Pricing Variance Swap with Heston Model

Student: Xinrong Gong  Technical University Eindhoven / BNPP Investment Partners
Student Nr.: 0560963
Supervisor: Harry van Zanten  Technical University Eindhoven
            Nail Shamsutdinov  BNPP Investment Partners
Date: September 2010
This page is left blank with purpose.
CONTENTS

Preface .................................................................................................................. 5

Acknowledgments ............................................................................................... 6

1 Introduction ...................................................................................................... 7
  1.1 Background .................................................................................................. 7
  1.2 Objectives ................................................................................................... 7
  1.3 Definition of Variance Swap ....................................................................... 7
  1.4 Usages of Variance Swap .......................................................................... 8

2 Pricing Theory and Models ........................................................................... 10
  2.1 Risk Neutral Pricing .................................................................................. 10
      2.1.1 Stocks and a Money Market ................................................................. 10
      2.1.2 Arbitrage and Market Viability ............................................................ 15
  2.2 European Contingent Claim and European Option ................................. 20
  2.3 Connection with Partial Differential Equation ........................................... 22
  2.4 Benchmark model: Black-Scholes model ............................................... 25
      2.4.1 Derivation of Black-Scholes PDE ....................................................... 29
      2.4.2 Black-Scholes Option Price Formula ............................................... 30
  2.5 Stochastic volatility model ........................................................................ 31
      2.5.1 Volatility Risk in PDE of a general stochastic volatility model ............. 31
      2.5.2 Hull and White Model ....................................................................... 37
      2.5.3 Scott’s Model ....................................................................................... 37
      2.5.4 Heston model ..................................................................................... 37

3 Analysis ........................................................................................................ 43
  3.1 Reason for choosing Heston model ......................................................... 43
  3.2 Option Price and Greeks in Heston model ............................................. 44
      3.2.1 How to calculate Price, Delta, Gamma and Vega in Heston model? ....... 45
3.2.2 How do Heston model specific parameters affect the price of an option? 54
3.2.3 How to fit the parameters for Heston model using option market price? 61
3.2.4 Pricing variance swap using market price of options 69

4 Conclusions ................................................................. 76

Symbols ........................................................................ 78

Appendix ......................................................................... 79

References ..................................................................... 82
PREFACE

This is a report for the research that I carried out in Multi Strategy Alternative Investments team at Fortis Investments (currently called BNP Paribas Investment Partners) in Amsterdam.

It is well observed that index level is negatively correlated with implied volatility levels. Heston model can describe such a relationship. It is interesting to see how such a model can be applied to price variance swap. Incorporating such relationship is also important for pricing the options as well as for risk management.

Chapter 1 is the introduction to variance swap and the objectives of this research. In Chapter 2, the pricing theory and models are reviewed to give the research a concrete ground. Relevant concepts and mathematical tools are presented. In Chapter 3, Heston model is discussed in details. Price and Greeks formula are presented. Efforts are also taken to tackle the difficulties in numerical implementation of Heston option price formula. Then the parameters in Heston model are examined to see how it reflects the correlation between index level and implied volatility. Finally the Heston model is fitted to market price to price variance swap.
ACKNOWLEDGMENTS

I am grateful to more people than I could possibly list here for their help, support and encouragement over the years.

First of all, I owe a debt of gratitude to Pf. van Zanten, my supervisor at Technical University Eindhoven. I also would like to thank him for his invaluable guidance throughout the whole project. He inspires me all the time and encourages me to find my way during the research. His patience and requirement of quality has motivated me to go through those difficult subjects.

I’m also deeply grateful to Dr. Nail Shamsutdinov, my supervisor at my company, for his commitment to my research. There are so many changes in the company during my research period. He always takes time to discuss my paper and gives me advice even after he moved from Netherlands to UK.

I deeply appreciate the support from my parents-in-law and my parents. They always have faith in me. They came over from China to Netherlands last year and this year to take care of my little family so that I can focus on this research.

Finally, this paper is dedicated to my wife Jianqing, my son Kai Wen and my daughter Kai Li.
1 INTRODUCTION

1.1 Background
Derivatives have gained more and more ground in financial markets. More and more derivatives are created in different markets, which add the complexities to pricing and risk management. Common derivative types are options, futures and swaps.

As the word derivative implies, the value of a derivative is derived from some financial asset, such as stock and index. The financial asset referred by a derivative is called an underlying.

Variance swap is a derivative which provides exposure to variance of a financial asset. It makes the hedging of volatility risk possible and therefore facilitates a complete market. Pricing and hedging variance swap are two main questions for institutions involved in variance swap trading. It is important for institutions to know what a fair price is for a variance swap. It is even more important for institutions to know how to hedge the variance exposure. When an institution sells the instrument, it has unlimited risk to variance theoretically. Therefore a seller needs to hedge the exposure away so that it can earn the bid-ask spread.

1.2 Objectives
This paper will investigate how much Heston model can reflect the market price of options. And both Black-Scholes and Heston Model will be applied to price variance swap and the corresponding prices will be compared.

1.3 Definition of Variance Swap
Variance swap is a forward contract on annualized variance, whose payoff at expiration is equal to

\[(\sigma_r^2 - K_{\text{var}}^2) \times N\]

where \(\sigma_r^2\) is the realized variance (quoted in annual terms) over a period specified by the contract; \(K_{\text{var}}\) is the agreed delivery price for variance between a buyer and a seller, which is quoted in volatility point; \(N\) is the notional amount that specifies how much each variance point worth. At maturity (end date of the contract), the seller must
pay N dollars for every point by which $\sigma_R^2$ exceeds $K_{var}$ to the buyer. When the payoff is negative, the seller receives money from the buyer.

A convention as follows is used to calculate $\sigma_R^2$.

$$\sigma_R^2 = \frac{252}{T} \sum_{i=1}^{T} \ln \left( \frac{S_i}{S_{i-1}} \right)^2$$

in which $S_i$ denotes the close levels at day during the period and there are totally $T+1$ levels.

The development of variance swap started in 1990's. In the beginning, the industry showed interests in volatility swap instead. However, it is not easy to replicate volatility. This limits the market growth of volatility swap. Meanwhile, a relatively simpler way is found to replicate variance swap. This discovery made variance swap more and more popular. By the end of 2006, variance swaps on various underlyings are available. In Europe, the most traded variance swaps are the variance swaps on EURO STOXX 50 index, followed by DAX and FTSE [4]. Variance swaps are also tradable on liquid stocks, indices of developed and developing markets and other asset classes such as commodities.

### 1.4 Usages of Variance Swap

The uses of variance swaps can be:

1. Trading a volatility view

   When a trader expects that the volatility of an underlying will increase, he can buy a variance swap. If the realized variance $\sigma_R^2$ exceeds the strike $K_{var}$, the trader makes profit. An example is the event timeline for a company is known but the likely outcome is unknown, for instance, a general shareholder meeting that will vote on the sale of the company. The event will trigger large movement in stock price in either direction depending on the outcome. In this case variance swap can be used to capture this movement in order to make profit.

2. Hedging

   Variance swap can also be useful for hedging out specific volatility exposures. For example, life assurance companies offer many products which give some form of guaranteed returns for which they try to generate by holding equities. Essentially,
these companies are in a position which holds equity and short put options to policy holders. To reduce the exposure of put option, delta hedging can be used. But delta hedging cannot hedge the risk in relation to volatility. Increasingly, insurers are looking into variance swap to hedge the volatility risk.

3. Diversification
Diversification is a risk management technique which limits the risk of an investment by using a portfolio of diversified investments. A portfolio of diversified investments reduces the risk. Since the return of variance swap has the negative correlation with equity, a variance swap can also be used to gain diversification.
2 PRICING THEORY AND MODELS

2.1 Risk Neutral Pricing

2.1.1 Stocks and a Money Market

Let us first put down a number of definitions. Here we will follow closely the notions in [13].

Let us begin with a complete probability space \((\Omega, \mathbb{F}, P)\) on which is given a standard \(D\)-dimensional Brownian motion \(W(t) = (W^{(1)}(t), ..., W^{(D)}(t))', \mathbb{F}(t), 0 \leq t \leq T\).

Here prime denotes transposition, so that \(W(t)\) is a column vector. We assume that \(W(0) = 0\) almost surely. All economic activity will be assumed to take place on a finite horizon \([0, T]\), where \(T\) is a positive constant. Define

\[
\mathbb{F}^W(t) \triangleq \sigma\{W(s); 0 \leq s \leq t\}, \quad \forall t \in [0, T],
\]

(2.1)

to be the filtration generated by \(W(\cdot)\) and let \(\mathbb{N}\) denote the \(P\)-null subsets of \(\mathbb{F}^W(T)\).

We shall use the augmented filtration

\[
\mathbb{F}(t) \triangleq \sigma(\mathbb{F}^W(t) \cup \mathbb{N}), \quad \forall t \in [0, T],
\]

(2.2)

We introduce now a money market and \(N\) stocks. A share of the money market has the price \(S_0(t)\) at time \(t\), with \(S_0(0) = 1\). The price process \(S_0(\cdot)\) is continuous, strictly positive, and \(\{\mathbb{F}(t)\}\)-adapted, with finite total variation on \([0, T]\). Being of finite variation, \(S_0(\cdot)\) can be decomposed into absolutely continuous (Definition see [1] in Appendix) and singularly continuous parts \(S_0^{ac}(\cdot)\) and \(S_0^{sc}(\cdot)\), respectively. We can then define

\[
\begin{align*}
    r(t) &\triangleq \frac{dS_0^{ac}(t)}{S_0(t)}, \\
    A(t) &\triangleq \int_0^t \frac{dS_0^{sc}(u)}{S_0(u)}
\end{align*}
\]

(2.3)

So that

\[
dS_0(t) = S_0(t)[r(t)dt + dA(t)], \forall t \in [0, T],
\]

(2.4)

Or equivalently,
Next we introduce $N$ stocks with prices-per-share $S_1(t), \ldots, S_N(t)$ at time $t$ and with $S_1(0), \ldots, S_N(0)$ positive constants. The processes $S_1(\cdot), \ldots, S_N(\cdot)$ are continuous, strictly positive, and satisfy stochastic differential equations

$$dS_n(t) = S_n(t)[b_n(t)dt + dA(t) + \sum_{d=1}^{0} \sigma_{nd}(t)dW^{(d)}(t)], \ \forall t \in [0, T]$$

(2.6)

The solution of the equation (2.6) is

$$S_n(t) = S_n(0)\exp\left\{\int_0^t \sum_{d=1}^{0} \sigma_{nd}(s)dW^{(d)}(s) + \int_0^t \left[ b_n(s) - \frac{1}{2} \sum_{d=1}^{0} \sigma_{nd}^2(s) \right]ds + A(t)\right\},$$

(2.7)

$$\forall t \in [0, T], \ n = 1, \ldots, N.$$

Consequently, the singularly continuous process $A(\cdot)$ does not enter the discounted stock prices

$$\frac{S_n(t)}{S_0(t)} = S_n(0)\exp\left\{\int_0^t \sum_{d=1}^{0} \sigma_{nd}(s)dW^{(d)}(s) + \int_0^t \left[ b_n(s) - r(s) - \frac{1}{2} \sum_{d=1}^{0} \sigma_{nd}^2(s) \right]ds \right\},$$

(2.8)

$$\forall t \in [0, T], \ n = 1, \ldots, N.$$

In some applications, the stocks have associated dividend rate processes. We model those as real-valued processes $\delta_n(\cdot)$, where $\delta_n(t)$ is the rate of dividend payment per dollar invested in the stock at time $t$. Adding the dividend rate process into (2.6), we can define the yield (per share) processes by $Y_n(0) = S_n(0)$ and

$$dY_n(t) = S_n(t)\left[ b_n(t)dt + dA(t) + \sum_{d=1}^{0} \sigma_{nd}(t)dW^{(d)}(t) + \delta_n(t)dt \right], \ n = 1, \ldots, N.$$

(2.9)

Or equivalently,

$$Y_n(t) = S_n(t) + \int_0^t S_n(u)\delta_n(u)du, \ n = 1, \ldots, N.$$

(2.10)

We set $Y_0(t) - S_0(t), 0 \leq t \leq T$.

We formalize this discussion with the following definition.

**Definition 2.1.** A financial market $M$ consists of

1. a probability space $(\Omega, \mathcal{F}, P)$;

2. a positive constant $T$, called the terminal time;
3. a D-dimensional Brownian motion \( \{ W(t), \mathbb{F}(t); 0 \leq t \leq T \} \) defined on \( (\Omega, \mathbb{F}, P) \), where \( \{ \mathbb{F}(t) \} _{0 \leq t \leq T} \) is the augmentation (by the null sets in \( \mathbb{F}^W(T) \)) of the filtration \( \{ \mathbb{F}^W(t) \} _{0 \leq t \leq T} \) generated by \( W(\cdot) \);

4. a progressively measurable risk-free rate process \( r(\cdot) \) satisfying \( \int_0^T |r(t)| \, dt < \infty \) almost surely (a.s.);

5. a progressively measurable N-dimensional mean rate of return process \( b(\cdot) \) satisfying \( \int_0^T \|b(t)\| \, dt < \infty \) almost surely (a.s.);

6. a progressively measurable, N-dimensional dividend rate process \( \delta(\cdot) \) satisfying \( \int_0^T \|\delta(t)\| \, dt < \infty \) almost surely (a.s.);

7. a progressively measurable, \((N \times D)\)-matrix-valued volatility process \( \sigma(\cdot) \) satisfying \( \sum_{n=1}^N \sum_{d=1}^D \int_0^T \sigma_{nd}^2(t) \, dt < \infty \) a.s.;

8. a vector of positive, constant initial stock prices \( S(0) = (S_1(0), ..., S_N(0))' \);

9. a progressively measurable, singularly continuous, finite-variation process \( A(\cdot) \) whose total variation on \([0,t]\) is denoted by \( \tilde{A}(t) \).

We refer to this financial market as \( M = (r(\cdot), b(\cdot), \delta(\cdot), \sigma(\cdot), S(0), A(\cdot)) \).

Given a financial market \( M \) as above, the money market and stock price are determined by (2.5), (2.7), and then (2.3), (2.4), (2.6) and (2.8) hold.

Let \( 0 = t_0 < t_1 < ... < t_M = T \) be a partition of \([0,T]\). For \( n = 0, 1, ..., N \) and \( m = 0, ..., M - 1 \), let \( \eta_n(t_m) \) denote the number of shares of stock \( n \) held by the investor over the time interval \([t_m, t_{m+1})\). Let \( \eta_0(t_m) \) denote the number of shares held in the money market. For \( n = 0, 1, ..., N \) the random variable \( \eta_n(t_m) \) must be \( \mathbb{F}(t_m) \)-measurable; in other words, anticipation of the future (insider trading) is not permitted.
Let us define the associated gains process by the stochastic difference equation

\[ G(0) = 0, \]  

\[ G(t_{m+1}) - G(t_m) = \sum_{n=1}^{N} \eta_n(t_m) S_n(t_m), \quad m = 0, ..., M. \]  

(2.11) \hfill (2.12)

We have

\[ G(t_m) = \sum_{n=1}^{N} \eta_n(t_m) S_n(t_m), \quad m = 0, ..., M. \]

if and only if there is no infusion or withdrawal of funds over \([0,T]\). In this case the trading is called “self-financed”. The holdings are shorting some stocks/money market to finance the long position on other stocks.

Now suppose that \(\eta(\cdot) = (\eta_0(\cdot), ..., \eta_N(\cdot))'\) is an \(\{\mathbb{F}(t)\}\)-adapted process defined on all of \([0,T]\), not just the partition points \(t_0, ..., t_M\). The associated gains process is now defined by the initial condition \(G(0) = 0\) and the stochastic differential equation

\[ dG(t) = \sum_{n=1}^{N} \eta_n(t) dY_n(t), \quad m = 0, ..., M. \]  

(2.13)

We take this equation as an axiom; references related to this point are cited in the Notes, Section 1.8 of [13].

Further we define \(\pi_n(t) \triangleq \eta_n(t) S_n(t), \quad \pi(\cdot) \triangleq (\pi_1(\cdot), ..., \pi_N(\cdot))'\).

**Definition 2.2.** Consider a financial market \(M = (r(\cdot), b(\cdot), \delta(\cdot), \sigma(\cdot), S(0), A(\cdot))\) as in Definition 2.1. A *portfolio process* \((\pi_0(\cdot), \pi(\cdot))\) for this market consists of an \(\{\mathbb{F}(t)\}\)-progressively measurable, real-valued process \(\pi_0(\cdot)\) and an \(\{\mathbb{F}(t)\}\)-progressively measurable, \(\mathbb{R}^N\)-valued process \(\pi(\cdot) \triangleq (\pi_1(\cdot), ..., \pi_N(\cdot))'\) such that

\[ \int_{0}^{T} \left| \pi_0(t) + \pi'(t) 1[\{ r(t) \} dt + dA(t) ] \right| < \infty, \]  

(2.14)

\[ \int_{0}^{T} \left| \pi'(t) \left( b(t + \delta(t) - r(t)) \right) dt < \infty, \]  

(2.15)

\[ \int_{0}^{T} \| \sigma'(t) \pi(t) \|^2 dt < \infty, \]  

(2.16)

hold almost surely.

The gains process \(G(\cdot)\) associated with \((\pi_0(\cdot), \pi(\cdot))\) is
\[ G(t) \triangleq \int_0^t \left[ \pi_0(s) + \pi'(s) \left( r(s)ds + dA(s) \right) + \int_0^s \pi'(s) \left[ b(s) + \delta(s) - r(s) \right] ds \right] ds + \int_0^t \pi'(s) \sigma(s) dW(s), 0 \leq t \leq T. \]  

(2.17)

The portfolio process \((\pi_0(\cdot), \pi(\cdot))\) is said to be self-financed if

\[ G(t) = \pi_0(t) + \pi'(t), \quad \forall t \in [0, T], \]  

(2.18)

In other words, the value of the portfolio at every time is equal to the gains earned from investments up to that time.

**Definition 2.3.** Define the \(N\)-dimensional vector of excess yield (over the interest rate) processes

\[ R(t) \triangleq \int_0^t \left[ b(u) + \delta(u) - r(u) \right] du + \int_0^t \sigma(u) dW(u), \quad 0 \leq t \leq T, \]  

(2.19)

And simplify (2.17) as

\[ G(t) = \int_0^t \left[ \pi_0(s) + \pi'(s) \left( r(s)ds + dA(s) \right) + \int_0^s \pi'(s) dR(s) \right] ds, 0 \leq t \leq T, \]  

(2.20)

If \((\pi_0(\cdot), \pi(\cdot))\) is self-financed, then (2.20) reads in differential form

\[ dG(t) = \frac{G(t)}{S_0(t)} dS_0(t) + \pi'(t) dR(t), \]  

(2.21)

and has the solution

\[ G(t) = S_0(t) \int_0^t \frac{1}{S_0(u)} \pi'(u) dR(u); 0 \leq t \leq T, \]  

(2.22)

**Definition 2.4.** An \(\{\mathbb{F}(t)\}\)-adapted, \(\mathbb{R}^N\) valued process \(\pi(\cdot)\) satisfying (2.15) and (2.16) is said to be tame if the discounted gains semimartingale

\[ \frac{G(t)}{S_0(t)} = M_0^\pi(t) \triangleq \int_0^t \frac{1}{S_0(u)} \pi'(u) dR(u), \quad 0 \leq t \leq T, \]  

(2.23)

is almost surely bounded from below by a real constant that does not depend on \(t\) (but possibly depends on \((\pi_0(\cdot), \pi(\cdot))\)). If \((\pi_0(\cdot), \pi(\cdot))\) is a portfolio process and \((\pi_0(\cdot), \pi(\cdot))\) is tame, we say that the portfolio process \((\pi_0(\cdot), \pi(\cdot))\) is tame.
**Definition 2.5.** Let $M$ be a financial market. A cumulative income process $\Gamma(t), 0 \leq t \leq T$, is a semimartingale, i.e., the sum of a finite-variation RCLL process and a local martingale.

**Definition 2.6.** Let $M$ be a financial market, $\Gamma(\cdot)$ a cumulative income process, and $(\pi_0(\cdot), \pi(\cdot))$ a portfolio process. The *wealth process* associated with $(\Gamma(\cdot), \pi_0(\cdot), \pi(\cdot))$ is

$$X(t) \triangleq \Gamma(t) + G(t),$$

where $G(\cdot)$ is the gains process of (2.17). The portfolio $(\pi_0(\cdot), \pi(\cdot))$ is said to be *$\Gamma(\cdot)$-financed* if

$$X(t) = \pi_0(t) + \pi'(t)1, \quad \forall t \in [0,T].$$

(2.25)

**Remark 2.12.** For a $\Gamma(\cdot)$-financed portfolio $(\pi_0(\cdot), \pi(\cdot))$, using the vector of excess yield process $R(\cdot)$ of (2.19) we may write the wealth equation (2.25) in differential form as

$$dX(t) = d\Gamma(t) + \frac{X(t)}{S_0(t)}dS_0(t) + \pi'(t)dR(t)
$$

$$= d\Gamma(t) + X(t)[r(t)dt + dA(t)] + \pi'(t)[b(t) + \delta(t) - r(t)1]dt,$$

$$+ \pi'(t)\sigma(t)dW(t)$$

by analogy with (2.21), and therefore the *discounted wealth process* is given by

$$\frac{X(t)}{S_0(t)} = \Gamma(0) + \int_{(0,t]} \frac{d\Gamma(u)}{S_0(u)} + \int_{(0,t]} \frac{1}{S_0(u)}\pi'(u)dR(u), \quad 0 \leq t \leq T.$$ (2.27)

### 2.1.2 Arbitrage and Market Viability

**Definition 2.7.** In a financial market $M$ we say that a given tame, self-financed portfolio process $\pi(\cdot)$ is an arbitrage opportunity if the associated gains process $G(\cdot)$ of (2.17) satisfies $G(T) \geq 0$ almost surely and $G(T) > 0$ with positive probability. A financial market $M$ in which no such arbitrage opportunities exists is said to be viable.
Theorem 2.1. First Fundamental Theorem of Asset Pricing

If a financial market $M$ is viable, then there exists a progressively measurable process $\theta(\cdot)$ with values in $\mathbb{R}^D$, called the market price of risk, such that for Lebesgue-almost-every $t \in [0, T]$ the risk premium $b(t) + \delta(t) - r(t)1$ is related to $\theta(t)$ by the equation

$$b(t) + \delta(t) - r(t)1 = \sigma(t)\theta(t) \text{ a.s.} \quad (2.28)$$

Conversely, suppose that there exists a process $\theta(\cdot)$ that satisfies the above requirements, as well as

$$\int_0^T \|\theta(s)\|^2 \, ds < \infty \text{ a.s.} \quad (2.29)$$

$$E \left[ \exp \left\{ -\int_0^T \theta'(s)dW(s) - \frac{1}{2} \int_0^T \|\theta(s)\|^2 \, ds \right\} \right] = 1. \quad (2.30)$$

Then the market $M$ is viable.

Proof: See page 12 till 15 of [13].

Definition 2.8. A financial market model $M$ is said to be standard if

1. it is viable;
2. the number $N$ of stocks is not greater than the dimension $D$ of the underlying Brownian motion;
3. the $D$-dimensional, progressively measurable market price of the risk process $\theta(\cdot)$ of (2.28) satisfies

$$\int_0^T \|\theta(t)\|^2 \, dt < \infty. \quad (2.31)$$

almost surely; and

4. The positive local martingale

$$Z_0(t) \triangleq \exp \left\{ -\int_0^T \theta'(s)dW(s) - \frac{1}{2} \int_0^T \|\theta(s)\|^2 \, ds \right\}, \quad 0 \leq t \leq T, \quad (2.32)$$

is in fact a martingale.

For a standard market, we define the standard martingale measure $P_0$ on $\mathcal{F}(t)$ by.

$$P_0(A) \triangleq E[Z_0(T)1_A], \quad \forall A \in \mathcal{F}(T) \quad (2.33)$$
Above definition is closely related to Girsanov’s Theorem. Let us look at the theorem in details.

Define a probability space \((\Omega, \mathbb{F}, P)\) and a \(d\)-dimensional Brownian motion \(W = \{W(t) = (W^{(i)}(t)), 0 \leq t \leq T\}\) defined on it, with \(P[W_0 = 0] = 1\). Assume that the filtration \(\{\mathbb{F}(t)\}\) satisfies the usual conditions. Let \(\theta = \{(\theta^{(i)}(t)), F(t); 0 \leq t \leq T\}\) be a vector of measurable, adapted process satisfying

\[
\int_0^T \|\theta^{(i)}(t)\|^2 \, dt < \infty, \text{ almost surely, } 1 \leq i \leq D, 0 \leq T < \infty. \tag{2.34}
\]

We set

\[
Z_0(t; \theta) \equiv \exp \left\{ -\int_0^T \theta(s) dW(s) - \frac{1}{2} \int_0^T \|\theta(s)\|^2 \, ds \right\}, \quad 0 \leq t \leq T, \tag{2.35}
\]

which is a local martingale.

**Theorem 2.2. Girsanov Theorem** (Girsanov (1960), Cameron and Martin (1944)).

Assume that \(Z_0(t; \theta)\) is a martingale. Define a process

\[
W(t) = (W^{(i)}(t)), 0 \leq t < \infty, \tag{2.36}
\]

For each fixed \(T \in [0, \infty)\), the process \(\{W(t), \mathbb{F}(t); 0 \leq t \leq T\}\) is a \(D\)-dimensional Brownian motion on \((\Omega, \mathbb{F}(T), P_0(T))\), where \(P_0(\cdot) \equiv E[\cdot | Z(T; \theta)]\), \(A \in \mathbb{F}(t)\).

Note: Here the \(Z\) is acted as a ratio between two measure \(P\) and \(P_0\), i.e. \(Z = \frac{dP_0}{dP}\). It is also called Radon–Nikodym derivative.

**Definition 2.9.** For a viable market, the excess-yield process \(R(t)\) of (2.19) is given by

\[
R(t) = \int_0^T \sigma(u) \left[ \theta(u) du + dW(u) \right], \quad 0 \leq t \leq T, \tag{2.37}
\]

where \(\theta(\cdot)\) satisfies (2.28).
Definition 2.10. According to Girsanov’s theorem the process
\[ W_0'(t) \overset{d}{=} W(t) + \int_0^t \theta(s) ds, \quad \forall t \in [0, T] \]  \tag{2.38}

is a D-dimensional Brownian motion under \( P_0 \), relative to the filtration \( \{ \mathcal{F}(t) \} \) of (2.2).
In terms of \( W_0' (\cdot) \), the excess yield process of (2.19) and (2.37) can be rewritten
as \( R(t) = \int_0^t \sigma(u) dW_0'(u) \), the discounted gains process becomes
\[ \frac{G(t)}{S_0(t)} = M_0^\pi(t) \overset{d}{=} \int_0^t \frac{1}{S_0(u)} \pi'(u) \sigma(u) dW_0'(u) \]  \tag{2.39}

and the discounted wealth process of (2.27) corresponding to a \( \Gamma(\cdot) \)-financed
portfolio is
\[ \frac{X(t)}{S_0(t)} = \Gamma(0) + \int_{[0,t]} d\Gamma(u) + \int_0^t \frac{1}{S_0(u)} \pi'(u) \sigma(u) dW_0'(u), \quad 0 \leq t \leq T. \]  \tag{2.40}

Definition 2.11. An \( \{ \mathcal{F}(t) \} \)-adapted, \( \mathbb{R}^N \)-valued process \( (\pi_0(\cdot), \pi(\cdot)) \) satisfying (2.15),
(2.16) is said to be martingale-generating if under the probability measure \( P_0 \) of (2.33),
the local martingale \( M_0^\pi(\cdot) \) of (2.39) is a martingale. If \( (\pi_0(\cdot), \pi(\cdot)) \) is a portfolio
process and \( (\pi_0(\cdot), \pi(\cdot)) \) is martingale-generating, we say that the portfolio
process \( (\pi_0(\cdot), \pi(\cdot)) \) is martingale-generating.
No we will come cross an important definition “Complete”.
Let us define further what a derivative is. A derivative can be defined as a financial
instrument whose value depends on the values of other, more basic underlying
variables\(^1\). This is intuitive but also loose enough. Let us look at the definition given
by [13]. In order to define the derivative, we also need concepts “Wealth Process” and
“financed portfolio”.
Having those definitions in hand, we can now introduce the definition of a derivative.

\(^1\) As defined in [5].
**Definition 2.12.** Let \( M \) be a standard financial market, and let \( B \) be an \( \mathbb{F}(t) \)-measurable random variable such that \( \frac{B}{S_0(T)} \) is almost surely bounded from below and

\[
x \triangleq E_0 \left[ \frac{B}{S_0(T)} \right] < \infty.
\]  

(2.41)

We say that \( B \) is financeable, if there is a tame, \( x \)-financed portfolio process \((\pi_0(\cdot), \pi(\cdot))\) whose associated wealth process satisfies \( X(T) = B \); i.e.,

\[
\frac{B}{S_0(T)} = x + \int_0^T \frac{1}{S_0(u)} \pi'(u)\sigma(u)dW_0(u)
\]  

(2.42)

almost surely.

We say that the financial market \( M \) is *complete* if every \( \mathbb{F}(t) \)-measurable random variable \( B \), with \( \frac{B}{S_0(T)} \) bounded from below and satisfying (2.41), is financeable.

Otherwise, we say that the market is *incomplete*.

\( B \) represents a derivative. Another alternative term for “Financeable” is “Hedgable”.

The definition of *complete* seems rather straightforward, even obvious. But it is abstract enough. Given a model, how to decide if a model is complete?

The following theorem makes the concept clearer.

**Theorem 2.3. Second Fundamental Theorem of Asset Pricing**

A standard financial market \( M \) is complete if and only if for every \( \mathbb{F}(T) \)-measurable random variable \( B \) satisfying

\[
E_0 \left[ \frac{|B|}{S_0(T)} \right] < \infty \tag{2.43}
\]

And with \( x \) defined by (2.41), there is a martingale-generating, \( x \)-financed portfolio process \((\pi_0(\cdot), \pi(\cdot))\) satisfying (2.42).

The completeness of a standard financial market can be judged using the following theorem.
Theorem 2.4. A standard financial market $M$ is complete if and only if the number of stocks $N$ is equal to the dimension $D$ of the underlying Brownian motion and the volatility matrix $\sigma(t)$ is nonsingular for Lebesgue-a.e. $t \in [0, T]$ almost surely.

Proof: See Page 24-27 in [13].

In another word, when a model defines processes for stocks/financial asset, the stock/financial asset must be tradable. If a financial asset is not tradable, it is not possible any more to hedge the uncertainty/risk with this financial asset.

Bear that in mind, we look at two fundamental theorems about asset pricing.

2.2 European Contingent Claim and European Option

There are a number of instruments available on financial markets. European option is the most relevant to this paper.

Definition 2.13. A European contingent claim (ECC) is an integrable cumulative income process $C(\cdot)$. To simplify the notation, we shall always assume that $C(0) = 0$ almost surely.

European contingent claims are bought and sold. The buyer, who is said to assume a long position in the claim, pays some nonrandom amount $\Gamma(0)$ at time zero and is thereby entitled to the cumulative income process $C(\cdot)$. The seller (writer, issuer), who is said to assume a short position, receives $\Gamma(0)$ at time zero and must provide $C(\cdot)$ to the buyer. Thus, the seller has cumulative income process

$$\Gamma(t) = \Gamma(0) - C(t), 0 \leq t \leq T.$$  \hspace{1cm} (2.44)

We are also interested in the price of the ECC at other times $t \in [0, T]$.

Definition 2.14. Let $C(\cdot)$ be a European contingent claim. For $t \in [0, T]$ the value of $C(\cdot)$ at $t$, denoted by $V^{ECC}(t)$, is the smallest (in the sense of a.s. domination) $\mathbb{P}(t)$-measurable random variable $\xi$ such that if $X(t) = \xi$ in the following equation

$$\frac{X(T)}{S_0(T)} = \frac{X(t)}{S_0(t)} - \int_t^T \frac{dC(u)}{S_0(u)} + \int_t^T \frac{1}{S_0(u)} \pi'(u)\sigma(u) dW_u(u), \quad 0 \leq t \leq T$$  \hspace{1cm} (2.45)
then for some martingale-generating portfolio process \( \pi(\cdot) \) we have \( X(T) \geq 0 \) almost surely. The value \( V^{ECC}(0) \) at time \( t = 0 \) is called the (arbitrage-based) price for the ECC at \( t = 0 \).

**Proposition 2.1.** The value at time \( t \) of a European contingent claim \( C(\cdot) \) is

\[
V^{ECC}(t) = S_0(t) \cdot E_0[ \int_{(t,\cdot]} \frac{dC(u)}{S_0(u)} | \mathbb{F}(t) ], \quad 0 \leq t \leq T. \tag{2.46}
\]

In particular,

\[
V^{ECC}(0) = x \Leftrightarrow E_0[ \int_{(0,T]} \frac{dC(u)}{S_0(u)} ] = x. \tag{2.47}
\]

Notice here that \( V^{ECC}(t) \) is \( \mathbb{F}(t) \) measurable.

**Example 2.1.** (European call option): A European call option on the first stock in our market is the ECC given by \( C(t) = 0, \ 0 \leq t < T \) and \( C(T) = (S_1(T) - q)^+ \). The nonrandom constant \( q > 0 \) is called the strike price, and \( T \) is the expiration date. The random variable \( (S_1(T) - q)^+ \) is the value at time \( T \) of the option to buy one share of the first stock at the (contractually specified) price \( q \). If \( S_1(T) > q \), this option should be exercised by its holder; the stock can be resold immediately at the market price, at a profit of \( S_1(T) - q \). If \( S_1(T) < q \), the option should not be exercised; it is worthless to its holder.

**Example 2.2.** (European put option): A European put option confers to its holder the right to sell a stock at a future time at a prespecified price. We model a put on the first stock as the ECC with \( C(t) = 0 \) for \( 0 \leq t < T \) and \( C(T) = (q - S_1(T))^+ \).

Because \( (q - S_1(T))^+ = -(S_1(T) - q) + (S_1(T) - q)^+ \), holding a long position is equivalent to holding the simultaneously a short position in a forward contract and a long position in a European call. This is the so-called put-call parity relationship.
2.3 Connection with Partial Differential Equation

At the moment we will focus on ECC. The value of ECC $V^{ECC}(t)$ is $\mathbb{F}(t)$ measurable according to (2.46). Under certain conditions, $V^{ECC}(t)$ is a Markov process and can be represented as a function $g(S,t)$ which satisfies a partial differential equation.

Let us lay down a few definitions before introducing the theorem. We start with Borel-measurable functions $\{b_i(t,x)\}_{1 \leq i \leq d}$ and the $(D \times n)$ dispersion matrix $\sigma(t,x) = \{\sigma_{ij}(t,x)\}, 1 \leq i \leq D, 1 \leq j \leq n$. Then we have stochastic differential equation

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \quad t \leq s < \infty,$$  \hspace{1cm} (2.48)$$

Written component wise as

$$dX^{(i)}(t) = b_i(t, X(t))dt + \sum_{j=1}^{D} \sigma_{ij}(t, X(t))dW^{(j)}(t), \quad 1 \leq i \leq n,$$  \hspace{1cm} (2.49)$$

The $(d \times d)$ matrix $a(t,x) \triangleq \sigma(t,x)\sigma^T(t,x)$ with elements

$$\sum_{j=1}^{d} \sigma_{ij}(t,x)\sigma_{ij}(t,x) ; 1 \leq i, k \leq d$$  \hspace{1cm} (2.50)$$

will be called the diffusion matrix.

**Definition 2.15.** A strong solution of the stochastic differential equation (2.48) on the given probability space $(\Omega, \mathbb{F}, P)$ and with respect to the fixed Brownian motion $W$ and initial condition $x_0$, is a process $X = \{X(t) : 0 \leq t < \infty\}$ with continuous sample paths and with the following properties:

1. $X$ is adapted to the filtration $\{\mathbb{F}(t)\}$ of (2.2),
2. $P[X(0) = x_0] = 1,$
3. $P\left[\int_0^t \left|b_i(s, X(s)) + \sigma_{ij}(s, X(s))\right| ds < \infty\right] = 1$ holds for every $1 \leq i \leq d, 1 \leq j \leq n$ and $0 \leq t < \infty$, and
4. the integral version of (2.48)

$$X(t) = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW(s), \quad 0 \leq t < \infty,$$  \hspace{1cm} (2.51)$$
Or equivalently,

\[ X^{(i)}(t) = X_0^{(i)} + \int_0^t b_i(s, X_s)ds + \sum_{j=0}^n \int_0^t \sigma_j(s, X_s)dW^{(j)}(s); \ 0 \leq t < \infty, 1 \leq i \leq d, \]  

(2.52)

holds almost surely.

**Definition 2.16.** A *weak solution* of the stochastic differential equation (2.48) is a triple \((X, W), (\Omega, F, P), \{\mathbb{F}(t)\}\), where

1. \((\Omega, F, P)\) is a probability space, and \(\{\mathbb{F}(t)\}\) is a filtration of sub-\(\sigma\)-fields of \(F\) satisfying the usual conditions,
2. \(X = \{X(t), \mathbb{F}(t); 0 \leq t < \infty\}\) is a continuous, adapted \(\mathbb{R}^d\)-valued process,
3. \(W = \{W(t), \mathbb{F}(t); 0 \leq t < \infty\}\) is a \(D\)-dimensional Brownian motion, and 3 and 4 of Definition 2.16 are satisfied.

Lipschitz and linear growth conditions

\[ \|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K\|x - y\|. \ 0 \leq t < \infty, 1 \leq i \leq d, \]  

(2.53)

\[ \|b(t, x)\|^2 + \|\sigma(t, x)\|^2 \leq K^2(1 + \|x\|^2). \]  

(2.54)

For every \(0 \leq t < \infty, x \in \mathbb{R}^d, y \in \mathbb{R}^d\), where \(K\) is a positive constant.

Let us suppose that \((X, W), (\Omega, F, P), \{\mathbb{F}(t)\}\) is a weak solution to the stochastic differential equation (2.48). For every \(t \geq 0\), we introduce the second-order differential operator

\[ \left( A, f \right)(x) \triangleq \frac{1}{2} \sum_{i=1}^d \sum_{k=1}^d a_{ik}(t, x) \frac{\partial^2 f(x)}{\partial x_i \partial x_k} + \sum_{j=1}^d b_j(t, x) \frac{\partial f(x)}{\partial x_j}; \ f \in C^2(\mathbb{R}^d), \]  

(2.55)

where \(a_{ik}(t, x)\) are the components of the diffusion matrix (2.50).

Considering a solution to the stochastic integral equation

\[ X^{(i, s)}_t = x + \int_t^s b(\theta, X^{(i, x)}_\theta)d\theta + \int_t^s \sigma(\theta, X^{(i, x)}_\theta)dW_\theta; \ t \leq s < \infty, \]  

(2.56)

Under the standing assumptions that

1. the coefficients \(b_j(t, x), \sigma_i(t, x): [0, \infty) \times \mathbb{R}^d \to \mathbb{R}\) are continuous

and satisfy the linear growth condition (2.54)
2. the equation (2.56) has a weak solution
\[(X^{(t,x)}, W), (\Omega, \mathbb{F}, \mathbb{P}), \{\mathbb{F}(t)\}\] for every pair \((t, x)\); and

\[\text{(2.58)}\]

3. this solution is unique in the sense of probability law.

\[\text{(2.59)}\]

With an arbitrary but fixed \(T > 0\) and appropriate constants \(L > 0, \lambda \geq 1\), we consider functions
\[f(x) : \mathbb{R}^d \to \mathbb{R}, g(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}\]
and
\[k(t, x) : [0, T] \times \mathbb{R}^d \to [0, \infty)\]
which are continuous and satisfy

(i) \(|f(x)| \leq L(1 + \|x\|^{2\lambda})\) or (ii) \(f(x) \geq 0; \ \forall x \in \mathbb{R}^d\)

As well as

(i) \(|g(t, x)| \leq L(1 + \|x\|^{2\lambda})\) or (ii) \(g(t, x) \geq 0; \ \forall 0 \leq t \leq T, x \in \mathbb{R}^d\).

We recall also the operator \(A(2.55)\).

**Theorem 2.5.** Under the preceding assumptions and (2.57)-(2.59), suppose that
\[v(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d\]
is continuous, is of class \(C^{1,2}([0, T] \times \mathbb{R}^d)\), and satisfies the Cauchy problem

\[\frac{\partial v}{\partial t} + kv = A^2 v + g; \text{ in } [0, T] \times \mathbb{R}^d,\]

\[v(T, x) = f(x); \ x \in \mathbb{R}^d,\]

As well as the polynomial growth condition

\[\max_{0 \leq t \leq T} |v(t, x)| \leq M(1 + \|x\|^{2\mu}); \ x \in \mathbb{R}^d,\]

For some \(M > 0, \mu \geq 1\). Then \(v(t, x)\) admits the stochastic representation

\[v(t, x) = E^{x, t}[f(X_T) \exp\left\{-\int_t^T k(\theta, X_\theta) d\theta\right\}]
+ \int_t^T g(s, X_s) \exp\left\{-\int_s^T k(\theta, X_\theta) d\theta\right\} ds]\]

\[\text{(2.61)}\]

On\([0, T] \times \mathbb{R}^d\); in particular, such a solution is unique.

Proof: See page 366 of [17].
An impact of the Feynman-Kac theorem is Markov property of price of a European option.

In a financial market $M$, right hand side of (2.61) can represent the price of a specific type of European Contingent Claim $V^{ECC}(t)$ of (2.47) whose cumulative process $C(\cdot)$ satisfies $C(t) = 0, 0 \leq t < T, C(T) = f(X(T))$. Let us denote it by $V^{ECC,X(T)}(t)$. In case $X(t)$ satisfy the conditions described in Feynman-Kac Theorem, the price of such a claim, such as a European Option, is a function of $t$ and $x$. To see this, we just need to set $k(t,x) = r(t), \ g(t,X(t)) = 0$ and take the expectation under the risk-neutral measure. Therefore the price is measurable with respect to $X(t)$. The price process is a Markov process.

As the Theorem states, the price satisfies a partial derivative equation. By writing down the SDE for stock price under risk neutral measure, a PDE can be obtained. Solving such a PDE with boundary conditions gives the price of a claim.

### 2.4 Benchmark model: Black-Scholes model

The framework described in the current section so far is abstract and general enough to accommodate many types of models. Now let us examine a specific case in which there is only one stock in the model so that we have some ideas about how the framework actually works.

In 1973, Fisher Black and Myron Scholes published a paper *The Pricing of Options and Corporate Liabilities* in *The Journal of Political Economy* [6]. The paper proposed an option pricing model. Since then it has become widely accepted and even a benchmark model.

The models are defined as follows:

- **Lognormal Property of Stock Prices**
  The stock price $S$ follows geometric Brownian motion with constant drift $\mu$ and volatility $\sigma$,
  \[
  dS(t) = \mu S(t) dt + \sigma S(t) dW(t),
  \]
  Then
  \[
  \ln S(t) = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW(t)
  \]
  \[ (2.62) \]
  That means the change in $\ln S$, between time $t$ and $T$ is normally distributed.
\[ \ln S(T) - \ln S(t) \sim N\left(\frac{\mu - \sigma^2}{2} \tau, \sigma \sqrt{\tau}\right) \]

where \( \tau = T - t \) is the time to maturity.

- It is possible to short sell the underlying stock.
- Trading in the stock is continuous.
- There are no transaction costs or taxes.
- All securities are perfectly divisible (e.g. it is possible to buy any fraction of a share).
- It is possible to borrow and lend cash at a constant risk-free interest rate \( r \).

**Black-Scholes Model and Standard Financial Market**

The Black-Scholes model can be considered to be a special case of the Framework as follows,

- The dimension of Brownian Motion \( D = 1 \)
- \( b(t) = \mu, r(t) = r, \sigma(t) = \sigma, \delta(t) = 0 \)

Let us examine the arbitrage-free and completeness of the model.

**Corollary 2.1.** Black-Scholes model is arbitrage-free.

Proof: According to Theorem 2.1, we just need to check three conditions one by one.

For the equation (2.28), \( \theta(t) = (\mu - r) / \sigma \) will suffice.

For the equation (2.29), we have

\[
\int_0^T \| \theta(s) \|^2 \, ds = \int_0^T \left( \frac{\mu - r}{\sigma} \right)^2 \, ds = \left( \frac{\mu - r}{\sigma} \right)^2 \cdot T < \infty.
\]

For the equation (2.30), we have,

\[
E[\exp\left\{ -\int_0^T \mu dW(t) - \frac{1}{2} \int_0^T \mu^2 ds \right\}] = E[\exp\left\{ -\mu W(T) - \frac{1}{2} \mu^2 T \right\}]
\]

\[
= \int_{-\infty}^\infty \exp\left\{ -\mu x - \frac{1}{2} \mu^2 T \right\} \cdot \frac{1}{\sqrt{2\pi T}} \exp\left\{ - \frac{x^2}{2T} \right\} dx
\]

\[
= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi T}} \exp\left\{ - \frac{x^2 + 2\mu x T + \mu^2 T^2}{2T} \right\} dx
\]

\[
= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi T}} \exp\left\{ - \frac{(x + \mu T)^2}{2T} \right\} dx = 1
\]
the last equation hold since the integrand of 2.47 is the distribution function of normal distribution with mean $-\mu T$ and variance $T$.

All three conditions are satisfied. Therefore Black-Scholes model is complete according to Theorem 2.1.

**Corollary 2.2.** Black-Scholes model is complete.

Proof: Since Black-Scholes model defines a standard finance market. We can use Theorem 2.4. In Black-Scholes model there is one stock and dimension of the Brownian Motion is one. And the volatility matrix $\sigma(t) = \sigma$ is non-negative and thus non-singular. According to Theorem 2.4, Black-Scholes model is complete.

**Corollary 2.3.** $V^{ECC,Y(T)}(t)$ under Black-Scholes model is a function $g(t, S(t))$.

Proof: According to Proposition 2.1, we have

\[
V^{ECC,Y(T)}(t) = S_0(t)E_0 \left[ \frac{f(X(T))}{S_0(u)} \big| \mathbb{F}(t) \right] = S_0(t)E_0 \left[ \frac{f(S(T))}{S_0(u)} \big| \mathbb{F}(t) \right] = S_0(t)E_0 \left[ \frac{f(S(T))}{S_0(T)} \big| \mathbb{F}(t) \right] = \frac{1}{S_0(T-t)} E_0 \left[ f(S(T)) \big| \mathbb{F}(t) \right]
\]

Since $r$ is deterministic

\[
= \frac{1}{S_0(T-t)} E_0 \left[ f(S(t)\exp(\sigma W(T) - W(t)) + (\mu - \frac{1}{2} \sigma^2)(T-t)) \big| \mathbb{F}(t) \right]
\]

Now use the Girsanov Theorem and change the measure using $\theta(t) = \frac{\mu - r}{\sigma}$.

\[ W_0(t) = \int_0^t \theta(t) dt + W(t) \] Then we have,

\[
V^{ECC,Y(T)}(t)
\]
\[= \frac{1}{S_0(T-t)} \mathbb{E}_0 \left[ f \left( S(t) \exp \left( \sigma(W_0(T) - W_0(t)) + (\mu - \frac{r}{\sigma})(T-t) \right) \right) \right] \]

\[= \frac{1}{S_0(T-t)} \mathbb{E}_0 \left[ f \left( S(t) \exp \left( \sigma(W_0(T) - W_0(t)) + (\mu - r - \frac{1}{2} \sigma^2)(T-t) \right) \right) \right]

under standard martingale measure \( P_0 \), \( W_0(\cdot) \) is a Brownian motion. Above equation holds due to the independent incremental property of Brownian motion \( W_0(\cdot) \). And \( W_0(T) - W_0(t) \) is a normal distribution with mean 0 and variance \( T-t \) under measure \( P_0 \).

\[= \frac{1}{S_0(T-t)} \int_{-\infty}^{\infty} f \left( S(t) \exp(\sigma(x)(T-t)) \right) \cdot \frac{1}{\sqrt{2\pi(T-t)}} \exp\left( \frac{x^2}{2(T-t)} \right) dx \]

\[= g(t, S(t)) \]

Therefore price process of \( V^{ECC, X(T)}(\cdot) \) is a function \( g(t, S(t)) \).

**Black-Scholes Model under Risk Neutral Measure**

Under the standard martingale measure \( P_0(A) \Delta E(1_A Z_0) \),

\( W_0(t) \triangleq W(t) + \int_0^t \frac{\mu - r}{\sigma} ds, \ \forall t \in [0,T] \) is a Brownian Motion. So we have,

\[dW_0(t) \triangleq dW(t) + \frac{\mu - r}{\sigma} dt, \ \forall t \in [0,T]. \]

Then the SDE describing Black-Scholes model becomes

\[dS(t) = \mu S(t) dt + \sigma S(t) (dW_0(t) - \frac{\mu - r}{\sigma} dt) \]

\[dS(t) = \mu S(t) dt + \sigma S(t) dW_0(t) - (\mu - r)S(t)dt \]

\[dS(t) = rS(t) dt + \sigma S(t) dW_0(t) \]
2.4.1 Derivation of Black-Scholes PDE

In Black-Scholes model, the only uncertainty comes from the underlying asset. This is also the only tradable asset in the model. So let us have a closer look how the uncertainty can be hedged.

First we will look through the definition of European Contingent Claim.

First define $V(\cdot)$ as the value process of the European Option. We have shown that $V(\cdot)$ is a function of $S$ and $t$. We define

$$\Delta \triangleq \frac{\partial V}{\partial S} \quad (2.65)$$

What follows is different from usual steps taken in [7]. In [7], the portfolio under discussion is $-V + \Delta S$, in which cash positions are not included. Instead we look at a portfolio $-V + \Delta S + \eta S_0$, where $\eta$ is the number of money market bond held. The holder of this portfolio is short one derivative, long an amount $\Delta$ of shares ($\Delta \geq 0$) and hold $\eta \in \mathbb{R}$ shares of bond. Be noticed that $\eta < 0$ means that the holder is selling money market bond to finance the portfolio. $V(t, S) = \Delta$ and $\eta = 0$ when $t = 0$. $-\Delta S + \eta S_0$ is itself a self-financed portfolio. Define $\Pi$ as the value of the portfolio

$$\Pi = -V + \Delta S + \eta S_0 \quad (2.66)$$

By applying the equation (2.13) that self-financed portfolio observes, we get

$$d\Pi = -dV + \Delta dS + \eta dS_0 \quad (2.67)$$

Write $V$ out using Itô’s formula, we have,

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} \right) dt + \frac{\partial V}{\partial S} dS \quad (2.68)$$

Now we can substitute (2.68) into (2.67), we get,

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} \right) dt - \frac{\partial V}{\partial S} dS + \Delta dS + \eta dS_0$$

$$= \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial^2 S} \right) dt + \eta dS_0 \quad (2.69)$$

Using again the self-financed portfolio argument, we have
\[ \eta = \frac{(\Pi + V - \Delta S)}{S_0} \] (2.70)

Substitute (2.70) into (2.69),
\[
d\Pi = -\left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{(\Pi + V - \Delta S)}{S_0} dS_0
\]
= \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + (\Pi + V - \Delta S) r dt
\] (2.71)

Note that the right hand side does not contain stochastic term. So it’s a deterministic drift. Now using the arbitrage argument, the holding \( \Pi \) must have a drift \( \Pi r \) to guarantee the arbitrage free of the financial market. If the holding has a drift rate larger than \( \Pi r \), borrowing money with risk-free rate \( r \) and buying the holding will create arbitrage opportunity. If it has a drift rate less than \( \Pi r \), selling the holding and buying money will create arbitrage opportunity. So we have,
\[
d\Pi = \Pi r dt = \left( \frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \Pi r + V r - \Delta S r \right) dt
\]

Now we get the following PDE,
\[
\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0
\] (2.72)

This partial differential equation is known as Black-Scholes partial differential equation.

Compared with the holding used by [5], the extra term \( \eta S_0 \) is added to \( \Pi \). When we look at the influence it bring to \( d\Pi \) and \( \Pi r dt \), the term contains \( \eta S_0 \) just cancel out.
That is due to the fact that \( \eta S_0 r dt = \eta dS_0 \), namely that \( S_0 \) satisfies the stochastic partial differential equation \( dS_0 = rS_0 dt \).

2.4.2 Black-Scholes Option Price Formula

To solve the PDE (2.72), let us specify now the boundary conditions. They are
\[ V(T) = \max(S - K, 0) \] for European Call Option
Now we can derive the European Option price formula in Black-Scholes model. Let \( p \) denote the price for a European Call option price and \( c \) denote a European put option price, \( S_0 \) denote the underlying level, \( q \) denote the annual dividend rate, \( T \) denote the time to maturity in years,

\[
c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2) \tag{2.73}
\]

\[
p = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1) \tag{2.74}
\]

Where

\[
d_1 = \frac{\ln(S_0 / K) + (r - q + \sigma^2 / 2)T}{\sigma \sqrt{T}}.
\]

\[
d_2 = \frac{\ln(S_0 / K) + (r - q - \sigma^2 / 2)T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}.
\]

### 2.5 Stochastic volatility model

#### 2.5.1 Volatility Risk in PDE of a general stochastic volatility model

In a stochastic volatility model, a partial differential equation can be derived which resembles the PDE as in (2.72). In this section, we will recap how the PDE can be derived in [14].

The framework laid down by Karatzas and Shreve is general enough since the parameters such as \( r(\cdot) \), \( b(\cdot) \) are progressively measurable. Such a general setup can make the derivative price depends on the whole path of underlying up to \( t \). We will focus ourselves on the Markovian type of model under the framework, one specialization of this setup. This specialization provides the simplicity such that the price depends only on the current state/underlying level. This is described by Feyman-Kac Theorem 2.5.

Let us first define a general stochastic volatility model.

**Definition 2.17** A stochastic volatility model \( M \) is a financial market which satisfies the follows,

Suppose that the stock price \( S \) and its variance \( \sigma \) satisfy the following SDEs:
\[ dS = S\mu(t, S)dt + S\sigma dZ_1 \]  
\[ d\sigma = p(t, \sigma)dt + q(t, \sigma)dZ_2 \]  
(2.75)  
(2.76)

Where \( \mu(t, S), p(t, \sigma), \) and \( q(t, \sigma) \) are Borel-measurable function.

With

\[ \langle dZ_1, dZ_2 \rangle = \rho dt . \]  
(2.77)

There are two stochastic variables in this model. Further we assume \( S \) is tradable while the volatility is not a traded asset.

We show now how this kind of model can fit into the setup of a Standard Financial Market.

**Corollary 2.4** The stochastic volatility model \( M \) is a Standard Financial Market if condition \( \int_0^T \frac{(\mu - r)^2}{\sigma^2} dt < \infty \) a.s. and Novikov condition \( E \left[ \exp \left\{ \frac{1}{2} \int_0^T \|\theta(t)\|^2 dt \right\} \right] < \infty \) are satisfied.

**Proof:** Let \( dZ_1 = \frac{\rho\sigma}{\sigma} dW_1 + \frac{\sigma^2 - \rho^2}{\sigma} dW_2 \), \( dZ_2 = dW_2 \). And we know \( dZ_1 \cdot dZ_2 = \rho dt \) satisfies the condition (2.77). Then the model can be formulated as follows,

\[ dS = \mu(t, S)Sdt + \sigma S(\rho dW_1 + \sqrt{1 - \rho^2} dW_2) \]  
(2.78)

\[ d\sigma = p(S, \sigma, t)dt + q(S, \sigma, t)dW_2 \]  
(2.79)

Now the equation (2.75) describes a stock process \( S \) with progressive measurable volatility under the notion of Standard Financial Market. We will check if it satisfies other conditions of a Standard Financial Market.

(i) Viable: We know that \( (\mu - r) = \rho \sigma \theta_1(t) + \sqrt{1 - \rho^2} \sigma \theta_2(t) \), \( \theta(t) = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \text{affine hull} \)

\[
\begin{pmatrix}
0 \\
\frac{\mu - r}{\sigma} \\
\frac{\rho \sigma}{\sqrt{1 - \rho^2}} \\
0
\end{pmatrix}
\begin{pmatrix}
\mu - r \\
\sigma \\
\frac{\rho \sigma}{\sqrt{1 - \rho^2}} \\
0
\end{pmatrix}
\]

\( \theta \) can also be written in the following form.
\[
\theta = (1-k) \left( \frac{0}{\sigma \sqrt{1-\rho^2}} \right) + k \left( \frac{\mu-r}{\sigma \rho} \right), \quad k \in \mathbb{R}
\]

So \( \int_0^T \|\theta(t)\|^2 dt = \int_0^T k^2 \left( \frac{\mu-r}{\sigma \rho} \right)^2 dt = \frac{k^2}{\rho^2} \int_0^T \left( \frac{\mu-r}{\sigma \rho} \right)^2 dt < \infty \) a.s. as \( \int_0^T \frac{(\mu-r)^2}{\sigma^2} dt < \infty \) a.s. is given in the conditions.

\[
\int_0^T \|\theta(t)\|^2 dt = \int_0^T \left( 1 - k \right)^2 \left( \frac{\mu-r}{\sigma \sqrt{1-\rho^2}} \right)^2 dt = \frac{1}{1-\rho^2} \int_0^T \left( \frac{\mu-r}{\sigma \rho} \right)^2 dt < \infty \quad \text{as}\quad \int_0^T \frac{(\mu-r)^2}{\sigma^2} dt < \infty \quad \text{a.s.}
\]

\[
\int_0^T \frac{(\mu-r)^2}{\sigma^2} dt < \infty \quad \text{a.s. is given in the conditions.}
\]

Let \( Z_0(T) = \exp \left\{ \theta(t) dW(s) - \frac{1}{2} \int_0^T \|\theta(s)\|^2 ds \right\} \). We know \( Z_0(0) = 1 \). When \( \theta(t) \) satisfies Novikov condition, namely \( E \left[ \exp \left\{ \frac{1}{2} \int_0^T \|\theta(t)\|^2 dt \right\} \right] < \infty \), \( Z_0(T) \) is a martingale and therefore \( Z_0(T) = E[Z_0(T) | \mathbb{F}(0)] = E[Z_0(0)] = 1 \). According to Theorem 2.1, the market \( M \) is viable.

(ii) Dimension of stocks:
Further we also know that the number \( N=1 \) of stocks in this case. It is not greater than the dimension \( D=2 \) of the underlying Brownian motion.

(iii) \( \int_0^T \|\theta(t)\|^2 dt < \infty \) almost surely is a given condition.

(iv) Under the Novikov condition, the positive local martingale \( Z_0(t) \) is a martingale.

Therefore the market \( M \) is a Standard Financial Market according to Definition 2.8.

Now we will show that the value of a European call option is a function \( V(S, \sigma, t) \).

The first attempt is to try what we did in Corollary 2.3.

\[
V_{ECC,X(T)}^0(t) = S_0(t) E_0 \left[ \frac{f(X(T))}{S_0(t)} | \mathbb{F}(t) \right]
\]

\[
= \exp(-r(T-t)) E \left[ f \left( S(t) \exp \left\{ -\frac{1}{2} \int_t^T \sigma^2(s) ds + \int_t^T \sigma(s) \rho dW_1 + \int_t^T \sigma(s) \sqrt{1-\rho^2} dW_2 \right\} \right) | \mathbb{F}(t) \right]
\]
The difficulty is that the direct calculation will face the distribution of \( \int_T^t \sigma(s) \, dW \) and \( \frac{1}{2} \int_T^t \sigma^2(s) \, ds \). Instead the direction calculation, we can use the following reasoning. The equations (2.78) (2.79) actually imply that if we know \( S \) and \( \sigma \) at time \( s \) then we know the expected value of \( S \) and \( \sigma \) at the time \( t \) for \( t \geq s \) due to the independent increment of Brownian motions. Therefore we can suppose that \( E_0(S_T | \mathcal{F}(t)) = E_0(S_T | S(t)) = g(S, \sigma, t) \). Then we also have

\[
E_0(f(S_T) | \mathcal{F}(t)) = u(S, \sigma, t).
\]

(2.80)

Now let us look at the price of a European call option. The price is a function \( V(S, \sigma, t) \) according to (2.80). We set up a portfolio that consists of the option, a quantity \( -\Delta \) of the stock and a quantity \( -\Delta_i \) of another option \( V_i(S, \sigma, t) \) and \( \eta \) shares of money market bond.

We have

\[
\Pi = V - \Delta S - \Delta_i V_i + \eta S_0
\]

Therefore

\[
\begin{align*}
    d\Pi &= dV - \Delta dS - \Delta_i dV_i + \eta dS_0 \\
    &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS \cdot dS + \frac{\partial V}{\partial \sigma} d\sigma + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} d\sigma \cdot d\sigma + \frac{\partial^2 V}{\partial S \partial \sigma} dS \cdot d\sigma \\
    &\quad - \Delta dS - \Delta_i \left( \frac{\partial V_i}{\partial t} dt + \frac{\partial V_i}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V_i}{\partial S^2} dS \cdot dS + \frac{\partial V_i}{\partial \sigma} d\sigma + \frac{1}{2} \frac{\partial^2 V_i}{\partial \sigma^2} d\sigma \cdot d\sigma + \frac{\partial^2 V_i}{\partial S \partial \sigma} dS \cdot d\sigma \right) \\
    &\quad + \eta dS_0 \\
    &= \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \right) dt \\
    &\quad - \Delta_i \left( \frac{\partial V_i}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_i}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V_i}{\partial S \partial \sigma} + \frac{1}{2} \frac{\partial^2 V_i}{\partial \sigma^2} \right) dt \\
    &\quad + \left( \frac{\partial V}{\partial S} - \Delta_i \frac{\partial V_i}{\partial S} - \Delta \right) dS + \left( \frac{\partial V}{\partial \sigma} - \Delta_i \frac{\partial V_i}{\partial \sigma} \right) d\sigma \\
    &\quad + \eta S_0 r dt
\end{align*}
\]
This implies the following: In order to maintain the portfolio risk free, we must choose $rac{\partial V}{\partial S} - \Delta - \Delta_1 \frac{\partial V_1}{\partial S} = 0$ to eliminate $dS$ terms and $rac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma} = 0$ to eliminate $d\sigma$ terms.

By solving those two equations we get $\Delta$ and $\Delta_1$,

$$\Delta_1 = \frac{\frac{\partial V}{\partial \sigma}}{\frac{\partial V}{\partial \sigma} - \frac{\partial V_1}{\partial \sigma}} ,$$

$$\Delta = \frac{\frac{\partial V}{\partial V} - \frac{\partial V_1}{\partial V_1}}{\frac{\partial V}{\partial \sigma} - \frac{\partial V_1}{\partial \sigma}}$$  \hspace{1cm} (2.81)

These two functions described how the hedging works. Be noticed that how those two functions, especially that $\Delta$ differs from the $\Delta$ defined in (2.65).

Now let us go back the reasoning line of Wilmott. So we get the following equation,

$$d\Pi = \left\{ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} \frac{q^2 \partial^2 V}{\partial \sigma^2} + \rho \sigma S q \frac{\partial^3 V}{\partial S \partial \sigma} \right\} dt$$

$$- \Delta_1 \left\{ \frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2} \frac{q^2 \partial^2 V_1}{\partial \sigma^2} \right\} dt$$

$$+ \eta r S_0 dt$$

$$= r \Pi dt$$

$$= r (V - \Delta S - \Delta_1 V_1 + \eta S_0) dt$$

where the arbitrage-free argument is used to set the return on the portfolio equal to the risk-free rate. We can clearly see that the term contains $S_0$ just cancel out. Collecting all terms contains $V$, all terms contains $V_1$ and plug (2.81) into it, it becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma S q \frac{\partial^3 V}{\partial S \partial \sigma} + \frac{1}{2} \frac{q^2 \partial^2 V}{\partial \sigma^2} + r S \frac{\partial V}{\partial S} - r V$$

$$\frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \sigma S q \frac{\partial^3 V_1}{\partial S \partial \sigma} + \frac{1}{2} \frac{q^2 \partial^2 V_1}{\partial \sigma^2} + r S \frac{\partial V_1}{\partial S} - r V_1$$

The left-hand side is a function of $V$ but not $V_1$ and the right-hand side is a function of $V_1$ but not $V$. Since the two options will typically have different payoffs, strikes or expiries, the only way for this to be possible is for both sides to be independent of the contract type. In other words, both sides must in some sense be equal to the same
‘universal’ constant. Except that really it is a universal function of all of the independent variables common to all options. Both sides can only be functions of the independent variables, $S$, $\sigma$ and $t$. Thus we have

$$\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + r S \frac{\partial V}{\partial S} - r V \\
= -(p - \lambda q) \frac{\partial V}{\partial \sigma} 
\end{align*}$$

(2.82)

for some function $\lambda(s, \sigma, t)$.

The function $\lambda(s, \sigma, t)$ is called the market price of (volatility) risk.

**The Market Price of Volatility**

When volatility is not traded, we find that the pricing equation contains a market price of risk term. What does this mean? Suppose we hold one option with value $V$, and satisfying the pricing equation (2.82). Delta hedged with the underlying asset only. We have

$$\Pi = V - \Delta' S + \eta S_0,$$

(2.83)

In which $\Delta' = \frac{\partial V}{\partial S}$

The change in this portfolio value is

$$d\Pi = dV - \Delta'dS + \eta dS_0$$

$$= \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} \right) dt$$

$$+ \left( \frac{\partial V}{\partial S} - \Delta^\prime \right) dS + \frac{\partial V}{\partial \sigma} d\sigma$$

$$+ \eta dS_0$$

(2.84)

Because we are delta hedging, the coefficient of $dS$ is zero. We find that

$$d\Pi - r\Pi dt = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + r S \frac{\partial V}{\partial S} - r V \right) dt$$

$$+ \frac{\partial V}{\partial \sigma} d\sigma$$

$$= q \frac{\partial V}{\partial \sigma} (\lambda dt + dZ_t)$$

(2.85)
This has used both the pricing equation (2.82) and the stochastic differential equation for \( \sigma \) (2.76). Observe that for every unit of volatility risk, represented by \( dZ_t \), there are \( \lambda \) units of extra return, represented by \( dt \). Hence the name ‘market price of risk’ for \( \lambda \).

The quantity \( p - \lambda q \) is called the **risk-neutral drift rate** of the volatility. Recall that the risk-neutral drift of the underlying asset is \( r \) and not \( \mu \). When it comes to pricing derivatives, it is the risk-neutral drift that matters and not the real drift, whether it is the drift of the asset or of the volatility.

### 2.5.2 Hull and White Model

John Hull and Alan White proposed a model with correlation in [7]:

\[
dS = \phi Sdt + \sigma Sdw \\
dV = \mu Vdt + \xi Vdz
\]

where \( S \) is the security price, \( V \) is its instantaneous variance and \( V = \sigma^2 \). \( \phi \) is a parameter that may depend on \( S, \sigma \) and \( t \). The variable \( \mu \) and \( \xi \) may depend on \( \sigma \) and \( t \), but it is assumed that they do not depend on \( S \). Wiener processes \( dw \) and \( dz \) have correlation \( \rho \). When \( V \) is uncorrelated with \( S \), the paper gave the price of option in a series form; when \( V \) and \( S \) is correlated with correlation \( \rho \), numerical procedure is used to simulate underlying so that the option price can be calculated.

### 2.5.3 Scott’s Model

Luis O. Scott proposed a model in [8]. The stock prices is given by the following stochastic process,

\[
dS = \phi Sdt + \sigma Sdw \\
d\sigma = \beta(\overline{\sigma} - \sigma)dt + \gamma dz
\]

In which \( \phi, \beta, \overline{\sigma} \) and \( \gamma \) are constant and \( w \) and \( z \) are Wiener process.

### 2.5.4 Heston model

Heston model laid out in [9] assumes that volatility is correlated with underlying level. The model is as follows,
\[ dS(t) = \mu \, S \, dt + \sqrt{v(t)} \, S \, dz_1(t) \]  
\[ dv(t) = \kappa [\theta - v(t)] \, dt + \sigma, \sqrt{v(t)} \, dz_2(t) \]

where \( z_1(t) \) has correlation \( \rho \) with \( z_2(t) \).

The variance process can reach zero if \( \sigma_v > 2\kappa \theta \). If \( 2\kappa \theta \geq \sigma_v^2 \) the upward drift is sufficiently large to make the zero not reachable. This condition is given by the page 391 of [10]. We will take it as given and omit the discussion of the technical details. From this point on, we will assume that \( 2\kappa \theta \geq \sigma_v^2 \) is satisfied so that variance process is always positive.

By analogy with the Black-Scholes formula, a solution of the following form is first proposed,

\[ C(S, v, t) = SP_1 - KP(t, T)P_2 \]  

where \( P(t, T) = e^{-r(T-t)} \), \( P_1 \) is interpreted as the conditional probability that the option expires in-the-money and \( x = \ln(S) \).

\[ P_1(x, v, T; \ln[K]) = \Pr([x(T) \geq \ln[K] | x(t) = x, \, v(t) = v]) \]

Using the Ito’s formula, we know that the price \( C(S, v, t) \) must satisfy the following PDE.

\[ \frac{1}{2} \sigma_v^2 \frac{\partial^2 C}{\partial S^2} + \rho \sigma_v S \frac{\partial^2 C}{\partial S \partial v} + \frac{1}{2} \sigma_v^2 v \frac{\partial^2 C}{\partial v^2} + rS \frac{\partial C}{\partial S} \]
\[ + \{\kappa [\theta - v(t)] - \lambda(S, v, t)\} \frac{\partial C}{\partial v} - rC + \frac{\partial C}{\partial t} = 0. \]

which subject to the following boundary conditions,
\[ C(S, v, T) = \max(0, S - K), \]
\[ C(0, v, T) = 0, \]
\[ \frac{\partial C}{\partial S}(\infty, v, t) = 1, \]  
\[ rS \frac{\partial C}{\partial S}(S, 0, t) + \kappa \theta \frac{\partial C}{\partial S}(S, 0, t) - rC(S, 0, t) + U_i(S, 0, t) = 0, \]
\[ C(S, \infty, t) = S. \]

Substituting the solution form (2.88) into above PDE shows that \( P_1 \) and \( P_2 \) must satisfy the PDE.
\[
\frac{1}{2} \sigma \frac{\partial^2 P}{\partial x^2} + \rho \sigma \frac{\partial^2 P}{\partial x \partial v} + \frac{1}{2} \sigma^2 \frac{\partial^2 P}{\partial v^2} + (r + u_j v) \frac{\partial P}{\partial x} \\
+ \left[ a_j - b_j v \right] \frac{\partial P}{\partial v} + \frac{\partial P}{\partial t} = 0. 
\] (2.90)

For \( j = 1, 2 \) where \( u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = k \theta, b_1 = \kappa + \lambda - \rho \sigma, b_2 = \kappa + \lambda \).

The condition in (2.89) requires that PDE (2.90) satisfies the terminal condition
\[
P_j(x,v,T; \ln[K]) = 1_{\{x \geq \ln[K]\}}.
\]

Then \( P_j \) can be interpreted as the probability that \( x \geq \ln[K] \), in which \( x \) follows the stochastic process
\[
dx(t) = [r + u_j v] dt + \sqrt{v(t)} dz_1(t),
\]
\[
dv = (a_j - b_j v) dt + \sigma \sqrt{v(t)} dz_2(t)
\]
The PDE (2.90) is difficult to solve. The trick is that the characteristics function of \( P_j \) also satisfy the PDE (2.90) and then the PDE can be reduced to ODEs which are easier to solve.

A guess of the functional form for characteristics function \( f(x,v,t) \) of \( P_j \) is
\[
f(x,v,t) = \exp[C(T - t) + D(T - t)v + i \phi x]. \quad (2.91)
\]

Using this functional form, the PDE can be reduced to two ordinary differential equations,
\[
-\frac{1}{2} \phi^2 + \rho \sigma \phi D + \frac{1}{2} \sigma^2 D^2 v + u_j \phi - b_j D + \frac{\partial D}{\partial t} = 0 \\
r \phi + a \phi + \frac{\partial D}{\partial t} = 0
\] (2.92)

Subject to
\[
C(0) = 0, D(0) = 0.
\]

Note that here (2.92) is different than the original ODE given by [9]. We derived the ODE and failed to reach the same result as [9]. Equation (2.92) is identical to the ODEs derived by [12]. Although the ODE looks different, they do give the same solution. It is non-trivial to solve such an SDE and we will simply present the result.

Then using the technique of characteristic function, \( P_j \) can be obtained in a closed form,
\[ P_j(x,v,T;\ln[K]) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( e^{i\phi \ln[K]} \frac{f_j(x,v,T;\phi)}{i\phi} \right) d\phi. \] (2.93)

where

\[ f_j(x,v,t;\phi) = e^{C(T-t;\phi)+D(T-t;\phi)v^2+i\phi v} \]

and

\[ C(\tau;\phi) = r\phi i\tau + \frac{a}{\sigma^2} \left\{ (b_j - \rho\sigma\phi)i\tau - 2\ln\left[ \frac{1-ge^{\rho\tau}}{1-g} \right] \right\}, \]

\[ D(\tau;\phi) = \frac{b_j - \rho\sigma\phi i + d}{\sigma^2} \left[ \frac{1-ge^{\rho\tau}}{1-g} \right], \]

and

\[ g = \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}, \]

\[ d = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)}. \]

\[ u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = \kappa\theta, b_1 = \kappa + \lambda - \rho\sigma, b_2 = \kappa + \lambda \]

**Heston model and Standard Financial Market**

Heston model is a specific case of Standard Financial Market. To see that, let us make the following setup,

Consider the 2-dimensional Brownian motion \( W(t) = (W^{(1)}(t), W^{(2)}(t))^\top, \mathbb{F}(t), 0 \leq t \leq T \)

and the \((1 \times 2)\)-matrix-valued volatility process \( \sigma(\cdot) \). Write \( \sqrt{v(t)}dW(t) \) out as the follows,

\[ \sqrt{v(t)}d\zeta_1 = \sqrt{v(t)} \left( \frac{\sigma_{11}(t)}{\sqrt{v(t)}} dW^{(1)}(t) + \frac{\sigma_{12}(t)}{\sqrt{v(t)}} dW^{(2)}(t) \right) \]

where

\[ \sigma_{11}^2(t) + \sigma_{12}^2(t) = v(t) \] (2.94)

The variance process itself can be formulated as the follows,

\[ dv(t) = \kappa(\theta - v(t))dt + \sqrt{v(t)}\sigma_v \left( \frac{\sigma_{v1}}{\sigma_v} dW^{(1)}(t) + \frac{\sigma_{v2}}{\sigma_v} dW^{(2)}(t) \right) \]

where
\[ \sigma_{\nu_1}^2(t) + \sigma_{\nu_2}^2(t) = \sigma_v^2 \]  \hspace{1cm} (2.95)

\[ dz_1 dz_2 = \lambda dt \text{ means} \]

\[ \frac{1}{\sigma_v \sqrt{\sigma_{\nu_1}^2 + \sigma_{\nu_2}^2}} (\sigma_{\nu_1} \sigma_{\nu_1} + \sigma_{\nu_2} \sigma_{\nu_2}) = \lambda. \]  \hspace{1cm} (2.96)

Putting every thing together, we have in total 3 equations (2.94), (2.95), (2.96) where 4 unknowns \( \sigma_{\nu_1}, \sigma_{\nu_2}, \sigma_v, \sigma_v \). This means we have infinite solutions. For the sake of simplicity, we choose \( \sigma_{\nu_1} = 0, \sigma_{\nu_2} = \sigma_v \) which gives us two correlated Brownian motions with correlation \( \lambda \). Then we have

\[ \frac{\sigma_{\nu_1}}{\sqrt{\sigma_{\nu_1}^2 + \sigma_{\nu_2}^2}} = \frac{\sigma_{\nu_2}}{\sqrt{\nu}} = \lambda. \]  \hspace{1cm} (2.97)

This setup just fit into the Standard Financial Market setup since \( \nu = \sigma_{\nu_1}^2 + \sigma_{\nu_2}^2 \) is adapted to the filtration \( \mathbb{F}(t) \), where \( \mathbb{F}(t) \) is the filtration generated by 2-dimensional Brownian motion.

When variance is not tradable, we consider it as a model with 1 stock and volatility matrix is \( 1 \times 2 \). When variance is tradable, we can view it as a model with 2 stocks and volatility matrix is \( 2 \times 2 \).

**Corollary 2.5.** Heston model is incomplete when variance is not tradable.

Proof: When variance is not tradable, we view the model with 1 stock and 2 dimensional Brownian motion. Using the Theorem 2.4, the dimension of Brownian motion is higher than number of stocks. Therefore Heston model is incomplete.

Similar to Black-Scholes model, it has also a risk neutralized price and therefore a corresponding SDE’s as follows,

\[ dS(t) = r S(t) dt + \sqrt{\nu(t)} S(t) dz_1(t) \]  \hspace{1cm} (2.98)

\[ dv(t) = \kappa^* [\theta - \nu(t)] dt + \sigma_v \sqrt{\nu(t)} dz_2(t) \]  \hspace{1cm} (2.99)
Where $\kappa^* = \kappa + \lambda$, $\theta^* = \kappa \theta / (\kappa + \lambda)$ \hfill (2.100)
3 ANALYSIS

3.1 Reason for choosing Heston model

What is implied volatility? It is the volatility implied using a model by option prices observed in the market. The following example illustrates how volatility can be implied using Black-Scholes model.

Example 3.1. Suppose that the value of a call option on a non-dividend-paying stock is 1.875 when $S_0 = 21, K = 20, r = 0.1$ and $T = 0.25$. The implied volatility is the value of $\sigma$ that, when substituted into equation (2.73), gives $c = 1.875$.

A well-known observation is that the implied volatility is not a constant but contains a degree of randomness. The implied volatility also exhibits a mean-reverting behavior. Heston model describes a stochastic process with mean-reverting, which can be used to describe the implied volatility process.

Empirical evidences indicate that the underlying is negatively correlated with the implied volatility. We examine Standard & Poor 500 Index and VIX. VIX is the index which represents the level of implied volatility of options of SP500 Index. The specification of the VIX index can be found in page 3 of [15]. In a word, VIX is measure formed by a weighted average of prices of options of different maturity and strikes. In this way, VIX reflects the 1-month implied volatility. Figure 3.1 illustrates the negative correlation of VIX with Standard & Poor 500 index. The index levels are divided by the index close level of 9 May 2000 and multiplied with 100 for standardization. The VIX levels, quoted in volatility points, are used. The data for Standard & Poor 500 Index and VIX are retrieved via historical function of Bloomberg between 9 May 2000 and 29 Oct, 2009. Then correlation calculation is applied to the two sequences using the worksheet function CORREL in Ms Excel. The correlation coefficient of two series is -0.55751.
It is evident in the figure that there is a negative correlation between two sequences. Heston model allows for correlation between underlying process and volatility process to be taken into account.

### 3.2 Option Price and Greeks in Heston model

Heston model gives a formula for European Call Option price. The price of European Put Option can then be calculated using Put-Call parity. Implementing the formula numerically is not so straight-forward as described in section 2.5.4. There are a few things have to be taken care of.

Greeks are partial derivatives of security prices with respect to parameters and variables. Understanding Greeks in Heston model will give us insight about how option price behaves and to what extent it reflects the reality. By comparing the
Greeks in Heston to those in Black-Scholes, the difference between models can be understood.

By applying differentiation under integral sign (See [2] in Appendix) and using the Euler’s formula, the formula for Greeks can be derived. (Some formula will be added to illustrate this). The formula is quite lengthy. For our purpose, it will be sufficient to calculate those Greeks numerically. Specifically, the following will be examined.

3.2.1 How to calculate Price, Delta, Gamma and Vega in Heston model?
Calculating the price given by Heston model seems to be straight-forward. Indeed it is more involved since the price formula contains a Fourier Inverse Transform. It will also be particular useful to gains some experience in it since it is also applicable in other areas. There are two ways which are commonly used to calculate the European option price in Heston model. Numerical Integration (NI) and Fast Fourier Transform (FFT). In this paper we choose NI since we can simply use the integrals given in Section 2.5.4.
First we calculate it using the formula laid out in section 2.5.4. NMath, a C# math library, provide us with the complex function calculation and an adaptive quadrature method to calculate integration. Let us first check the price at maturity. The option price at maturity is as Table 3.1,

<table>
<thead>
<tr>
<th>Strik e</th>
<th>Price</th>
<th>Numerical Calculated Price</th>
<th>Strike</th>
<th>Price</th>
<th>Numerical Calculated Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>50</td>
<td>50.31541</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>55</td>
<td>45</td>
<td>46.94504</td>
<td>105</td>
<td>0</td>
<td>-0.00244</td>
</tr>
<tr>
<td>60</td>
<td>40</td>
<td>40.95157</td>
<td>110</td>
<td>0</td>
<td>-0.01656</td>
</tr>
<tr>
<td>65</td>
<td>35</td>
<td>36.67554</td>
<td>115</td>
<td>0</td>
<td>-0.00157</td>
</tr>
<tr>
<td>70</td>
<td>30</td>
<td>32.09322</td>
<td>120</td>
<td>0</td>
<td>-0.03473</td>
</tr>
<tr>
<td>75</td>
<td>25</td>
<td>24.99385</td>
<td>125</td>
<td>0</td>
<td>0.035529</td>
</tr>
<tr>
<td>80</td>
<td>20</td>
<td>20.02842</td>
<td>130</td>
<td>0</td>
<td>-0.00136</td>
</tr>
<tr>
<td>85</td>
<td>15</td>
<td>14.98061</td>
<td>135</td>
<td>0</td>
<td>-0.00586</td>
</tr>
<tr>
<td>90</td>
<td>10</td>
<td>9.996754</td>
<td>140</td>
<td>0</td>
<td>-6.19655</td>
</tr>
<tr>
<td>95</td>
<td>5</td>
<td>4.983463</td>
<td>145</td>
<td>0</td>
<td>-1.84969</td>
</tr>
</tbody>
</table>
We can see that the calculation for in-the-money option is close to the payoff while the out-the-money option shows a slightly negative price. For the out-the-money options, it gives a value close to zero but not accurate enough. There is even a jump in the numerical calculated price on the strike 140.

If we just implement the Heston formula, it is working fine for short maturity up to 2 year and 4 month. For longer maturities, the price explodes and even gives an overflow error. It is shown in Table 3.2.

After some investigation, we find out that the overflow is due to the exponential function. Another difficulty is that the upper bound is infinity and it will require us to choose a large number for numerical calculation. Change of variable can be applied so that the integration interval can be changed from \([0, \infty]\) to \([0,1]\). It will solve the overflow problems.

Let \(\phi(u) = -\ln u\). It is easy to see that \(\phi \in [0, \infty]\) implies \(x \in [0,1]\). Change the variable in the formula (2.93)

\[
P_j(x,v,T;\ln[K]) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\ln[K]} \text{Re}[e^{-i\phi\ln[K]} f_j(x,v,T;\phi)] d\phi
\]

\[
= \frac{1}{2} + \frac{1}{\pi} \int_0^{\ln[K]} \text{Re}[e^{-i\phi\ln[K]} f_j(x,v,T;u(y))] du(y)
\]

\[
= \frac{1}{2} + \frac{1}{\pi} \int_0^{\ln[K]} \text{Re}\left[ e^{-i(-\ln[y]\ln[K])} f_j(x,v,T;-\ln[y]) \right] d\ln[y]
\]

\[
= \frac{1}{2} + \frac{1}{\pi} \int_0^{\ln[K]} \text{Re}\left[ e^{-i(-\ln[y]\ln[K])} f_j(x,v,T;-\ln[y]) \right] \frac{1}{y} dy
\]

Now the calculation shows a better result. This can be seen in Table 3.3. For the maturity such as one year, it gives a price close to the one without changing of variable. The revised computation gives a reasonable price up to 2 year and 6 months.

What catch our attention is that the option price first goes up then down. In Black-
Scholes model, the option price only goes up when time to maturity is increasing since $\theta := \frac{\partial C}{\partial \tau}$ ($\tau$ is the time to maturity) is non-positive. The reason of oscillation is discussed in the paper [16]. It is caused by the discontinuity of the complex power function. Let us get some insight about it.

We rewrite the function

$$C(\tau, u) = R(\tau, u) - 2\alpha \ln G(\tau, u) \quad (3.1)$$

$$R(\tau, u) := \alpha(\kappa - \rho \sigma, ui + d)\tau$$

$$\alpha := \frac{\kappa \theta}{\sigma^2}$$

$$G(\tau, u) := \frac{ce^{\alpha \tau} - 1}{c - 1}$$

$$f_i = G(\tau, u)^{-2\alpha} e^{r(\tau, u) + D(\tau, u) V_i + iu \ln S}$$

$G(\tau, u)$ is a complex number. It can be written as

$$z = a_x + ib_x = r_x e^{\theta_x} \tau_x \in [ -\pi, \pi )$$

Whenever the $G(\tau, u)$ cross the negative real axis along its path (as $u$ varies), the phase of $G(\tau, u)$ changes from $-\pi$ to $\pi$ and therefore the phase of $G(\tau, u)^\alpha$ changes from $-\pi \alpha$ to $\pi \alpha$. This is the cause of the jump since

$$\begin{cases} e^{\pi \alpha} \neq e^{-i\pi \alpha} & \text{if } \alpha \notin \mathbb{Z} \\ e^{i\pi \alpha} = e^{-i\pi \alpha} & \text{if } \alpha \in \mathbb{Z} \end{cases}$$

So when $G(\tau, u)$ cross the negative real axis, a discontinuity is caused.

**Table 3.3 Call Option Price with Different Maturities after Applying Change of Variable.**

<table>
<thead>
<tr>
<th>Year</th>
<th>Month</th>
<th>Price</th>
<th>Year</th>
<th>Month</th>
<th>Price</th>
<th>Year</th>
<th>Month</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>15.77466</td>
<td>2</td>
<td>2</td>
<td>10.79995</td>
<td>3</td>
<td>4</td>
<td>-18643821.56</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>15.95160</td>
<td>2</td>
<td>3</td>
<td>9.48004</td>
<td>3</td>
<td>5</td>
<td>-95302977.25</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>16.04626</td>
<td>2</td>
<td>4</td>
<td>7.94845</td>
<td>3</td>
<td>6</td>
<td>-464799846.2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>16.08044</td>
<td>2</td>
<td>5</td>
<td>5.89813</td>
<td>3</td>
<td>7</td>
<td>-2395195944</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>16.04327</td>
<td>2</td>
<td>6</td>
<td>2.37170</td>
<td>3</td>
<td>8</td>
<td>-12301198859</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>15.93160</td>
<td>2</td>
<td>7</td>
<td>-9.10843</td>
<td>3</td>
<td>9</td>
<td>-5924105504</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>15.75080</td>
<td>2</td>
<td>8</td>
<td>-61.68170</td>
<td>3</td>
<td>10</td>
<td>-2.92983E+11</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>15.48552</td>
<td>2</td>
<td>9</td>
<td>-307.21816</td>
<td>3</td>
<td>11</td>
<td>-1.31093E+12</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>15.13540</td>
<td>2</td>
<td>10</td>
<td>-1550.45213</td>
<td>4</td>
<td>0</td>
<td>-5.56796E+12</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>14.70893</td>
<td>2</td>
<td>11</td>
<td>-7329.18688</td>
<td>4</td>
<td>1</td>
<td>-1.82403E+13</td>
</tr>
</tbody>
</table>
Let us set the parameters so that \( \alpha \) becomes integer so that we can focus on the behavior of the model without being distracted by the discontinuity problem. We get Table 3.4. We can see that that the discontinuity indeed disappeared.

### Table 3.4 Call Option Price with Different Time to Maturity (TtM) without Discontinuity

\[
S = 100, \kappa = 1, \theta = 0.25, \lambda = 1.0, \rho = 0.3, \sigma = 0.5, \nu = 0.25, \ r = 0.03. \ So = -2
\]

<table>
<thead>
<tr>
<th>TtM</th>
<th>Price</th>
<th>TtM</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.344872</td>
<td>11</td>
<td>17.96877</td>
</tr>
<tr>
<td>2</td>
<td>8.02088</td>
<td>12</td>
<td>18.67621</td>
</tr>
<tr>
<td>3</td>
<td>9.908177</td>
<td>13</td>
<td>19.37717</td>
</tr>
<tr>
<td>4</td>
<td>11.39324</td>
<td>14</td>
<td>20.03052</td>
</tr>
<tr>
<td>5</td>
<td>12.61221</td>
<td>15</td>
<td>20.68323</td>
</tr>
<tr>
<td>6</td>
<td>13.72287</td>
<td>16</td>
<td>21.31593</td>
</tr>
<tr>
<td>7</td>
<td>14.69374</td>
<td>17</td>
<td>21.91131</td>
</tr>
<tr>
<td>8</td>
<td>15.61569</td>
<td>18</td>
<td>22.51089</td>
</tr>
<tr>
<td>9</td>
<td>16.47289</td>
<td>19</td>
<td>23.07744</td>
</tr>
<tr>
<td>10</td>
<td>17.20223</td>
<td>20</td>
<td>23.64999</td>
</tr>
</tbody>
</table>

The delta of a European call option is defined as,

\[
\Delta = \frac{\partial C}{\partial S} = \partial (S P_1 - K P_2) / \partial S = P_1 + S \frac{\partial P_1}{\partial S} - K \frac{\partial P_2}{\partial S}
\]  \hspace{1cm} (3.2)

So now let us calculate \( \frac{\partial P_i}{\partial S}, i = 1, 2 \)

\[
\frac{\partial P_i}{\partial S} = \frac{1}{\pi} \int_{0}^{\infty} \partial (\text{Re}[\frac{e^{i\phi \ln[K]} e^{-C(T-i\phi)+D(T-i\phi)+i\phi x}}{i\phi}]) / \partial \phi d\phi
\]  \hspace{1cm} (3.3)

\[
\frac{\partial P_i}{\partial S} = \frac{1}{\pi} \int_{0}^{\infty} \text{Re}[\partial (\frac{e^{i\phi \ln[K]} e^{-C(T-i\phi)+D(T-i\phi)+i\phi x}}{i\phi})] / \partial \phi d\phi
\]  \hspace{1cm} (3.4)

Equation (3.3) holds due to the Differentiation under integral sign\(^2\). And (3.4) holds since \( S \) is real. Notice that the only term in exponents involve \( S \) is \( i\phi x \), it is quite straightforward to take the derivative. Strictly speaking, the function \( \text{Re}[z] \) might not

---

\(^2\) See Appendix for the formula.
be differentiable with respect to $z \in C$ where $C$ denotes complex number. But the

$$\lim_{z \to z_0} \frac{\text{Re}[z-z_0] - \text{Re}[z_0]}{z}$$

does exists with respect to change of $S$.

$$\frac{\partial P}{\partial S}$$

$$= \frac{1}{\pi} \int_0^\infty \partial(\text{Re}[\frac{e^{-i\phi \ln[K]+C(T-t,\phi)+D(T-t,\phi)+i\phi S}{i\phi}])/\partial S d\phi$$

$$= \frac{1}{\pi} \int_0^\infty \Re[\frac{e^{-i\phi \ln[K]+C(T-t,\phi)+D(T-t,\phi)+i\phi S}{i\phi} \cdot i\phi \cdot \frac{1}{S}d\phi$$

$$= \frac{1}{\pi} \int_0^\infty \Re[\frac{e^{-i\phi \ln[K]+C(T-t,\phi)+D(T-t,\phi)+i\phi S}{S}]d\phi$$

(3.5)

Plug the above result into (3.2), the delta is,

$$\Delta = P_1 + \frac{S}{\pi} \int_0^\infty \Re[\frac{e^{-i\phi \ln[K]} f_1(x,\nu,T;\phi)}{S}]d\phi - K \int_0^\infty \Re[\frac{e^{-i\phi \ln[K]} f_1(x,\nu,T;\phi)}{S}]d\phi$$

(3.6)

When evaluating $\Delta$ at maturity, the value of $\Delta$ is as Table 3.5.

<table>
<thead>
<tr>
<th>Strike</th>
<th>Delta</th>
<th>Calculated Delta</th>
<th>Strike</th>
<th>Delta</th>
<th>Calculated Delta</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>1</td>
<td>1.3172</td>
<td>100</td>
<td>0</td>
<td>0.5000</td>
</tr>
<tr>
<td>55</td>
<td>1</td>
<td>1.1107</td>
<td>105</td>
<td>0</td>
<td>0.1483</td>
</tr>
<tr>
<td>60</td>
<td>1</td>
<td>0.8215</td>
<td>110</td>
<td>0</td>
<td>-0.1292</td>
</tr>
<tr>
<td>65</td>
<td>1</td>
<td>0.7042</td>
<td>115</td>
<td>0</td>
<td>-0.2941</td>
</tr>
<tr>
<td>70</td>
<td>1</td>
<td>0.8132</td>
<td>120</td>
<td>0</td>
<td>-0.3375</td>
</tr>
<tr>
<td>75</td>
<td>1</td>
<td>1.0418</td>
<td>125</td>
<td>0</td>
<td>-0.2749</td>
</tr>
<tr>
<td>80</td>
<td>1</td>
<td>1.2354</td>
<td>130</td>
<td>0</td>
<td>-0.1379</td>
</tr>
<tr>
<td>85</td>
<td>1</td>
<td>1.2813</td>
<td>135</td>
<td>0</td>
<td>0.0349</td>
</tr>
<tr>
<td>90</td>
<td>1</td>
<td>1.1446</td>
<td>140</td>
<td>0</td>
<td>0.2061</td>
</tr>
<tr>
<td>95</td>
<td>1</td>
<td>0.8594</td>
<td>145</td>
<td>0</td>
<td>0.3446</td>
</tr>
</tbody>
</table>

The delta is correct at-the-money strike. But it is incorrect for in-the-money and out-the-money strike. And an oscillating behavior comes back again. Remember that we have used the specific parameters to avoid the discontinuity. What are causing this? Let us check analytically what is going here.
\[
\frac{\partial P_i}{\partial S} = \frac{1}{\pi} \int_0^\infty \text{Re}\left[ \frac{e^{i\phi \ln(K)}}{S} \right] d\phi
\]

\[
= \frac{1}{\pi} \int_0^\infty \frac{1}{S} \text{Re}\left[ e^{i\phi \ln(S/K)} \right] d\phi
\]

\[
= \frac{1}{\pi} \int_0^\infty \frac{1}{S} \text{Re}\left[ \cos(\phi \ln(S/K)) + i \sin(\phi \ln(S/K)) \right] d\phi
\]

\[
= \frac{1}{\pi} \int_0^\infty \frac{1}{S} \cos(\phi \ln(S/K)) d\phi
\]

The last equation does not converge. Therefore the second term and the third term in (3.6) will fluctuate when \( S \) is not equal to \( K \). That is why \( \Delta \) is oscillating when the strike is changing. Since even a perfect replicating of a European Option will not be differentiable at maturity, we will leave it as it is and look into delta of longer maturities.

Now let us look at the delta of an option when \( S \) is changing.
What we can see is that the shapes of the prices are quite similar. The prices given by Heston model are negative for some index levels around 60. This is due to some numerically error. Black-Scholes model is numerically more robust. For our analysis purpose, we will use the price and will not examine the numerical error at the moment.

In the Figure 3.2, the volatility in Black-Scholes model is assumed to be equal to the instantaneous volatility. Compared to Black-Scholes model, the call option price in Heston model is less when spot level is near the strike and is larger when spot level is larger and far away from the strike.

Let us put more maturities in one picture to get a clear overview of Heston price of different maturities.
As shown in Figure 3.3, the price of an at-the-money European call option increases when time to maturity increases for both Heston model and Black-Scholes model. For longer maturity, the price difference between Heston model and Black-Scholes model become more significant with longer time to maturity. For maturity longer than 0.5 year, the difference become larger when index level increases.

For an out-the-money European call option
For an in-the-money European call option

Now let us look at the delta of options of different strike at a date.

For longer maturity, the delta is shown in Table 3.6.
Table 3.6 Delta of European Call Option with Strike =100

<table>
<thead>
<tr>
<th>Year</th>
<th>Month</th>
<th>Delta</th>
<th>Year</th>
<th>Month</th>
<th>Delta</th>
<th>Year</th>
<th>Month</th>
<th>Delta</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S=80</td>
<td></td>
<td></td>
<td>S=100</td>
<td></td>
<td></td>
<td>S=120</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-0.199131</td>
<td>0</td>
<td>0</td>
<td>0.500000</td>
<td>0</td>
<td>0</td>
<td>1.254703</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.021469</td>
<td>0</td>
<td>1</td>
<td>0.532423</td>
<td>0</td>
<td>1</td>
<td>0.970672</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0.135059</td>
<td>0</td>
<td>2</td>
<td>0.541419</td>
<td>0</td>
<td>2</td>
<td>0.864551</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>0.198915</td>
<td>0</td>
<td>3</td>
<td>0.548306</td>
<td>0</td>
<td>3</td>
<td>0.818181</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>0.237025</td>
<td>0</td>
<td>4</td>
<td>0.553779</td>
<td>0</td>
<td>4</td>
<td>0.795720</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>0.264375</td>
<td>0</td>
<td>5</td>
<td>0.558541</td>
<td>0</td>
<td>5</td>
<td>0.781892</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
<td>0.283914</td>
<td>0</td>
<td>6</td>
<td>0.562412</td>
<td>0</td>
<td>6</td>
<td>0.773067</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
<td>0.299276</td>
<td>0</td>
<td>7</td>
<td>0.565749</td>
<td>0</td>
<td>7</td>
<td>0.766709</td>
</tr>
<tr>
<td>0</td>
<td>8</td>
<td>0.311017</td>
<td>0</td>
<td>8</td>
<td>0.568494</td>
<td>0</td>
<td>8</td>
<td>0.762201</td>
</tr>
<tr>
<td>0</td>
<td>9</td>
<td>0.319677</td>
<td>0</td>
<td>9</td>
<td>0.570650</td>
<td>0</td>
<td>9</td>
<td>0.759094</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>0.326326</td>
<td>0</td>
<td>10</td>
<td>0.572414</td>
<td>0</td>
<td>10</td>
<td>0.756867</td>
</tr>
<tr>
<td>0</td>
<td>11</td>
<td>0.330849</td>
<td>0</td>
<td>11</td>
<td>0.573711</td>
<td>0</td>
<td>11</td>
<td>0.755475</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.333762</td>
<td>1</td>
<td>0</td>
<td>0.574661</td>
<td>1</td>
<td>0</td>
<td>0.754701</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.335022</td>
<td>1</td>
<td>1</td>
<td>0.575240</td>
<td>1</td>
<td>1</td>
<td>0.754533</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.334799</td>
<td>1</td>
<td>2</td>
<td>0.575462</td>
<td>1</td>
<td>2</td>
<td>0.754875</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.333057</td>
<td>1</td>
<td>3</td>
<td>0.575392</td>
<td>1</td>
<td>3</td>
<td>0.755806</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>9.93E+141</td>
<td>1</td>
<td>4</td>
<td>-1.11E+142</td>
<td>1</td>
<td>4</td>
<td>-1.24E+142</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>NaN</td>
<td>1</td>
<td>5</td>
<td>NaN</td>
<td>1</td>
<td>5</td>
<td>NaN</td>
</tr>
</tbody>
</table>

For in-the-money and at the money option, there is a trend that delta increases when time to maturity. For out-the-money options, the trend is that the delta decrease when the time to maturity increase. Both trends make sense according to our experience in Black-Scholes model. A robust implementation of numerical integration can be itself a paper and let us focus on the behavior of the model from now on.

Similarly, we can also derive Vega of a call option.

\[
vega = \frac{\partial C}{\partial \nu} = \frac{\partial(SP_1 - Kp_1)}{\partial \nu} = S \frac{\partial P_1}{\partial \nu} - K \frac{\partial P_2}{\partial \nu} \quad (3.7)
\]

Where

\[
\frac{\partial P_1}{\partial \nu} = \frac{1}{\pi} \int_0^\infty \text{Re}[\frac{e^{-i\phi \ln[K]}}{i\phi} f_1(x, \nu, T; \phi) \cdot D_1(T-t; \phi)] d\phi
\]

\[
\frac{\partial P_2}{\partial \nu} = \frac{1}{\pi} \int_0^\infty \text{Re}[\frac{e^{-i\phi \ln[K]}}{i\phi} f_1(x, \nu, T; \phi) \cdot D_1(T-t; \phi)] d\phi
\]

Vega contains terms \( D_1(T-t; \phi) \) in the integrand. Since \( D_1(T-t; \phi) \) contains also the complex exponential function, it will also result in discontinuities. A workaround of
the discontinuities is to choose an integer for time to maturity. The workaround would be sufficient for the purpose of understanding Vega. But for the portfolio analysis it requires that the calculation should handle all maturities. And how those Greeks change with response to spot level and time will be discussed. Then the Greeks will be compared to those in Black-Scholes model. The comparison will be done on options whose maturity is short term (2 weeks and 4 weeks).

3.2.2 How do Heston model specific parameters affect the price of an option?

Parameters in Heston model such as market price of volatility, mean-reverting speed, volatility of volatility and correlation are not present in Black-Scholes model. The sensitivity of option price to those parameters is the dynamics of the model. Those parameters can reflect the market price better. This subsection will focus on those dynamics. The understanding of the parameters lays down the fundamentals for more complicated analysis, such as how the Heston model can fit the whole implied volatility surface. The option price formula in Heston model is quite complicated, which makes it difficult to check the sensitivity in the analytical way. We therefore will use the numerical method. The parameters can be classified into two categories, the parameters that control the shape of volatility curve and the parameters that control the behavior of the curve over time. Market price of volatility $\lambda$, volatility of volatility $\sigma_v$ and correlation $\rho$ belong to the first category. Mean reverting speed $\kappa$ belongs to the second category. We will now look at the parameters in first category.

3.2.2.1 Market price of volatility

First let us look at the sensitivity of price to market price of volatility. We can see in Figure 3.4 that the implied volatility curve shift down when $\lambda$ increases. Changing $\lambda$ can influences all volatility curves since $\theta^* = \kappa \theta / (\kappa + \lambda)$. It will decrease the long term mean level that variance process will revert to. Therefore the implied volatility in Black-Scholes model shift down when $\lambda$ increases.
Figure 3.4 Sensitivity of implied volatility from Heston Price to market price of volatility risk,
$\kappa=1$, $\theta=0.05$, $\sigma_v=0.01$, $\rho=-0.5$, $V=0.05$, $r=0.0$. $\tau=1$ year.
3.2.2.2 Volatility of Volatility

Figure 3.5 Sensitivity of Heston Price to volatility of volatility in terms of B-S implied volatility.

S=1, κ=1, 0=0.05, ρ=0, λ=0.1, S=100, V=0.05, r=0, τ = 1 year, ρ = 0.0001.

The effect of volatility of volatility is shown in Figure 3.5. When volatility of volatility is small, the variance process stays constant and the implied volatility curve is constant at all strike levels, which resembles Black-Scholes model. It can be seen that the convexity increases as volatility of volatility increases. The reason is that the volatility of volatility increases the tail of the distribution of the option return. Therefore implied volatility increases for the options that are far away from the ATM level.
3.2.2.3 Correlation

Figure 3.6 Sensitivity of Heston Price to correlation, $S=1$, $\kappa=1$, $\theta=0.05$, $\lambda=0.1$, $S=100$, $V=0.05$, $r=0$, $\sigma=0.01$, $\tau=1$ year,

Figure 3.6 shows that the slope of the implied volatility curve becomes more negative when the correlation becomes more negative. This can be explained as follows: when correlation become more negative, the variance and stock price will have the tendency to move in the opposite directions. When the variance increases, stock price will more likely decrease due to the more negative correlation. This is reflected directly in the above figure.

When the voloVol is close to zero such as 0.0001, the correlation will have no significant effect on volatility curve.

We have illustrated the effect of first category parameters on volatility curve. Now we will examine the effect of mean reverting speed.
3.2.2.4 **Mean reverting speed**

Mean reverting speed $\kappa$, as the name implies, defines how quick the variance process reverting to its long term mean. When $\kappa$ is large, the variance process will stick to long term mean and the diffusion term will have less influence. This makes the variance stay close to the long mean level and the model behaves similar to Black-Scholes model.

When $\kappa$ is small, the variance process have a very small drift term. $\kappa$ will determine how quick the volatility curve move over time towards the long term mean level. If we consider using the ATM volatility as a proxy for the volatility curve, the ATM volatility will move towards the long term mean level in time.

When the mean reverting speed is non-zero, the variance process moves from initial value to the long term mean. When it reaches the level near the long term mean, the mean will oscillate around that level in a speed defined by $\kappa$. The larger $\kappa$ is, the sooner it takes for variance to go back to the long term mean level.

If we consider the ATM volatility levels over maturities a function of $v(t)$, $v(t)$ in Heston model will be a monotone function. When $\kappa$ is positive, the volatility curves will shift over maturities towards the long term mean $\theta$ and starts oscillating after reaching $\theta$. The above situation is illustrated in Figure 3.7. We can see that the convexity of curve is decreasing as the maturity varies from 2 months to 1 year. When $\kappa$ is negative, the volatility curves will shift over maturities away from the current variance level. The original paper [10] focuses on a positive $\kappa$. For our purpose, a negative $\kappa$ is required in order to model the case of a volatility rise from long term mean or a volatility fall from the long term mean. The situation is illustrated in Figure 3.7. We can see that the volatility curve is shifting away from the long term mean. In addition to that, the convexity of curve is increasing as the maturity varies from 2 months to 1 year.
Figure 3.7 Sensitivity of Heston Price to Kappa in terms of Black-Scholes implied volatility curve,
$S=1$, $\kappa=1$, $\theta=0.05$, $\rho=0.0001$, $\lambda=0.1$, $S=100$, $V=0.03$, $r=0$, $\sigma=0.5$. 
Figure 3.8 Sensitivity of Heston Price to Kappa in terms of Black-Scholes implied volatility curve., S=1, κ=1, θ=0.05, ρ=0.0001, λ=0.1, S=100, V=0.03,r=0, σ=0.5,

Such a behavior defined by κ has made it possible to fit one Heston model to each volatility curve. It also implies that Heston model is not capable of modeling the case that the volatility drifts towards the long term mean and drift away subsequently. In [18], it considers fitting the Heston model to the whole volatility surface/available maturities. The analysis suggests the Heston model cannot give a satisfactory fit for long maturity as well as short maturity. This can also be seen as evidence that Heston model is not capable of fitting the whole surface. In the following subsection we will see some examples about this.

As we discussed, we can summarize the effect of parameters as follows: 3 parameters defines the shape of each volatility curve and the kappa is left to control the evolution of volatility curves along the time. Those understanding will help us to grasp how the Heston model is capable of fitting option prices of different maturities.
3.2.3 How to fit the parameters for Heston model using option market price?

We consider calibrating the parameters using market price directly. In this paper, we will focus on the European Options by looking into European Option on EURO STOXX 500 Index. The market prices of European options are obtained from Equity Derivative Strategy Group of Deutsche Bank. The average of ask and bid price, so called mid price are used. The prices are quoted in terms of implied volatility per strike. The strikes are given in the form of percentage of spot levels. The volatilities of maturities such as 1, 2, 3, 6, 12 months are available. The data is from 16 December, 1999 to 29 January, 2010.

Other observable parameters such as levels of EURO STOXX 50 Index, interest rate are obtained from Bloomberg. Using those parameters and Black-Scholes formula, we can calculate the market prices of European options.

Now we start by leaving all parameter empty and feed the prices of a given date to a optimizer to get a parameters settings which minimize the error. Initially the following error function is used.

\[
\sum_{i \in O(i)} \left( P(i) - H(i; \theta) \right)^2
\]

Where \( O(i) \) denote the all the option whose market price are known at date \( t \), \( P(i) \) denotes the price of a European call option, \( H(i; \theta) \) denotes the price using a parameter setting \( \theta \). Such an error function results in a parameter setting which have relative less error for in-the-money options than out-the-money call options. For instance, 1 euro for an option priced 100 euro is a small fitting error. But it is a large error for an option priced 0.50 euro. Therefore we choose the error function which considers the relative error compared to the market price.

\[
\sum_{i \in O(i)} \left( \frac{P(i) - H(i; \theta)}{P(i)} \right)^2
\]

which makes the relative errors small for options of all strikes.

We now use a number of different optimization methods to find the parameter setting. First we use default method provided by \texttt{optim} function in R. The default algorithm is Nelder-Mead method (see [3] in Appendix). As the Heston pricing formula is implemented with C#, R cannot call our pricing formula directly. We expose the Heston pricing formula as COM interface so that R can make calls. This does takes
some effects in programming. Having the ability to calculate option price using Heston pricing formula, the optimization in R is an easy task. Parameter settings and related error is shown as setting 1 in Table 3.7.

### Table 3.7 Parameter settings after fitting the market prices of all maturities on 17 Nov 2003, which is a median point for ATM implied volatility. SSE stands for sum of squared error. AE stands for average error which is the square root of SSE/number of option quote.

<table>
<thead>
<tr>
<th>Settings</th>
<th>(\kappa)</th>
<th>(\theta)</th>
<th>(\lambda)</th>
<th>(\rho)</th>
<th>(\sigma)</th>
<th>(\nu)</th>
<th>SSE/AE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.0625</td>
<td>0.1</td>
<td>-0.5</td>
<td>0.1</td>
<td>0.25</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>1.4054</td>
<td>0.0043</td>
<td>-0.5287</td>
<td>-0.2525</td>
<td>0.3035412</td>
<td>0.049283</td>
<td>2.6354/27.44%</td>
</tr>
<tr>
<td>2</td>
<td>1.0452</td>
<td>0.0104</td>
<td>0.1503</td>
<td>-0.5034</td>
<td>0.0011</td>
<td>0.0469</td>
<td>2.6921/27.73%</td>
</tr>
<tr>
<td>3</td>
<td>3.9965</td>
<td>0.0628*</td>
<td>0.000054</td>
<td>-0.5391</td>
<td>1.1736</td>
<td>0.0487*</td>
<td>1.3610/19.72%</td>
</tr>
</tbody>
</table>

The function does not work with constraints explicitly. The constraints has been added using the penalty multiplier. Without the constraints, the `optim` function gives negative \(\sigma\), which is invalid for Heston model. The constraints are,

\[
\kappa > 0, \theta > 0, \rho < 0, \sigma > 0, \nu > 0
\]

Setting 0 gives the initial value of the optimization. Setting 1 is the setting given by `optim` function using Nelder-Mead method. Nelder-Mead method is a common-used heuristic method for non-linear optimization problem. The advantage is that it is robust and simple to implement. The disadvantage is that it can be slow for simple problem and can converge to a non-stationary point. The algorithm is described in the appendix.

Other methods provided by `optim`, such as CG, BFGS, are also used to fit the market prices but the fit gives larger error and the iteration of those methods do not converge. In setting 2, we used the optimizer provided by Microsoft Excel, which implement Generalized Reduced Gradient method. Another advantage of this optimizer is that constraints can be added. For instance, the variance must be non-negative can be added as a constraint to this optimizer. The result of optimization is close to the result given by Nelder-Mead method.

In setting 3, we reduce the dimension of the parameter setting space by setting \(\theta = 0.0628\). The value is estimated from volatility of EURO STOX 50 levels. The reason we set \(\theta\) is that the observation indicates that the minimization is sensitive to the initial conditions. The optimization gives a local optimum sometimes. Fixing \(\theta\) indeed reduces the fitting error.
In Table 3.8, we can see the relative errors produced by parameter setting 1 is largest (-99.87%) for the call option with 1 month maturity and strike 3361.865, which is deep out-the-money. We can observe that the errors tend to increase when the option is deeper out-the-money. This tendency seems to become weaker when time to maturity is increasing. In the Figure 3.9, the relative error is less for 1-month maturity and more for 6-month maturity. The detailed table of market price and Heston price is given as Table 3.8.
Relative difference of prices on 11/17/03 produced by Heston model

Figure 3.9 Absolute Relative difference of parameter setting 3. Abs(Difference Error) is used. Relative Difference=(Heston Price-Market Price)/Market Price.

In Table 3.8, Black-Scholes price produced by fitting the market prices of all maturities is also shown. By minimizing the fitting error, we get a volatility parameter and it is used to calculate the Black-Scholes prices. We can see that the average of absolute relative difference of Heston prices is 11.03%, which is less than 17.56% of Black-Scholes prices. Heston model fits the whole volatility surface, represented by the market prices of all maturities, better than Black-Scholes model.
Table 3.8 Market Prices and Heston Price based on parameter setting 1 and on 17 03 2003. With average absolute relative difference 11.03% for Heston Price and 17.56% for Black-Scholes Price.

<table>
<thead>
<tr>
<th>TtM</th>
<th>Strike</th>
<th>MarketPrice</th>
<th>HestonPrice</th>
<th>Relative Diff Heston</th>
<th>Relative Diff BS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.083333</td>
<td>1810.235</td>
<td>780.1309852</td>
<td>-0.14%</td>
<td>-0.15%</td>
</tr>
<tr>
<td>2</td>
<td>0.083333</td>
<td>2068.84</td>
<td>523.906116</td>
<td>-0.44%</td>
<td>-0.58%</td>
</tr>
<tr>
<td>3</td>
<td>0.083333</td>
<td>2327.445</td>
<td>274.0303495</td>
<td>-1.10%</td>
<td>-3.53%</td>
</tr>
<tr>
<td>4</td>
<td>0.083333</td>
<td>2586.05</td>
<td>67.95968908</td>
<td>-4.70%</td>
<td>-9.95%</td>
</tr>
<tr>
<td>5</td>
<td>0.083333</td>
<td>2844.655</td>
<td>3.315206294</td>
<td>-27.11%</td>
<td>-0.59%</td>
</tr>
<tr>
<td>6</td>
<td>0.083333</td>
<td>3103.26</td>
<td>0.136371119</td>
<td>0.43%</td>
<td>-74.51%</td>
</tr>
<tr>
<td>7</td>
<td>0.083333</td>
<td>3361.865</td>
<td>0.064997723</td>
<td>-41.71%</td>
<td>-99.87%</td>
</tr>
<tr>
<td>8</td>
<td>1.666667</td>
<td>1810.235</td>
<td>785.4815994</td>
<td>-0.29%</td>
<td>-6.89%</td>
</tr>
<tr>
<td>9</td>
<td>1.666667</td>
<td>2068.84</td>
<td>534.0105914</td>
<td>-0.79%</td>
<td>-12.19%</td>
</tr>
<tr>
<td>10</td>
<td>1.666667</td>
<td>2327.445</td>
<td>14.19342672</td>
<td>-26.02%</td>
<td>-1.54%</td>
</tr>
<tr>
<td>11</td>
<td>1.666667</td>
<td>2586.05</td>
<td>1.83716512</td>
<td>-40.47%</td>
<td>-41.36%</td>
</tr>
<tr>
<td>12</td>
<td>1.666667</td>
<td>2844.655</td>
<td>0.639737453</td>
<td>-75.41%</td>
<td>-93.19%</td>
</tr>
<tr>
<td>13</td>
<td>1.666667</td>
<td>3103.26</td>
<td>0.064997723</td>
<td>-41.71%</td>
<td>-99.87%</td>
</tr>
<tr>
<td>14</td>
<td>0.25</td>
<td>1810.235</td>
<td>790.9865899</td>
<td>-0.30%</td>
<td>-0.71%</td>
</tr>
<tr>
<td>15</td>
<td>0.25</td>
<td>2068.84</td>
<td>543.1747977</td>
<td>-0.71%</td>
<td>-2.62%</td>
</tr>
<tr>
<td>16</td>
<td>0.25</td>
<td>2327.445</td>
<td>311.4847234</td>
<td>-2.03%</td>
<td>-7.89%</td>
</tr>
<tr>
<td>17</td>
<td>0.25</td>
<td>2586.05</td>
<td>123.2004539</td>
<td>-8.36%</td>
<td>-11.72%</td>
</tr>
<tr>
<td>18</td>
<td>0.25</td>
<td>2844.655</td>
<td>26.70247711</td>
<td>-23.04%</td>
<td>-1.12%</td>
</tr>
<tr>
<td>19</td>
<td>0.25</td>
<td>3103.26</td>
<td>4.08595688</td>
<td>-17.00%</td>
<td>0.45%</td>
</tr>
<tr>
<td>20</td>
<td>0.25</td>
<td>3361.865</td>
<td>0.939691684</td>
<td>-27.83%</td>
<td>-54.31%</td>
</tr>
<tr>
<td>21</td>
<td>0.5</td>
<td>1810.235</td>
<td>809.2821474</td>
<td>-0.29%</td>
<td>-1.73%</td>
</tr>
<tr>
<td>22</td>
<td>0.5</td>
<td>2068.84</td>
<td>572.0284766</td>
<td>-0.74%</td>
<td>-4.68%</td>
</tr>
<tr>
<td>23</td>
<td>0.5</td>
<td>2327.445</td>
<td>354.4895539</td>
<td>-1.96%</td>
<td>-9.15%</td>
</tr>
<tr>
<td>24</td>
<td>0.5</td>
<td>2586.05</td>
<td>176.8335814</td>
<td>-5.88%</td>
<td>-10.92%</td>
</tr>
<tr>
<td>25</td>
<td>0.5</td>
<td>2844.655</td>
<td>63.71498857</td>
<td>-12.00%</td>
<td>-1.03%</td>
</tr>
<tr>
<td>26</td>
<td>0.5</td>
<td>3103.26</td>
<td>16.87440907</td>
<td>-5.92%</td>
<td>23.83%</td>
</tr>
<tr>
<td>27</td>
<td>0.5</td>
<td>3361.865</td>
<td>3.73948793</td>
<td>30.64%</td>
<td>56.89%</td>
</tr>
<tr>
<td>28</td>
<td>0.5</td>
<td>3361.865</td>
<td>3.73948793</td>
<td>30.64%</td>
<td>56.89%</td>
</tr>
<tr>
<td>29</td>
<td>1</td>
<td>1810.235</td>
<td>847.5233582</td>
<td>-0.40%</td>
<td>-3.46%</td>
</tr>
<tr>
<td>30</td>
<td>1</td>
<td>2068.84</td>
<td>624.730116</td>
<td>-0.67%</td>
<td>-6.51%</td>
</tr>
<tr>
<td>31</td>
<td>1</td>
<td>2327.445</td>
<td>422.255349</td>
<td>-0.81%</td>
<td>-9.27%</td>
</tr>
<tr>
<td>32</td>
<td>1</td>
<td>2586.05</td>
<td>254.8296504</td>
<td>-1.46%</td>
<td>-9.77%</td>
</tr>
<tr>
<td>33</td>
<td>1</td>
<td>2844.655</td>
<td>133.7281058</td>
<td>-2.65%</td>
<td>-5.17%</td>
</tr>
<tr>
<td>34</td>
<td>1</td>
<td>3103.26</td>
<td>59.27953799</td>
<td>0.21%</td>
<td>9.40%</td>
</tr>
<tr>
<td>35</td>
<td>1</td>
<td>3361.865</td>
<td>22.58830301</td>
<td>13.85%</td>
<td>37.60%</td>
</tr>
</tbody>
</table>
The optimization so far illustrates well the problems we face in calibration of such a model. It explores different optimization methods and showed what are possible with those optimization tools. The average absolute relative difference is 13.85%. This makes the fitting far from usable for practical use. The unsatisfactory comes with a reason. Parameters in Heston model such correlation and volatility of volatility is constant for all maturities. However, in reality the skewness and convexity does change over maturities. This lack of fitness verified our conclusion that Heston model is incapable of fitting the whole volatility surface.

What is still possible is that Heston model has enough parameters to fit the volatility curve of a given maturity. Figure 3.10 shows implied volatility based on market prices compared to the implied volatility curve of based on fitted Heston model. For 1-month maturity, the curve is off the implied volatility curve based on market price at the strikes that are around 2000. This is due to the numerical error in our implementation of Heston option price formula.

**Fitness compared to fitting the whole surface**

Comparing with fitting the whole volatility surface, fitting per maturity gives less fitting errors. The fitted error is shown in Table 3.9. Fitting per maturity produces average errors ranging from to 15.75% to 0.99%. We now can compare the average fitting error with setting 3 in Table 3.7 since both have the same fixed $\theta$ and variance $\nu$. For 1-month maturity, the fitting error is 15.75% and is less than 19.72%. The average fitting error is decreasing with the maturity. For 12-month the average fitting error is 0.99%.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Kappa</th>
<th>Theta</th>
<th>Lambda</th>
<th>Rho</th>
<th>Volofvol</th>
<th>Variance</th>
<th>SSE</th>
<th>AE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.3776</td>
<td>0.0628</td>
<td>0.1090</td>
<td>-0.5639</td>
<td>0.9036</td>
<td>0.0487</td>
<td>0.1737</td>
<td>15.75%</td>
</tr>
<tr>
<td>2</td>
<td>4.6100</td>
<td>0.0628</td>
<td>0.0000</td>
<td>-0.4432</td>
<td>1.6214</td>
<td>0.0487</td>
<td>0.1348</td>
<td>13.87%</td>
</tr>
<tr>
<td>3</td>
<td>0.7969</td>
<td>0.0628</td>
<td>0.0000</td>
<td>-0.4232</td>
<td>0.6583</td>
<td>0.0487</td>
<td>0.0299</td>
<td>6.53%</td>
</tr>
<tr>
<td>6</td>
<td>0.9005</td>
<td>0.0628</td>
<td>0.0000</td>
<td>-0.5998</td>
<td>0.3653</td>
<td>0.0487</td>
<td>0.0020</td>
<td>1.70%</td>
</tr>
<tr>
<td>12</td>
<td>0.7964</td>
<td>0.0628</td>
<td>0.0000</td>
<td>-0.6784</td>
<td>0.2974</td>
<td>0.0487</td>
<td>0.0007</td>
<td>0.99%</td>
</tr>
</tbody>
</table>
Fitness compared to the market data

Comparing with the implied volatility curve from market price, the implied volatility curve of fitting per maturity still has a gap for the strikes that are lower than the spot level. This can be seen in Figure 3.10. Such a gap can be reduced by running additional GRG optimization. Figure 3.11 shows the implied volatility curve after additional optimization. For longer maturity from 3 to 12 month, the fitting are much better. But for 1-month and 2-month maturity the gap still remains.
Figure 3.11 Fit Heston model to volatility curve of each maturity using additional optimization.

See file FitPerCurveResultAdditionalOptimization.csv

Table 3.10 provides the fitting parameters and errors and we can conclude the fitting are much better by looking at the average fitting error.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Kappa</th>
<th>Theta</th>
<th>Lambda</th>
<th>Rho</th>
<th>Volofvol</th>
<th>Variance</th>
<th>SSE</th>
<th>AE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0000</td>
<td>0.0628</td>
<td>0</td>
<td>-0.5728</td>
<td>1.0391</td>
<td>0.0595</td>
<td>0.0014</td>
<td>1.43%</td>
</tr>
<tr>
<td>2</td>
<td>0.0000</td>
<td>0.0628</td>
<td>0</td>
<td>-0.7402</td>
<td>1.8063</td>
<td>0.0980</td>
<td>0.0238</td>
<td>5.83%</td>
</tr>
<tr>
<td>3</td>
<td>0.7969</td>
<td>0.7563</td>
<td>0</td>
<td>-0.6523</td>
<td>1.4962</td>
<td>0.0002</td>
<td>0.0076</td>
<td>3.29%</td>
</tr>
<tr>
<td>6</td>
<td>0.8609</td>
<td>0.1919</td>
<td>0</td>
<td>-0.6527</td>
<td>0.5471</td>
<td>0.0262</td>
<td>0.0004</td>
<td>0.79%</td>
</tr>
<tr>
<td>12</td>
<td>0.8189</td>
<td>0.1614</td>
<td>0</td>
<td>-0.6764</td>
<td>0.4827</td>
<td>0.0093</td>
<td>0.0002</td>
<td>0.47%</td>
</tr>
</tbody>
</table>

We pushed Heston model to its extreme to see how good it can reflect the market price. It does have the capacity to model the skewness and convexity to certain extend.
This makes Heston model an attractive model in case the correlation and volatility of volatility needs to be modeled. However, Heston model also has its limitations: accuracy in numerical calculation is essential and it is not trivial.

### 3.2.4 Pricing variance swap using market price of options

Our approach of pricing variance swap is two fold. First the available market price of options is used to derive option price that are not available in the market. Secondly the price of the variance swap is determined using the value of the replicating portfolio. The replicating portfolio is discussed in [2] and [3]. In short, the variance swap can be replicated using a strip of out-the-money call and put options whose weights are proportional to inverse of the its strike. The exact expression for fair price of variance swap is given as follows.

\[
K_{var} = \frac{2}{T} \left( e^{\frac{1}{2} \sigma^2 T} - \frac{e^{\sigma \sqrt{T} - 1}}{\sqrt{S_0}} \right) - \log \frac{S_0}{S_a} \\
+ e^{\sigma \sqrt{T}} \int_0^{S_a} \frac{1}{K^2} C(K) dK \\
+ e^{\sigma \sqrt{T}} \int_{S_a}^{\infty} \frac{1}{K^2} P(K) dK 
\]

(3.8)

Where \( K_{var} \) is the fair price of variance swap, \( T \) is the time to maturity, \( S_0 \) is the current index level, \( S_a \) is the at-the-money forward level which separate in-the-money and out-the-money option and \( K \) denotes the strike of an option. A perfect replicating portfolio requires infinite number of strikes. To simplify this replicating portfolio, only a number of strikes are used. We will use 10 strikes in total for the replicating portfolio.

We will calculate the variance swap price using two models, Black-Scholes model and Heston model. In that manner, we can illustrate how a model impacts the variance swap price. The steps for calculation can be outlined as follows. In Black-Scholes model, the volatility skew is implied from market price of options. For any strikes that are not available in the market price, the volatility will be given by linear interpolation. We will also investigate how good linear interpolation is. In Heston model, by fitting the Heston option price to market prices, the parameters of the Heston model will be estimated. Using those parameters, the option price which are not available from the market can be calculated. Both models use the same number of strikes to construct the replicating portfolio. As the portfolio also requires the price of put option, we will use
Put-Call parity to derive the put price. For the illustration, we will use 7 strikes to replicate the variance swap. The strikes are available in the market price we used.

### 3.2.4.1 Variance Swap Price in Black-Scholes Model

![Variance swap price comparison](image)

Figure 3.12 Variance swap market price and price derived from replication portfolio based on market price of European options on 7 Feb, 2005, 25 Jun, 2008, 21 Nov, 2008, of which the 3-month ATM volatility are minimum, median and maximum respectively.

In the Figure 3.12, the market prices of variance swap are compared to the replicating price using the market price of European options. The 3-month ATM volatilities for three days are 0.1119, 0.1993 and 0.7402 respectively. Those are the minimal, median and maximal 3-month volatility levels. First we can see that the replicating prices tends to decrease when maturity increase while the market price can show different slope. The replicating prices seem to deviate from the market prices.

Since the replicating method only use 7 strikes to approximate the variance swap payoff, the approximation will produce an upper bound for the fair price. We consider increasing the number of strikes to improve the approximation. This indeed lowers the replicating price and reduced the difference with market prices as shown in Figure 3.13.
Comparing Figure 3.12 to Figure 3.13, we can also see that increasing the strikes reduced the difference most for short maturity such as 1 month. For longer maturity the influence is less. Also it brings less improvement when the 3-month ATM volatility is high, which can be seen as there is almost no prices changes in the 3rd chart. We insert additional strikes which are mid point between known market strikes. We expect this will shift the replicating price further down as using more strikes in replicating portfolio will approximate the variance swap price better. The implied volatility for the additional strikes are linear interpolated from the Black-Scholes implied volatility curves. Figure 3.14 shows the prices with the number of strikes increased to 67. This reduced the difference further for low and mid 3-month ATM volatility day (the 1st and the 2nd chart) and does not bring much difference for the high 3-month ATM volatility day.

Now we can draw some conclusions about the convexity of implied volatility curve.

1. When the volatility is high, the convexity of implied volatility curve becomes relatively low according to 3rd chart in Figure 3.12, Figure 3.13 and Figure 3.14. The rationale is that when the volatility is high, the replication price given by linear interpolation shows a consistent difference compared to the market price. The consistent difference can be explained as margin for selling variance swap. When the implied volatility is low, the implied volatility curve is more convex.

2. And the shorter the maturity is, the more convex the volatility curve is. The message is that we could consider using more strikes to calculate replicating price for short maturity variance swap.
3.2.4.2 Variance Swap Price in Heston Model

In Heston model, the market prices are fitted to derive the model parameters by minimizing fitting error for each maturity. Then the option prices of additional strikes are calculated using the parameters. We will check how those parameters reflect the volatility skew of implied volatility and form the prices compared to the market prices.
Figure 3.15 shows the replication price based on Heston model. Comparing to the price generated by linear interpolation of volatility curve and Black-Scholes model, we can see the follows. In case of low volatility (chart 1), 1-month and 12-month prices are lower as we expect. Other maturities, such as 2, 3 and 6 month, are higher, which is in conflict with the fact that convexity offered Heston model will give a lower implied volatility and therefore a lower replicating price. In another word, Heston model makes it possible to reflect the convexity of volatility curve and it means we should be able to shift the price curve down. By looking at the implied volatility curve of the fitted model we can have some idea about the reason. We can see in Figure 3.16 that the fitted curves (blue lines) are above the curve from the market price (black one) for 2, 3 and 6 month maturities. That pushes also the price of replicating portfolio higher. For 1 and 6 month, the curve of Heston price is below the curve of market price. Therefore the price is lower than the one in Black-Scholes model.
Recall that in the previous section, we expect that increasing the number of strikes further will decrease the variance swap price. Be noticed that this will shift all price down and will not be able to solve the problem of 2-month peak price in the first chart of Figure 3.15. In case of mid volatility and high volatility, 2\textsuperscript{nd} and 3\textsuperscript{rd} charts all shows now a lower price for all strikes.

As the final step in this study, we increase the number of strikes to 67. The result is presented in Figure 3.17.

Figure 3.16 Implied Volatility Curve of fitted Heston model per Maturity
Figure 3.17 Variance swap market price and price derived from replication portfolio based on Heston model. The number of strikes is increased to 67.

Figure 3.17 shows the replication price based on Heston model by increasing the number of strikes in replication portfolio to 67. Comparing with the prices given by 13 strikes, we can see the follows. In case of low volatility (chart 1), the price of 1-month variance swap is now closer to the market price market price. 12-month price is even lower. 2, 3, 6 months prices are not changed. In the case of mid volatility (chart 2), all prices are lowered. The prices for 1, 2, 3, 6 months have been lowered and 1-month is most lowered. In the case of high volatility (chart 3), all prices are lowered. The fitted Heston models are of the same as in Figure 3.16. The error can be attributed to the fitting error of the near the ATM stroked options since those option contributes most to the replication price.
4 CONCLUSIONS

So far we have explored quite some different topics to reach our final destination: pricing variance swap with Heston model.

Our starting point is a mathematical model for a financial market. The model is general but well-defined framework: it is so general that a wide range of models fall under this framework; it is so well-defined that the model can be analyzed with the help of stochastic differential equation. In the framework, viable and complete are two important concepts. Using those concepts, we can derive the price of a claim under the risk neutral measure.

We also have reviewed the common mathematical tools in tackling the problem, such as Black-Scholes PDE and Characteristic functions. Those techniques can be applied to find the solution of more complicated model. We consider Black-Scholes model as our benchmark model. Although the volatility is constant, which is far from the reality, the simplicity of Black-Scholes model allows us to study the model thoroughly without much difficulty. The Black-Scholes implied volatility curve exhibits skewness and convexity which reflect the actual market. A satisfactory model must also be able to describe those two characteristics. After reviewing a few stochastic volatility models, we focus on Heston model.

Heston model captures two important features of the market: the variance process is a mean reverting stochastic process and the variance is negatively correlated with the index. The features have an exciting consequence: It makes Heston model capable of describing skewness and convexity of implied volatility curve with parameter correlation and volatility of volatility respectively. The detailed discussion of parameters in Heston model gives an insight about how an implied volatility curve can be modeled. Some efforts are spent on the numerical calculation of Heston formula and it illustrates what the difficulties are in implementation. Finally we fit the Heston model to market prices of European options per maturity to find the parameters. Using those parameters, European option price at any strike is calculated. The price of variance swap can then be determined by the value of the replication portfolio composed of European options.

Although Heston model is promising in describing the skewness and convexity, it does have limitations. Firstly, an accurate numerical implementation of Heston price
formula is non-trivial and involves quite some attentions. Secondly, it is observed from the implied volatility curve of market price that the skewness and convexity of implied volatility curve are changing over time. However, the controlling parameters (correlation and volatility of volatility) in Heston model are defined as constant. This makes it impossible for Heston model to describe the behavior of those parameters over time. Thirdly, an efficient method to fit the model is another challenge we face. The limitations result in a close but not ideal fit of Heston option price with market option price. The replication price of variance swap reflects the market price to some extent. For some maturity, the replication price is still quite different from the market price.

This also reveals the area for further research. One possibility is investigating other type of model which can reflect change of skewness and convexity over time. Other possible further research include modeling the volatility surface directly which allows easy fitting and reduced fitting error, better numerical calculation of Heston price or improving optimization method for a better fit.
SYMBOLS

$\left(\Omega, \mathbb{F}, P\right)$

$\mathbb{F}(t)$

$W = \{W(t) = (W^{(1)}(t), \ldots, W^{(D)}(t))' \mid \mathbb{F}(t), 0 \leq t \leq T\}$: $D$-dimensional Brownian motion
APPENDIX

[1]. Absolute Continuous
Let \((X, d)\) be a metric space and let \(I\) be an interval in the real line \(\mathbb{R}\). A function \(f: I \to X\) is **absolutely continuous** on \(I\) if for every positive number \(\varepsilon\), there is a positive number \(\delta\) such that whenever a (finite or infinite) sequence of pairwise disjoint subintervals \([x_k, y_k]\) of \(I\) satisfies
\[
\sum_k |y_k - x_k| < \delta
\]
Then
\[
\sum_k d(f(y_k), f(x_k)) < \varepsilon.
\]
The collection of all absolutely continuous functions from \(I\) into \(X\) is denoted \(AC(I; X)\).

[2]. Differentiation under integral sign
Suppose that it required to differentiate with respect to \(x\) the function
\[
F(x) = \int_{a(x)}^{b(x)} f(x,t)dt,
\]
Where the functions \(f(x,t)\) and \(\frac{\partial}{\partial x} f(x,t)\) are both continuous in both \(t\) and \(x\) in some region of the \((t,x)\) plane, including \(a(x) \leq t \leq b(x), x_0 \leq x \leq x_1\), and the functions \(a(x)\) and \(b(x)\) are both continuous and both have continuous derivatives for \(x_0 \leq x \leq x_1\). Then for \(x_0 \leq x \leq x_1\):
\[
\frac{d}{dx} F(x) = \frac{\partial F}{\partial b} \frac{db}{dx} + \frac{\partial F}{\partial a} \frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt
\]
\[
= f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt
\]
[3]. Nelder-Mead Method

The method uses the concept of a simplex, which is a special polytope of \( N + 1 \) vertices in \( N \) dimensions. Examples of simplices include a line segment on a line, a triangle on a plane, a tetrahedron in three-dimensional space and so forth.

1. **Order** according to the values at the vertices:
   \[
   f(x_1) \leq f(x_2) \leq \cdots \leq f(x_{n+1})
   \]

2. Calculate \( x_o \), the center of gravity of all points except \( x_{n+1} \).

3. **Reflection**
   
   Compute reflected point \( x_r = x_o + \alpha(x_o - x_{n+1}) \)

   If the reflected point is better than the second worst, but not better than the best, i.e.:
   \[
   f(x_1) \leq f(x_r) < f(x_n),
   \]
   
   then obtain a new simplex by replacing the worst point \( x_{n+1} \) with the reflected point \( x_r \), and go to step 1.

4. **Expansion**
   
   If the reflected point is the best point so far, \( f(x_r) < f(x_1) \),
   
   then compute the expanded point \( x_e = x_o + \gamma(x_o - x_{n+1}) \)

   If the expanded point is better than the reflected point, \( f(x_e) < f(x_r) \)
   
   then obtain a new simplex by replacing the worst point \( x_{n+1} \) with the expanded point \( x_e \), and go to step 1.
   
   Else obtain a new simplex by replacing the worst point \( x_{n+1} \) with the reflected point \( x_r \), and go to step 1.
   
   Else (i.e. reflected point is not better than second worst) continue at step 5.

5. **Contraction**

   Here, it is certain that \( f(x_r) \geq f(x_n) \)

   Compute contracted point \( x_c = x_{n+1} + \rho(x_o - x_{n+1}) \)
If the contracted point is better than the worst point, i.e., \( f(x_c) < f(x_{n+1}) \)
then obtain a new simplex by replacing the worst point \( x_{n+1} \) with the contracted point \( x_c \), and go to step 1.

Else go to step 6.

6. Reduction

For all but the best point, replace the point with

\[ x_i = x_1 + \sigma(x_i - x_1) \quad \text{for all } i \in \{2, \ldots, n + 1\} \]

**Note:** \( \alpha, \gamma, \rho \) and \( \sigma \) are respectively the reflection, the expansion, the contraction and the shrink coefficient. Standard values are \( \alpha = 1, \gamma = 2, \rho = 1/2 \) and \( \sigma = 1/2 \).

For the **reflection**, since \( x_{n+1} \) is the vertex with the higher associated value among the vertices, we can expect to find a lower value at the reflection of \( x_{n+1} \) in the opposite face formed by all vertices point \( x_i \) except \( x_{n+1} \).

For the **expansion**, if the reflection point \( x_r \) is the new minimum along the vertices we can expect to find interesting values along the direction from \( x_n \) to \( x_r \).

Concerning the **contraction**: If \( f(x_r) > f(x_n) \) we can expect that a better value will be inside the simplex formed by all the vertices \( x_i \).

The initial simplex is important, indeed, a too small initial simplex can lead to a local search, consequently the NM can get more easily stuck. So this simplex should depend on the nature of the problem.
REFERENCES


[15]. VIX: CBOE Volatility Index, Chicago Board Options Exchange,

[16]. C. Kahl, P. Jäckel, Not-so-complex logarithms in the Heston model, 
Wilmott Magazine, September 2005, pp. 94-103

[17]. Karatzas and Shreve, Brownian motion and Stochastic Calculus, 

[18]. Pavel Cizek, Wolfgang Haerdle, Rafal Weron, Statistical Tools for 