1 Assumptions

A portfolio consists of $n$ obligors, for which the following assumptions hold:

- An obligor has a loan of size ($EAD_i$), which remains the same until default or maturity;
- An obligor has to pay interest on his loans.
- When an obligor defaults, he can only pay back a certain amount of its $EAD$. The fraction that of $EAD$ that will not be recovered following default is called the loss given default of an obligor ($LGD_i$). We assume that this fraction is constant during the obligors term;
- Each loan has a maturity ($M_i$), which is assumed to be in whole years;
- Every obligor has a correlation with the market ($R^2_i$);
- The horizon over which we will determine several risk measures is equal to one year (see the figure below).

![Figure 1.1: The time line for a loan.](image)

For the calculation of the loss distributions, we assume the following:

- The loss distribution is determined, using Mark-to-Market Valuation;
- Concentrations of obligor are allowed;
- Credit migration is modeled via Merton’s one-factor model;
- The horizon over which the loss distribution is determined is equal to one year.
Chapter 2

Merton Default Model

In this second chapter we will introduce the theory of Merton [12]. This theory proposes that a firm’s asset value is the process which drives its default.

It is evident that the value of a company’s assets determines the ability and willingness to pay its debt holders. We may suppose that there is a specific asset value level such that if the company’s assets fall below that value in the next year, it will be unable to meet its contractual obligations and will default.

Firstly, we describe how we can write the probability of default in terms of the asset value. Next, we derive standardized asset thresholds, such that if the company falls below that threshold, it defaults.

2.1 The default probability

In this chapter we assume that the loan defaults if the value of the obligor’s asset at the horizon, which is assumed to be 1 year in this thesis, falls below the contractual value $B_i$. Let $A_i$ denote the value of the i-th obligor’s asset. The associated process can then be described as:

$$dA_i = \mu_i A_i dt + \sigma_i A_i dx_i,$$  \hspace{1cm} (2.1)

where $\mu_i$ represents the drift of the asset value, $\sigma_i$ represents the volatility and $x_i$ is a Brownian motion on $[0,1]$. Below, we suppress the $t$ in our notation, because we use a horizon of one year. Using Itô’s Lemma (see [16]) on log $A_i$ we get

$$d \log A_i = \frac{1}{A_i} dA_i - \frac{1}{2} \frac{1}{A_i^2} (\sigma_i A_i)^2 dt$$
$$= \frac{dA_i}{A_i} - \frac{1}{2} \sigma_i^2 dt$$
$$= \mu_i dt + \sigma_i dx_i - \frac{1}{2} \sigma_i^2 dt$$
$$= (\mu_i - \frac{1}{2} \sigma_i^2) dt + \sigma_i dx_i$$
$$= (\mu_i - \frac{1}{2} \sigma_i^2) dt + \sigma_i dX_i,$$  \hspace{1cm} (2.2)

where $X_i$ is a standard normal variable, since a Brownian motion on $[0,1]$ corresponds by definition to a standard normal distribution (see [14]). If we now solve this equation, we get the following expression for the asset value at the horizon

$$\log A_i = \log A_i^0 + (\mu_i - \frac{1}{2} \sigma_i^2) + \sigma_i X_i,$$  \hspace{1cm} (2.3)
where $A_i^0$ is the asset value at time 0. From this equation, we can derive the relation between probability of default of the $i$-th obligor and the asset value:

$$p_i = \mathbb{P}[A_i < B]$$
$$= \mathbb{P}[\log A_i < \log B]$$
$$= \mathbb{P}[\log A_i^0 + (\mu_i - \frac{1}{2}\sigma_i^2) + \sigma_i X_i < \log B]$$
$$= \mathbb{P}\left[X_i < \frac{\log B - \log A_i^0 - (\mu_i - \frac{1}{2}\sigma_i^2)}{\sigma_i}\right]$$
$$= \Phi\left(\frac{\log B_i - \log A_i^0 - (\mu_i - \frac{1}{2}\sigma_i^2)}{\sigma_i}\right),$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function. The following figure gives a graphical interpretation of the probability of default:

![Figure 2.1: Merton Default Model.](image)

**Remark 2.1.** From Figure 2.1 we see that blue (dark grey) asset value $A_i$ at the horizon is smaller than the threshold $B$. Therefore, the company defaults, while another possible path of the asset value, the red (light grey) one) does not lead to a default. We also see the probability density function (PDF) of a standard normal distribution. In the next section we explain the connection to this PDF.

### 2.2 Standardized Asset Thresholds

In this section we consider a portfolio, which consists of $n$ loans. Let the probability of default of any loan be $p$, and assume that the asset values of the borrowing companies are correlated with the market with coefficient $R^2$. An $R^2$ of 1 means that the movement of the obligor is completely explained by the movement of the market.
2.2 Standardized Asset Thresholds

The variables $X_i$ in (2.2) are jointly standard normal with equal pair-wise correlation $R_i^2$ and can therefore be written as:

$$X_i = Y \sqrt{R_i^2} + Z_i \sqrt{1 - R_i^2},$$

where $Y, Z_1, \ldots, Z_n$ are mutually independent standard normal variables. The variable $Y$ can be interpreted as the portfolio common factor, such as an economic index. So $Y \sqrt{R_i^2}$ represents the company’s exposure to the common factor and $Z_i \sqrt{1 - R_i^2}$ represents the company’s specific risk.

From (2.4) it directly follows that

$$p_i = P\{i\text{-th obligor defaults}\} = 1 = P[X_i < \Phi^{-1}(p_i)].$$

Therefore, if a company’s value of $X$ falls below the standard normal inverse of the default probability, it goes into default. So the standardized default threshold equals:

$$\tilde{X}_K = \Phi^{-1}(p_i).$$

**Remark 2.2.** In Figure 2.1 we see that the orange area under the standard normal PDF equals $p_i$. The displayed PDF is a mirrored image of the PDF of the standardized asset values. This PDF is displayed in Figure 2.2. From this figure, we see that the company defaults if its standardized asset value $X_i$ falls below $\Phi^{-1}(p_i)$.

Thus, given the probability of default, the correlation with the market and some scenario (a realization of the common and specific factor), we are able to determine whether or not the company defaults. Probabilities of default are readily available within Rabobank for each individual obligor or pool of retail facilities. Because we know these probabilities, we do not need to estimate the unknown parameters $B_i, A_i^0, \mu_i$ and $\sigma_i$ in order to determine the default threshold $\tilde{X}_K$.

---

1. In [10] a methodology is given for approximating the asset correlation.

2. The split up is done in this way, because then $X_i$ is also standard normally distributed.
Chapter 3

Merton Migration Model

In this chapter we extend Merton’s default model to a migration model. This model proposes that a firm’s asset value is the process which drives defaults as well as credit rating migration.

Since the credit rating is a measure for the capacity and willingness of the obligor to meet its financial commitments on time and in accordance with its contract, it is evident that the value of a company’s asset also determines the rating. We may suppose that there are specific asset value levels such that if the company’s asset value lies in between these levels in the next year, it will get the corresponding rating.

Firstly, we describe how we can write the migration probabilities in terms of the asset value. Subsequently, we derive standardized asset thresholds.

3.1 The migration probabilities

In order to obtain a relation between the migration probabilities and the asset value, we make the same assumption as in Section 2.1. Again, the asset value can be described by the following process:

\[ dA_i = \mu_i A_i dt + \sigma_i A_i dx_i, \]  

(3.1)

and using Itô’s Lemma, we obtain:

\[ \log A_i = \log A_i^0 + (\mu_i - \frac{1}{2} \sigma_i^2) + \sigma_i X_i. \]  

(3.2)

Next, we assume that the obligor gets a rating of at most \( j \) if the asset value falls below \( B_{i,j} \). Let \( q_{i,j} \) denote the probability that obligor \( i \) with some rating becomes an obligor with rating of at most \( j \). Then similar to Equation (2.4), we obtain the following relation between \( q_{i,j} \) and the asset value:

\[ q_{i,j} = \Phi\left( \log B_{i,j} - \log A_i^0 - (\mu_i - \frac{1}{2} \sigma_i^2) \right). \]  

(3.3)

Remark 3.1. Suppose that we have \( K \) ratings, where rating 1 is the best rating and \( K \) represents the default status. Then the actual migration probability \( p_{i,j} \) of obligor \( i \) becoming a \( j \) rated obligor, can be written as follows:

\[ p_{i,j} = \begin{cases} 
q_{i,j}, & \text{if } j = K; \\
q_{i,j} - q_{i,j+1}, & \text{if } 2 \leq j \leq K - 1; \\
1 - q_{i,j+1}, & \text{if } j = 1.
\end{cases} \]
The following figure gives a graphical representation of the migration probabilities, supposing that there are four ratings and a default status ($K = 5$):

Figure 3.1: Merton Migration Model.

**Remark 3.2.** From Figure 3.1 we see that the red (light grey) asset value $A_i$ at the horizon lies between threshold $B_{i,2}$ and $B_{i,3}$. Therefore, the company gets rating 3, while the blue (dark grey) asset value indicates that the company defaults. In Section 3.2, we explain the connection with the PDF of the standard normal distribution displayed in the figure.

### 3.2 Standardized Asset Thresholds

Let us again write the asset value as:

$$X_i = Y \sqrt{R_i^2} + Z_i \sqrt{1 - R_i^2}.$$  

From (3.3) it follows that

$$q_{i,j} = P[X_i < \Phi^{-1}(q_{i,j})]$$  

Again, if a company’s value $X_i$ falls below the standard normal inverse of the cumulative migration probability, it goes into default. So the standardized migration thresholds equal:

$$\tilde{X}_j = \Phi^{-1}(q_{i,j}), \quad \text{for all } 2 \leq j \leq K.$$  

From this it follows that

$$p_{i,K} = \Phi(\tilde{X}_K)$$

$$p_{i,K-1} = \Phi(\tilde{X}_{K-1}) - \Phi(\tilde{X}_K),$$

and so on.
Remark 3.3. Again the PDF of the standardized asset value is the mirror image of the PDF in Figure 3.1. From Figure 3.2, we see that the company becomes 3-rated if its standardized asset value $X_i$ falls between $\tilde{X}_3$ and $\tilde{X}_4$.

Thus, given the migration probabilities, the correlation with the market and some scenario of asset values (a realization of the common and specific factor), we are able to determine the new rating of an obligor. The migration probabilities can be extracted from the migration matrix, which will be formally introduced in the next section. Also, the migration matrix is available within Rabobank. Therefore, we do not need to estimate the unknown parameters $A^0_i$, $B_i$, $\mu_i$ and $\sigma_i$ in order to determine the migration thresholds $\tilde{X}_j$.

![Figure 3.2: Standardized Migration Threshold.](image)

3.3 Migration Matrix

Every obligor has a rating, which indicates his credit quality. Over time, ratings can change or an obligor can default. This can be captured in a migration matrix. In (3.2) it is assumed that credit rating migration is a discrete, time-homogeneous Markov chain on a finite state space $S = \{1, \ldots, K\}$, where the states 1 to $K - 1$ represent the different ratings (ordered from best rating to worst) and $K$ represents the default status (which is an absorbing state). Hence, the $K \times K$ one-period migration matrix $T^{(1)}$ looks as follows:

$$
T^{(1)} = \begin{pmatrix}
t_{1,1} & t_{1,2} & \cdots & t_{1,K} \\
t_{2,1} & t_{2,2} & \cdots & t_{2,K} \\
\vdots & \vdots & \ddots & \vdots \\
t_{K-1,1} & t_{K-1,2} & \cdots & t_{K-1,K} \\
0 & 0 & \cdots & 1
\end{pmatrix},
$$

where $t_{i,j}$ represents the probability of becoming a $j$-rated obligor for being initially a $i$-rated obligor.

Remark 3.4. Suppose that obligor one has an initial rating of R3 and we want to know the probability that he becomes a R5-rated obligor. Then

$$p_{1,5} = t_{3,5}.$$
3.3.1 Link with the asset value model

There is a clear link between the probabilities obtained via the asset model and via the migration matrix. Suppose we have a R2-rated obligor and 6 different ratings, including the default status.

Then the link looks as follows:

<table>
<thead>
<tr>
<th>Rating</th>
<th>Probability from the migration matrix</th>
<th>Probability according to the asset value model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$t_{2,1}$</td>
<td>$1 - \Phi(\tilde{X}_2)$</td>
</tr>
<tr>
<td>2</td>
<td>$t_{2,2}$</td>
<td>$\Phi(\tilde{X}_2) - \Phi(\tilde{X}_3)$</td>
</tr>
<tr>
<td>3</td>
<td>$t_{2,3}$</td>
<td>$\Phi(\tilde{X}_3) - \Phi(\tilde{X}_4)$</td>
</tr>
<tr>
<td>4</td>
<td>$t_{2,4}$</td>
<td>$\Phi(\tilde{X}_4) - \Phi(\tilde{X}_5)$</td>
</tr>
<tr>
<td>5</td>
<td>$t_{2,5}$</td>
<td>$\Phi(\tilde{X}_5) - \Phi(\tilde{X}_6)$</td>
</tr>
<tr>
<td>6</td>
<td>$t_{2,6}$</td>
<td>$\Phi(\tilde{X}_6)$</td>
</tr>
</tbody>
</table>

Table 3.1: The 1-year migration probabilities for a R2-rated obligor.
Chapter 4

Analytical Portfolio Loss Distribution

Now that we have discussed the basic idea behind credit migration, we are able to determine the portfolio loss distribution. In this chapter we assume that we have a credit portfolio which contains only two obligors. The purpose of this small portfolio is to get a clear view on the methodology. The goal of this methodology is to obtain several useful risk measures at both portfolio and obligor level, e.g. the expected loss or the economic capital. That is, we want to obtain the loss distribution from which all risk measures can be derived.

The portfolio loss distribution gives the probability of a certain amount of loss. Let us define

\[ L_i = \frac{\mathbb{E}[MtM_{i,ND,H}^i] - MtM_{i,H}^i}{1 + r^f} \]  

(4.1)

where \( L_i \) is a random variable describing the loss of obligor \( i \), \( \mathbb{E}[MtM_{i,ND,H}^i] \) is the expected Mark-to-Market value (discounted to the horizon)\(^2\) of obligor \( i \) given that he does not default between the as-of-date and the horizon, \( MtM_{i,H}^i \) is the actual discounted Mark-to-Market value of obligor \( i \) at the horizon, which is a random variable\(^3\) and \( r^f \) is the risk-free rate. Since the portfolio loss is equal to the sum of all obligor losses, the portfolio loss is also a random variable and can be written as:

\[
L = \sum_{i=1}^{2} L_i = \sum_{i=1}^{2} \left( \frac{\mathbb{E}[MtM_{i,ND,H}^i] - MtM_{i,H}^i}{1 + r^f} \right) = \frac{\mathbb{E}[MtM_{ND,H}] - MtM_{H}}{1 + r^f},
\]  

(4.2)

where

\[
\mathbb{E}[MtM_{ND,H}] = \sum_{i=1}^{2} \mathbb{E}[MtM_{i,ND,H}], \quad MtM_{H} = \sum_{i=1}^{2} MtM_{i,H}
\]

\(^1\)By defining the loss as in (4.1), we assume that the expected spread (which is the amount a bank wants to receive on top of the risk-free rate in order to compensate for expected risks in the portfolio) is not used as a buffer for losses. I.e., this is a more conservative view on losses then when we assume that expected spread is used as a buffer.

\(^2\)In Section 4.2 we will elaborate more on discounted Mark-to-Market values.

\(^3\)The corresponding state space is \( \{MtM_{i,1}^i, MtM_{i,2}^i, \ldots, MtM_{i,K}^i\} \).
From this it follows that the loss distribution can be written as:

\[ P[L \leq x] = \mathbb{P} \left[ \sum_{i=1}^{2} \left( \mathbb{E}[MtM_{i}^{ND,H}] - MtM_{i}^{H} \right) \leq x \right]. \tag{4.3} \]

There are three main steps in calculating the portfolio loss distribution and its corresponding risk measures:

1. The ratings of the obligors at the as-of-date together with the asset correlation determine the joint probability of either defaulting or migrating to some new rating at the horizon.
2. The value of a loan at horizon (the Mark-to-Market value) is obtained via risk-free discounting of the future cash flows from the horizon to maturity, which depends on the rating at the horizon. The loss of an obligor is then determined by the discounted expected Mark-to-Market value given non default minus the realized discounted Mark-to-Market value.
3. The probabilities from Step 1 and the losses from Step 2, should then be combined in our calculation of the loss distribution an its corresponding risk measures.

We will elaborate on each of these steps in the next sections.

### 4.1 Credit Rating Migration

Every obligor has a rating, which indicates his credit quality. Over time, ratings can change or an obligor can default. Such a migration process can, in the two-obligor portfolio case, easily be displayed in a tree. Suppose that there are three different ratings R1, R2 and R3 and that there is one default state. Then the migration tree looks as follows:

![Figure 4.1: The migration tree for a portfolio.](image)

Each of the leafs of the tree represents a different ‘state of the world’. In order to determine the loss distribution we need to know the probabilities of each state of the world. This involves two steps:

1. For each obligor we need to know the transition probabilities. This is captured in the migration matrix.
2. The second and final step is to determine the joint transition probabilities. Because each obligor has some correlation with the market, the transition probabilities are not independent.

Recall that the migration matrix looks as described in Section 3.3. In this thesis, we assume that this matrix is known.

4.1.1 Migration Probabilities

For valuation purposes we are interested in the rating of each obligor at the horizon. Therefore, we want to construct a table in which the migration probabilities of each obligor are stated.

**Example 4.1.** Suppose we use a rating system with ratings R1 to R3 and the we use a D for the default status, then the one-year matrix $T^{(1)}$ could look as follows: 

<table>
<thead>
<tr>
<th>Rating</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>0.90</td>
<td>0.06</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>R2</td>
<td>0.04</td>
<td>0.82</td>
<td>0.04</td>
<td>0.10</td>
</tr>
<tr>
<td>R3</td>
<td>0.05</td>
<td>0.10</td>
<td>0.65</td>
<td>0.20</td>
</tr>
<tr>
<td>D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 4.1: The 1-year migration matrix.

Furthermore, suppose that we have a portfolio which consists of an R1-rated obligor and an R2-rated obligor. Then we obtain the following migration table and cumulative migration table:

<table>
<thead>
<tr>
<th>Rating</th>
<th>Obligor 1</th>
<th>Obligor 2</th>
<th>Obligor 1 Cumulative</th>
<th>Obligor 2 Cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>0.90</td>
<td>0.04</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>R2</td>
<td>0.06</td>
<td>0.82</td>
<td>0.10</td>
<td>0.96</td>
</tr>
<tr>
<td>R3</td>
<td>0.02</td>
<td>0.04</td>
<td>0.04</td>
<td>0.14</td>
</tr>
<tr>
<td>D</td>
<td>0.02</td>
<td>0.10</td>
<td>0.02</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Table 4.2: The 1-year (cumulative) migration table.

4.1.2 Joint Migration Probabilities

Each obligor in the portfolio has a correlation with the market $R^2_i$ and therefore obligors themselves are correlated with one common factor. The correlation between two obligors is given by:

$$\rho_{i,j} = \sqrt{R^2_i R^2_j}.$$ 

From the probabilities $p_{i,j}$ we can derive the joint migration probabilities. From Merton’s migration model it follows that these joint probabilities follow a multivariate standard normal distribution. Let

$$\Phi_2(a, b, c, d) = \mathbb{P}[a \leq X_1 \leq b, c \leq X_2 \leq d],$$

where $(X_1, X_2) \sim N_2(0, \Sigma)$, with

$$\Sigma = \begin{pmatrix} 1 & \rho_{1,2} \\ \rho_{2,1} & 1 \end{pmatrix}.$$ 

Note that $\rho_{1,2} = \rho_{2,1}$. Let $\hat{p}_{i_1, j_1, i_2, j_2}$ be the joint probability that obligor $i_1$ becomes a $j_1$-rated obligor and obligor $i_2$ becomes a $j_2$-rated obligor. Then these joint probabilities are given by
Let $\Phi^{-1}(\cdot)$ denote the inverse standard normal distribution.

\[
\hat{p}_{i,j_1,j_2} = \begin{cases} 
\Phi_2(-\infty, \Phi^{-1}(q_{i_1,K}), -\infty, \Phi^{-1}(q_{i_2,K})), & \text{if } j_1 = K, j_2 = K; \\
\Phi_2(\Phi^{-1}(q_{i_1,j_1}), -\infty, \Phi^{-1}(q_{i_2,K})), & \text{if } j_1 < K, j_2 = K; \\
\Phi_2(-\infty, \Phi^{-1}(q_{i_1,K}), \Phi^{-1}(q_{i_2,j_2+1}), \Phi^{-1}(q_{i_2,j_2})), & \text{if } j_1 = K, j_2 < K; \\
\Phi_2(\Phi^{-1}(q_{i_1,j_1+1}), \Phi^{-1}(q_{i_1,j_1}), \Phi^{-1}(q_{i_2,j_2+1}), \Phi^{-1}(q_{i_2,j_2})), & \text{if } 1 \leq j_1 < K, 1 \leq j_2 < K. 
\end{cases}
\]

Example 4.2. Again, suppose that we have a portfolio with an R1-rated obligor with $R_1^2 = 0.20$ and an R2-rated obligor with $R_2^2 = 0.30$, which means that the correlation between the obligors equals $\rho_{1,2} = 0.24$. Furthermore, we assume that we have the same migration matrix as in Example 4.1. Now suppose we want to calculate the joint probability that obligor 1 becomes an R2-rated obligor and that obligor 2 becomes R1-rated. If migration of obligors was an independent process this probability was just the product of both migration probabilities (obtained from Table 4.2),

\[
\hat{p}_{1,2,1} = p_{1,2}p_{2,1} = 0.06 \cdot 0.04 = 0.0024.
\]

However, migration is not an independent process. Therefore, the joint probability is calculated as follows:

\[
\hat{p}_{1,2,1} = \Phi_2(\Phi^{-1}(0.04), \Phi^{-1}(0.10), \Phi^{-1}(0.96), \Phi^{-1}(1)) = 0.009.
\]

We can conclude from this that due to the positive correlation, it is more unlikely that the grade changes of two obligors go in different directions; this fits intuition.

Next, we discuss risk-neutral migration probabilities, which are used for risk-free valuation.

### 4.1.3 Risk-neutral Migration Probabilities

In order to value an obligor’s loan at the horizon, we use risk-free discounting (which will be explained in more detail in Section 4.2). This is only allowed if we use risk-neutral migration probabilities. Therefore, the migration matrix has to be transformed into a risk-neutral migration matrix. For this purpose, we use (3.3) and make a small adjustment. Recall that $q_{i,j}$ denotes the 1-year probability that obligor $i$ with some rating becomes an obligor with a rating of at most $j$ and can be written as

\[
q_{i,j} = \Phi \left( \frac{\log B_{i,j} - \log A_i - (\mu_i - \frac{1}{2} \sigma_i^2)}{\sigma_i} \right). \tag{4.4}
\]

Now, we must adjust the actual probabilities (c.f. [18]) represented above, such that we account for the risk aversion of investors buying and selling these risky assets. The adjustment relies on estimates of the market price of risk. Therefore, the expected return $\mu_i$ in (4.4) will be replaced by the risk-free rate $r^f$. This is because investors refuse to hold risky assets with expected return less than the risk-free rate. I.e., the expected return has to be at least as large as the risk-free rate. Let $\tilde{q}_{i,j}$ denote the risk-neutral probability that obligor $i$ becomes an obligor with rating at most $j$, then

\[
\tilde{q}_{i,j} = \Phi \left( \frac{\log B_j - \log A_i - (r^f - \frac{1}{2} \sigma_i^2)}{\sigma_i} \right). \tag{4.5}
\]

Substituting equation (4.4) into (4.5) gives:

\[
\tilde{q}_{i,j} = \Phi \left( \Phi^{-1}(q_{i,j}) + \frac{\mu_i - r^f}{\sigma_i} \right), \tag{4.6}
\]

where $\Phi^{-1}(\cdot)$ denotes the inverse standard normal distribution.
Remark 4.1. The fraction $\frac{\mu_i - r_f}{\sigma_i}$ in (4.6) is also known as the Sharpe ratio of an asset, which is a measure of the mean excess return per unit of risk.

In [2], the following relationship between the market’s and obligor’s Sharpe ratio is derived:

$$\frac{\mu_i - r_f}{\sigma_i} = R_i \lambda,$$

where $R_i$ is the square root of the correlation between asset $i$ and the market and $\lambda$ is the Market Sharpe ratio. The risk-neutral cumulative migration probabilities can now be written as

$$\tilde{q}_{i,j} = \Phi \left( \Phi^{-1}(q_{i,j}) + R_i \lambda \right).$$

Remark 4.2. According to [11], the Market’s Sharpe ratio generally varies around 0.4 to 0.5 in value and is relatively stable over time. Therefore, from here on we assume that the Market’s Sharpe ratio equals 0.4.

We are now able to determine the risk free migration probabilities:

$$\tilde{p}_{i,j} = \begin{cases} \tilde{q}(i, j), & \text{if } j = K; \\ \tilde{q}(i, j) - \tilde{q}(i, j+1), & \text{if } 1 < j < K; \\ 1 - \tilde{q}(i, j+1), & \text{if } j = 1. \end{cases}$$

Example 4.3. Assume that we have the same portfolio and (cumulative) migration table as in Example 4.2 and suppose that these obligors have market correlations of $R_1^2 = 0.20$ and $R_2^2 = 0.30$. Furthermore, assume that the Market Sharpe Ratio equals 0.4. If we transform the cumulative migration probabilities according to Equation (4.8), we obtain the following tables for $\tilde{q}_{i,j}$ and $\tilde{p}_{i,j}$:

<table>
<thead>
<tr>
<th>Rating</th>
<th>Obligor 1 $\tilde{q}_{i,j}$</th>
<th>Obligor 2 $\tilde{q}_{i,j}$</th>
<th>Obligor 1 $\tilde{p}_{i,j}$</th>
<th>Obligor 2 $\tilde{p}_{i,j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.8649</td>
<td>0.0244</td>
</tr>
<tr>
<td>R2</td>
<td>0.1351</td>
<td>0.9756</td>
<td>0.0771</td>
<td>0.7810</td>
</tr>
<tr>
<td>R3</td>
<td>0.0580</td>
<td>0.1946</td>
<td>0.0276</td>
<td>0.0506</td>
</tr>
<tr>
<td>D</td>
<td>0.0304</td>
<td>0.1440</td>
<td>0.0304</td>
<td>0.1440</td>
</tr>
</tbody>
</table>

Table 4.3: The risk-neutral (cumulative) migration probabilities.

Remark 4.3. We see that all downgrade probabilities have increased and upgrade probabilities have decreased, which is consistent with our intuition. The bank now has a more negative view of its obligors.

4.2 Valuation

The second step in the determination of the loss distribution is the determination of the value of the obligor’s loan at the horizon. Because each loan has a maturity of at least one year there are three possibilities at the horizon:

1. The obligor defaulted before the horizon.
2. The loan reached its maturity at the horizon.
3. The obligor did not default before the horizon and the loan did not reach its maturity.

The market Sharpe Ratio is the expected excess return of the market portfolio (which is a portfolio of a weighted sum of every asset in the market, with weights in the proportions that they exist in the market) divided by the standard deviation of the portfolio.
In the first case the bank recovers only a fraction of the loan by, so the MtM-value equals the recovery rate times the exposure at default:

\[ \text{MtM}^H_{i,K} = (1 - \text{LGD}_i) \text{EAD}_i. \]  
(4.9)

If the obligor reached its maturity at the horizon, he has to pay back the loan plus interest over one year, therefore for all \( j \leq K - 1 \) the Mark-to-Market value equals

\[ \text{MtM}^H_{i,j} = (1 + r_i) \text{EAD}_i, \]  
(4.10)

where \( r_i \) is the interest rate. The third case is somewhat more difficult. In the first year the obligor had to pay interest over his loan. Because the obligor did not default and did not reach its maturity, there are future cash flows. Therefore, for all \( j \leq K - 1 \) the MtM-value equals the interest plus the expected discounted future cash flows, which (as it will turn out) depend on the rating at the horizon:

\[ \text{MtM}^H_{i,j} = r_i \text{EAD}_i + \mathbb{E}[F^H_{i,j}], \]  
(4.11)

where \( \mathbb{E}[F^H_{i,j}] \) is the expected future cash flow (discounted to the horizon) of obligor \( i \) with rating \( j \) at the horizon. Note that (4.10) is a special case of (4.11). This fits intuition, because if the loan reached its maturity at the horizon, there are no future cash flows.

The Mark-to-Market value of obligor \( i \), which becomes \( j \)-rated after one year, can now be written as:

\[ \text{MtM}^H_{i,j} = \begin{cases} 
(1 - \text{LGD}_i) \text{EAD}_i, & \text{if } j = K; \\
(1 + r_i) \text{EAD}_i, & \text{if } j \leq K - 1 \text{ and } M_i = 1; \\
r_i \text{EAD}_i + \mathbb{E}[F^H_{i,j}], & \text{if } j \leq K - 1 \text{ and } M_i \geq 2,
\end{cases} \]  
(4.12)

where \( M_i \) denotes the maturity of the loan of obligor \( i \).

The determination of \( \mathbb{E}[F^H_{i,j}] \) consists of the following four steps, which will be discussed in the next subsections:

1. We need to determine all possible future paths of the obligor’s level of credit worthiness.
2. Next, we need the calculate the possibility of each path, which has to be done in a risk-neutral way, because of the risk-free discounting.
3. In order to obtain the expected future cash flow, we need to determine the value of each path.
4. Finally, the expected future cash flow is the weighted average of the path values over all possible trajectories of path.

### 4.2.1 Future Paths of Obligors

For the determination of the expected future cash flows, we are only interested in the years to default after the horizon. This can also be visualized in a tree. If we suppose that obligor two from the previous examples has a maturity of \( M_2 = 4 \), then the path tree looks as follows:
4.2 Valuation

Figure 4.2: The path tree for an obligor.

In path one, the obligor reaches its maturity of four years. In path two the obligor defaults in the third year after the horizon. In path three, the obligor defaults in the second year after the horizon, and so on. Next, we need to determine the risk-neutral probabilities of each path.

4.2.2 Risk-Neutral Cumulative Probabilities

In order to find the risk-neutral cumulative probabilities we need to go through the following steps:

1. Determine, from the migration matrix, the necessary cumulative default probabilities.
2. Use the cumulative default probabilities to determine the marginal default probabilities.
3. Map the marginal default probabilities to risk-neutral marginal default probabilities.
4. Determine the risk-neutral cumulative probabilities from the risk-neutral marginal default probabilities.

Let $c_{i,j,k}$ denote the $k$-year (after the horizon) cumulative default probability of obligor $i$ with rating $j$ at the horizon. Then

$$c_{i,j,k} = T^{(k)}[j,K],$$

i.e. the $k$-year cumulative default probability can be found in the $j$-th row and $K$-th column (recall that in this column the default probabilities are stated) of the $k$-th power of the migration matrix.

Next, let $m_{i,j,k}$ denote the marginal default probability of obligor $i$ with rating $j$ at the horizon in the $k$-th year after the horizon. Then $m_{i,j,k}$ can be written as a function of the cumulative default probability:

$$m_{i,j,k} = \begin{cases} c_{i,j,k}, & \text{if } k = 1; \\ (c_{i,j,k} - c_{i,j,k-1})/(1 - c_{i,j,k-1}), & \text{if } 1 < k \leq M_i - 1. \end{cases} \quad (4.13)$$

**Remark 4.4.** Recall that the path tree displays the possible paths after the horizon. Therefore, the equation holds for $k \leq M_i - 1$.

If we use a simplified version of Equation (4.8), we can map the marginal default probabilities $m_{i,j,k}$ to risk-neutral marginal default probabilities $\tilde{m}_{i,j,k}$:

$$\tilde{m}_{i,j,k} = \Phi \left( \Phi^{-1}(m_{i,j,k}) + R_i \lambda \right). \quad (4.14)$$
Finally, we are able to determine the risk-neutral probability of each path. Let $\tau$ denote the number of years to default, starting at the horizon. Furthermore, let $\delta_{i,j}$ denote a path for obligor $i$ with rating $j$ at horizon, then

$$P[\delta_{i,j} = d] = \begin{cases} \prod_{k=1}^{M_i-1} (1 - \tilde{m}_{i,j,k}), & \text{if } d = 1 \text{ and the obligor thus reaches its maturity;} \\ \tilde{m}_{i,j,\tau} \prod_{k=1}^{\tau-1} (1 - \tilde{m}_{i,j,k}), & \text{if } 1 < d < M_i \text{ and thus } M_i - 1 \geq \tau \geq 2; \\ \tilde{m}_{i,j,1}, & \text{if } d = M_i \text{ and thus } \tau = 1. \end{cases}$$

(4.15)

**Example 4.4.** Suppose that we have the same portfolio and migration matrix as in the previous examples and suppose that we want to know the risk-neutral probability that obligor 2 (with a maturity of $M_2$, an initial rating of $R_2$ and a correlation of $R_{2,2} = 0.30$) defaults in the second year after the horizon, given that he has rating $R_3$ at the horizon, i.e. $P[\delta_{2,3} = 3]$. Firstly, we need to determine the 1-year and 2-year cumulative default frequency from the migration matrix. Therefore, we also need the square of the migration matrix:

<table>
<thead>
<tr>
<th>Rating</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>0.8134</td>
<td>0.1052</td>
<td>0.0334</td>
<td>0.0480</td>
</tr>
<tr>
<td>R2</td>
<td>0.0708</td>
<td>0.6788</td>
<td>0.0596</td>
<td>0.1908</td>
</tr>
<tr>
<td>R3</td>
<td>0.0815</td>
<td>0.1500</td>
<td>0.4275</td>
<td>0.3410</td>
</tr>
<tr>
<td>D</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 4.4: The 2-year migration matrix.

From this it follows that $c_{2,3,1} = 0.20$ (see Table 4.1) and $c_{2,3,2} = 0.3410$. Using Equation (4.9) gives the following marginal default probabilities: $m_{2,3,1} = 0.20$ and $m_{2,3,2} = 0.1763$. The next step is to map these probabilities to risk free probabilities according to Equation (4.10). This leads to, given that the Market’s Sharpe Ratio equals 0.4, $\tilde{m}_{2,3,1} = 0.2668$ and $\tilde{m}_{2,3,2} = 0.2387$. Now we are able to determine the risk-neutral probability that obligor 2 defaults in the second year after the horizon:

$$P[\delta_{2,3} = 3] = \tilde{m}_{2,3,2}(1 - \tilde{m}_{2,3,1}) = 0.2387 \cdot (1 - 0.2668) = 0.1750.$$

The other cumulative probabilities for this obligor are given by:

$$P[\delta_{2,3} = 1] = 0.4393 \quad P[\delta_{2,3} = 2] = 0.1188 \quad P[\delta_{2,3} = 4] = 0.2668$$

Note that the risk-neutral cumulative probabilities sum to one for a given obligor with a given rating at the horizon.

### 4.2.3 Values of the Paths

If the obligor’s loan reaches maturity, then each year he has to pay interest and in the last year he has to pay back his loan plus the interest over that year. All these interest payments have to be risk-free discounted to the horizon. For example, the interest paid in the $t$-th year after the horizon is discounted to

$$r_i EAD_i / (1 + r_f)^t,$$

where $r_f$ is the risk-free rate.

If the obligor defaults before reaching maturity he has to pay interest in the years that he did not
default and he has to pay back a certain amount of his loan (the recovery rate times the exposure at default). Again, all these payments have to be risk free discounted. The discounting of the interest is the same as in the case that the obligor reaches maturity.

The payments can be displayed in the path tree from before:

Let \( V[\delta_{i,j} = d] \) denote the value of path \( d \) of obligor \( i \). From Figure 4.3 we can conclude that the values are equal to:

\[
V[\delta_{i,j} = d] = \begin{cases} 
(1 + r_i EAD_i) EAD_i & \text{if } d = 1; \\
(1 - LGD_i) EAD_i (1 + r_f)^{M_i - 1} & \sum_{t=1}^{M_i-2} \frac{r_i EAD_i}{(1 + r_f)^t} + \sum_{t=1}^{\tau-1} r_i EAD_i & \text{if } 1 < d \leq M_i \text{ and thus } M_i - 1 \geq \tau \geq 1.
\end{cases}
\]

(4.16)

**Example 4.5.** Suppose that obligor 2 (which became R3 rated) from Example 4.4 has the following characteristics:

\( EAD_2 = 10,\quad r_2 = 0.05,\quad r_f = 0.035,\quad LGD_2 = 0.30. \)

Suppose we want to determine the value of path 3, which means that the obligor defaults in the second year after the horizon. Then in the first year after the horizon the discounted interest equals 0.05 \cdot 10 / (1 + 0.035) = 0.4831. The discounted recovery in the second year equals \((1 - 0.3) \cdot 10 / (1 + 0.035)^2 = 6.5346.\) If we sum these two figures we obtain \( V[\delta_{2,3} = 3] = 0.4831 + 6.5346 = 7.0177.\) The other three possible path values are given by:

\( V[\delta_{2,3} = 1] = 10.4203,\quad V[\delta_{2,3} = 2] = 7.2634,\quad V[\delta_{2,3} = 4] = 6.7633.\)

### 4.2.4 The Value Matrix

Now that we have determined the probabilities and values of each path, we are able to determine the expected future cash flows:

\[
E[F_{i,j}^H] = \sum_{d=1}^{M_i} \mathbb{P}[\delta_{i,j} = d] V[\delta_{i,j} = d].
\]

Next, we can construct a value matrix in which for each obligor and every possible rating at the horizon the MtM-values are displayed. The next example shows such a valuation matrix.
Example 4.6. Assume that we have a portfolio which consists of a one-year and the four-year loan from the previous examples, then the valuation matrix (in which the MtM-values are actually calculated for obligor 2) looks as follows:

<table>
<thead>
<tr>
<th>Rating at the horizon</th>
<th>MtM(_{1,j}^H)</th>
<th>MtM(_{2,j}^H)</th>
<th>MtM(_{2,j}^H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>((1 + r_1)EAD_1)</td>
<td>(r_2EAD_2 + E[F_{2,j}^H])</td>
<td>10.4841</td>
</tr>
<tr>
<td>R2</td>
<td>((1 + r_1)EAD_1)</td>
<td>(r_2EAD_2 + E[F_{2,j}^H])</td>
<td>9.6371</td>
</tr>
<tr>
<td>R3</td>
<td>((1 + r_1)EAD_1)</td>
<td>(r_2EAD_2 + E[F_{2,j}^H])</td>
<td>8.9740</td>
</tr>
<tr>
<td>D</td>
<td>((1 - LGD_1)EAD_1)</td>
<td>((1 - LGD_2)EAD_2)</td>
<td>7.0000</td>
</tr>
</tbody>
</table>

Table 4.5: The valuation matrix.

4.2.5 The Expected Mark-to-Market Value Given Non-Default

In order to determine the loss in a state of the world, we need to determine the expected Mark-to-Market value of an obligor given that the obligor does not default \(E[MtM_{i,ND,H}^i]\). For this purpose, we first need to determine the conditional migration probabilities. Let \(s_{i,j}\) denote the risk-neutral probability that obligor \(i\) becomes a \(j\)-rated obligor, given that he does not default, then

\[
s_{i,j} = \frac{p_{i,j}}{1 - p_{i,K}}, \quad \forall j < K.
\]

Next, we combine the valuation matrix with these probabilities. Then we obtain

\[
E[MtM_{i,ND,H}^i] = \sum_{j=1}^{K-1} s_{i,j} \cdot MtM_{i,j}^H. \tag{4.17}
\]

Example 4.7. Again, we take a closer look at obligor 2. The conditional risk-neutral probabilities equal:

<table>
<thead>
<tr>
<th>Rating at the horizon</th>
<th>Conditional risk neutral probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>(s_{2,1} = \frac{p_{2,1}}{1 - p_{2,4}} = 0.94)</td>
</tr>
<tr>
<td>R2</td>
<td>(s_{2,2} = \frac{p_{2,2}}{1 - p_{2,4}} = 0.82)</td>
</tr>
<tr>
<td>R3</td>
<td>(s_{2,3} = \frac{p_{2,3}}{1 - p_{2,4}} = 0.90)</td>
</tr>
</tbody>
</table>

Table 4.6: The conditional transition probabilities for obligor 2.

The Mark-to-Market value of obligor 2 given that the obligor does not default is now given by

\[
E[MtM_{i,ND,H}^i] = \sum_{j=1}^{K-1} s_{i,j} \cdot MtM_{i,j}^H
\]

\[
= \sum_{j=1}^{3} s_{2,j} \cdot \left( (1 + r_2)(EAD_2) + E[F_{2,j}^H] \right)
\]

\[
= 9.6443.
\]

4.2.6 The Loss Matrix

From the valuation matrix and all obligors’s MtM-values given that they do not default, we can derive the loss matrix (in which also the losses are calculated for obligor 2 from Example 4.7):
### 4.3 The Loss Distribution

<table>
<thead>
<tr>
<th>Rating at the horizon</th>
<th>Obligor 1</th>
<th>Obligor 2</th>
<th>Obligor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>(\frac{\text{E}[\text{MtM}<em>{1}^{\text{ND,H}}] - \text{MtM}</em>{1}^{H}}{(1 + r^f)})</td>
<td>(\frac{\text{E}[\text{MtM}<em>{2}^{\text{ND,H}}] - \text{MtM}</em>{2}^{H}}{(1 + r^f)})</td>
<td>-0.8106</td>
</tr>
<tr>
<td>R2</td>
<td>(\frac{\text{E}[\text{MtM}<em>{1}^{\text{ND,H}}] - \text{MtM}</em>{1}^{H}}{(1 + r^f)})</td>
<td>(\frac{\text{E}[\text{MtM}<em>{2}^{\text{ND,H}}] - \text{MtM}</em>{2}^{H}}{(1 + r^f)})</td>
<td>0.0079</td>
</tr>
<tr>
<td>R3</td>
<td>(\frac{\text{E}[\text{MtM}<em>{1}^{\text{ND,H}}] - \text{MtM}</em>{1}^{H}}{(1 + r^f)})</td>
<td>(\frac{\text{E}[\text{MtM}<em>{2}^{\text{ND,H}}] - \text{MtM}</em>{2}^{H}}{(1 + r^f)})</td>
<td>0.6486</td>
</tr>
<tr>
<td>D</td>
<td>(\frac{\text{E}[\text{MtM}<em>{1}^{\text{ND,H}}] - \text{MtM}</em>{1}^{H}}{(1 + r^f)})</td>
<td>(\frac{\text{E}[\text{MtM}<em>{2}^{\text{ND,H}}] - \text{MtM}</em>{2}^{H}}{(1 + r^f)})</td>
<td>2.5558</td>
</tr>
</tbody>
</table>

Table 4.7: The loss matrix.

**Remark 4.5.** Note that in Table 4.7 negative losses (gains) can be found. That is, if obligor 2 becomes an R1 rated obligor after one year, the bank gains on that obligor.

As said before, each state of the world consists of a new rating for each obligor. Each new rating corresponds to a loss, given in the loss matrix. Therefore, given a state of the world and the loss matrix, we have a possible portfolio loss (which is simply the sum of the obligor losses).

### 4.3 The Loss Distribution

If we determine the joint probabilities in the migration tree (see Figure 4.1) for each state of the world and fill in the loss matrix, we can determine the portfolio loss distribution. Each state of the world represents a possible value of the loss distribution. Let \(L_s\) be the loss in a certain state \(s\) of the world. If we then assume that obligor 1, with a maturity of 8 years, becomes R2-rated and obligor 2, with a maturity of four years, goes in default, then the loss is given by

\[
L_s = \text{E}[\text{MtM}_{1}^{\text{ND,H}}] - \text{MtM}_{1}^{H} + \text{E}[\text{MtM}_{2}^{\text{ND,H}}] - \text{MtM}_{2}^{H}. 
\]

Analogously, we can determine the losses of all other possible states of the world. If we now sort all the possible portfolio losses (with their corresponding probabilities), we obtain the loss distribution.

**Remark 4.6.** Note that in the years to the horizon, in this thesis the first year, we do not use risk-neutral probabilities. Only in the years after the horizon, we do this. This is because both \(EL\) and \(EC\) are being risk-free invested.

### 4.4 Loss Distribution Risk and Performance Measures

From the loss distribution we can derive risk measures, like the expected loss (\(EL\)), the unexpected loss (\(VaR_\alpha\)) and the economic capital \(EC\). In the next subsections all the relevant risk measures are introduced.

#### 4.4.1 Expected Loss

In the two-obligor portfolio the number of states of the world is equal to:

\[
N_s = K^2, \tag{4.18}
\]

where \(K\) is again the number of ratings, including the default status. The expected loss\(^\text{\footnote{The expected loss can also be determined without the loss distribution. In Appendix \ref{appendix_b} this determination can be found for the two-obligor portfolio.}}\) is then given by:

\[
EL = \text{E}[\frac{\text{E}[\text{MtM}_{1}^{\text{ND,H}}] - \text{MtM}_{1}^{H}}{1 + r^f}] = \frac{\text{E}[\text{MtM}_{1}^{\text{ND,H}}] - \text{E}[\text{MtM}_{1}^{H}]}{1 + r^f}. \tag{4.19}
\]
This equation is equivalent to the following expression, given that we have all possible losses (with their corresponding probabilities):

\[ EL = \sum_{s=1}^{N_s} \mathbb{P}[L = L^s]L^s. \]  

(4.20)

From this we can also derive the obligor specific expected losses \( EL_i \):

\[ EL_i = \sum_{s=1}^{N_s} \mathbb{P}[L = L^s]L_i^s. \]  

(4.21)

### 4.4.2 Value-at-Risk

The \( VaR_\alpha \)-level\(^7\) is the portfolio loss which will not be exceeded with a probability of \( \alpha \). So

\[ VaR_\alpha = \frac{\mathbb{E}[MtM^{ND,H}] - MtM^H_\alpha}{1 + r_f}, \]  

(4.22)

where \( MtM^H_\alpha \) is the Mark-to-Market level which will not be exceeded with probability \( 1 - \alpha \). Given all \( L^s \), the Value-at-Risk can be determined by

\[ VaR_\alpha = \max_s \{ L^s | \mathbb{P}[L < L^s] \leq \alpha \}. \]  

(4.23)

From this it follows that the obligor specific value-at-risk (also known as the risk contribution) equals:

\[ VaR_{i,\alpha} = \{ L_i^s | L^s = VaR_\alpha \}. \]  

(4.24)

### 4.4.3 Economic Capital

The economic capital is defined as the unexpected loss minus the expected loss. It is the capital a bank wants to reserve to capture unexpected losses. The economic capital on portfolio level looks thus as follows:

\[ EC = VaR_\alpha - EL. \]  

(4.25)

The obligor specific economic capital or capital contribution of an obligor equals:

\[ EC_i = VaR_{i,\alpha} - EL_i. \]  

(4.26)

Note that

\[ EC = \sum_{i=1}^{n} EC_i. \]

A graphical interpretation of the risk measures \( EL \), \( VaR_\alpha \) and \( EC \) is given in the following figure, in which a typical loss distribution is displayed:

\(^7\)This is sometimes referred to as the unexpected loss.
4.4 Loss Distribution Risk and Performance Measures

4.4.4 Economic Value Added

Economic Value Added (EVA) is a risk adjusted performance measure for calculating the true economic profit of a bank. It is defined as the return minus the cost of capital minus the expected loss. So in case of the two-obligor portfolio EVA looks as follows

\[ EVA = \sum_{i=1}^{2} \frac{r_i EAD_i}{1 + r_f} - r^c EC - EL. \]  

The economic value added contribution of obligor \( i \) equals

\[ EVA_i = \frac{r_i EAD_i}{1 + r_f} - r^c EC_i - EL_i. \]  

Similar to the economic capital, EVA is the sum of all EVA-contributions.

4.4.5 Risk Adjusted Return on Capital

Risk Adjusted Return on Capital (RAROC) is a risk adjusted performance measure for measuring risk adjusted financial performance. It is defined as the return minus the expected loss divided by the economic capital, so in case of the two-obligor portfolio RAROC looks as follows

\[ RAROC = \frac{\sum_{i=1}^{2} \frac{r_i EAD_i}{1 + r_f}}{EC} - EL. \]  

The risk adjusted return on capital contribution of obligor \( i \) equals

\[ RAROC_i = \frac{\frac{r_i EAD_i}{1 + r_f} - EL_i}{EC_i}. \]  

Note that, differently to EVA, RAROC is the weighted sum of all RAROC-contributions:

\[ RAROC = \sum_{i=1}^{n} RAROC_i \frac{EC_i}{EC}. \]
4.5 A Stylized Two-obligor Portfolio Example

In this section we use the methodology described in the previous sections to determine the loss distribution and some corresponding risk measures.

Example 4.8. Suppose we have the following portfolio and migration matrix:

![Figure 4.5: The portfolio.](image)

![Figure 4.6: The 1-year migration matrix.](image)

Furthermore, assume that market Sharpe ratio equals 0.4. From Figure 4.6, we obtain the (cumulative) migration probabilities $p_{i,j}$ and $q_{i,j}$. Because we have three different ratings (R1 to R3), we have the same migration tree as in Figure 4.1. Now we are able to determine the probability of each state of the world, using the methodology in Section 4.1.3.

<table>
<thead>
<tr>
<th>State of the world</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R1, R1)</td>
<td>$p_{1,2,1,1} = 0.0387$</td>
</tr>
<tr>
<td>(R1, R2)</td>
<td>$p_{1,2,1,2} = 0.7463$</td>
</tr>
<tr>
<td>(R1, R3)</td>
<td>$p_{1,2,1,3} = 0.0339$</td>
</tr>
<tr>
<td>(R2, R1)</td>
<td>$p_{1,2,2,1} = 0.0099$</td>
</tr>
<tr>
<td>(R2, R2)</td>
<td>$p_{1,2,2,2} = 0.0454$</td>
</tr>
<tr>
<td>(R2, R3)</td>
<td>$p_{1,2,2,3} = 0.0035$</td>
</tr>
<tr>
<td>(R3, R1)</td>
<td>$p_{1,2,3,1} = 0.0002$</td>
</tr>
<tr>
<td>(R3, R2)</td>
<td>$p_{1,2,3,2} = 0.0146$</td>
</tr>
<tr>
<td>(R3, R3)</td>
<td>$p_{1,2,3,3} = 0.0012$</td>
</tr>
<tr>
<td>(R3, D)</td>
<td>$p_{1,2,3,4} = 0.0039$</td>
</tr>
<tr>
<td>(D, R1)</td>
<td>$p_{1,2,4,1} = 0.0002$</td>
</tr>
<tr>
<td>(D, R2)</td>
<td>$p_{1,2,4,2} = 0.0138$</td>
</tr>
<tr>
<td>(D, R3)</td>
<td>$p_{1,2,4,3} = 0.0014$</td>
</tr>
<tr>
<td>(D, D)</td>
<td>$p_{1,2,4,4} = 0.0047$</td>
</tr>
</tbody>
</table>

Table 4.8: The probability of each state of the world.

Now we want to construct the loss matrix. First we need to determine the risk-neutral path probabilities of each obligor. For this we need to determine the risk-neutral marginal default probabilities for each obligor in each year after the horizon, using (4.14). If we then substitute these values into Equation (4.15), we obtain the risk-neutral path probabilities given in Table 4.9 below. For each path we also need the corresponding value. Using (4.16) gives

<table>
<thead>
<tr>
<th>Path</th>
<th>Obligor 1</th>
<th>Obligor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R1</td>
<td>R2</td>
</tr>
<tr>
<td>1</td>
<td>0.6711</td>
<td>0.3705</td>
</tr>
<tr>
<td>2</td>
<td>0.0521</td>
<td>0.0531</td>
</tr>
<tr>
<td>3</td>
<td>0.0532</td>
<td>0.0624</td>
</tr>
<tr>
<td>4</td>
<td>0.0530</td>
<td>0.0735</td>
</tr>
<tr>
<td>5</td>
<td>0.0515</td>
<td>0.0864</td>
</tr>
<tr>
<td>6</td>
<td>0.0478</td>
<td>0.1013</td>
</tr>
<tr>
<td>7</td>
<td>0.0411</td>
<td>0.1178</td>
</tr>
<tr>
<td>8</td>
<td>0.0304</td>
<td>0.1351</td>
</tr>
</tbody>
</table>

Table 4.9: The risk-neutral path probabilities and path values.
4.5 A Stylized Two-obligor Portfolio Example

Note that there are white spaces in the table. This is because obligor 2 has a maturity of 4 years, while obligor 1 has a maturity of 8 years. Combining the two tables, we arrive at the following valuation matrix:

<table>
<thead>
<tr>
<th>Rating at the horizon</th>
<th>Obligor 1</th>
<th>Obligor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>5.6019</td>
<td>10.4841</td>
</tr>
<tr>
<td>R2</td>
<td>5.0963</td>
<td>9.6371</td>
</tr>
<tr>
<td>R3</td>
<td>4.8139</td>
<td>8.9740</td>
</tr>
<tr>
<td>D</td>
<td>4.0000</td>
<td>7.0000</td>
</tr>
</tbody>
</table>

Table 4.10: The valuation matrix.

The only thing left to determine is the expected Mark-to-Market value of each obligor given that the obligors do not default in the first year. Analogue to Example 4.6, we obtain the following values:

\[ E[MtM_{1}^{{\text{ND,H}}}] = 5.5550, \quad E[MtM_{2}^{{\text{ND,H}}}] = 9.6443. \]

This results in the following loss matrix:

<table>
<thead>
<tr>
<th>Rating at the horizon</th>
<th>Obligor 1</th>
<th>Obligor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>-0.0455</td>
<td>-0.8106</td>
</tr>
<tr>
<td>R2</td>
<td>0.4431</td>
<td>0.0079</td>
</tr>
<tr>
<td>R3</td>
<td>0.7160</td>
<td>0.6486</td>
</tr>
<tr>
<td>D</td>
<td>1.5024</td>
<td>2.5558</td>
</tr>
</tbody>
</table>

Table 4.11: The loss matrix.

Substituting these losses into Table 4.8 gives the following loss distribution

<table>
<thead>
<tr>
<th>State of the world</th>
<th>( L_{1}^{a} )</th>
<th>( L_{2}^{a} )</th>
<th>( L^{a} )</th>
<th>( P[L = L^{a}] )</th>
<th>( P[L \geq L^{a}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D, D)</td>
<td>1.5024</td>
<td>2.5558</td>
<td>4.0582</td>
<td>0.0047</td>
<td>0.0047</td>
</tr>
<tr>
<td>(R3, D)</td>
<td>0.7160</td>
<td>2.5558</td>
<td>3.2718</td>
<td>0.0039</td>
<td>0.0086</td>
</tr>
<tr>
<td>(R2, D)</td>
<td>0.4431</td>
<td>2.5558</td>
<td>2.9989</td>
<td>0.0102</td>
<td>0.0188</td>
</tr>
<tr>
<td>(R1, D)</td>
<td>-0.0455</td>
<td>2.5558</td>
<td>2.5103</td>
<td>0.0811</td>
<td>0.0999</td>
</tr>
<tr>
<td>(D, R3)</td>
<td>1.5024</td>
<td>0.6486</td>
<td>2.1510</td>
<td>0.0014</td>
<td>0.1013</td>
</tr>
<tr>
<td>(D, R2)</td>
<td>1.5024</td>
<td>0.0079</td>
<td>1.5103</td>
<td>0.0138</td>
<td>0.1151</td>
</tr>
<tr>
<td>(R3, R3)</td>
<td>0.7160</td>
<td>0.6486</td>
<td>1.3646</td>
<td>0.0012</td>
<td>0.1163</td>
</tr>
<tr>
<td>(R2, R3)</td>
<td>0.4431</td>
<td>0.6486</td>
<td>1.0917</td>
<td>0.0035</td>
<td>0.1198</td>
</tr>
<tr>
<td>(R3, R2)</td>
<td>0.7160</td>
<td>0.0079</td>
<td>0.7239</td>
<td>0.0146</td>
<td>0.1344</td>
</tr>
<tr>
<td>(D, R1)</td>
<td>1.5024</td>
<td>-0.8106</td>
<td>0.6918</td>
<td>0.0002</td>
<td>0.1346</td>
</tr>
<tr>
<td>(R1, R3)</td>
<td>-0.0455</td>
<td>0.6486</td>
<td>0.6031</td>
<td>0.0339</td>
<td>0.1685</td>
</tr>
<tr>
<td>(R2, R2)</td>
<td>0.4431</td>
<td>0.0079</td>
<td>0.4510</td>
<td>0.0454</td>
<td>0.2139</td>
</tr>
<tr>
<td>(R1, R2)</td>
<td>-0.0455</td>
<td>0.0079</td>
<td>-0.0376</td>
<td>0.7463</td>
<td>0.9602</td>
</tr>
<tr>
<td>(R3, R1)</td>
<td>0.7160</td>
<td>-0.8106</td>
<td>-0.0946</td>
<td>0.0002</td>
<td>0.9604</td>
</tr>
<tr>
<td>(R2, R1)</td>
<td>0.4431</td>
<td>-0.8106</td>
<td>-0.3675</td>
<td>0.0009</td>
<td>0.9613</td>
</tr>
<tr>
<td>(R1, R1)</td>
<td>-0.0455</td>
<td>-0.8106</td>
<td>-0.8561</td>
<td>0.0387</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 4.12: The loss distribution.

The corresponding histogram looks as follows:
From the loss distribution, we can derive the following risk measures, given that the cost of capital equals $r^c = 0.02$:

<table>
<thead>
<tr>
<th>Measures</th>
<th>Portfolio</th>
<th>Obligor 1</th>
<th>Obligor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$EL$</td>
<td>0.2856</td>
<td>0.0300</td>
<td>0.2556</td>
</tr>
<tr>
<td>$EL^{AN}$</td>
<td>0.2856</td>
<td>0.0300</td>
<td>0.2556</td>
</tr>
<tr>
<td>$VaR_{0.99}$</td>
<td>2.9989</td>
<td>0.4431</td>
<td>2.5558</td>
</tr>
<tr>
<td>$EC_{0.99}$</td>
<td>2.7133</td>
<td>0.4131</td>
<td>2.3002</td>
</tr>
<tr>
<td>$EVA_{0.99}$</td>
<td>0.4632</td>
<td>0.2627</td>
<td>0.2005</td>
</tr>
<tr>
<td>$RAROC_{0.99}$ (%)</td>
<td>18.96</td>
<td>65.36</td>
<td>10.63</td>
</tr>
</tbody>
</table>

Table 4.13: The loss distribution risk measures.

If we look at this table we see that a bank has to reserve 2.7133 million euro to protect itself from an unexpected loss. Furthermore, we see that the Economic Value Added is equal to 0.4632 million euro. I.e., the bank makes a net profit of million 0.4632 euro.
Chapter 5

Simulation of the Portfolio Loss Distribution

Rabobank’s credit portfolio for corporate clients (large clients), consists of hundreds of obligors. Therefore, we need a framework for the determination of the loss distribution of \( n \)-obligor portfolios. The methodology for this is roughly the same as in the 2-obligor portfolio case. However, when \( n \) becomes large, some difficulties arise if we want to calculate the loss distribution.

Firstly, we discuss these difficulties. Subsequently, we explain how we deal with these difficulties. We give two simulation methods which determine the loss distribution and their corresponding risk measures. Finally, we give illustrative examples.

5.1 The Difficulties of \( n \) Obligors

Until this point, the loss distribution was determined analytically. This has two major advantages:

1. **Speed.** For small portfolios, direct calculations require fewer operations than simulation. Therefore, the analytical approach is faster in this case.
2. **Precision.** In simulation, errors (the exact loss distribution is no longer calculated explicitly) are introduced in the calculations, whereas this is not the case in the analytical approach. Therefore the analytical approach is more precise.

However, when the number of obligors becomes large, the number of states becomes even larger. Recall from Section 4.4 that the number of states in the two-obligor case is given by \( K^2 \). In general the number of states is given by:

\[
N_s = K^n.
\]

So if we assume a portfolio of 100 obligors and 20 grades, which is even a small portfolio in Rabobank terms, the number of states is approximately \( 10^{130} \). In such cases analytical approaches are no longer suitable and we need the simulation based approaches, described in the following sections.

In these sections we give the simulation methods for the determination of the loss distribution. Firstly, we discuss Monte Carlo Simulation, which determines the loss distribution by randomly generating loss scenarios. Secondly, we discuss an accelerated Monte Carlo simulation (Importance Sampling). This method also randomly generates loss scenarios, but now more focused on the area of interest (in this case the tail of the loss distribution). Details of both methods will be discussed in the next two sections.\footnote{More information on both methods can be found in e.g. [3] or [20].}
For a better understanding of the methodology, we will use the same portfolio as in the final example of the previous chapter.

## 5.2 Monte Carlo Simulation

The purpose of this section is to provide the methodology of Monte Carlo simulation. The goal of Monte Carlo simulation is to obtain an approximation of the loss distribution. We now use the following procedure to obtain this approximation:

1. **Generating Scenarios.** Each scenario corresponds to a possible state of the world. Because we want to use the standardized thresholds to determine the new rating at the horizon (see Section 3.2), we only have to generate a market factor $Y$ and for each obligor an obligor specific factor $Z_i$.

2. **Determine losses.** For each scenario, we determine the losses corresponding to the new credit ratings. For this we use the standardized asset value $X_i$, the Merton framework from Chapter 3 and the loss matrix from Section 4.2. This step gives us a large number of possible portfolio losses.

3. **Determine the loss distribution and risk measures.** Given the loss scenarios generated in the previous steps, we have an estimate for the loss distribution. From this distribution, we can derive several risk measures.

In the next sections all of these steps will be explained in more detail.

### 5.2.1 Generating Scenarios

Here, we will discuss how to generate scenarios of future credit ratings for the obligors in our portfolio. We will rely heavily on the asset value model discussed in Chapter 3. The steps to scenario generation are as follows:

1. Establish asset return thresholds for the obligors in the portfolio.
2. Generate scenarios of asset returns.
3. Map the asset return scenarios to credit rating scenarios.

If we use the migration matrix from Section 4.5 and determine the standardized thresholds according to Section 3.2, we obtain the following table:

<table>
<thead>
<tr>
<th>Rating</th>
<th>Obligor 1</th>
<th>Obligor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{R2}$</td>
<td>-1.2816</td>
<td>1.7507</td>
</tr>
<tr>
<td>$X_{R3}$</td>
<td>-1.7507</td>
<td>-1.0803</td>
</tr>
<tr>
<td>$X_D$</td>
<td>-2.0537</td>
<td>-1.2816</td>
</tr>
</tbody>
</table>

**Table 5.1:** The risk neutral migration thresholds.

**Remark 5.1.** Recall that if the asset return is between $X_{R2}$ and $X_{R3}$, then the rating at the end of the horizon is $R2$.

From Chapter 3 it also follows that the standardized asset values can be written as

$$X_i = \sqrt{R_i^2 Y} + \sqrt{1 - R_i^2} Z_i,$$

where $R_i^2$ is again the correlation of obligor $i$ with the market. So a scenario is generated by generating a market factor $Y$ and for each obligor an obligor specific factor $Z_i$. To fully specify our scenarios, it is only necessary to assign ratings to the asset return scenarios. For example, consider scenario 5 in the table below. The asset return for Obligor 1 is -1.2907, which lies between
\(\tilde{X}_{R2}\) and \(\tilde{X}_{R3}\) for this obligor. This corresponds to a new rating of R2. For obligor 2, the return is -0.1846, which also falls between \(\tilde{X}_{R2}\) and \(\tilde{X}_{R3}\) for this obligor, corresponding to a new rating of R2. Continuing this process, we may construct the following table which completes the process of scenario generation.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Market Factor</th>
<th>Obligor Specific Factor</th>
<th>Asset Return</th>
<th>New Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.5971</td>
<td>0.6992</td>
<td>0.3583</td>
<td>R1</td>
</tr>
<tr>
<td>2</td>
<td>-1.8870</td>
<td>-1.2302</td>
<td>-1.9442</td>
<td>R3</td>
</tr>
<tr>
<td>3</td>
<td>-0.7220</td>
<td>1.2681</td>
<td>0.8113</td>
<td>R1</td>
</tr>
<tr>
<td>4</td>
<td>-1.2006</td>
<td>1.1892</td>
<td>0.5267</td>
<td>R1</td>
</tr>
<tr>
<td>5</td>
<td>-0.6780</td>
<td>-1.1041</td>
<td>-1.2907</td>
<td>R2</td>
</tr>
<tr>
<td>6</td>
<td>-0.4444</td>
<td>0.4240</td>
<td>0.1805</td>
<td>R1</td>
</tr>
<tr>
<td>7</td>
<td>-1.3014</td>
<td>1.3221</td>
<td>0.6005</td>
<td>R1</td>
</tr>
<tr>
<td>8</td>
<td>2.4360</td>
<td>2.1017</td>
<td>2.9692</td>
<td>R1</td>
</tr>
<tr>
<td>9</td>
<td>-0.7831</td>
<td>0.4121</td>
<td>0.0184</td>
<td>R1</td>
</tr>
<tr>
<td>10</td>
<td>0.2584</td>
<td>0.2494</td>
<td>0.3386</td>
<td>R1</td>
</tr>
</tbody>
</table>

Table 5.2: Mapping scenarios to new ratings.

5.2.2 Determine losses

The portfolio value in the Monte Carlo Simulation is the same as in the previous chapter. For each scenario and each obligor, the new rating maps directly to a loss. So in order to value the portfolio, we need the loss matrix. Recall from Section 4.3 that the valuation matrix looks as follows:

<table>
<thead>
<tr>
<th>Rating at the horizon</th>
<th>Obligor 1</th>
<th>Obligor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>-0.0455</td>
<td>-0.8106</td>
</tr>
<tr>
<td>R2</td>
<td>0.4431</td>
<td>0.0079</td>
</tr>
<tr>
<td>R3</td>
<td>0.7160</td>
<td>0.6486</td>
</tr>
<tr>
<td>D</td>
<td>1.5024</td>
<td>2.5558</td>
</tr>
</tbody>
</table>

Table 5.3: The loss matrix.

Given the scenarios from Table 5.1 and the loss matrix, we obtain a portfolio loss for each scenario. These results are stated in the table below.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>New Rating</th>
<th>Obligor Loss</th>
<th>Portfolio Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>R1</td>
<td>R2</td>
<td>-0.0455 0.0079</td>
</tr>
<tr>
<td>2</td>
<td>R3</td>
<td>R2</td>
<td>0.7160 0.0079</td>
</tr>
<tr>
<td>3</td>
<td>R1</td>
<td>R3</td>
<td>-0.0455 0.6486</td>
</tr>
<tr>
<td>4</td>
<td>R1</td>
<td>R2</td>
<td>-0.0455 0.0079</td>
</tr>
<tr>
<td>5</td>
<td>R2</td>
<td>R2</td>
<td>0.4431 0.0079</td>
</tr>
<tr>
<td>6</td>
<td>R1</td>
<td>R2</td>
<td>-0.0455 0.0079</td>
</tr>
<tr>
<td>7</td>
<td>R1</td>
<td>D</td>
<td>-0.0455 2.5558</td>
</tr>
<tr>
<td>8</td>
<td>R1</td>
<td>R2</td>
<td>-0.0455 0.0079</td>
</tr>
<tr>
<td>9</td>
<td>R1</td>
<td>D</td>
<td>-0.0455 2.5558</td>
</tr>
<tr>
<td>10</td>
<td>R1</td>
<td>R2</td>
<td>-0.0455 0.0079</td>
</tr>
</tbody>
</table>

Table 5.4: Valuation of the portfolio.

Remark 5.2. Note that for a given obligor, its loss is the same in scenarios with the same credit rating.
5.2.3 Determine the Loss Distribution and Risk Measures

Given a number of scenarios \( N_{sc} \) the loss distribution is given by the portfolio losses sorted in descending order \( L^{(1)} \geq L^{(2)} \geq \ldots \geq L^{(N_{sc})} \). The cumulative density function is then given by:

\[
P[L \leq x] = \mathbb{E}[\mathbb{1}_{\{L \leq x\}}] = \frac{1}{N_{sc}} \sum_{i=1}^{N_{sc}} \mathbb{1}_{\{L^{(i)} \leq x\}}. \tag{5.1}
\]

**Expected Loss**

From the sorted losses, we can also determine the expected loss

\[
EL = \sum_{j=1}^{N_{sc}} L^{(j)} \frac{1}{N_{sc}}. \tag{5.2}
\]

The obligor specific expected loss is given by:

\[
EL_i = \sum_{j=1}^{N_{sc}} L^{(j)} \frac{1}{N_{sc}}. \tag{5.3}
\]

**Value-At-Risk**

Again, the expected loss can also be determined analytically (see Appendix B). Recall from Section 4.4 that the \( VaR_\alpha \)-level is the portfolio loss which will not be exceeded with probability \( \alpha \). In this case it is given by

\[
VaR_\alpha = \min_i \{L^{(i)} | P[L < L^{(i)}] > \alpha\} = L^{(\lceil (1-\alpha)N_{sc} \rceil)}. \tag{5.4}
\]

We also want to calculate the risk contributions of each obligor. Because there is no closed form solution for this, we had some discussions within Rabobank, which lead to the following approximation for the risk contribution:

\[
VaR_{i,\alpha} = \frac{1}{k_2 - k_1 + 1} \sum_{j=k_1}^{k_2} L^{(j)}_i, \tag{5.5}
\]

where \( k_1 \) and \( k_2 \) are such that

\[
L^{(k_1)} = (1 - \beta)VaR_\alpha, \quad L^{(k_2)} = (1 + \beta)VaR_\alpha,
\]

and where \( \beta \) is a small number. I.e., the risk contribution is determined by calculating the mean obligor specific loss in a certain region around the portfolio Value-at-Risk. If we choose a relatively large \( \beta \), the estimation of the risk contribution becomes more precise but also more biased, while if we choose a relatively small \( \beta \), the estimation becomes more unbiased but also less precise. Choosing the optimal \( \beta \) is portfolio dependent and can only be done heuristically.

**Economic Capital**

Analogue to Section 4.4 the economic capital \( EC \) and capital contribution \( EC_i \) are defined as:

\[
EC = VaR_\alpha - EL, \quad EC_i = VaR_{i,\alpha} - EL_i. \tag{5.6}
\]

**Economic Value Added**

Again, \( EVA \) and \( EVA_i \) are given by

\[
EVA = \frac{\sum_{i=1}^2 r_i EAD_i}{1 + r^f} - r^f EC - EL, \quad EVA_i = \frac{r_i EAD_i}{1 + r^f} - r^f EC_i - EL_i. \tag{5.7}
\]

\(^2\)Each scenario is equally likely.
5.3 Importance Sampling

The purpose of this section is to provide the methodology of Importance Sampling. The goal of Importance Sampling is to speed up the Monte Carlo algorithm by sampling losses in a certain region of interest. Since EL can be calculated analytically, the region of interest lies around VaRα. To generate more samples in this region, we shift the mean \( \mu \) of the market factor with \( \mu \). This causes smaller values of \( Y \) and therefore more downgrades and defaults, thus higher losses. Firstly, we give a theorem (see e.g. [7]) about the loss distribution. Secondly we give the Importance Sampling algorithm to determine this loss distribution.

**Theorem 5.1. Huang et al (2007).** With mean shifting, the market factor \( Y \) is sampled under probability measure \( Q_2 \) which is equivalent to the original measure \( Q_1 \) such that under \( Q_2 \), \( Y \) is normally distributed with mean \( \mu \leq 0 \) and variance \( \sigma^2 = 1 \). The tail probability is then given by

\[
P[L > x] = E_{Q_2}[I(Y)1_{\{L > x\}}],
\]

where \( l(Y) = e^{-\mu Y + \mu^2/2} \) denotes the likelihood ratio, which is necessary to correct for the change in distribution of the market factor.

**Proof.** Let us define \( f_y(x) \) as the conditional loss probability. Now, Equation (5.1) can be written as

\[
P[L > x] = E_{Q_1}[I_{\{L > x\}}]
        = \int_{-\infty}^{\infty} P[L > x | Y] dP[Y < y]
        = \int_{-\infty}^{\infty} f_y(x) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy,
\]

(5.10)

where \( Y \) represents the market factor and assumed to be standard normally distributed. Analogously, we can write (5.9) as

\[
P[L > x] = E_{Q_2}[l(y)1_{\{L > x\}}]
        = \int_{-\infty}^{\infty} f_y(x) \frac{1}{\sqrt{2\pi}} e^{-(y-\mu)^2/2} dy.
\]

(5.11)

Since both equations should represent the same loss distribution, we can derive an expression for the likelihood ratio. Equating (5.10) and (5.11) leads to:

\[
l(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \frac{1}{\sqrt{2\pi}} e^{-(y-\mu)^2/2}
    = e^{-\mu Y + \mu^2/2}.
\]

(5.12)

This statement holds for all \( y \). Therefore, the likelihood ratio, given that \( Y \sim N(\mu, 1) \), is given by:

\[
l(Y) = e^{-\mu Y + \mu^2/2}.
\]

(5.13)

\[\square\]
Analogue to Monte Carlo, Importance Sampling consists of the following three main steps:

1. Generate Scenarios.
2. Determine losses.

The only difference is that we perform a shift when generating scenarios. Because step 2 in Importance Sampling is analogue to step 2 in the Monte Carlo simulation, we refer to the previous section. In the next subsections, we will elaborate more on step 1 and 3, since both steps are slightly different than in the Monte Carlo simulation.

5.3.1 Generate Scenarios

The generation of scenarios of future credit ratings again heavily relies on the Merton Migration Model. Therefore, the steps to scenario generation are again as follows:

1. Establish asset return thresholds for the obligors in the portfolio.
2. Generate scenarios of asset returns, but now more focussed on the region of interest.
3. Map the asset return scenarios to credit rating scenarios.

The determination of the risk neutral asset return thresholds, can be found in Section 5.2.1. Again, the standardized asset values can be written as

\[ X_i = \sqrt{R_i^2} Y + \sqrt{1 - R_i^2} Z_i, \]

where \( Y \) is now normally distributed with mean \( \mu \). However, we now get an over-concentration of extreme events, which needs to be compensated for properly. Following Theorem 5.1, we have to calculate a likelihood ratio for each scenario and use it as the weight of the scenario in statistical calculations. How we use this likelihood in order to obtain the portfolio risk measures is explained further on.

So a scenario is again generated by generating a shifted market factor \( Y \) and for each obligor an obligor specific factor \( Z_i \). Below, we list ten scenarios which might be produced by such a procedure. To fully specify our scenarios, it is only necessary to assign ratings to the asset return scenarios. This may lead to the following table of scenarios

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Market Factor</th>
<th>Likelihood Ratio</th>
<th>Obligor Specific Factor Obligor 1 Obligor 2</th>
<th>Asset Return Obligor 1 Obligor 2</th>
<th>New Rating Obligor 1 Obligor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2.5971</td>
<td>0.0409</td>
<td>0.6992 -0.2033</td>
<td>-0.5361 -1.5926</td>
<td>R1 D</td>
</tr>
<tr>
<td>2</td>
<td>-3.8870</td>
<td>0.0031</td>
<td>-1.2302 0.8298</td>
<td>-2.8386 -1.4347</td>
<td>D D</td>
</tr>
<tr>
<td>3</td>
<td>-2.7220</td>
<td>0.0319</td>
<td>1.2681 -0.6051</td>
<td>-0.0831 -1.9972</td>
<td>R1 D</td>
</tr>
<tr>
<td>4</td>
<td>-3.2006</td>
<td>0.0122</td>
<td>1.1892 -0.0120</td>
<td>-0.3677 -1.7631</td>
<td>R1 D</td>
</tr>
<tr>
<td>5</td>
<td>-2.6780</td>
<td>0.0348</td>
<td>-1.1041 0.2232</td>
<td>-2.1852 -1.2801</td>
<td>D R3</td>
</tr>
<tr>
<td>6</td>
<td>-2.4444</td>
<td>0.0556</td>
<td>0.4240 -0.0024</td>
<td>-0.7139 -1.3409</td>
<td>R1 D</td>
</tr>
<tr>
<td>7</td>
<td>-3.3014</td>
<td>0.0100</td>
<td>1.3221 -3.4301</td>
<td>-0.2939 -4.6781</td>
<td>R1 D</td>
</tr>
<tr>
<td>8</td>
<td>0.4360</td>
<td>17.672</td>
<td>2.1017 -0.8924</td>
<td>2.0748 -0.5078</td>
<td>R1 R2</td>
</tr>
<tr>
<td>9</td>
<td>-2.7831</td>
<td>0.0282</td>
<td>0.4121 -0.9298</td>
<td>-0.8760 -2.3023</td>
<td>R1 D</td>
</tr>
<tr>
<td>10</td>
<td>-1.7416</td>
<td>0.2269</td>
<td>0.2494 0.3946</td>
<td>-0.5558 -0.6238</td>
<td>R1 R2</td>
</tr>
</tbody>
</table>

Table 5.5: Mapping scenarios to new ratings.

Remark 5.3. If we compare Table 5.3 and 5.5, we see that shifting the mean indeed leads to more downgrades and defaults. We can do this comparison, because the realization of the market factor in the Importance Sampling scheme in this example is exactly the same as the market factor from the Monte Carlo simulation plus a shift of -2.
5.3 Importance Sampling

5.3.2 Determine Losses

In this section we only state the losses in each scenario. Notice that indeed the portfolio losses are larger than in Table 5.4.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Likelihood Ratio</th>
<th>New Rating Obligor 1</th>
<th>Obligor 2</th>
<th>Obligor Loss Obligor 1</th>
<th>Obligor 2</th>
<th>Portfolio Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0409</td>
<td>R1</td>
<td>D</td>
<td>-0.0455</td>
<td>2.5558</td>
<td>2.5103</td>
</tr>
<tr>
<td>2</td>
<td>0.0031</td>
<td>D</td>
<td>D</td>
<td>1.5024</td>
<td>2.5558</td>
<td>4.0582</td>
</tr>
<tr>
<td>3</td>
<td>0.0319</td>
<td>R1</td>
<td>D</td>
<td>-0.0455</td>
<td>2.5558</td>
<td>2.5103</td>
</tr>
<tr>
<td>4</td>
<td>0.0122</td>
<td>R1</td>
<td>D</td>
<td>-0.0455</td>
<td>2.5558</td>
<td>2.5103</td>
</tr>
<tr>
<td>5</td>
<td>0.0348</td>
<td>D</td>
<td>R3</td>
<td>1.5024</td>
<td>0.6486</td>
<td>2.1510</td>
</tr>
<tr>
<td>6</td>
<td>0.0556</td>
<td>R1</td>
<td>D</td>
<td>-0.0455</td>
<td>2.5558</td>
<td>2.5103</td>
</tr>
<tr>
<td>7</td>
<td>0.0100</td>
<td>R1</td>
<td>D</td>
<td>-0.0455</td>
<td>2.5558</td>
<td>2.5103</td>
</tr>
<tr>
<td>8</td>
<td>17.672</td>
<td>R1</td>
<td>R2</td>
<td>-0.0455</td>
<td>0.0079</td>
<td>-0.0376</td>
</tr>
<tr>
<td>9</td>
<td>0.0282</td>
<td>R1</td>
<td>D</td>
<td>-0.0455</td>
<td>2.5558</td>
<td>2.5103</td>
</tr>
<tr>
<td>10</td>
<td>0.2269</td>
<td>R1</td>
<td>R2</td>
<td>-0.0455</td>
<td>0.0079</td>
<td>-0.0376</td>
</tr>
</tbody>
</table>

Table 5.6: Valuation of the portfolio.

5.3.3 The Loss Distribution and Risk and Performance Measures

Given a number of scenarios $N_{sc}$, the loss distribution is given by the portfolio losses sorted in descending order $L(1) \geq L(2) \geq \ldots \geq L(N_{sc})$ with their corresponding likelihood ratios $l(1), \ldots, l(N_{sc})$. The cumulative density function is then given by:

$$
P[L \leq x] = \frac{1}{N_{sc}} \sum_{j=1}^{N_{sc}} l(j) 1_{\{L(j) \leq x\}}
$$

(5.14)

Expected Loss

The expected loss is again calculated analytically, as explained in Appendix E.

Value-at-Risk

We used Importance Sampling in order to generate more scenarios at the tail of the loss probability. Therefore, the $VaR_{\alpha}$-level can be determined more accurately (this will be shown in the next section). From the cumulative density function it follows that the associated percentile is given by the cumulative likelihood ratio $\bar{l}(s)$ divided by the total number of scenarios. Consequently,

$$
VaR_{\alpha} = \min_s \{L(s) \bar{l}(s) \leq (1 - \alpha) \cdot N_{sc}\}.
$$

(5.15)

We also want to determine the risk contributions of each client in the portfolio. The determination of the risk contributions is similar to the Monte Carlo simulation, except that we now take the weighted average of the obligor loss over a certain range:

$$
VaR_{i,\alpha} = \frac{\sum_{j=k_1}^{k_2} L(j)l(j)}{\sum_{j=k_1}^{k_2} l(j)},
$$

(5.16)

where $k_1$ and $k_2$ are again defined as in the previous subsection.

Economic Capital, Economic Value Added and Risk Adjusted Return on Capital

Analogue to Subsection 5.2.3 we are now able to determine $EC$, $EC_i$, $EVA$, $EVA_i$, $RAROC$ and $RAROC_i$. 
5.4 Validation

In this section we will validate both simulation methods. Firstly, we compare the simulated results with the analytical results. For this purpose we use the same portfolio as in the concluding example of the previous chapter. Furthermore, we will also show that the resulting risk measures of both methods converge and that the Importance Sampling scheme is both faster and more accurate for determining $\text{VaR}_\alpha$. Finally, we will show the influence of the shift parameter on the estimation of Value-at-Risk. For this, we construct a new portfolio, in which more obligors are comprised.

5.4.1 A Stylized 2-obligor Portfolio Example (Revisited)

In this subsection we use the same portfolio as in Section 4.5. We will show that both simulation methods give similar results.

Example 5.1. Recall that we have the following portfolio and migration matrix:

<table>
<thead>
<tr>
<th>Obligor</th>
<th>FAD</th>
<th>LGD</th>
<th>Maturity</th>
<th>Correlation</th>
<th>Rating</th>
<th>Interest Rate</th>
<th>Risk-free rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>0.2</td>
<td>8</td>
<td>0.2</td>
<td>R1</td>
<td>0.08</td>
<td>0.08</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>0.3</td>
<td>4</td>
<td>0.3</td>
<td>R2</td>
<td>0.05</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Figure 5.1: The portfolio.

Figure 5.2: The 1-year migration matrix.

Since Importance Sampling only focusses on the right tail of the loss distribution, only the Monte Carlo scheme can produce a full loss distribution. In order to obtain this distribution, we use $5 \cdot 10^4$ scenarios. This results in the following loss distribution:

Figure 5.3: The simulated loss distribution.
Note that this distribution is almost similar to the analytical one. If we take \( \mu = 2 \), \( N_{SC} = 2500 \) and \( \beta = 0.02 \) in the Importance Sampling scheme, we get the following risk measures, given that \( c = 2 \):

<table>
<thead>
<tr>
<th>Measures</th>
<th>Analytical</th>
<th>Monte Carlo</th>
<th>Importance Sampling</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Portfolio</td>
<td>Obli 1</td>
<td>Obli 2</td>
</tr>
<tr>
<td>( EL )</td>
<td>0.2856</td>
<td>0.0300</td>
<td>0.2556</td>
</tr>
<tr>
<td>( EL^{AN} )</td>
<td>0.2856</td>
<td>0.0300</td>
<td>0.2556</td>
</tr>
<tr>
<td>( VaR_{0.99} )</td>
<td>2.9989</td>
<td>0.4431</td>
<td>2.5558</td>
</tr>
<tr>
<td>( EC_{0.99} )</td>
<td>2.7133</td>
<td>0.4131</td>
<td>2.3002</td>
</tr>
<tr>
<td>( EV_{A_{0.99}} )</td>
<td>0.4632</td>
<td>0.2627</td>
<td>0.2005</td>
</tr>
<tr>
<td>( RAROC_{0.99} ) (%)</td>
<td>18.96</td>
<td>65.36</td>
<td>10.63</td>
</tr>
</tbody>
</table>

*Here, both \( EC \) and \( EV_A \) are based on the simulated \( EL \).

*Here, both \( EC \) and \( EV_A \) are based on the analytical \( EL^{AN} \).

Table 5.7: Risk and performance measures for the different methods (2-obligor portfolio).

From the example we can conclude that both simulation methods perform well for small portfolios. All risk measures are relatively close to each other. In this case, we can attribute this to the fact that we used \( 5 \cdot 10^5 \) scenarios, while the state of the world with the smallest joint probability occurs once in \( 5 \cdot 10^3 \) times (see Table 4.8). Therefore, on average, each state of the world should occur approximately 100 times in the Monte Carlo simulation.

### 5.4.2 A Stylized n-obligor Portfolio Example

In this subsection, we show that both methods are accurate. Furthermore we will compare both methods and finally we will elaborate more on the optimal shift in Importance Sampling. For these purposes, we use the following example:

**Example 5.2.** Assume that we have the following portfolio and migration matrix:

<table>
<thead>
<tr>
<th>Oblige</th>
<th>EAD</th>
<th>LCD</th>
<th>Maturity</th>
<th>Correlation</th>
<th>Rating</th>
<th>Interest Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.48</td>
<td>4</td>
<td>0.28</td>
<td>4</td>
<td>1</td>
<td>0.125</td>
</tr>
<tr>
<td>2</td>
<td>0.58</td>
<td>9</td>
<td>0.23</td>
<td>2</td>
<td>1</td>
<td>0.056</td>
</tr>
<tr>
<td>3</td>
<td>0.56</td>
<td>5</td>
<td>0.25</td>
<td>6</td>
<td>1</td>
<td>0.211</td>
</tr>
<tr>
<td>4</td>
<td>0.59</td>
<td>6</td>
<td>0.24</td>
<td>4</td>
<td>1</td>
<td>0.396</td>
</tr>
<tr>
<td>5</td>
<td>0.87</td>
<td>7</td>
<td>0.45</td>
<td>5</td>
<td>1</td>
<td>0.119</td>
</tr>
<tr>
<td>6</td>
<td>0.38</td>
<td>2</td>
<td>0.94</td>
<td>6</td>
<td>1</td>
<td>0.094</td>
</tr>
<tr>
<td>7</td>
<td>0.70</td>
<td>3</td>
<td>0.39</td>
<td>4</td>
<td>1</td>
<td>0.036</td>
</tr>
<tr>
<td>8</td>
<td>0.66</td>
<td>2</td>
<td>0.49</td>
<td>2</td>
<td>1</td>
<td>0.075</td>
</tr>
<tr>
<td>9</td>
<td>0.70</td>
<td>1</td>
<td>0.30</td>
<td>0</td>
<td>1</td>
<td>0.035</td>
</tr>
<tr>
<td>10</td>
<td>0.70</td>
<td>5</td>
<td>0.45</td>
<td>4</td>
<td>1</td>
<td>0.175</td>
</tr>
<tr>
<td>11</td>
<td>0.39</td>
<td>3</td>
<td>0.30</td>
<td>6</td>
<td>1</td>
<td>0.021</td>
</tr>
<tr>
<td>12</td>
<td>0.71</td>
<td>3</td>
<td>0.43</td>
<td>4</td>
<td>1</td>
<td>0.135</td>
</tr>
<tr>
<td>13</td>
<td>0.27</td>
<td>6</td>
<td>0.27</td>
<td>1</td>
<td>1</td>
<td>0.106</td>
</tr>
<tr>
<td>14</td>
<td>0.62</td>
<td>4</td>
<td>0.24</td>
<td>6</td>
<td>1</td>
<td>0.063</td>
</tr>
<tr>
<td>15</td>
<td>0.47</td>
<td>4</td>
<td>0.16</td>
<td>1</td>
<td>1</td>
<td>0.028</td>
</tr>
<tr>
<td>16</td>
<td>0.75</td>
<td>7</td>
<td>0.47</td>
<td>2</td>
<td>1</td>
<td>0.065</td>
</tr>
<tr>
<td>17</td>
<td>0.27</td>
<td>9</td>
<td>0.36</td>
<td>3</td>
<td>1</td>
<td>0.121</td>
</tr>
</tbody>
</table>

**Figure 5.4:** The portfolio. **Figure 5.5:** The 1-year migration matrix.
If we do Monte Carlo Simulation with $2.5 \cdot 10^6$ scenarios we obtain the following loss distribution:

![Figure 5.6: The simulated loss distribution.](image)

If we also do Importance Sampling with a shift of 2 and $10^5$ scenarios and take $\beta = 0.02$, $\lambda = 0.4$ and $\gamma = 0.035$, we obtain the following loss risk measures:

<table>
<thead>
<tr>
<th>Measures</th>
<th>Monte Carlo</th>
<th>Importance Sampling</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Portfolio</td>
<td>Obligor 7</td>
</tr>
<tr>
<td>$EL$</td>
<td>5.5141</td>
<td>0.9932</td>
</tr>
<tr>
<td>$EL^{AN}$</td>
<td>5.5174</td>
<td>0.9933</td>
</tr>
<tr>
<td>$VaR_{0.99}$</td>
<td>22.2930</td>
<td>1.5279</td>
</tr>
<tr>
<td>$EC_{0.99}$</td>
<td>16.7789</td>
<td>1.4347</td>
</tr>
<tr>
<td>$EVA_{0.99}$</td>
<td>2.6683</td>
<td>-0.5899</td>
</tr>
<tr>
<td>$RAROC_{0.99}$ (%)</td>
<td>17.90</td>
<td>-39.12</td>
</tr>
</tbody>
</table>

Table 5.8: Risk and performance measures for the different methods (18-obligor portfolio).

From this table we conclude that both methods give similar results. However, this is mainly because of the large number of simulations and a good choice of the shift parameter $\mu$. Therefore, we conclude this section with a discussion on the number of simulations, the choice of $\mu$ and the choice of $\beta$. For this purpose, we use the same portfolio and migration matrix as in this subsection and the same shift unless stated otherwise.

**Convergence of the Methods**

From Table 5.8, we can conclude that given the number of scenarios for both methods, the results are comparable. Here we will show that indeed the risk measures generated by both methods converge as the number of scenarios increases. Firstly, we show this for the expected loss $EL$ in the Monte Carlo simulation.
5.4 Validation

We see that $EL$ (the mean over all 50 simulations) is approximately equal to 5.5174, which is the same as $EL^{AN}$. If we look at $N_{SC} = 170 \cdot 10^3$ it seems that $EL$ is more accurate than at $N_{SC} = 200 \cdot 10^3$. However, if we determine the standard deviation of $EL$ in the 50 simulations, we see that it decreases as the number of scenarios becomes larger. Below, we show that the $VaR_{\alpha}$ converge in both simulation methods. From both figures and other examples we studied, we conclude that $VaR_{\alpha}$ converges as $N$ increases. Subsequently, we conclude that if $\alpha$ increases, the corresponding Value-at-Risk level becomes converges more slowly. Finally, we conclude that all $VaR_{\alpha}$-values converge faster in Importance Sampling. For example, if we look at $VaR_{0.9999}$ we see that the graph at $2 \cdot 10^4$ scenarios in Importance Sampling is much more narrow than the graph at $2 \cdot 10^5$ scenarios in the Monte Carlo simulation.

Comparison of both Methods

In order to make a fair comparison between Monte Carlo and Importance Sampling, we perform 50 simulations with $2 \cdot 10^4$ scenarios for both simulation methods. Plotting the $VaR$-levels against


$\alpha$ results in

*Figure 5.10: Comparison between Monte Carlo and Importance Sampling.*

We see that indeed, Importance Sampling is more efficient than Monte Carlo. For all $VaR$ levels, Importance Sampling outperforms the Monte Carlo simulation.

**Choosing $\mu$**

As mentioned before, in Importance Sampling the mean of the market factor is shifted. Until now, we assumed a shift of 2, which was a somewhat arbitrary choice. We choose a shift of $\mu = 2$, because if the shift is too small, there are still too less scenarios around the $VaR$-level (most of the scenarios will be below $VaR_\alpha$) and if the shift is too large most scenarios will be above $VaR_\alpha$. Thus, there exists an optimal value for the shift, which can be determined heuristically and depends on the portfolio and $\alpha$. If we plot the $VaR$-levels against the shift $\mu$, we are able to determine the optimal shift.

*Figure 5.11: The Optimal Choice of $\mu$.***
Looking at Figure 5.11, we see that for each $\alpha$, the corresponding VaR-levels first become more accurate and after some value of $\mu$ gradually become less accurate. We conclude that a shift of 2 (as it lies between the lines indicating the region in which optimal lies for all $\alpha$) was indeed not a bad choice. Furthermore, we conclude that the optimal shift increases as $\alpha$ increases. This fits intuition because if $\alpha$ increases, the area of interest lies further in the tail.

**Remark 5.4.** For the default/non-default case (i.e. where migration is ignored), Glasserman [3] derives an expression for the optimal choice of $\mu$. For the migration model no explicit formula is known. Therefore, we chose to investigate the influence of $\mu$ from empirical data.

**Choosing $\beta$**

In order to determine risk contributions ($VaR_{i,\alpha}$) we need to construct an interval around the portfolio $VaR_{\alpha}$. Recall that the width of this interval equals $2\beta VaR_{\alpha}$. The choice of $\beta$ is arbitrary. Choosing $\beta$ too small leads to an unbiased but imprecise estimation of $VaR_{i,\alpha}$, while choosing $\beta$ too large leads to a biased but precise estimator. Therefore, we need to find a $\beta$ which is as unbiased but as precise as possible. Here, we will only focus on choosing a good $\beta$ in Importance Sampling, since we have shown it is superior to Monte Carlo. Plotting $VaR_{i,\alpha}$ for obligors 7 and 14 against $\beta$ gives the figures below. Looking at Figure 5.12 we see that for both obligors, the risk contributions converge. Comparing the asymptotical value with Table 5.8 we see that the risk contributions converge to the obligor’s expected loss. This fits intuition because as $\beta$ goes to infinity, the risk contribution is determined by taking the average of the whole loss distribution.

![Figure 5.12: The Optimal Choice of $\beta$.](image)

In Figures 5.13 and 5.14 we zoomed in on Figure 5.12. Indeed we see that if $\beta$ is small, the estimation of the risk contribution is very imprecise and that it stabilizes around $\beta = 0.04$ for obligor 7 and around $\beta = 0.05$ for obligor 14. Therefore, we conclude that $\beta = 0.05$ is the optimal choice for this portfolio, given that we have $2 \cdot 10^4$ scenarios.
Figure 5.13: The Optimal Choice of $\beta$ for Obligor 7.

Figure 5.14: The Optimal Choice of $\beta$ for Obligor 14.
Chapter 6
Comparative Study

In the previous chapters, we presented a framework for analyzing credit portfolios. We are now able to investigate the influence of some parameters on the loss distribution and risk measures. For this purpose, we use the following portfolio and migration matrix\(^1\):

### Table: Portfolio

<table>
<thead>
<tr>
<th>Obligor</th>
<th>EAD (%)</th>
<th>GDP</th>
<th>Maturity</th>
<th>Correlation</th>
<th>Rating</th>
<th>Interest Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>14</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>16</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>17</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
<tr>
<td>18</td>
<td>5</td>
<td>5.2</td>
<td>0.7</td>
<td>0.2</td>
<td>BBB</td>
<td>0.060</td>
</tr>
</tbody>
</table>

Furthermore, we assume that all other parameters are the same as in the previous examples. Firstly, we investigate the effects on the loss risk measures if we move from a default/non-default model to a Mark-to-Market model. Secondly, we investigate the effect of name concentration (relatively high EAD for some client) on \(EVA\) and \(RAROC\). Subsequently, we investigate the influence of respectively the rating, the loss given default, the correlation and the maturity on \(EVA\) and \(RAROC\).

### 6.1 Default/Non Default vs. Mark-to-Market

In this subsection we investigate the effects of moving from a default/non-default model to a Mark-to-Market model. For this purpose, we conduct an analysis if all obligors from the portfolio (see Figure 6.1) have rating AAA (a good portfolio) and if all obligors from the portfolio have rating CCC/C (a bad portfolio). In order to determine the loss risk measures in the default/non-default model, we only need to make a small adjustment in the migration matrix:

\[^1\]This matrix is taken from Standard and Poors’s Corporate Default Database CreditPro 7.50
Note that the default probabilities have not changed. Importance Sampling, with $2.5 \cdot 10^5$ scenarios, leads to the following results:

<table>
<thead>
<tr>
<th>Measures</th>
<th>Good Portfolio</th>
<th>Bad Portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>D/ND MtM Ratio</td>
<td>D/ND MtM Ratio</td>
</tr>
<tr>
<td>$EL$</td>
<td>0.0074</td>
<td>3.0897</td>
</tr>
<tr>
<td>$VaR_{0.9999}$</td>
<td>4.1123</td>
<td>37.4090</td>
</tr>
<tr>
<td>$EC_{0.9999}$</td>
<td>4.7799</td>
<td>34.3192</td>
</tr>
<tr>
<td>$EV_A_{0.9999}$</td>
<td>131.37</td>
<td>6.74</td>
</tr>
<tr>
<td>$RAROC_{0.9999}$ (%)</td>
<td>1.0356</td>
<td>1.0356</td>
</tr>
</tbody>
</table>

Table 6.1: Default/non-default vs. Mark-to-Market.

To explain the results, we first rewrite (4.19) as follows:

$$EL = \frac{\mathbb{E}[MtM^{ND,H} - MtM^H]}{1 + r_f} = \frac{\sum_{i=1}^{n} (\mathbb{E}[MtM_{i}^{ND,H}] - \mathbb{E}[MtM_{i}^{H}])}{1 + r_f} = \frac{\sum_{i=1}^{n} \left( \mathbb{E}[MtM_{i}^{ND,H}] - \left( p_{i,K} \mathbb{E}[MtM_{i}^{D,H}] + (1 - p_{i,K})[MtM_{i}^{ND,H}] \right) \right)}{1 + r_f} = \frac{\sum_{i=1}^{n} p_{i,K} \left( \mathbb{E}[MtM_{i}^{ND,H}] - \mathbb{E}[MtM_{i}^{D,H}] \right)}{1 + r_f}$$

Note that $p_{i,K}$ and $\mathbb{E}[MtM^{D,H}]$ are the same for both the default/non-default model and the Mark-to-Market model. Therefore, if we have a good portfolio (where all obligors have a high rating) and move to the Mark-to-Market model, $\mathbb{E}[MtM^{ND,H}]$ decreases for all $i$ because of downgrades. On the other hand, if we have a bad portfolio and move to the Mark-to-Market model, $\mathbb{E}[MtM^{ND,H}]$ increases for all $i$ because of upgrades. Therefore, if we move to a Mark-to-Market model, the expected loss should decrease in case of a good portfolio and increase in case of a bad portfolio, which is exactly what we see in Table 6.1.

Recall that the value-at-risk is given by:

$$VaR_\alpha = \frac{\mathbb{E}[MtM^{ND,H}] - MtM^H}{1 + r_f}.$$
6.2 The Influence of Name Concentration

One source of credit risk is name concentration, when a portfolio is not very well diversified, i.e. when an obligor has a large \(EAD\). A one obligor portfolio is more risky than a four-obligor portfolio with a total \(EAD\) equal to the total \(EAD\) of the one-obligor portfolio. Assuming that all other parameters are the same for all five obligors, it’s easy to see that the one-obligor portfolio is more risky: suppose that the obligor from the one-obligor portfolio defaults then the loss equals the loss of the four-obligor portfolio if and only if all four obligors default. Because the characteristics of the obligors in both portfolios are the same, the probability that all four obligors default is much smaller than the probability that the obligor in the one-obligor portfolio defaults. Therefore, large losses occur with higher probabilities in the one-obligor portfolio and are thus more risky.

In this section we investigate the influence of name concentration on both \(EVA\) and \(RAROC\). For this purpose, we increase \(EAD_1\). This results in

![Figure 6.4: Name concentration versus \(EVA\).](image1)

![Figure 6.5: Name concentration versus \(RAROC\).](image2)

From these figures we conclude that both risk adjusted return parameters show an optimum and decrease after that optimum. Recall that both \(EVA\) and \(RAROC\) depend on the total margin, \(EL\) and \(EC\). The optimum can be explained by the fact that due to an increase of the \(EAD\) the total margin minus the expected loss increases faster than the economic capital and even faster than the cost of economic capital. After the optimum, both \(EVA\) and \(RAROC\) decrease, because of the fast increase of \(EC\). This can be explained by the fact that the total margin minus the expected loss increases linearly, while \(EC\) increases more than linear.

6.3 The Influence of the Ratings

An other source of credit risk is the rating of an obligor. It is evident that a lower rating implies more risk, given that all other parameters remain the same. Here, we investigate the influence of the rating on both \(EVA\) and \(RAROC\). For this purpose we use the portfolio from Figure 6.1 and decrease the rating of obligor 1, resulting in:
In the figures above, we see a decrease of both $EVA$ and $RAROC$, which can be attributed to the increase of both $EC$ and $EL$. The numerator in the expression for $RAROC$ decreases while the denominator increases, which results in a decrease of $RAROC$. In the expression for $EVA$, the interest does not compensate for the increase in credit risk and thus $EVA$ decreases.

### 6.4 The Influence of the $LGD$

A third source of risk is the loss given default. The higher the $LGD$, the higher the loss due to a default. Therefore, we examine the effect of the $LGD$ on the risk adjusted return parameters. We keep all parameters for all obligors constant, except for $LGD_1$. We now get the following relation between $LGD_1$ and $RAROC$ and $EVA$:

Again, we see a decrease of both $EVA$ and $RAROC$. This is explained by the same reasoning as in Section 6.4.
6.5 The Influence of the Correlation

A fifth source of credit risk is the correlation of the obligors with the market \( R^2 \). Note that the correlation only affects \( EC \) (and not \( EL \)). More specific, if an obligor has a large correlation, \( \text{VaR}_\alpha \) increases. This is because of the influence of the market factor \( Y \) on the loss distribution. Most of the tail scenarios of the loss distribution happen when there are bad market conditions. Due to the large correlation, the corresponding obligor now has an even higher probability to downgrade or default and thus a larger probability of a larger loss. Therefore, there is a higher probability of a larger portfolio loss and thus \( \text{VaR}_\alpha \) increases. As a consequence, \( EC \) increases.

Next, we show the relation between \( R^2 \) and the risk adjusted return parameters:

Figure 6.10: Correlation versus \( \text{EVA} \).

Figure 6.11: Correlation versus \( \text{RAROC} \).

Indeed we see that the return parameters decrease, which is caused by the increase of \( EC \).

6.6 The Influence of Maturity

The last source of credit risk we discuss is the maturity of a loan. A loan with a higher maturity is more risky than a loan with a lower maturity, because there are more default possibilities for the obligor (in year 1, 2, \ldots, \( M \)). Therefore we increase the maturity of obligor 1, so that we can explore the influence on \( \text{EVA} \) and \( \text{RAROC} \).

In the figures below, we see that \( \text{EVA} \) and \( \text{RAROC} \) are descending in \( M_1 \). The explanation is analogue to Sections 6.3 and 6.4.
6.7 Conclusion

In the previous sections, we investigated the influence of a number of sources of credit risk. We conclude that all sources eventually have a negative impact on the risk adjusted return parameters $EVA$ and $RAROC$. We also conclude that there are three main sources of risk, which have the most negative effect on $EVA$ and $RAROC$:

- Name Concentration
- Rating
- Correlation
Chapter 7

Conclusion and Recommendations

7.1 Conclusion

In this thesis we have extended Rabobank’s current in-house credit portfolio loss model. The two most important vendor models (CreditMetrics and KMV Portfolio Manager) formed the basis for this. Rabobank is now able to determine the loss distribution on a Mark-to-Market basis, taking migration, maturities and cash flows into account. Furthermore, Rabobank is now able to investigate portfolios not only on an aggregated level, but also on a client level.

We showed that Merton’s default model, and Merton’s migration model (which is an extension of the default model) forms the backbone of the new credit portfolio loss model.

The main goal of the credit portfolio loss model is to achieve the loss distribution of a portfolio and to derive several useful risk measures from that distribution. The main steps in determining the loss distribution are:

1. The rating of an obligor determines the probability of either defaulting or migrating to some new rating at the horizon.
2. The value of a loan at horizon (the Mark-to-Market value) is obtained via risk-free discounting of the future cash flows, which depends on the rating at horizon. The loss of an obligor is then determined by the discounted expected Mark-to-Market value given non default minus the realized discounted Mark-to-Market value.
3. The probabilities from Step 1 and the losses from Step 2, should then be combined in our calculation of the loss distribution and its corresponding risk measures.

It is possible to determine the loss distribution analytically. However, in practice portfolios are large. Consequently, the analytical determination is no longer suitable and we have to rely on simulation methods.

If we have determined the loss distributions we are able to determine the following (new) risk measures both on portfolio level and on obligor level:

- Expected Loss
- Value-at-Risk
- Economic Capital
- Economic Value Added (new w.r.t. the current loss model)
- Risk Adjusted Return on Capital (new w.r.t. the current loss model)

7.2 Recommendations

We conclude this thesis with some recommendations for future research.
7.2.1 Multi-Factor Model

One of the main assumptions in this thesis was the one-factor representation of the standardized asset returns. Recall that this representation looks as follows:

\[ X_i = Y \sqrt{R^2_i} + Z_i \sqrt{1 - R^2_i}. \]

This model assumes that obligors have only one factor in common. However, there can be more common factors, such as industry or region. Therefore, in future research, the model can be expanded to a multi-factor model in which the standardized asset returns look as follows:

\[ X_i = Y_1 \sqrt{R^2_{1,i}} + \ldots + Y_{n_f} \sqrt{R^2_{n_f,i}} + Z_i \sqrt{1 - \tilde{R}^2_i}, \]

where

\[ \tilde{R}^2_i = \sum_{j=1}^{n_f} R^2_{j,i}. \]

There arise some difficulties when we move to a multi-factor model, e.g. in Importance Sampling. In the one-factor model, we just shifted the mean of the only common factor in order to increase the portfolio losses. In a multi-factor model, different combinations of common factors can result in large losses. Therefore, mean shifting is not an evident step in Importance Sampling. In [5], a method is given for Importance Sampling in a multi-factor default/non-default model.

7.2.2 Portfolio Optimization

Another interesting topic for future research is to develop a methodology for optimizing a credit portfolio. Since most probably this will be done within Rabobank, we will elaborate more on this topic.

Rabobank is also interested in optimizing the return of a portfolio, in particular in optimizing the risk-adjusted return Economic Value Added (EVA) or Risk Adjusted Return on Capital (RAROC). More specific, Rabobank is interested in solving the following optimization problems:

\[
\begin{align*}
\text{max} & \quad \text{EVA}, \\
\text{s.t.} & \quad l_i \leq EAD_i \leq u_i.
\end{align*}
\]

(7.1)

and

\[
\begin{align*}
\text{max} & \quad \text{RAROC}, \\
\text{s.t.} & \quad l_i \leq EAD_i \leq u_i.
\end{align*}
\]

(7.2)

First of all, it is clear that the problem is static, stochastic and large. Next, we present a theorem, which proves that the optimization problem is also non-convex.

**Theorem 7.1.** EVA and RAROC are not convex.

**Proof.** Both EVA and RAROC depend on the economic capital, which is a function of EL and VaR\(\alpha\). We will show that the latter is not convex and thus that both EVA and RAROC are not convex. Firstly, we will show that VaR\(\alpha\) is not subadditive. A function \(f\) is subadditive if for all \(x\) and \(y\) the following holds:

\[ f(x + y) \leq f(x) + f(y). \]

Suppose that portfolio \(X\) consists of one obligor. If he defaults after one year the loss equals 1 and if he does not default, the loss equals 0. Furthermore assume that the probability of default equals 0.005. Then it is easy to see that \(\text{VaR}_{0.05} = 0\). Now, let portfolio \(Y\) be the same portfolio
7.2 Recommendations

as portfolio $X$, then also $VaR_{0.95}^Y = 0$. If we combine portfolio $X$ and $Y$ to a portfolio $Z$ and assume that the default probabilities are independent, then $VaR_{0.95}^Z = 1$. So

$$f(x + y) = VaR_{0.95}^Z > VaR_{0.95}^X + VaR_{0.95}^Y = f(x) + f(y).$$

Therefore, $VaR_\alpha$ is not subadditive. It is clear that $VaR_\alpha$ is positive homogeneous:

$$VaR_{0.95}^{2X} = 2VaR_{0.95}^X.$$ 

Next we show that $VaR_\alpha$ cannot be convex. A function $f$ is convex if for all $x$ and $y$ and for all $l \in [0,1]$ the following holds:

$$f(lx + (1-l)y) \leq lf(x) + (1-l)f(y).$$

From this it follows that a function is convex if

$$f(x/2 + y/2) \leq f(x)/2 + f(y)/2. \tag{7.3}$$

Combining the positive homogeneity and the lack of subadditivity, we arrive at the conclusion that $VaR_\alpha$ is not convex. Therefore, $EVA$ and $RAROC$ cannot be convex.

Theorem 7.1 shows that both optimization problems are very hard to solve, because of the non-convex objective function.

7.2.3 Credit Default Swaps

When a bank has a certain risky obligor in its portfolio, it is possible that the bank sells this risk to a third party (e.g. an other bank). Such a construction is called a credit default swap (CDS). Now, there are four possible scenarios:

- The obligor defaults but the other bank does not. In this case, the other bank pays back an amount and the CDS stops.
- The obligor defaults and so does the other bank. In this case the bank receives only the loss given default of the obligor and maybe some other amount from the other bank.
- The obligor does not default but the other bank does. In this case the CDS stops and with that also the payments to the other bank.
- The obligor does not default and neither does the other bank. The bank receives interest and maybe repayments from the obligor, but has to pay the other bank.

It is possible to incorporate CDS’s into the model presented in this thesis. Therefore, a valuation tree needs to be constructed in which the above four steps are incorporated.

7.2.4 Other topics

There are many more topics that are interesting for future research. Without going too much into detail we would like to propose the following topics:

- Retail Obligors
  Rabobank’s total portfolio consists of corporate clients as well as millions of retail clients. Because the retail part of the portfolio is well diversified, fast approximations (see e.g. [7]) can be used to determine the risk contributions of those clients.
- Parent-Child structures
- Other products
  In this thesis we only considered bullet loans. This is the most simple type of loan. It is also possible to incorporate other types of loans, such as credit lines and term loans. The latter is discussed in Appendix C.
• Fee Structure
   In this thesis we did not consider fees. There are several fees, an obligor has to pay, such as:
   – Upfront Fee
     A fee a client has to pay when he starts a loan contract with the bank.
   – Commitment Fee
     A client has to pay a fee for the amount of commitment.

We see that there are a lot of extensions possible to the model presented in this thesis. By incorporating the suggestions in this sections, the credit portfolio loss model can handle more financial products and resembles reality even more close.
Appendix A

Project Justification

During my internship at Rabobank, I also participated in another project titled “Methodology of mapping external to internal ratings. Application to S&P-data (1981-2006)”. First, I will give a short summary of this project. Subsequently, I will describe my role in this project.

A.1 Summary

For (corporate) portfolios for which insufficient number of obligors are available but sufficient external ratings do exist, Rabobank resorts to the so-called 'Shadow-bond' methodology to assess internal ratings. A major step in the development of a shadow-bond rating is the mapping of the external rating to the internal ratings based on the estimated default probabilities of the external ratings.

The purpose of this study is to develop a methodology for mapping external ratings to internal ratings and for estimating the underlying default probabilities of external ratings. In this pilot, the methodology will be applied to S&P’s rating data (CreditPro 7.5).

The request for an update of methodology, documentation and mapping has been prompted by Basel-II regulations that require that the documentation and methodology should be based on recent data, transparent and avoiding any bias and inconsistencies (see par 449). In particular, De Nederlansche Bank (DNB) states that ‘The new shadow-bond methodology should incorporate recent external default experience, at the start and on an ongoing basis’. DNB expects ‘Rabobank to incorporate new information as it becomes available into probability of default (PD) or loss given default (LGD) estimates on an annual basis, especially in the period during which the data requirements are building up’. See Appendix 7 for more detailed DNB comments.

The new methodology and mapping differ from the previous ones as they are based on the default history 1981 - 2006 and have been tested for:

1. differences in default definition between Basel-II and S&P;
2. the optimum unbiased default history;
3. precision (weighted or unweighted estimate);
4. the impact of removing non-rated obligors (at rating date);
5. the presence of regional or industrial bias.

A.2 Role

In this section I will describe my role in the project. First of all, I was involved in the development of steps 2 to 5 in the previous section. Together with Mâcé Mesters, I developed a framework for
performing the tests in these steps.

I wrote a program in Excel which reads the database and returns tables with the number of obligors and number of defaults for each external rating. From this we are able to calculate default frequencies and asset correlations. Furthermore, I implemented all the tests in Matlab and provided tables and figures for the report (which was written by Mâcé).

Besides the development and implementation, I was also involved in defending the methodology for Rabobank’s validation team. This team has to approve our document, before it is implemented in the organization.
Appendix B

Analytical Derivation of $EL$

The expected loss can be determined either from simulation or analytically. In this appendix, we show how the analytical expected loss ($EL_{AN}$) is determined. It is well known that the expected value of the sum of random variables is equal to the sum of the expected values of each of the random variables. This is even true when the variables are correlated. Therefore, we first derive an expression for the expected loss of an obligor.

B.1 The Component $EL$

Suppose that the loss matrix and the risk neutral migration matrix (as described in Sections 4.1.2 and 4.2.6) are known. Then, these vectors of length $K$ look like:

$$v_L^i = \begin{pmatrix} l_{i,1} \\ l_{i,2} \\ \vdots \\ l_{i,K-1} \\ l_{i,K} \end{pmatrix}, \quad v_{RNM}^i = \begin{pmatrix} \tilde{p}_{i,1} \\ \tilde{p}_{i,2} \\ \vdots \\ \tilde{p}_{i,K-1} \\ \tilde{p}_{i,K} \end{pmatrix},$$

where $v_L^i$ is the loss matrix and $v_{RNM}^i$ is the risk neutral migration matrix for obligor $i$. Then the expected loss for obligor $i$ simply equals the inner product of the two vectors:

$$EL_{AN}^i = v_{RNM}^i \cdot v_{loss}^i = \sum_{j=1}^{K} \tilde{p}_{i,j} l_{i,j}. \quad (B.1)$$

B.2 The Portfolio $EL$

The portfolio expected loss is by definition the sum of the obligor expected losses. Therefore, if there are $n$ obligors, the portfolio expected loss is given by:

$$EL_{AN} = \sum_{i=1}^{n} EL_{AN}^i = \sum_{i=1}^{n} \sum_{j=1}^{K} \tilde{p}_{i,j} l_{i,j}. \quad (B.2)$$
Analytical Derivation of $EL$
Appendix C

Valuation for a Term Loan

In this thesis we only considered a bullet loan, i.e., a loan without repayments. Here, we will show the valuation in the case there are repayments so in the case of a term loan. Again there are three possibilities:

1. The obligor defaulted before the horizon.
2. The loan reached its maturity at the horizon.
3. The obligor did not default before the horizon and the loan did not reach its maturity.

In the first case the MtM-value equals the recovery rate times the exposure at default:

\[ MtM_{i,K}^H = (1 - LGD_i)EAD_i. \]  \hspace{1cm} (C.1)

If the obligor reached its maturity at the horizon, he has to pay back the loan plus interest over one year. Therefore, for all \( j \leq K - 1 \) the Mark-to-Market value equals

\[ MtM_{i,j}^H = (1 + r_i)EAD_i, \]  \hspace{1cm} (C.2)

where \( r_i \) is the interest rate. The third case is somewhat more difficult. In the first year the obligor had to pay interest over his loan. Because the obligor did not default and did not reach its maturity, there are future cash flows. Therefore, for all \( j \leq K - 1 \) the MtM-value equals the interest plus the expected discounted future cash flows, which (as it will turn out) depend on the rating at the horizon:

\[ MtM_{i,j}^H = r_i EAD_i + \xi_i + E[F_{i,j}^H], \]  \hspace{1cm} (C.3)

where \( \xi_i \) is the amount obligor \( i \) repays every year and \( E[F_{i,j}^H] \) is the expected future cash flow (discounted to the horizon) of obligor \( i \) with rating \( j \) at the horizon. It is evident that the following must hold:

\( (M_i - 1)\xi_i < EAD_i. \)

C.1 Expected Cash Flows

Again we have the following path tree
Valuation for a Term Loan

Figure C.1: The path tree for an obligor.

The probability of each path is the same as in 4.2.2, thus
\[
\mathbb{P}[\delta_{i,j} = d] = \begin{cases}
\prod_{k=1}^{M_i-1} (1 - \bar{m}_{i,j,k}), & \text{if } d = 1 \text{ and the obligor thus reaches its maturity;}

\bar{m}_{i,j,\tau} \prod_{k=1}^{\tau-1} (1 - \bar{m}_{i,j,k}), & \text{if } 1 < d < M_i \text{ and thus } M_i - 1 \geq \tau \geq 2;

\bar{m}_{i,j,1}, & \text{if } d = M_i \text{ and thus } \tau = 1.
\end{cases}
\] (C.4)

The only thing that differs from Section 4.2 is the value of each path. The valuation tree in case of a term loan looks as follows

From this figure we see that the values of the paths are given by
\[
V[\delta_{i,j} = d] = \begin{cases}
(1+r_i)(EAD_i-\xi_i) + \sum_{t=1}^{M_i-2} \frac{r_i(EAD_i-\xi_i) + \xi_i}{(1+r_f)^t}, & \text{if } d = 1;

(1-LGD_i)(EAD_i - \xi_i) \left(1 + \frac{1}{r_f}\right)^{\tau} + \sum_{t=1}^{\tau-1} \frac{r_i(EAD_i-\xi_i) + \xi_i}{(1+r_f)^t}, & \text{if } 1 < d \leq M_i \text{ and thus } M_i - 1 \geq \tau \geq 1.
\end{cases}
\] (C.5)
Bibliography


