Dutch House Price Derivatives
A new Perspective on Pricing and Applications

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Management summary

Motivation

Real estate is one of the last large asset classes with only few hedging possibilities, despite the fact that the wealth of Dutch households is largely dominated by the value of their house. Moreover, it is often leveraged with a mortgage. Whereas people generally feel that house prices always rise, we have seen that it is very well possible to see house price decline. In times of house price declines the house price risks are substantial, especially for homeowners with a high mortgage. Apart from households, we find that also mortgage providers experience high house price risk. A market in house price derivatives such as futures and options could facilitate the needed hedging possibilities. Institutional investors could then take over the risk for diversification benefits.

Project description

In this study we wish to describe the opportunities that a market in house price derivatives would yield for both mortgage providers and households. We will give an overview of possible difficulties that may arise developing products for this particular market. The main focus of this study lies in developing and pricing house price derivatives, options in particular.

Option pricing models

We start by studying the Black-Scholes option pricing model. We find that there are some major problems with the assumptions of the Black-Scholes model with regard to the underlying house price movements. Firstly, we discuss the price movements. Whereas Black-Scholes assumes this to be a standard random process (Brownian motion), we conclude that a price process with sudden downfalls is more appropriate for the house price movements. This leads us to study Merton’s jump-diffusion models, which is very similar to the Black-Scholes model for it only adds a jump process to the geometric Brownian motion. Theoretically, Merton’s model seems very suitable, but it has some disadvantages from a practical point of view, the biggest disadvantage being that limited data leads to difficulties with parameter estimation.

Apart from the assumptions concerning the movements of the underlying price movements, another problem for both Black-Scholes and Merton is the assumption of the underlying asset to be tradable. Since the underlying asset is the Dutch housing market and since it is very difficult to have a position on the entire Dutch housing market, this assumption does not hold. This, together with data analysis leads us to studying expectation pricing. This pricing method includes market expectations as a variable in the option pricing formula. For this we need to estimate a risk premium. We find that this leads to no problems if there are derivative prices available with the same underlying price movement. However, for the Dutch housing market this is not the case, so we must estimate the risk premia using expert views on future price movements.
Practicalities
There are some hurdles to be taken when introducing a market in house price derivatives. Most importantly, we have seen in similar markets in the UK and US that liquidity grows very slowly. This is because a too strong consensus among market players may disturb the balance between supply and demand.

Conclusions
We found that the possibilities of a market in house price derivatives are great. Regarding the pricing issues of house price derivatives, we find that both Merton’s model and expectation pricing can be considered suitable, since they both take into account specific characteristics of the housing market. Based on historical house prices, we find that downfalls can not be ignored. This is a strong incentive to use Merton’s model. However, based on the typical low liquidity of the housing market, together with high transaction costs and times, we find that it is necessary to use expectation pricing. A combination of the two seems to be ideal, although it must follow from further research whether this is possible.
Chapter 1

Introduction

The study considers house price derivatives as a hedge vehicle for both households and companies with a high exposure to house price risk. Pricing methods and practical issues for the introduction of a market in house price derivatives are the main focuses of our project. In Section 1.1 we consider our motives for investigating house price derivatives. In Section 1.2 we will give a description of the project, in terms of research questions and the general outline of this document.

1.1 Project motivation

In this section we start with discussing four characteristics of the housing market in Section 1.1.1. Secondly, in Section 1.1.2 we study the effect of price changes on a household’s wealth. Within this framework we find that there is much room for improvement, i.e. there are possibilities of improving the risks that households face as a result of decreasing house prices, for instance by providing hedge possibilities for house price changes.

1.1.1 Typical characteristics of the housing market

The housing market is a difficult market to describe. House prices can be influenced by a wealth of factors, not only financial ones. In fact, sentimental arguments play a big role. An illustrative example is given in Gautier et al. (2009), which studies the effect of the murder of Theo van Gogh (a Dutch film producer who made a controversial movie which was perceived offensive by large parts of the ethnic minorities in the Netherlands) on the house prices in Amsterdam. The study showed that following the murder, neighborhoods with more than 25% of the people belonging to an ethnic minority showed an extra house price decrease 3% in 10 months. They find that segregation increased as people belonging to the Muslim minority are more likely to buy and less likely to sell a house in such a neighborhood than before the murder. Furthermore, natives became more reluctant in buying houses in these neighborhoods, resulting in a negative effect on the house prices.

The obvious difficulties arising when trying to measure sentiment are probably the reason for the lack in consensus between researchers on naming the explanatory effects (see Verbruggen et al. (2005)). We find that macro economic factors, such as interest rates, inflation and GDP have a great impact on the housing market. As stated above we have apart from the economic climate also strong sentimental factors. We will now consider several factors in this section, which we find interesting and necessary for understanding the fluctuations of house prices.

First, we consider affordability. The purchase of a house is associated with a relatively large transaction. This generally means that it cannot be financed with just personal equity, but that a mortgage is necessary. The mortgage costs depend on the principle value (the amount that is borrowed initially), the mortgage rate and fiscal arrangements. Both the mortgage costs and mortgage lending criteria substantially affect the price a buyer is willing to pay for a house. So, if the affordability of the mortgage costs increases, the buyer is willing to spend more money on a house, which of course has a
positive effect on the house prices.
Now, suppose the total income of households increases. Then we have higher affordability of mortgage costs and as a result the house prices will increase. Similar for the family equity: a positive change will lead to higher house prices.
The fiscal arrangements also influence the mortgage costs. The better the fiscal arrangements, the higher the mortgage can be and thus households can afford more. In the Netherlands the most influential arrangement is the home mortgage interest reduction. A home mortgage interest deduction allows taxpayers who own their house to reduce their taxable income by the interest paid on the mortgage. Especially in case of abolition of the interest reduction, the net costs for home owners will (dramatically) increase. As a result the affordability will decrease and the house prices will drop.
In 1991 the Swedish government decided to reduce the mortgage interest reduction to a maximum amount of 30%, whereas it once was 80%. The housing market crashed. The mortgage interest reduction was not the only cause of the crash. Jaffee (1994) argues in his study of the ‘boom and bust’ of the Swedish real estate market that the following factors also played a significant role in the recession: a decrease in real Gross Domestic Product (GDP), a growing level of unemployment and increasing real interest rates, and the high rate for interest deduction. In the second half of the 1980's, these factors all experienced a positive development, whereas after 1990, all of these factors were reversed. So the affordability changed, but changes of the other factors, that indicate the global state of the economy, also contributed to the recession and it is impossible to quantify the effect of each separate factor.

Another factor that we wish to discuss is the rule of supply and demand. This factor greatly impacts all markets, whether it concerns cows, oil or stock options. In general there should be some sort of equilibrium between supply and demand. If the demand goes up, the prices go up, the supply is increased and as a result the prices go down again. In the housing market however, there is no such mechanism. The housing market is a supply driven market.
A house usually has a long lifespan. On average, a house in the Netherlands lasts for about 110 years (Priemus (2000)). Furthermore, the process of creating new houses takes a long time, due to legislation and regulations and the extensive building process itself. Hence, the supply level will adjust to the level of demand very slowly if at all. To illustrate this point, the demand in 1995-2005 tripled, whereas the level of building new houses decreased. Since the supply cannot meet the demand in a supply driven market, the price is generally determined by the demand.
Because of the slow building process, there are not only insufficient houses, but the actual wishes of the potential buyers cannot be met within a short period of time. This means that the newly built houses match needs (concerning location, quality and size) that are no longer there (Verbruggen et al. (2005)). So the supply and demand are not in line with each other for large periods of time. This results into either surpluses or shortages in certain segments of the housing market.
Another effect of the long lifespan of houses is the division between using and owning, i.e. renting and buying. Priemus (2000) argues that this division leads to two separate markets with ‘separate’ price developments. Of course we have that if the rents increase, buying a house becomes a more attractive alternative and hence the house prices will increase.

Another essential factor in house price development is the effect of expectation. In contrast to the purchase of most consumer goods, the purchase of a home is both an emotionally and a financially substantial event. As a consequence of the purchase, the financial situation of the home owner is highly related to future value of the house. As a result, the inclination to buy a house is strongly correlated with the expectation of price changes. Suppose the consensus among potential buyers is that house prices will increase. The increase in demand that will follow, will have a positive effect on the house prices. The opposite effect is also true.
The effect of expectation in general implies that house price indices are autocorrelated, since the expectation is based on past performance. Brown and Matysiak (1995) investigated autocorrelation of house prices. They found that it is close to zero for single objects whereas for indices the autocorrelation is high. A positive autocorrelation causes cyclic behavior (e.g. rising house prices stimulate the housing market which causes the prices to rise even more, until it reaches an upper limit. House
1.1. PROJECT MOTIVATION

prices above this limit are no longer realistic, so at this point, the demand will decrease, and hence house prices decrease, which will discourage potential buyers. The latter decrease in demand causes prices to fall even harder. This continues until the prices are so low that the incentive to buy is again great after which the prices will rise again.

The NVM house price index for all house types in the Netherlands is shown in Figure 1.1. From the returns shown in the figure it is very clear that we have a yearly periodicity. We find both seasonal behavior and a trend line which might suggest a more general cyclic behavior such as described above. As for the seasonal behavior, this is mostly due to the construction method of the index. Since the NVM index uses the median of transaction prices a decrease in transactions in the higher segment leads to a decrease of the index whereas the house prices perhaps did not change. Furthermore we find in Figure 1.1 that the average increase in the second quarter lies around 4%. For the third quarter we find that the index often stays at the same level or even decreases with 1% to 2%. It seems unrealistic that the value of a house fluctuates so much over six months.

As for the ‘long-term’ periodicity suggested in Figure 1.1 by the red trend line, we find that the existence of such house price cycles is acknowledged in literature. Tsatsaronis and Zho (2004) describe house price cycles with an average period of 17 years, but they also argue that the length of these cycles vary over the different countries. Girouard and Blöndal (2001) also find these differences, but suggests an average cycle of 10 years. There are numerous studies concerning the explanatory effects of the house price cycles. Edelstein and Tsang (2007) for example argue that employment growth and interest rates are key determinants of the residential real estate cycles. Quigley (1999) evaluates the effect of economic conditions as well as lagged property prices upon property prices. This agrees with the suggested autocorrelation due to the effect of expectation.

The phenomena of seasonal periodicity and cycles in general indicate that returns are not independently distributed but that they are partly predictable, Myer et al. (1997). This degree of randomness is relevant when looking at the suitability of an index for derivatives.

Figure 1.1: The NVM house price index. Top: The index developments 1985-2008. Bottom: Quarterly returns (gray) and general trendline (red)

Eichholtz (1997a) studied the price development of houses at the Herengracht in Amsterdam. The quality of these houses has been on a constant high level throughout time. This makes the Herengracht a suitable sample for a house price index for the very long run. The nominal and real values of the index are shown in Figure 1.2. We find that whereas in nominal terms, so without taking inflation into account, the index increases more than tenfold over the sample period, this increase almost disappears when looking at the real-valued index. Eichholtz argues that since the period after WWII is the most
prosperous in history and since most existing studies of real estate are based on this period, they may well overstate the long run performance. In addition we find that most downfalls can be explained by historical events. Periods of war of course have a highly negative impact on house prices, as well as economical factors, e.g. Napoleon closing the port of Amsterdam, the great depression in the 1930’s and crisis due to subprime mortgages in 2008. This implies that the cyclic behavior of real estate prices caused by autocorrelation alone does not explain the downturns, but rather that downturns are caused by external factors.

So far we described some factors which influence house prices, being the general macro economic factors, the effects of changing affordability, supply and demand, the effect of expectation, i.e. the general idea of the public about future price changes and the impact of historical events.

Probably the most evident effect in the past year is the effect of expectation. In 2009 we see that the fear of falling house prices is holding back potential buyers. Potential sellers are reluctant to offering their houses for sale, for they fear it will not be sold with sufficient profit or will not be sold at all. Some already bought a new house, but are unable to sell their old one and are facing double mortgage costs. Accepting a loss on their old house may be necessary in their case. We see that these kinds of developments greatly impact the house prices and hence also the wealth of households. But how large is the risk that home-owners face? In order to answer that question we take a look at household portfolios.

1.1.2 The influence of housing on household portfolios

In the US, half of the national wealth lies in residential real estate and in land (FED (1991)). A significant part of this value lies with the households, whether or not leveraged by a mortgage. The average US homeowner holds about 88% of his non-pension wealth in home equity. In Europe, the situation is not very different. In Western Europe and North America, the average household spends 25 to 35 percent of its income on housing, and this is an even higher percentage for younger households (Englund et al. (2002)).

Several studies have been published considering the effects of housing on households portfolios, e.g. Flavin and Yamashita (2002), Brueckner (1997), Eichholtz et al. (2000) for the U.S., Le Blanc and Lagarenne (2004) for France and Quigley (2006) from the European perspective.

An important issue in all these studies is the duality between consumption and investing motives for
1.1. PROJECT MOTIVATION

There is little freedom in the amount of money we wish to invest and for which period of time. Transaction costs are high and houses are generally expensive. This leads to ‘overinvestment’ in housing, leaving little room for diversification.

Dutch household portfolios

We believe that the situation is not different for Dutch households. We examined the average Dutch household’s portfolio and used Markowitz portfolio optimization to quantify the overinvestment. Markowitz provides a method which gives optimal portfolios. That is, for some given risk level Markowitz gives asset weights which optimize the expected return of the portfolio. Basics on Markowitz portfolio optimization, the financial situation for the average Dutch household and the actual application of Markowitz to the households situation can be found in Appendix A. The balance sheet of the average household is shown in Table A.1. The non-pension related assets consists for 86% of housing. We also see that we can improve our risk-return ratio considerably by holding a more diversified portfolio. In Figure 1.3 three portfolios are shown. First, in Figure 1.3(a) we see the housing dominated portfolio of the average household. This portfolio gives an expected yearly return of about 6% at a risk level of 3.6%. Second we see the asset contribution for a portfolio with the same risk level, but higher return (7%). Another option, shown in Figure 1.3(c) is to reduce the risk level for an equally expected return. The risk level in this case would be only 2.4%. In both cases we can see that the contribution of housing to the portfolio is close to 40%. Of course, this contribution is dependent on the amount of risk we would be willing to take on. In Figure A.4, the contribution of housing to optimal portfolios is shown for risk levels between 2% and 7%. The maximal contribution of housing to the optimal portfolios does not exceed 50%. We can safely conclude that an 85% contribution to a portfolio is too large, hence we can speak of overinvestment. Ideally, we would like to reduce this contribution to around 40%.

(a) The average Dutch household  
(b) Optimal Portfolio (higher return, same risk)  
(c) Optimal Portfolio (same return, lower risk)

Figure 1.3: Asset allocation for (optimal) household portfolios
Clearly, housing is of too large an influence on the asset side of the households balance sheet. But this is also the case for the liabilities side. Houses can rarely be bought without mortgage funding. This is also clear in Table A.1, where we see that a mortgage corresponds to no less than 95% of the liabilities. Almost 50% of the house value is leveraged with a mortgage. This is of course an average and is it clear that the leverage is much higher for younger home owners. This percentage is also known to be the LtV (Loan to Value) ratio.

![Figure 1.4: Change in invested equity over time](image)

Suppose a young couple decides to buy a house for €300,000 and they finance it for 95% with a mortgage. Personal equity invested now corresponds to 5% of the house value being €15,000. In the first years they decide to pay only the mortgage rent. Now the value of the house is not constant. Suppose it changes over time with the same rate as the NVM house price index. The change over time of the personal equity invested in the house is shown in Figure 1.4. In the second quarter of 2007 the house prices rose with 3.6%, hence the house value rose to over €311,000. Since the mortgage does not change, the personal equity increased with €11,000. This continues and after little over a year we see that our equity has doubled. Now the decrease in house prices starts in 2008 and the expected movement for 2009 is shown by the dotted line. Three years after the buy, we see that the house value decreased by more than €15,000 and our mortgage is now higher than the collateral value, i.e. an LtV value of over 100%. The percentage change is far less dramatic than for instance on the stock market, but the high house prices make that even small percentages have great effect, especially in cases of high leverage.

This is no problem as long as the couple is able to pay for the mortgage costs, because they can decide not to sell their house until it rises in value again. However, events such as sudden unemployment or divorce can lead to dramatic changes in the financial situation of the family. Forced auction leaves the family with debts and the mortgage provider with a loss. Rabobank is the largest mortgage provider in the Netherlands with a market share of 30%. The risks of house price changes are thus of great interest to Rabobank as well as to the individual households.

**Rabobank mortgage portfolio**

The combination of a number of extreme scenarios could lead to dramatic losses, even for the typically low risk mortgage portfolio of Rabobank. Such scenarios are characterized by rapidly increasing mortgage interest rates and a rising unemployment rate as a result of a decay in economical growth, and falling house prices due to political risks, such as abolition of the mortgage interest tax reduction
1.1. PROJECT MOTIVATION

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Table 1.1: Overview of traditional investments and derivatives

or a lessened affordability by a stricter mortgage issuance policy. The possibilities of hedging the risks of the mortgage portfolio are thus of great interest for Rabobank.

We already saw the risks incorporated with the LtV ratio. In periods of decreasing house prices the bank suffers a higher loss if a customer defaults. Moreover, an increasing number of forced auctions leads to an increasing amount of houses displayed for sale, which in its turn will have a negative effect on the house prices. It may trigger a snowball effect as recently seen in the US.

Such a scenario provides a strong motivation for examining the possibilities of hedging or mitigating the risk of increasing LtV ratios. One solution could be to adjust the mortgage issuance policy, for example to avoid an LtV ratio of over 100%. Another solution, but still a theoretical one, is to hedge the risk of decreasing house prices (hence increasing LtV ratios) via house price derivatives.

1.1.3 Possibilities for hedging house price risk

Hedging is in essence finding a party to take over (a part of) the risk at some price. So we are looking at possible investors in the housing market. Investing in house price derivatives is different from traditional investments in housing. These traditional investments can be direct investment, i.e. buying a house, or indirect investment by which we mean investing in real estate companies or real estate funds.

Syz (2008) gives an overview of the differences between traditional investments and derivatives, which we show in Table 1.1. Because of the object specific risk of direct investment, it takes time to gain all necessary information to decide whether it is a solid investment. It does give an opportunity for investors to try to outperform the market. But high transaction costs, property taxes and regulatory restrictions are problems that would not be encountered when using house price derivatives. As for indirect investments, we still have specific risks, e.g. business risk. Furthermore the performance of a real estate company is not necessarily a good hedge for residential house price movements. In short, the introduction of derivatives would create possibilities for quick investment in housing with low transaction costs. So if for instance a Japanese pension fund wishes to alter its portfolio, investing in the derivatives of the Dutch housing market could give high diversification benefits without entailing high transaction costs and maintenance problems.

This suggests that the introduction of a hedging medium for house price risk would be desirable and that financial derivatives would be ideal for creating a hedge. Case et al. (1991) stated:

“Futures and option markets should be established that are cash settled based on indexes of real estate prices ... Individual and corporate owners of real estate might then hedge away most of the real estate risk that they bear, risk that has caused them enormous concern and trepidation in past years. That risk would be more efficiently borne by large institutional investors who can diversify over many regions and types of real estate, as well as over financial assets... Yet futures and options markets devoted to real estate are nowhere in evidence today.” This leads us to the question of why such derivatives were not available in the 1990s and whether they are available now?

We find that in the last 25 years there have been developments on house price derivatives, mainly concentrated in the UK and the USA. Attempts to create a property derivatives market emerged in
the 1990s. Transaction volumes and sizes however were very low. They went up around 2005, a period in which house price risk became more apparent and the incentive to hedge thus rose. Most of the deals concerned over-the-counter (OTC) traded futures, swaps and bonds and were created for institutional investors. Low liquidity prevented businesses to create suitable hedge possibilities for households. In Chapter 2 we describe these developments in more detail and discuss the obstacles encountered.

1.2 Project description

In the previous section we laid out our motives for introducing house price derivatives in the Netherlands. In this section we will give an overview of which questions we would like to answer during our study.

First, we state the main question that we wish to answer:

Main Question

*Which possibilities will house price derivatives yield for the Rabobank and her clients and how can we determine their price?*

1.2.1 Option pricing

To price house price index (HPI) linked derivatives, we first need to decide what kind of product we wish to use. However, for many financial products, the price can be decomposed into a combination of option prices. Hence we first focus on option pricing on the house price index. Pricing methods for options are discussed in Chapter 3. The most common model for option pricing is the Black-Scholes model. Therefore we choose to start with discussing the suitability of Black-Scholes pricing for options on house price indices. This leads to the following subquestion.

What assumptions must be made to use Black-Scholes for option pricing and could this pricing method be suitable for options on a house price index?

This issue is discussed in Section 3.1.

Several difficulties arise when looking at the assumptions. We choose in Chapter 3 to focus on the behavior of the underlying index. Black-Scholes assumes it to follow a geometric Brownian motion. However, looking at the index, we see that the volatility in ‘good times’ is very small, but at some point in time the housing market crashes and there is a sudden steep decline. The overall volatility is thus for most part accounted for by these short but heavy price declines. Hence it could be interesting to look at the process in two parts. First there is a geometric Brownian motion as in the Black-Scholes model, but now with a somewhat higher return and much lower volatility. To this process we add crashes. The combined process now has the same return and volatility as the Black-Scholes process, but now the volatility is mainly due to the crashes and the relatively high returns in periods of economical prosperity are tempered with the occasional crash, hence we model a boom-burst effect. This model in fact is Merton’s mixed jump diffusion model. We can now state the next question analogous to the latter.

What assumptions must be made to use Merton’s Mixed Jump Diffusion model for option pricing and could this pricing method be suitable for options on a house price index?

We discuss this topic in Section 3.2 and see that apart from the assumption of geometric Brownian motion with jumps, the Merton model has the same assumptions as the Black-Scholes model and of course the same objections hold for them. So we ‘improved’ the model with respect to the underlying model for the index, but there are still difficulties with the other assumptions.
Is either Black-Scholes or Merton’s model suitable for pricing options on the housing model?

As discussed above, Merton’s model is the more accurate model with respect to the behavior of the underlying index. However, being the most accurate model, does not mean that it is wisest to use it in practice. We find that it is hard to estimate the parameters which are needed to price options with Merton’s model, and of course estimation errors cause poor option prices.

So theoretically, Merton’s model turns out to be a more accurate model for practice. However in practise, the Black-Scholes model has some advantages.

Whereas in Chapter 3 we focussed on the price movements of the underlying index, in Chapter 4 we wish to focus on other, more practical problems. As already discussed, we found that both models need assumptions which are not consistent with the characteristics of the housing market. The major problem is the assumption that the underlying asset to the option is tradable. Due to high transaction costs, it is virtually impossible to take on a position in the Dutch housing market. Yet both methods use the construction of a risk free portfolio as the basis for their calculations and a risk free portfolio cannot be constructed if the Dutch housing market is not tradable.

We find that it is possible to alter the Black-Scholes problem such that this assumption is no longer necessary. This is done by including a risk premium as discussed in Section 4.1.2. The use of a risk premium basically means that market expectations are taken into account when pricing options.

Should we use a risk premium for option pricing and how can we determine the size of the risk premium?

To answer this question we take a look at house price derivatives on a US house price index and the UK all property index. We find that the risk premium explains differences in movements between the derivatives prices and the spot prices. Theoretical, i.e. according to both the Black-Scholes model and Merton’s model, these price movements should be perfectly correlated. We find that the ‘lack’ of correlation can be explained by looking at the market expectations, which are represented in the risk premium. So a risk premium is necessary to explain historical prices of derivatives. This is a strong indication that the use of a risk premium for Dutch HPI derivatives pricing is not only good, but also necessary.

Estimating the risk premium can be done based on risk premia for (other) derivatives on the same underlying index. However, we find that for the Dutch housing market no such derivatives exist. In that case, it is necessary to use experts opinions on the future movements of the index to determine the risk premium.

1.2.2 Practicalities

So far we discussed the pricing difficulties for options on a house price index. If we are able to price the options, we should be able to price other, more complicated financial products. This leads us to the following question.

What kind of financial products can we price based on options prices and what possibilities do they yield for possible investors?

One of these products is an index-linked bond. This is a bond with returns corresponding to the performance of the house price index. A strong advantage of a bond is that it is typically traded in the OTC market. This means that there is no need for a liquid market. We find that pricing the bonds is easily done by decomposing it in a risk free savings and a combination of long and short positions on call options. This is shown in Section 4.2.

Another type of product is designed especially for households. Since households are not likely to invest in bonds, options or futures, other products may be offered to them to than give to opportunity to reduce the specific risk of their housing dominated portfolios. In Section 4.2 we price two such
products, being an index-linked mortgage and a house price insurance policy. We find that there are numerous possibilities for the design of such products. For instance the maturity of the products and the amount of consumer risk can be varied. Higher maturities and higher consumer risk of course lead to lower prices. We find that the index-linked mortgage in particular is an interesting product for households to reduce their house price risk. As for the insurance policy, we encounter some problems with pricing it, for its pricing method is based on the law of large numbers, which is applicable for more traditional insurances, but not for housing since there is a high correlation between individual policies.

If we speak about introducing house price derivatives in the Netherlands we should consider the characteristics of the housing market and take a look at whether they can cause problems.

*How do the specific characteristics of the housing market influence the introduction of house price derivatives in the Netherlands?*

Chapter 2 describes the former developments of house price derivatives. One of the biggest issues encountered was a lack of liquidity. However, we have seen in the past fifteen years that liquidity grew slowly, and that it continued growing even in times of downfall, Syz and Vanini (2009). We discuss liquidity in Section 4.3 together with other main issues for introducing a market in house price derivatives. Another important issue is the index. Of course, if we wish to base derivatives on an index, the index should be representative of the entire market and it should be trustworthy. We discuss the advantages and disadvantages of three Dutch indices.

The issues that we discuss should be viewed as points of attention, for none of these issues is a definite deal breaker. In general we find that the introduction of house price derivatives in the Netherlands is definitely possible, but we must approach it with caution, choose a reliable index and give the market some time to get used to this new product so that liquidity can grow.

### 1.3 Conclusions

In this chapter we discussed our motives to introduce house price derivatives in the Netherlands. We find that these derivatives yield possibilities to hedge house price risk and that it is very interesting to have such a hedge possibility for both households and institutional investors. Households typically have a house dominated portfolio, hence they experience a high specific risk. Markowitz portfolio optimizations suggests that the households portfolio should consist for about 40% in housing, whereas it is now about 80%. Especially households with a high mortgage rate experience greater risks, because a decrease of their house value could result in a loan to value ratio of over 100%. We have seen that high LtV ratios entail risks for the mortgage provider as well, which makes a hedging medium very interesting for them as well.

As for institutional investors, we find that if they wish to invest in the Dutch housing market, they now face high transaction times and costs since there is only the possibility of direct investments. With house price derivatives, being an indirect investment, they can avoid these costs which makes it easier for them to diversify their portfolio by participating in the Dutch housing market.

All in all we find that since multiple parties can benefit from housing derivatives, both on the sellers and buyers side. This makes it interesting to take a closer look at the possibilities of house price derivatives and at their pricing methods. In this study, we try to give an overview of the developments of property derivatives, possible pricing methods and points of attention when introducing the market in the Netherlands. It is build up as follows.

In Chapter 2 we briefly go over the basics of financial derivatives and introduce some terminology. We also consider former developments on property derivatives, both residential and commercial. These developments are mostly situated in the UK and the USA. In addition we describe the initiatives for financial products designed for households to protect them from house price risk.
Chapter 3 considers option pricing on the house price index. We discuss the Black-Scholes option pricing method and its shortcomings with respect to the housing market. In Chapter 3 we also focus on the problems with the assumptions considering the movements of the house price index. We introduce a model that includes jumps, Merton’s model. These two models are then compared. Finally, we give a procedure to deal with parameter estimation and a practical example of this procedure for the Dutch real estate market.

In Chapter 4 we deal with problems caused by the specific characteristics of the housing market. We find that both models discussed in Chapter 3 are built on assumptions that do not match with practise. We discuss the effect of expectation on derivatives prices and introduce a variant of the Black-Scholes model that includes this effect. Next we consider three, more complicated, financial products which we can price based on option prices. Finally, we discuss the possible difficulties when introducing a market in house price derivatives in the Netherlands.

In Chapter 5 we draw conclusions and make suggestions for further research.
Chapter 2

Developments on house price derivatives

In this chapter we discuss developments of property derivatives as seen in the past. We find that there have been some difficulties with the introduction of these kind of derivatives, which is curious since real estate is the last major asset class without a hedge vehicle. Therefore the incentive to create a market should be high. We will try to explain the difficulties by linking them to the housing characteristics as discussed in Chapter 1. We will first recall these characteristics. In Section 2.1 we will introduce some basics on financial derivatives, after which we will go into the actual developments of house price derivatives in both the UK and the US.

The analysis of housing characteristics in Chapter 1 gave us four explanatory effects for house price changes, namely

Affordability: Mostly the affordability of mortgage costs. The possibility of taking on higher mortgages increases house prices. A big influence on this are fiscal arrangements such as the home mortgage interest reduction. Abolition of this interest reduction would have a dramatic effect on house prices. On the other hand for instance, an increase of GDP leads to a higher affordability and hence to higher house prices. We thus find that the affordability factor is influenced by both political and macro economic factors.

Supply and demand: We found that the housing market is a supply driven market, which means that house prices are generally determined by the demand.

Effect of expectation: The inclination to buy a house is strongly correlated with the expectation of price changes. Since the expectation is based on past performance of the housing market, the housing market is a slow market, typically autocorrelated. Autocorrelation leads to periodicity and partly predictable future returns.

Historical events: We find that downturns observed in the past were caused by dramatic historical events. Note here that since the housing market is a slow, illiquid market, it responds rather slowly to these events, although the impact may be substantial. We again emphasize that this leads to some kind of ‘predictability’.

We can now discuss the former developments of house price derivatives, mostly concentrated in the UK and US. We will discuss the impact of the housing market characteristics and, if possible, we give the pricing methods. We will show some data and draw conclusions for the possibilities of house price derivatives in the Netherlands.

Before we can go into specifics of which derivatives are traded when, however, we now give a brief introduction on characteristics of financial derivatives.
CHAPTER 2. DEVELOPMENTS ON HOUSE PRICE DERIVATIVES

2.1 Basics on financial derivatives

The two basic characteristics of derivatives are the way in which they are traded and the structure of the product. This provides us a basis for the discussion of which types of derivatives we have seen based on property indices. We will start by explaining the difference between over-the-counter traded and exchange traded contracts.

**OTC trading:** Over-the-counter (OTC) derivatives are contracts that are traded directly between two parties. This type of trading is almost always used for derivatives such as swaps, forward rate agreements and exotic options. The OTC market is the largest market for derivatives. The most important players are banks and hedge funds. It is difficult to get information about the size of the market and the prices that are agreed upon, since the activity is not visible on any exchange and most trades occur in private.

**Exchange trading:** Exchange-traded derivatives are products that are traded on specialized derivatives exchanges or other exchanges. The exchange acts as an intermediary to all transactions, thus there are no private negotiations between the buyer and seller as for OTC contracts. So if we wish to sell, the intermediary finds a buyer. The risk of not finding a counterparty for a transaction is taken on by the intermediary. He acts as a guarantee, and for this he takes a margin from both sides of the trade. The most important derivatives exchanges are the Korea Exchange, the Eurex, and the CME group (formerly the Chicago Mercantile Exchange and the Chicago Board of Trade). Some derivatives may be traded on traditional exchanges, mostly convertible bonds and warrants.

Next to the way in which derivatives are traded, the structure of the contract is of course very important. We will discuss the three most important classes of derivatives, being futures/forwards, options and swaps.

**Futures/Forwards:** A future is in essence the same as a forward, only the future is typically traded on an exchange and forwards are OTC traded. It is a contract between two parties to buy/sell an asset at some prespecified specified time and price.

**Options:** Options are contracts that give the owner the right, but not the obligation to buy or sell an asset at some prespecified time and price.

**Swaps:** Swaps are contracts in which two counterparties agree to exchange one stream of cash flows against another stream. These streams are called the legs of the swap. The contract has specified dates on which the cash flows are paid and when they are calculated. Usually one or both of the cash flows are uncertain. We can for instance use interest rates, foreign exchange rates or equity prices.

Of course there are many other derivatives, but in general we can say that they are constructed from these three derivatives (e.g. the holder of a ‘swaption’ has the right, but not the obligation, to enter into a swap on a prespecified future date) or that they are basically the same, but with subtle differences (e.g. exotic options are options for which the rules of the contract are altered in a non standard way, for instance that the payoff at maturity depends not just on the value of the underlying asset at maturity, but at its value at several times during the contract’s life).

2.2 Experience in property derivatives

The cost of buying and selling physical property are estimated to be between 5% and 8% of the property investment (Syz (2008)). This is an international estimate. For the Dutch case the transaction costs lie around 7.5%. By using derivatives one can strongly reduce these costs. Also, the investment in derivatives is far more flexible. One can buy and sell more quickly and easily in a liquid market. Furthermore, the investor avoids the idiosyncratic risk of single objects. These motives for introducing
2.2. EXPERIENCE IN PROPERTY DERIVATIVES

property derivatives, but mostly the first, seem to have triggered the development of property derivatives. Over the past decades there have been attempts to create a (liquid) market in property derivatives. We have seen deals in countries as Australia, France, Japan, Switzerland, but mostly in the UK and the US. Most traded contracts are OTC, with the price determined through negotiations. A liquid market however is not yet established, leaving property to be the last major asset class without a liquid derivatives market.

The UK has been a pioneer in the developments of property derivatives in Europe, so we will focus first on the experience in the UK. Then we will go into the experience with property derivatives in the US.

2.2.1 Property derivatives in the UK

In the UK, we have seen the earliest attempts to create a liquid property derivatives market. Most developments concerned property index futures. They are discussed in e.g. Gemmill (1990) (overview of the conditions which are necessary for a futures market in housing and concludes that a futures market could succeed if banks and building societies offered insurance to purchasers, offsetting the risk on the futures market). Thomas (1996) proposes that financial institutions should offer house-buyers insurance policies related to the level of a house price index. To facilitate a hedge for the risk of a portfolio of such policies, a market in ‘perpetual futures’ on house price indices is proposed. We thus see that the main incentive to establish a futures market is to create a hedge so that institutions can provide some kind of insurance for house price declines for households.

The London Fox experiment was the first major attempt to introduce property futures. It did not achieve this goal, but it did yield valuable insights for future developments. Detailed information about the project can be found in Roche (1995). London Fox launched property futures in May 1991. They offered four contracts based on commercial property capital value, commercial rent, residential property and mortgage rates. The volume rose in some months to only 4662 contracts. This disappointing figure was due to several reasons. First the housing industry proved to be rather conservative, and was simply not ready for such a modern and newly launched product. Essentially, everyone stood back and waited for someone else to use it. Furthermore, they did not believe that hedging was necessary. They had a perception of inadequate volatility. This means that the investors did not believe in big risks in real estate prices, which resulted in a low incentive to partake in the futures market. In addition, the property world believed in concurrence among themselves about the future movements of indices. That is, they thought that all parties on the futures market were having the same view on future index developments. The basis of both problems is the autocorrelation of the index. This autocorrelation makes future returns partially predictable and it softens the impact of current price changes, leading to seemingly small volatility.

The most important issue however was liquidity. An inadequate system for market information caused that the prices were not continuously monitored, making it impossible for investors to have constant information on futures prices. This prohibits constant trading, causing insufficient liquidity. To create liquidity in the market, a higher trading volume must be obtained. Trading volumes were artificially boosted using deals that in the end produce neither gain nor loss. The property industry saw the trading volumes rise, while they were still not using the market. Meanwhile the regulators found out that the market was artificial and London Fox was forced to stop the trading. In summary, the market was open only from May to October.

Throughout the 1990s, some other initiatives were launched. In spite of the London Fox failure, the incentive to create a liquid market was still high. The UK real estate market had just been through a crash. Barclays had some outstanding loans to property developers with a high risk. The idea of hedging this property exposure was Barclays’ motive to collaborate with Aberdeen Property Investors. They structured a tradable bond that pays out index returns on the IPD all property index
CHAPTER 2. DEVELOPMENTS ON HOUSE PRICE DERIVATIVES

(predominantly commercial property). The bonds were called Property Index Certificates (PICs). Barclays’ hope was that a large and liquid market in PICs would be created, which would enable them to offset the risks of their lending portfolio as well as the interest income when the next downturn occurs. Sadly, they underestimated the subtlety of institutional investors. PICs enabled investors to bet on the market, but not against it. So the investors will certainly buy the fundamentally illiquid PICs so long as the market is almost certain to rise, but they will not do so as soon as we expect a downturn, especially in the absence of proper market makers. Again, autocorrelation and partial predictability of the index causes this problem.

Around the same time as the introduction of the PICs, Barclays launched a series of Property Index Forwards (PIFs), again based on the IPD all property index. In contrast to futures, the forwards are not exchange traded but OTC, so the bank takes the role of market maker. The market was not very liquid, so it basically came down to Barclays finding matching buyers and sellers. In the period 1996-1998 the market grew to £400m. This was a definite growth, although it was not until 2005 that the market experienced a much faster growth.

![Graph showing IPD UK trading volumes](image)

**Figure 2.1: IPD UK trading volumes (Table 2.1)**

Most of the property derivatives in the UK and in Europe are based on IPD indices. Figure 2.1 shows us the notional value of derivative trades executed each quarter on the IPD index in the UK. The total outstanding value increased from 260 million pounds in 2004 to more than 11 billion pounds ultimo 2008. The trade volumes and such are shown in Table 2.1. We see that the number and sizes of trades went down in the second half of 2008 and kept declining in 2009. We did see a positive change in the notional of trades executed in the second quarter of 2009 which suggests that the market is able to withstand serious house price declines. This supports the idea of investors regaining trust after the credit crunch.

The major parties in the OTC deals are shown in Table 2.2. We can see that banks and property companies contribute to 45% of the market volume. They use the derivatives in their hedging strategies and wish to insure against price declines. This risk is bought by companies such as pension funds and insurance funds. They use this asset class for diversification of their portfolio. Basically we can see that the buy side consists of pension funds, insurance funds and so on. On the sell side we have institutions with large property portfolios being banks, property funds and property companies. Typically, the sell side is interested in larger volume trades and the buy side in smaller ones. In times of prosperity a broad range of institutions is willing to buy. In 2006, investors were keen to invest in a property index, while the owners of large property portfolios were less willing to sell.
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<tr>
<td>Total Outstanding Notional (m£)</td>
<td>260</td>
<td>485</td>
<td>806</td>
<td>927</td>
<td>1100</td>
<td>1963</td>
<td>2466</td>
<td>4124</td>
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<td>Notional of Trades Executed each Quarter (m£)</td>
<td>260</td>
<td>225</td>
<td>321</td>
<td>121</td>
<td>183</td>
<td>853</td>
<td>513</td>
<td>1658</td>
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<td>Total Outstanding Number of Trades</td>
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<td>14</td>
<td>42</td>
<td>56</td>
<td>80</td>
<td>139</td>
<td>189</td>
<td>279</td>
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<td>Number of Trades Executed each Quarter</td>
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<td>4</td>
<td>28</td>
<td>14</td>
<td>24</td>
<td>59</td>
<td>54</td>
<td>90</td>
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<td>Average Outstanding Deal Size (m£)</td>
<td>26</td>
<td>35</td>
<td>19</td>
<td>17</td>
<td>14</td>
<td>13</td>
<td>15</td>
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<td>Average Deal Size each Quarter (m£)</td>
<td>26</td>
<td>56</td>
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<td>Total Outstanding Notional (m£)</td>
<td>6769</td>
<td>7266</td>
<td>7916</td>
<td>9032</td>
<td>9069</td>
<td>10140</td>
<td>10364</td>
<td>11182</td>
<td>7795</td>
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<tr>
<td>Notional of Trades Executed</td>
<td>2927</td>
<td>970</td>
<td>1660</td>
<td>1662</td>
<td>3441</td>
<td>1628</td>
<td>1028</td>
<td>979</td>
<td>554</td>
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<tr>
<td>Total Outstanding Number of Trades</td>
<td>428</td>
<td>473</td>
<td>543</td>
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<td>990</td>
<td>1054</td>
<td>1185</td>
<td>1038</td>
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<td>Number of Trades Executed each Quarter</td>
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<td>96</td>
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<td>13</td>
<td>12</td>
<td>10</td>
<td>10</td>
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<td>8</td>
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<tr>
<td>Average Deal Size each Quarter (m£)</td>
<td>19</td>
<td>11</td>
<td>17</td>
<td>8</td>
<td>13</td>
<td>6</td>
<td>8</td>
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Table 2.1: Trading volumes of property derivatives on the IPD indices (UK), source: www.ipd.com.
CHAPTER 2. DEVELOPMENTS ON HOUSE PRICE DERIVATIVES

<table>
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<tr>
<th>Type of user</th>
<th>Purpose</th>
<th>Estimated share of volume (%)</th>
</tr>
</thead>
</table>
| Banks             | · Proprietary trading
|                   | · Hedging strategies                      | 40                            |
| Pension funds     | · Tactical asset allocation
|                   | · Strategic long-term asset allocation    | 20                            |
| Insurance funds   | · Tactical asset allocation
|                   | · Strategic long-term asset allocation    | 10                            |
| Property funds    | · Strategic long-term allocation
|                   | · Short-term proxy to direct property exposure | 10                         |
| Hedge funds       | · Relative value plays/arbitrage
|                   | · Long/short strategies                   | 10                            |
| Property companies| · Hedging strategies
|                   | · Short-term proxy to direct property exposure | 5                           |
|                   | · Strategic long-term asset allocation    | 10                            |
| Other             | · Strategic long-term asset allocation    | 5                             |

Table 2.2: Estimated share of volume by type of user, source: IPF (2008)

However in 2007, the prospects for the UK property market were less good, leading to large investors worrying about their property exposure and willing to hedge. But it was no longer clear who would take over the risk. Again, the problem of autocorrelation plays a role here.

In general we find that trades on property indices experience good demand. There were also some sector trades, based on ‘subindices’, so for instance the index for office buildings in London. They however proved to be not so successful in the UK, Syz (2008). The sector specific derivatives provide a better hedge or more specific diversification, but investors rather choose for the more liquid all property indices. But as the market evolves, we can expect trade volumes to go up and more and more contracts on regional indices can be introduced, providing in its turn the required liquidity. The increasing volume and variety of property derivatives suggests that the emergence of more contingent claims, such as options may be possible.

2.2.2 Property derivatives in the US

The United States have a very large potential for trading property derivatives. The extreme increase in housing prices in the 1990s and the following decline in house prices in 2007 brought a lot of attention to house price derivatives, beginning of course with transferring the high returns to professional investors and leading to speculations about house price insurances and methods to hedge the house price risk.

Financial derivatives based on real estate have a long history in the US. Stocks and bonds based on real estate began trading in 1929 on the New York Real Estate Securities Exchange (NYRESE). In 1941, with a collapse in real estate and securities prices, the Securities and Exchange Commission (SEC) decertified the NYRESE (Bertus et al. (2008)). After WWII, trading did not restart. The first trade in the US, intermediated by Morgan Stanley, was conducted in 1993, being a property swap linked to the National Council of Real Estate Investment Fiduciaries (NCREIF) Property Index. A US pension fund wanted to reallocate assets from (commercial) property to equity. The buyer, a life insurance company, agreed to pay US$ LIBOR (London Interbank Offered Rate), in exchange for income payments generated by property (Syz (2008)).

In 2005, NCREIF gave Credit Suisse Investment Bank (CSIB) a mandate to develop derivatives based
on the index. CSIB marketed those products and completed the first transactions mid 2005. Most trades involved swaps. A Credit Suisse-related entity was counter party to all trades. Initially, CSIB had an exclusive right to trade such derivatives, but it renounced its rights in October 2006 so that the market could gain liquidity. The exclusive agreement expired in April 2007. Ultimo 2007 seven banks were licensed to trade NCREIF based derivatives. The traded volume reached US$ 300 million at that time, while the Financial Times published on 3 May 2007 that Credit Suisse only conducted two trades, worth US$ 50 million. So in six months the trade volume increased with 500%.

Besides commercial property, the US have seen initiatives for trading derivatives based on residential house price indices. The literature proposed several derivatives to hedge housing exposure. Case et al. (1991) propose a futures market based on regional indices and Englund et al. (2002) emphasize that there are large potential gains from policies or institutions that would permit households to hedge their lumpy investments in housing. It turns out that the potential reduction of risk to households, yielding the same return is surprisingly large, especially to poorer homeowners.

As a result of the efforts of among others Case and Shiller, the Chicago Mercantile Exchange (CME) offers futures contracts designed to follow home prices in 10 US cities as well as on a composite index, the cities being Boston, Chicago, Denver, Las Vegas, Los Angeles, Miami, New York, San Diego, San Francisco and Washington DC. The indices used are the S&P/Case-Shiller Home Price indices, shown in Figure 2.2.

Trading started on 22 May 2006. The trading volume was relatively low in the first year with an average of about 50 contracts a day. The volumes remained rather low and the CME stated that there is a “huge educational need” for this new derivatives market. Another problem was addressed, being that the contracts had a maturity of one year, whereas most investors wish to hedge for a longer period of time. As a result CME extended the terms up to 60 months.

If we take another look at Figure 2.2, we see that all ten cities in the US followed some sort of boom burst process over the last few years. This effect is very clear for Miami. Boston sees a similar movement but the effect is smaller. In Figure 2.3 the development of futures on the indices for Boston and Miami is shown. We can see that the Futures prices for Miami show great reaction to the dropping house prices. The index lost over 40% of its value in two years, so it is only logical that
futures prices decrease as well. We see that decrease in Figure 2.3(b) although it is not as steep as the index itself. We find that in May 2009 the index is lower than the futures prices for maturities of more than one year. We may explain this as being an indication for a general believe that the index will reach its lowest point before the end of 2010. The lowest future price in May 2009 is the one with maturity November 2009. This indicates the believe that the index will show a further decrease over the summer. For Boston the futures prices show a more constant behavior. It is possible that a higher volatility of the index results in a higher volatility of the futures price. The decrease of the house price index is not as dramatic as for Miami. In fact, we find that the prices rise at some points. These rises are probably caused by seasonal effects, similar as for the Dutch house price index in Figure 1.1. Miami also has these seasonal effects, but they do not weigh up to the downfall which is occurring, although we see a slightly less steep decrease around May of each year, which would be consistent with the period in Boston. We do not see these seasonal effects in the futures prices, which can be explained by the market accepting these effects and simply neglecting them for it is known in which season the maturity lies.

We must be careful however with drawing conclusions, for these figures are based on a small amount of transactions. Moreover, futures prices are not to be confused with indicative prices. Theoretically, that is under Black-Scholes assumptions as discussed in Chapter 3, futures prices should equal $e^{rT}S_0$ with $S_0$ the current index price, $T$ the time until maturity and $r$ the risk free interest rate for discounting. This is the theoretical price when using a risk free portfolio. This kind of derivative pricing will be discussed extensively in Chapter 3. For now we just conclude that the futures prices do not follow their theoretical path. In Chapter 4 we will go further into this and link these observations to their implications for the theory of option pricing for house price derivatives.

2.2.3 Initiatives for protecting home owners from house price risk

So far we discussed the type of derivatives that are suitable for institutional investors. One of the main objectives to create such a market however is to protect home owners from house price risk. It seems unlikely that homeowners will enter the futures market on house prices. So how can we make such a ‘hedge’ accessible for average Joe?

There have been several initiatives, some of which we will discuss in this section.

Syz et al. (2008) propose house price index-linked mortgages which would protect home owners from price declines. They too stress the fact that the existing property derivatives have been targeted to meet the needs of institutional investors. Index-linked mortgages however could be tailored to the needs of home owners specifically. The mortgage providers can pass on parts of the risk of the mortgages to institutional investors through the use of property derivatives.

Two basic designs are proposed for the index-linked mortgages. First, the interest payments of the mortgages depends on the performance of the housing market. The second possibility is linking the principal to house price developments which comes down to adding a put-option. The put premium can be added to the periodic interest payments. In both cases, the volatility of a household’s home equity is lessened.

To price the mortgage, they divide the mortgage into an unsecured loan, a credit derivative for the collateral and a put option for the the index-linked property. Details on pricing index-linked mortgages can be found in Akguen and Vanini (2006). For the put-option they use the same altered version of Black-Scholes as in Shiller and Weiss (1999), which takes the influence of expectation into account.

There are still some difficulties however for these mortgages. Namely, it is unclear whether there is a sufficiently large amount of transactions on the derivatives to create derivatives on regional indices, which are crucial for the mortgages. This is caused by the idiosyncratic risk, i.e. the price of one house does not follow the exact movements of the house price index. For instance, it may increase because of strong improvements in local facilities or it can decrease because a highway is being build next to it.

Shiller and Weiss (1999) propose a home equity insurance policy which insures homeowners against
2.2. EXPERIENCE IN PROPERTY DERIVATIVES

Figure 2.3: Future prices for various maturities over time.

declines in the prices of their homes. One form could be a life-event-triggered insurance policy, in which the homeowner pays regular fixed insurance premia and is entitled to a claim if both a sufficient decline in the real estate price index and a specified life event (such as a move beyond a certain geographical distance) occur. They mention two fundamental problems with home price insurance. First, if an insurance policy covers the value of an individual home, a homeowner who knows that the losses in value of his home are borne by the insurance company could have a reduced inclination to maintain the home. This is called the moral hazard problem. Another form of moral hazard is selling home in a hurry and to a low bidder, knowing that the loss of value on resale is covered by the insurance. The second problem which they address is the selection-bias problem. Again, if the policies insure the value of an individual home, a homeowner who feels that he paid too much for the home and could not sell it for the same price, would have a special incentive to buy home equity insurance. This would lead to a high risk portfolio for the insurance companies, because only high risk home owners will buy the insurance. A way of dealing with these problems is to offer insurance on the change of a regional index rather then on the individual house. But doing this would lead to a higher risk of mismatching, that is a higher risk that the index declines when the home price does not or vise versa. The index must thus have a sufficiently small range to reduce the risk of a mismatch, but a sufficiently large range to avoid moral hazard and selection bias problems.

With the life-event triggered insurance policy, the homeowner will receive payment from the insurer only when there is a loss experienced by the homeowner. It can thus be seen as a put option on the
value of the home at the time it is bought with the exercise date contingent on the sale of the house. To price put options Shiller and Weiss (1999) use a variant of the Black-Scholes option pricing formula. They conclude that this insurance policy is similar to other insurance policies which makes it marketable for the general public and acceptable for insurance regulators. Problems lie in the great uncertainty about the willingness of homeowners to strategically move when they qualify for a claim. Furthermore, there is uncertainty about the strategic cancelations of clients in response to price movements. However, they claim that these uncertainties are very similar to the uncertainties of regular insurance companies.

Eichholtz (1997b) considers house price risk in the Netherlands. He describes the problem of insufficient diversification for Dutch households and considers the possibility of creating an insurance policy. He describes the same problems as in Shiller and Weiss (1999) namely moral hazard, adverse selection and the problem of mismatching when using indices. He states that the recent developments concerning the construction of house price indices makes it possible to create reliable indices for small regions. Again the similarity to put-options is mentioned. However, where Shiller and Weiss (1999) compare house price insurance to regular insurances, Eichholtz stresses that there is great difference, namely that for regular insurances, the law of large number can be used. For instance, the probability that all tourists in Rome in the summer of 2009 are being mugged is rather small, but the probability that all the apartments in Rome have a decrease in value over the summer of 2009 is much higher, because of the high correlation. The law of large numbers is thus not applicable for house price insurances, hence it is much harder for the insurer to hedge its risk. In times of house price declines, the number of claims will be extremely high. But since the house prices are autocorrelated, no investor will be willing to take over the risk. Even though diversification benefits for institutional investors can be great (especially for regional indices), the low liquidity of house prices hinders the development of a derivatives market. Eichholtz concludes that a more liquid product should be created, in order to make the hedge possible. As for this product, Eichholtz proposes ‘woonfondsen’ (living funds). In general such a fund would consist of a well diversified portfolio of houses. Stocks of the fund could be traded on the stock market, providing the wanted liquidity.

All these things considered, it seems that the incentive to create hedge possibilities for home owners is great although we find ambiguity in how to arrange it. We have seen proposals for house price insurances and index-linked mortgages. Both methods would lead to a strongly reduced house price risk for home owners. Similar for both methods is the problem of mismatching. The use of regional indices is probably the best solution. It is unclear however whether the insurers or mortgage providers can pass on parts of the risk to institutional investors. As the liquidity of the housing market is typically low, the index is autocorrelated, resulting in predictability of future returns on the index. This prohibits investors to buy the risk through index derivatives in times of price declines, which causes the liquidity on the derivatives market to be low as well. If for instance a bank sells products to home owners to reduce their house price risk, the bank takes over the risk. So the bank needs to sell this risk again to third parties, e.g. through a derivatives market. We can conclude that a liquid derivatives market must be developed before we can create ‘hedging’ products for house owners.

2.3 Conclusions

In the past few years, we have seen a shift in perception toward the housing markets. Rapidly increasing house values and no decrease seen in decades lead a lot of people to believe that there is no house price risk. The credit crunch starting in 2008 however made it very clear that they were wrong. Looking at house prices over a large period of time we can understand this. The period after WWII, and specifically after the price decrease in 1980, was a very prosperous one, and our perception was based on data from this period. Large downfalls are thus rare and mostly the result of external factors. In between downfalls the risk is typically low, but it is important not to forget that downfalls do occur.
Although the risk perception used to be very low, the extreme amounts of money invested in real estate made the development of derivatives interesting. We have seen some attempts of creating a derivatives market in the 1990s, but the transaction volumes and sizes remained low until around the year 2005. The bulk of the deals concerned OTC traded futures, swaps and bonds. This makes it hard to find information on the market prices of these deals. Also, the still insufficient liquidity of the market may be preventing institutions from creating home owner accessible products to hedge their house price risk.

The effect of expectation on house prices causes autocorrelation, and hence partial predictability of index returns. Furthermore we find several forms of periodicity. This all makes it hard to believe that option pricing is straightforward. Since option pricing lies at the heart of pricing both bonds and the products for homeowners, we find that further research in this area is necessary. Although there are numerous papers about option pricing in general, it is very hard to find information about option pricing for real estate specifically. We find some indications of Black-Scholes pricing and an altered form of it. Furthermore Syz (2008) suggests the possibility of including jumps in pricing models. Since we found that sudden downfalls induced by external factors characterize the housing market, this seems plausible.

In this study we will first consider the Black-Scholes model for option pricing. We find that the specific characteristics of the housing market cause problems with the required assumptions for Black-Scholes. We found that the house prices are not of a ‘contant’ behavior, that is, it seems to include jumps. Merton’s mixed jump-diffusion model is a variant to Black-Scholes that includes jumps. In Chapter 3 we discuss both Black-Scholes and Merton. Besides the non-constant behavior of the index we also found autocorrelation due to the effect of expectation. Furthermore we find that the underlying index is not tradable, which causes problems with the assumptions for Black-Scholes and Merton. A possible solution for this problem is given in Chapter 4 by the use of a risk premium. Furthermore we will discuss issues we might encounter when introducing a market in house price derivatives and we will price more complicated products such as bonds and the index-linked mortgage.
CHAPTER 2. DEVELOPMENTS ON HOUSE PRICE DERIVATIVES
Chapter 3

Pricing models

In this chapter we will take a closer look at option pricing on house price indices. There are numerous methods for option pricing of which we will discuss two in this chapter. The most commonly used method for option pricing is the Black-Scholes method, discussed in Section 3.1. It is questionable however whether this is an accurate model for house price indices. It has been suggested that jumps should be added, which leads us to a discussion of Merton’s Jump-Diffusion model in Section 3.2. In Section 3.3 the models are compared and Section 3.4 covers option pricing in practise. This is followed by a discussion and conclusions in Section 3.5.

3.1 Black - Scholes option pricing formula

Before we go into the Black-Scholes method for option pricing we will discuss what options are and give some basic definitions. An option is a financial derivative, basically a contract to buy or sell an underlying asset against some fixed price at some time in the future. This price is called the strike price, denoted by $K$ and the specified time of buying or selling is called maturity, denoted by $T$. An option to buy is called a call option and an option to sell is called a put option. The buyer of the contract is the party going long. The seller of the contract goes short. Now the party going long has the option of buying or selling, so he will only do so if profitable. For instance, suppose that we have a long position on a call option with a strike price of $A100. Now suppose that the underlying asset is worth $A110 at maturity. In this case we will exercise the option, buy the underlying asset for $A100 and can immediately sell it for $A110 which leaves us with a profit of $A10. In case of a lower value of the underlying asset, say $A90, we simply do not exercise the option, leaving us with a zero profit.

Another example is having a short position on a put option with strike price $A100. Then the long party has the option of selling the underlying for $A100 if profitable. We thus must buy if the underlying has a value less than $A100, say $A90. In that case the short position suffers a loss of $A10. In general we have that the long party makes a profit on the call option if the asset has a higher value than the strike price and on the put option in case of a lower value of the underlying asset. The profit of the long position is the loss of the short. This is shown for calls and puts and for short and long positions in Figure 3.1.

The basic idea underlying the Black and Scholes (1973) theory, is that one is able to hold a riskless portfolio. Such a portfolio would consist of a position on the index and a position on its derivative. Since the derivative has the index as underlying price movements, we have that there is a perfect correlation between the index and its derivative in any short period of time. We assume that there are no arbitrage opportunities, i.e. no opportunities to buy an asset at a low price and then immediately selling it on a different market for a higher price which gives a risk free profit. Then we have a theoretical return equal to the risk free interest rate $r$. 

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The riskless portfolio can be constructed easily if we look at the dependence of price developments of for instance stocks, and call options on the stocks. This is illustrated in Figure 3.2. If we look at an infinitely small time interval, we have that

\[ \Delta C = q \cdot \Delta S \]

where \( \Delta C \) and \( \Delta S \) are the price changes over some period \( \Delta t \) for a call option and a stock, respectively, and \( q \) is some constant characterizing this time period. Now if we have a long position in \( q \) shares and a short position on a call option, we have a riskless portfolio. For instance, if the stock price increases with 10 cents and we have the long position on \( q \) shares, we have a profit of \( \varepsilon q \cdot 10 \) on the long position. The price of the call option of course increases with \( q \cdot 10 \) cents, and since we own a short position on one call option, we have a loss of \( \varepsilon q \cdot 10 \), which is equal to our profit. In general we have that

\[ P = q \cdot S - C. \]

\[ \Delta P = \left( q \cdot (S_0 + \Delta S) - (C_0 + \Delta C) \right) - \left( q \cdot S_0 - C_0 \right), \]

\[ = q \cdot S_0 + q \cdot \Delta S - C_0 - q \cdot \Delta S - q \cdot S_0 + C_0, \]

\[ = 0. \]
3.1. BLACK - SCHOLES OPTION PRICING FORMULA

Figure 3.2: Price changes of $S$, $C$ and the riskless portfolio $P$ consisting of $q$ stocks and 1 short call option.

We must keep in mind that $q$ is not constant during longer time intervals. In fact, it can change rather rapidly. With this change, we must also adjust our portfolio to maintain the riskless property. Nevertheless, for small time periods, we do have that a certain portfolio is riskless and that the return is thus $r$. This is the basic idea behind the Black-Scholes analysis.

3.1.1 Assumptions

In this section we will give an overview of the necessary conditions for using the Black-Scholes model for option pricing. We will not give an elaborate description of the assumptions, more information and the applicability to the housing market is discussed in Section 3.5.1.

1. There are no arbitrage opportunities. Arbitrage opportunities would lead to a so called ‘free lunch’. It means that it is possible to get a risk free profit when holding a certain portfolio in which some assets are priced incorrectly.

2. The price development $S$ of the underlying asset is a geometric Brownian motion, characterized by

$$dS = \mu S dt + \sigma S dz,$$

$$\log(S_T) \sim N\left(\log(S_0) + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right),$$

where $\mu$ and $\sigma$ constant, and $z$ is a Brownian motion. This means that we have a constant (exponential) drift with rate $\mu$ and to this drift a standard Brownian motion is added to create volatility. Basics on Brownian motion can be found in Appendix B.

3. Naked short selling of securities is permitted. Short selling in general is a trading strategy where an investor believes that a stock is likely to decline. Basically, he borrows a stock and sells it on the open market. He then buys it back at a later date, hopefully at a lower price and making a profit. Naked short selling is short selling without owning the underlying asset or principal value.

4. There are no transaction costs or taxes. All securities are perfectly divisible.

5. There are no dividends during the life span of the derivative.

6. Security trading is continuous, so one is always able to buy or sell the derivatives or their underlying assets.

7. The risk-free interest rate $r$, is constant and the same for all maturities.

8. The underlying asset is tradeable, i.e. the asset can be sold in any location at any time. Stocks are for instance perfectly tradable. Products however with high transportation costs and short lifespan such as flowers are generally considered non-tradable.
3.1.2 Derivation of formula

The price process of the underlying asset is given in (3.1). Let $f$ represent the price process of the call option on $S$. Then $f$ must be a function of $S$ and $t$. Ito’s Lemma, see Theorem B.3.2 in Appendix B, then yields

$$df = \left( \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz.$$  \hspace{1cm} (3.3)

Note that (3.1) and (3.3) have the same underlying Brownian motion, which means that the Brownian motion can be eliminated by choosing a portfolio in a clever way.

In the introduction we saw that a portfolio with $q$ shares and a short position on a call option, where $q = \Delta C/\Delta S$. We now choose a similar portfolio, of course having $f$ as the option price instead of $C$. As we assumed $S$ and thus $f$ to be continuous, we can let $t \to 0$ and use the derivative. This gives us the portfolio construction with portions $-1$ of the derivative and $+\partial f/\partial S$ of shares.

We now define $\Phi$ as the value of the portfolio. By definition we have

$$\Phi = -f + \frac{\partial f}{\partial S} S$$  \hspace{1cm} (3.4)

and thus

$$d\Phi = -df + \frac{\partial f}{\partial S} dS.$$  \hspace{1cm} (3.5)

Using (3.1) and (3.3) in (3.5) gives

$$d\Phi = \left( -\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt.$$  \hspace{1cm} (3.6)

This equation no longer contains the Brownian motion $z$, and is therefore riskless during very small time intervals. Hence it must have a rate of return $r$, otherwise borrowing and investing would give a risk free profit, which is of course not allowed since we assumed that there are no arbitrage opportunities. Hence, we have

$$d\Phi = r\Phi dt$$ \hspace{1cm} (3.7)

and substituting (3.4) and (3.6) into (3.7) gives

$$\left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt = r \left( f - \frac{\partial f}{\partial S} S \right) dt,$$  \hspace{1cm} (3.8)

and thus

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$  \hspace{1cm} (3.9)

Equation (3.9) is known as the Black-Scholes differential equation. It can be used for all sorts of derivatives. The situation is characterized by the boundary conditions. These are very different for e.g. call options and futures. This difference in boundary conditions leads to the different prices for the derivatives. In case of a European call option with strike price $K$ on maturity $T$, the most important boundary condition is

$$f = \max(S - K, 0), \quad \text{when } t = T.$$  \hspace{1cm} (3.10)

Black-Scholes pricing formulas for call and put options

In this section we will derive the pricing formula for a European call option. Furthermore we will discuss its relation with the European put option. We already gave the boundary condition on the call option in (3.10) and similarly for the put-option this will be

$$f = \max(K - S, 0), \quad \text{when } t = T.$$
The European call option

One can derive the pricing formula for the call option by solving the Black-Scholes differential equation with the given boundary condition. Another method, which we will use, is risk-neutral valuation. We denote the expected value of some random variable in a risk neutral world, to be $E_n$. So for the value of a call option at maturity we have

$$E_n[\max(S_T - K, 0)].$$ (3.11)

We must have that the call option price $c$ is simply the discounted value of this expected maturity value, because that would be the fair price that an investor or trader would pay for it. Hence,

$$c = e^{-rT}E_n[\max(S_T - K, 0)].$$ (3.12)

Now $S_T$ is lognormal, and from Equations (B.23) and (B.22) we obtain $E_n[S_T] = S_0e^{rT}$ and the standard deviation of $\ln S_T$ is of course $\sigma \sqrt{T}$. Now we will use the following result to further examine (3.12).

**Proposition 3.1.1** If $X$ is lognormally distributed and the standard deviation of $\ln X$ is $\sigma$ and $C$ is some constant, then

$$E[\max(X - C, 0)] = E(X)N(d_1) - CN(d_2),$$ (3.13)

where

$$d_1 = \frac{\ln(E[X]/C) + \sigma^2/2}{\sigma \sqrt{T}},$$

$$d_2 = \frac{\ln(E[X]/C) - \sigma^2/2}{\sigma \sqrt{T}},$$

with $N(x)$ the cumulative distribution function of a standard normal random variable.

The proof can be found e.g. in Hull (2008), p.307. The basic idea behind it is to calculate the expected value as an integral which allows explicit evaluation.

Using Lemma 3.1.1, we can rewrite Equation (3.12) as

$$c = e^{-rT}[S_0e^{rT}N(d_1) - K N(d_2)] = S_0N(d_1) - Ke^{-rT}N(d_2),$$ (3.14)

(3.15)

where

$$d_1 = \frac{\ln\left(\frac{E_n[S_T]}{K}\right) + \sigma^2T/2}{\sigma \sqrt{T}} = \frac{\ln\left(\frac{S_0}{K}\right) + (r + \sigma^2/2)T}{\sigma \sqrt{T}},$$ (3.16)

$$d_2 = \frac{\ln\left(\frac{E_n[S_T]}{K}\right) - \sigma^2T/2}{\sigma \sqrt{T}} = \frac{\ln\left(\frac{S_0}{K}\right) + (r - \sigma^2/2)T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}. (3.17)

The put-call parity and put option pricing

It is of course possible to derive a price for the put option directly from the Black-Scholes differential equation using other boundary conditions, but it appears to be easier to derive it directly from the call option price. In this section we will show how this can be done. A necessary condition for doing this is that the option is not exercised before maturity, which is always the case for European options.

We consider a call and a put option with the same maturity at time $T$ and with the same strike price $K$ on the same underlying asset, which pays no dividend. $S_T$ again denotes the underlying value
at time $T$. Now consider the following assets: a share, a risk-free bond that pays 1 at time $T$, a put and a call option. Table 3.1 shows the value of different ‘derivatives’ at maturity for the two cases being a smaller or greater strike price $K$ than the value $S_T$ of the underlying asset. Now consider two different portfolios. First, a portfolio consisting of a put option and one share. At maturity the value of the portfolio equals $\max(S_T, K)$. Second, a portfolio consisting of a call option and $K$ bonds. Again at maturity the value is $\max(S_T, K)$. So whatever the eventual price $S_T$, the portfolios always have the same value. Now of course the portfolios must have the same value at time $T$, as well as at any time $0 \leq t \leq T$, for else there would be arbitrage opportunities.

We now have that

$c + Ke^{-rT} = p + S_0$,

(3.18)

where $c$ is again our call option price, $p$ the put option price, and $S_0$ the value of the underlying share. Furthermore, $Ke^{-rT}$ is the discounted value of the $K$ risk free bonds. This gives for $p$:

\[
p = c + Ke^{-rT} - S_0 \\
= S_0 N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT} - S_0 \\
= S_0(N(d_1) - 1)Ke^{-rT}(1 - N(d_2))p \\
= Ke^{-rT}N(-d_2) - S_0N(-d_1).\]

(3.19)

\[\Delta = \frac{\partial f}{\partial S_0} = N(d_1).\]

We can see in (3.16) that this is a constant when $K/S_0$ constant.

3.1.3 Results, robustness

In this section we will investigate the sensitivity of the Black-Scholes pricing formula with respect to its parameters. This will give us insight in how robust the pricing method is. It is of course not easy to estimate all the necessary parameters and information on robustness, but it allows us to make some comments on the impact of errors in estimation. All option prices in this section refer to call options. We use a Matlab function ‘bsf’ to calculate the Black-Scholes prices. The source code is displayed in Appendix C, File C.1.1.

We wish to get an idea of how the option price changes if we change the parameters. We focus on the impact of the strike price (as a function of $S_0$ and on the impact of $r$ and $\sigma$). The results can be compared with the ‘Greeks’. The Greeks are measures for the sensitivity of the value of an asset to a small change in a given underlying parameter. For more information on the Greeks we refer to Hull (2008).

We will first discuss the linearity of option prices when $K/S_0$ is constant. This basically means that when we have $S_0 = 100$ and $K = 110$ the option price is twice as big as when $S_0 = 50$ and $K = 55$. The linearity is shown in Figure 3.3 for various values of $K/S_0$. The result is intuitively correct, since in this example when $S_0 = 100$ is the same as buying two options when $S_0 = 50$. Furthermore it immediately follows from the second assumption, that the stock price movements are linear with respect to the stock price. The linearity in $S_0$ if $K/S_0$ constant follows directly from one of the Greeks, $\Delta = \frac{\partial f}{\partial S_0} = N(d_1)$. We can see in (3.16) that this is a constant when $K/S_0$ constant. More interesting is now the impact of $K/S_0$. When $K/S_0 > 1$ we have a strike price that is higher than the current stock price, which means that we will only exercise the option if the stock price has increased. Similarly for $K/S_0 < 1$, we may exercise it when the stock price has decreased (until it reaches K), so there is a higher probability of making a profit. This of course leads to higher option

<table>
<thead>
<tr>
<th>Asset</th>
<th>$S_T \leq K$</th>
<th>$S_T &gt; K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Share</td>
<td>$S_T$</td>
<td>$S_T$</td>
</tr>
<tr>
<td>Bond</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Put</td>
<td>$(K - S_T)$</td>
<td>0</td>
</tr>
<tr>
<td>Call</td>
<td>0</td>
<td>$(S_T - K)$</td>
</tr>
</tbody>
</table>

Table 3.1: Asset value at maturity date $T$ for the two cases $S_T \leq K$ and $S_T > K$. 

More interesting is now the impact of $K/S_0$. When $K/S_0 > 1$ we have a strike price that is higher than the current stock price, which means that we will only exercise the option if the stock price has increased. Similarly for $K/S_0 < 1$, we may exercise it when the stock price has decreased (until it reaches K), so there is a higher probability of making a profit. This of course leads to higher option
prices for lower $K/S_0$. This effect is shown in Figure 3.4 for various levels of $S_0$. Besides the decreasing prices, two other characteristics stand out. First we see that for $K/S_0 = 0$ the option price equals the asset price at time zero. We can explain this as follows. This contract gives us the option of getting the asset for free at maturity. The profit thus always equals $S_T$, but now the option price must be $S_0$ because this kind of option has the same effect as buying the asset at time zero.

Secondly we see that the option prices converge rather rapidly to zero as $K/S_0$ increases. Now the decreasing property of the option price and of course the fact that we cannot have negative option prices immediately gives us this result. It is due to the decreasing probability of exercising the option. For high $K/S_0$ we will rarely exercise, for it is not probable that the stock price increases that much. Not exercising the option leads to no changes in wealth for both the short als long side and thus the option price must go to zero.

Figure 3.3: Linearity of option prices in the stock price $S_0$ with $K/S_0$ constant, where $K$ is the strike price.

Figure 3.4: Option prices as a function of the ratio $K/S_0$, with $K$ strike price and $S_0$ the stock price.

After the impact of $S_0$ and $K$, we will now focus on the impact of $r$ and $\sigma$, starting with $r$. The Greeks give us for the derivative of $f$ to $r$ the following formula, $\rho = \frac{\partial f}{\partial r} = KT e^{-rT} N(d_2)$, which is clearly positive, so we expect to see an increase in the option price as $r$ increases. In Figure 3.5 the call option prices are shown as a function of $r$. Figure 3.5(a) displays a large interval for $r$ and Figure 3.5(b) gives the values of $r$ that are more relevant in practise. We immediately see that option prices
are increasing with $r$, which we expected. This can be explained by taking a closer look at equation (3.18). There we can see that the call and put price is equal if $S_0$ is the discounted value of $K$. We can interpret this result as having $r$ as the expected rate of return for the stock price. Now suppose we have a risk-free rate of $10\%$ per year. Then if we have a stock price $S_0 = 100$, the call and put price are the same for a strike price $K = 110$ and a maturity of one year. This means that we could buy the stock, take on a short position on the call and a long position on the put, which costs us €100. At maturity the call would be exercised if the stock was above 110. When the stock price is below 110 we exercise the put. Either way we have a profit of €10. Of course this is the same return as we would have made on holding a risk-free asset. Now suppose that the risk-free rate drops to 5%. If the price of the call and the put would not change we could still make 10% with the same riskless portfolio as described above, but this cannot happen, for we assumed to have no arbitrage opportunities. The market would have to adjust and the call price will go down and the put price will go up.

For very high values of $r$ we see that the option price approaches the stock price. This means that when the interest rate is very high, buying a call option is just as expensive as buying the stock. This is because in cases of a very high interest rate, the probability of not exercising the call option goes to zero. Hence the effect is the same as buying the stock at time zero which means that the option price equals the stock price.

The extreme values of $r$ are not too important for the practise of option pricing. We will use values of $r$ which lie within the interval $[2\%, 5\%]$, which is shown in Figure 3.5(b). The option price is close to a straight line. This means that the impact of errors in estimating $r$ is small as long as the error is small and will increase as the error increases. Now we can probably give an estimate of $r$ within a relatively small confidence interval, which means that the impact on the quality of the option price is manageable.

For various $\sigma$ we see rather similar behavior as for $r$, shown in Figure 3.6. The corresponding Greek is Vega, denoted by $\nu$, given by $\nu = \frac{\partial^2 C}{\partial \sigma^2} = S_0 N'(d_1) \sqrt{T}$. More information can be found in Hull (2008). To give an idea of the influence of $\sigma$, consider a call option and look ahead to the expiration date. If the option expires in the money, then the higher the price of the underlying, the greater the payoff of the option. On the other hand, if the option expires out of the money, then no matter how low the price of the underlying, the option always pays zero. Since the volatility is defined to be the standard deviation of the Brownian motion, we have that a higher volatility impacts both upward and downward movements. And as we have a bonus on high stock prices and no great loss for low stock prices, we find that a higher volatility makes an option more valuable.

We will now discuss the extreme case $\sigma = 0$, which thus should be the minimum option price with respect to $\sigma$. The other parameters are $S_0 = 100$, $K = 100$, $r = 0.03$ and $T = 1$. If we have no volatility, the stock is a risk-free asset, which means that it has return $r$. So in one year the stock is worth $100 \cdot e^{0.03} = 103.05$. The option must thus have a value of 3.05 at maturity. Discounting leads us to a value at time zero of $3.05 \cdot e^{-0.03} = 2.96$, which is also the result of the pricing formula. As $\sigma$ increases we can see that the option price increases as well. This is as expected. The current stock price is an upper bound for the option price, for if the option price is higher than the stock price, it is always more profitable to simply buy the stock.

Now for relevant values of $\sigma$, the option prices are shown in Figure 3.6(b). Again we see that the line is close to straight, which means that the same conclusions hold as for the robustness in $r$.

### 3.2 Merton’s mixed jump diffusion model

In the previous section we discussed the Black-Scholes model for option pricing. An important assumption there is the ‘constant’ behavior of the underlying price process. That is, the volatility and the expected rate of return are constant through time. As we discussed in Chapter 2, the housing market is partly characterized by sudden events with a radical impact on the price process. This suggests a less ‘constant’ process, which can be constructed by adding jumps. Merton’s Jump-Diffusion model includes such jumps. In this section we present Merton’s model, and we will first discuss assumptions, then the derivations of the price formulas and we conclude with a robustness analysis.
3.2. MERTON’S MIXED JUMP DIFFUSION MODEL

Merton’s model is based on the same assumptions as Black-Scholes option pricing, given in Section 3.1.1. The only difference lies in modeling the price development of the underlying asset. In the Black-Scholes case we assumed this to be a Geometric Brownian motion. As for Merton, the geometric Brownian motion is still the basic underlying process, but we add jumps to it. The second assumption is now as follows:

2. The price development $S$ of the underlying asset is characterized as a composition of two types of changes. First, the normal fluctuation in price, and secondly the abnormal changes (jumps). The normal fluctuations correspond to small continuous price changes, modelled by a geometric Brownian motion. The jumps are instantaneous, they arrive according to a Poisson process and the impact is stochastic. This process can be written formally as the stochastic differential equation

$$dS = (\mu - \lambda k)Sdt + \sigma Sdz + Sdq,$$

(3.21)

where $\mu$ is the continuously compounded rate of return on stock (unconditional on jumps), $\sigma$ is the instantaneous variance of return (conditional on no jump), $z(t)$ is a standard Brownian motion and $q(t)$ is an independent Poisson process with arrival rate $\lambda$. Now let $\vartheta_0 := 0$ and $\vartheta_1, \vartheta_2, \ldots$ denote the event times of this Poisson process, i.e. $\vartheta_{k+1} - \vartheta_k \sim \text{Exp}(\lambda)$. With each event of the Poisson process we associate a random variable $Y_k$ that represents the jump of the price process. That is

$$S(\vartheta_{k+} = S(\vartheta_k-)Y_k.$$

The random variables are i.i.d. and $Y_i \overset{d}{=} Y, (i = 1, 2, \ldots)$. The distribution of $Y$ is still free.
CHAPTER 3. PRICING MODELS

to choose, except that we want \( Y \) to be a nonnegative random variable (to model downward jumps). We define \( k = E[Y - 1] \) to be the expected price change due to a jump.

Equation (3.21) can be written as

\[
dS = \begin{cases} 
(\mu - \lambda k)S \, dt + \sigma S \, dz, & \text{if no jumps occur,} \\
(\mu - \lambda k)S \, dt + \sigma S \, dz + (Y - 1)S, & \text{if a jump occurs,}
\end{cases}
\]

(3.22)

where the occurrence of jump \( k \) is in the infinite small time interval \([\vartheta_k^-, \vartheta_k^+])\) and corresponds to an instantaneous jump of size \((Y_k - 1)S\), since \( Y_k \) is multiplicative. Furthermore, we have with probability one that no more than one Poisson event occurs in this interval. Note that we can look at the jump as adding an impulse function to the Brownian motion as shown in Figure 3.17.

**Remark** The jump component represents ‘non-systematic’ risk. This means that the jump component is uncorrelated with the market. We assume that the Capital Asset Pricing Model (CAPM) holds, hence the expected return on portfolios with ‘non-systematic’ risk equals the riskless rate, because it can be diversified away. CAPM was derived in Sharpe (1964) and is basically a model for pricing individual securities or portfolios. We will not go into the theory, assumptions and shortcomings of CAPM, but just assume that we have a riskless rate return from portfolios which consist only of ‘non-systematic’ risk, such as the jumps.

### 3.2.2 Derivation of price formulas

In this section we will discuss the derivation of a formula for option pricing, as shown in Merton (1975). We find that a solution is given by

\[
f(S_0, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda \tau} (\lambda \tau)^n}{n!} \left( E[c(S_0 X_n e^{-\lambda k \tau}, \tau, K, r, \sigma^2)] \right),
\]

(3.23)

with \( c = c(S_0 X_n e^{-\lambda k \tau}, \tau, K, r, \sigma^2) \) the price of a European call option in the Black-Scholes model in (3.15). Formula (3.23) can thus be seen as a weighted average over Black and Scholes option prices. We need a specification of the distribution of \( Y \) in order to turn (3.23) into an explicit form.

To derive (3.23) we take the following approach. First, we analyze the option price \( f \). Secondly, we try to construct a riskless portfolio and find that there exists no riskless portfolio. However, since the risk is non-systematic, we can conclude that the portfolio return must be the risk-free rate. This will eventually lead to a difference equation, of which (3.23) is a solution.

In this section, if we mention options, we always mean European call options.

**The option price process**

As we defined the stock price dynamics in equations (3.21) and (3.22), we will now examine the option price \( f \) and suppose that this is a twice differentiable function of \( S \) and \( t \). Similar to the Black-Scholes analysis, the option price process can be written in the same way as the stock price process. Now since the stock price follows (3.21), we can write the option price in a similar form as

\[
df = (\mu_c - \lambda k_c) f \, dt + \sigma_c f \, dZ + f \, dq_c,
\]

(3.24)

where we define \( \mu_c \) to be the rate of return and \( \sigma_c^2 \) the variance of return conditional on no jumps. The option price consists of three parts. First we have the (exponential) drift, denoted by \((\mu_c - \lambda k_c) f \, dt\), thus with drift rate \( \mu_c - \lambda k_c \). The entire underlying process has expected return rate \( \mu \), similarly the option price has expected return rate \( \mu_c \). The return consists of two parts: the exponential drift and the return due to jumps. The expected return caused by jumps (per year) is \( \lambda k_c \) (will be explained
3.2. MERTON’S MIXED JUMP DIFFUSION MODEL

later). The exponential drift must thus be the difference between the total and jumps related expected rates of return.

The second part of the right-hand side of (3.24) is the influence of the ‘noise’ factor, corresponding to the normal changes, which can again be written as a Brownian motion (parameter \(\sigma_c\)). Thirdly the jump component in the underlying stock relates to a similar Poisson jump process in the option price. We have \(q_c(t)\) as an independent Poisson process with parameter \(\lambda\) and \(k_c := E[Y_c - 1]\), where \(Y_c - 1\) is the random variable of the change in the option price due to an event (that is a jump). This explains that the impact of jumps on the expected return equals \(\lambda k_c\). (3.21) is of no use if we know nothing about the variables \(\mu_c, \sigma_c\) and \(k_c\). We will now give some characteristics of these variables and will use Itô’s lemma to give formulas for them.

An event for the option price occurs if and only if the Poisson event for the stock price occurs. Now if \(Y_k\) takes on the value \(y\), if follows that \(Y_c\) takes on the value \(y_c := f(S_y, t)/f(S, t)\), hence

\[
Y_c := \frac{f(S_y, t)}{f(S, t)},
\]

(3.25)

This of course agrees with the definition of \(k_c\), being \(k_c = E[Y_c - 1] = E[f(S_y, t)/f(S, t) - 1]\).

The parameters \(\mu_c\) and \(\sigma_c\) can be obtained from applying Itô’s lemma and the counterpart of Itô’s lemma for Poisson processes (see Merton (1975)). This gives

\[
\mu_c = \frac{1}{f(S, t)} \left( \frac{\partial f(S, t)}{\partial S} (\mu - \lambda k) S + \frac{\partial f(S, t)}{\partial t} \right) + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 + \lambda (E[f(S_y, t) - f(S, t)])
\]

(3.27)

and

\[
\sigma_c = \frac{\partial f(S, t)}{f(S, t)} \sigma S.
\]

(3.28)

**Construction of a risk-free portfolio**

Now analogous to the stock price and the option price, we wish to give a formulation of the portfolio value, in which we hold stocks and options, in proportions \(w_1\) and \(w_2\), respectively, and with \(w_1 + w_2 = 1\). Now if \(P\) is the value of the portfolio, we can write

\[
dP = (\mu_p - \lambda k_p) P dt + \sigma_p P dz + P dq_p.
\]

(3.29)

From (3.21) and (3.24), and substitution of \(w_2 = 1 - w_1\) we obtain

\[
\mu_p = w_1 \mu + w_2 \mu_c,
\]

(3.30)

\[
\sigma_p = w_1 (\sigma - \sigma_c) + \sigma_c
\]

(3.31)

and

\[
Y_p - 1 = w_1 (Y - 1) + w_2 (f(S_y, t) - f(S, t))/f(S, t).
\]

(3.32)

In the Black-Scholes analysis we would choose a riskless portfolio which essentially means that we choose weights such that \(w_1^* \sigma + w_2^* \sigma_c = 0\). There are no arbitrage opportunities, hence such a portfolio would have return rate \(r\). This and using equations (3.30) and (3.31) gives us

\[
\frac{\mu - r}{\sigma} = \frac{\mu_c - r}{\sigma_c}.
\]

(3.33)

In the Black-Scholes analysis, we had \(\lambda = 0\). In this case however, because of the addition of a jump process, the portfolio with weights \(w_1^*\) and \(w_2^*\) is no longer riskless. The ‘jump risk’ corresponds to \(Y_p\) and a closer look at equation (3.32) tells us that there is no portfolio which gives \(Y_p = 1\). The reason
sign $w_2$ & sign $(Y_p^*-1)$ & sign $k_p^*$ & stocks & options & anticipated return & unanticipated return \\
- & - & - & long & short & $\mu_p^* + \lambda|k_p^*|$ & $-|Y_p^*-1|$ \\
+ & + & + & short & long & $\mu_p^* - \lambda|k_p^*|$ & $+|Y_p^*-1|$ \\

Table 3.2: anticipated and unanticipated returns for Black-Scholes portfolios.

is that portfolio mixing is a linear process whereas the option price is a non-linear function of $S$. More information on the subject can be found in Merton (1975), p. 131.

In spite of the jump risk, we can still work out the characteristics of the Black-Scholes hedge. Similar to in the Black-Scholes analysis, the Brownian motion is hedged out. We thus obtain a pure jump process.

$$P^* = (\mu_p^* - \lambda k_p^*)P^* \, dt + P^* \, dq_p^*.$$  \hfill (3.34)

Similar to (3.22) we can write this in the following form

$$dP^* = \begin{cases} (\mu_p^* - \lambda k_p^*)P^* \, dt, & \text{if the Poisson event does not occur,} \\ (\mu_p^* - \lambda k_p^*)P^* \, dt + (Y_p^*-1)P^*, & \text{if the Poisson event occurs.} \end{cases}$$  \hfill (3.35)

We now examine $Y_p^*-1$. Starting from (3.32), substitution of $w_1^* = -w_2^* \sigma_c / \sigma$ and using (3.28) yields

$$Y_p^*-1 = w_1^*(Y-1) + w_2^*(f(SY,t) - f(S,t))/f(S,t),$$

$$= -w_2^*(\sigma_c(Y-1)/\sigma - (f(SY,t) - f(S,t))/f(S,t)),$$

$$= w_2^* \left( f(SY,t) - f(S,t) - \frac{\partial f(S,t)}{\partial S}(SY-S) \right).$$ \hfill (3.36)

The option price is convex in the stock price. For the Black-Scholes model this can be seen in Figure 3.4. More general cases are discussed in e.g. Merton (1973) and Jagannathan (1984). For convex functions we have that the function value never lies below an arbitrary tangent line of the function, i.e. for a convex function $c$, $c(y) \geq c(x) + c'(x)(y-x)$ $\forall y$. The same holds for $f$ as a function of $S$, hence $f(SY,t) - f(S,t) - \frac{\partial f(S,t)}{\partial S}(SY-S) \geq 0$ for all values of $Y$.

Table 3.2 shows the signs of $(Y_p^*-1)$ and $k_p^*$ for the two possible portfolios. The anticipated and unanticipated returns follow directly from (3.35). We can see that the investor going short on options has a somewhat higher return if there is no jump. But, in case of a jump an investor suffers a much larger loss. The other way around, an investor going long on the option benefits greatly from jumps, but has a smaller anticipated return. So in periods of many jumps the investor going long on options has the advantage, but in quiet periods, the investor going short benefits. Of course, the unexpected nature of the Poisson process will level out the benefits and there is thus no way of knowing whether one should write or buy an option for some period.

Since we cannot hold a riskless portfolio, we can not use the same method as for the Black-Scholes pricing formula. We find however that the option pricing formula can be derived when we know the required expected return on the option as a function of the stock price $S$ and the time to expiration $\tau := T - t$. Define $g(S,\tau)$ to be the equilibrium, instantaneous expected rate of return on the option. Then (3.27) yields

$$0 = \frac{1}{2} \frac{\partial^2 f(S,\tau)}{\partial S^2} \sigma^2 S^2 + \frac{\partial f(S,\tau)}{\partial S}(\mu - \lambda k)S - \frac{\partial f(S,\tau)}{\partial \tau} - g(S,\tau)f(S,\tau) + \lambda (E[f(SY,\tau) - f(S,\tau)]).$$ \hfill (3.37)

To this, we must add the following boundary conditions:

$$f(0,\tau) = 0,$$ \hfill (3.38)

$$f(S,0) = \max(0, S - K),$$ \hfill (3.39)
3.2. MERTON'S MIXED JUMP DIFFUSION MODEL

where $K$ is the strike price of the option. Equation (3.37) is a mixed partial differential-difference equation and hard to solve. Another downside is the necessity of knowing $\mu$ and $g(S, \tau)$, whereas the power of Black-Scholes lies in not needing to know these parameters.

We assumed that the jump component represents ‘non-systematic’ risk. As for the portfolios, we can see in Table 3.2 that the only source of uncertainty in the return is the jump component. Now since we assume that CAPM holds, the expected return on portfolios with ‘non-systematic’ risk must equal the riskless rate, for it can be diversified away. Hence we have a riskless return from our portfolio $P^*$, so $\mu^*_n = r$. (3.30) now gives us (3.33).

Using (3.33) together with (3.27) and (3.28) yields that $f$ must satisfy

$$0 = \frac{1}{2} \frac{\partial^2 f(S, \tau)}{\partial S^2} \sigma^2 S^2 + \frac{\partial f(S, \tau)}{\partial S} (r - \lambda k) S - \frac{\partial f(S, \tau)}{\partial \tau} - rf(S, \tau) + \lambda (E[f(SY, \tau) - f(S, \tau)]).$$

subject again to the boundary conditions (3.38) and (3.39). Now (3.40) no longer depends on $\mu$ and $g(S, T)$, but as in the Black-Scholes case only depends on $r$ and $\sigma$. Moreover, if we let $\lambda = 0$ we have the Black-Scholes formula (3.9).

A complete closed-form solution to (3.40) cannot be given without a further specification of $Y$. However, we can give a partial solution.

Recall the function $c$ for Black-Scholes option pricing (3.15)

$$c(S_0, \tau, K, r, \sigma^2) = S_0 N(d_1) - Ke^{-\tau} N(d_2).$$

Now define the random variable $X_n := \prod_{k=1}^{n} Y_k$ to be the product of the jumps (with $(Y_k)_{k=1}^{n}$ i.i.d. and $Y_1 \overset{d}{=} Y$) for the first $n$ events ($n = 1, 2, \ldots$). Set $X_0 = 1$. The solution to (3.40) with current stock price $S_0$ and remaining time until maturity $\tau$ is given by

$$f(S_0, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda \tau} (\lambda \tau)^n}{n!} \left( E[c(S_0 X_n e^{-\lambda \tau}, \tau, K, r, \sigma^2)] \right).$$

A verification of (3.41) being a solution of (3.40) is given in the appendix of Merton (1975). There are two special cases for which (3.41) can be simplified. First, we have the situation where there is no immediate ruin, that is when the Poisson event occurs, the stock price goes to zero. We thus have $Y = 0$ with probability one. This means that $X_n = 0$ for $n \neq 0$ and of course $k = -1$. This gives us the option price

$$f(S_0, \tau) = e^{-\lambda \tau} c(S_0 e^{\lambda \tau}, \tau, K, r, \sigma^2),$$

which is identical to the Black-Scholes option price but with an increased interest rate. Now the Black-Scholes option price is increasing with the interest rate, so the positive probability of ruin leads to a higher option price. This seems to be counter intuitive, but is in fact quite logical. We will discuss this in detail in Section 3.2.3.

The second special case is the case of log-normally distributed jump sizes. Let $\gamma$ and $\delta^2$ denote the mean and variance of the associated normal distribution. The distribution of $X_n$ is again log-normal with $\text{Var}(\log(X_n)) = \delta^2 n$ and $E[\log(X_n)] = n \gamma$. We then obtain for the option price the following formula

$$f(S_0, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda \tau} (\lambda \tau)^n}{n!} c_n,$$

where $\lambda^* := \lambda (1 + k)$ and $c_n$ is the Black-Scholes option price with variance rate $\sigma^2 + \frac{\lambda \delta^2}{\tau}$ and the risk free rate is $r - \lambda k + \frac{\lambda \delta^2}{\tau}$. 
### 3.2.3 Results, robustness

In this section we will examine the behavior of Merton’s option prices. We will do this for two distributions of $Y$: $Y$ being a constant and $Y$ lognormal.

#### Deterministic jumps

For $Y$ being deterministic and equal to $b$, we use Matlab function C.1.2 to determine the option prices.

Now the behavior of the prices as a function of $S$ and $K$ is the same as for the Black-Scholes model, so we will skip those parameters. This leaves $b$ and $\lambda$, the impact and the frequency of jumps as the interesting parameters. Of course for $b = 1$ or $\lambda = 0$ we have that the Merton price equals the Black-Scholes price.

We start by looking at the impact of $b$, shown in Figure 3.7. We can see that $b = 1$ gives the minimum of the option price over $b$. Now this seems logical for $b > 1$, since we have positive jumps and thus a higher expected stock price at maturity. But for $b < 1$ the expected stock price is smaller, so why is the call option more expensive?

Recall that the Brownian motion has a drift rate of $\mu - \lambda k$, in this deterministic case $\mu - \lambda (b - 1)$. Furthermore, we have that the probability of no jumps is rather high (depending on $\lambda$ of course). Now look at the sum of the option price conditional on the number of jumps weighed by the probability of having that number of jumps. If we have a jump, the jump has such a high impact that the option price quickly goes to zero. But, conditional on no jumps and for $b < 1$, we find that the Brownian motion has a ‘high’ drift rate, resulting in a high option price. So, when Merton’s price is determined by the weighted average of option prices conditional on the number of jumps, we find that the first term corresponding to zero jumps is dominant.

The sum components are shown in Table 3.3. For $b > 1$ it is just the other way around. There we have higher option prices if there are jumps, and the no jump component has a minimal contribution to the option price.

The impact of $b$ is highly dependent on $\lambda$. For instance, when $\lambda = 1$ we have that a 10% change of $b$ can lead to a change of €10 in the option price.

Figure 3.8 depicts the option prices as a function of $\lambda$. When $b$ moves further from 1, option prices become more sensitive to changes in $\lambda$.

If we consider estimating $b$ and $\lambda$, we must keep in mind that we have very limited data and that the confidence interval for the estimate will be rather high. This means that we must be very careful with choosing our parameters. In this regard it is fortunate that we are typically interested in low values for $\lambda$, since this allows us some space in choosing $b$ without changing the option price dramatically. But either way, we must try to give some confidence interval so that we have an idea of the possible error.
### 3.2. Merton's Mixed Jump Diffusion Model

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<th>Price</th>
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<td>0.90</td>
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<td>0.00</td>
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<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th># Jumps</th>
<th>BS part</th>
<th>Prob</th>
<th>Product</th>
<th>Price</th>
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</tr>
</tbody>
</table>

Table 3.3: Construction of Merton prices with \( P[Y = y] = 1 \).
CHAPTER 3. PRICING MODELS

In this section we take a look at the option prices for stochastic $Y$. This means that not only the time between jumps is stochastic but the impact of the jumps as well. In this case we define $b = E[Y]$ and $\delta$ as the standard deviation of $\log(Y)$. We use the Matlab function C.1.3 to determine the prices.

The impact of $b$ and $\lambda$ for various $\delta$ is shown in Figures 3.9 and 3.10 respectively. Of course we have that for $\delta = 0$ the prices are equal to the deterministic case where $P[Y = b] = 1$ as discussed above. As in the Black-Scholes case for $\sigma$ we expect that higher $\delta$ leads to higher option prices. This effect is clear from the figures, although the impact of $\delta$ is not as dramatic as the impact of $b$ and $\lambda$. In general we must conclude that as in the case of deterministic $Y$ we must be very careful in choosing $b$ and $\lambda$. Higher values of $\delta$ may result in an even bigger impact of estimation errors. In this case again, when choosing the parameters we must check confidence intervals.

Figure 3.7: Option prices as a function of (multiplicative) jump size $b$.

Figure 3.8: Option prices as a function of jump frequency $\lambda$.

Lognormal jumps

In this section we take a look at the option prices for stochastic $Y$. This means that not only the time between jumps is stochastic but the impact of the jumps as well. In this case we define $b = E[Y]$ and $\delta$ as the standard deviation of $\log(Y)$. We use the Matlab function C.1.3 to determine the prices.

The impact of $b$ and $\lambda$ for various $\delta$ is shown in Figures 3.9 and 3.10 respectively. Of course we have that for $\delta = 0$ the prices are equal to the deterministic case where $P[Y = b] = 1$ as discussed above. As in the Black-Scholes case for $\sigma$ we expect that higher $\delta$ leads to higher option prices. This effect is clear from the figures, although the impact of $\delta$ is not as dramatic as the impact of $b$ and $\lambda$. In general we must conclude that as in the case of deterministic $Y$ we must be very careful in choosing $b$ and $\lambda$. Higher values of $\delta$ may result in an even bigger impact of estimation errors. In this case again, when choosing the parameters we must check confidence intervals.
3.3 Comparison of pricing methods

In the previous section we discussed the robustness and the response to parameter changes for option prices according to Merton’s model. We now wish to give a comparison between Merton’s option prices and the Black-Scholes option prices as discussed in Section 3.1. We saw that the assumed underlying process is very similar for both models, being a geometric Brownian motion, but in Merton’s model we added jumps to this process. The addition of jumps to the process led in most cases to higher option prices, even in case of a positive probability of immediate ruin, as discussed in Section 3.2.3. But this is of course not a fair comparison, for the addition of jumps causes a positive change in the volatility of the entire process. Hence, in this section we will compare the option price methods with comparable volatility in the underlying processes. To do this, we must first find a method which gives us the volatility of the Black-Scholes process that is consistent with the volatility of Merton’s process.

3.3.1 Comparison of volatility in the underlying price processes

In the Black-Scholes model we saw that the index change over some time interval $T$ is lognormally distributed. To estimate the volatility of the price process $\hat{\sigma}_{BS}$, we can fit either a lognormal distribution to yearly index changes or a normal distribution to the continuously compounded rates of return as described in Appendix B.3.2.
For Merton’s model this is not as easy as for the Black-Scholes model. We denote volatility as $\sigma_M$, instead of $\sigma$ to avoid confusion. $\sigma_M$ is defined as the volatility of the geometric Brownian motion, so it gives the volatility of the underlying process conditionally on no jumps. But the jumps add volatility to the entire process. To illustrate this, we plotted some histograms of index changes in Figure 3.11, where the index follows a process as in Merton’s model. In Figure 3.11(a) the histogram is depicted for a positive probability of immediate ruin. The bell shape on the right corresponds to the index changes conditional on no jumps. This is of course lognormally distributed. The bar on the left corresponds to having one or more jumps, where the index jumped to zero with probability one. Similar we have for Figures 3.11(b) and 3.11(c) several separated peaks, where the peak around 100 corresponds to no jump, around 50 to one jump and around 25 to two jumps. In these cases the jumps are of size 0.5, but now in Figure 3.11(c) the jumps are not deterministic, but have a lognormal distribution with $\delta = 0.2$. This results in lower kurtosis, i.e. lower peaks and a wider bell shape.

\begin{figure}[h]
\centering
\subfigure[$\mu = 0.05, \sigma_M = 0.05, b = 0, \delta = 0, \lambda = 1$]{\includegraphics[width=0.45\textwidth]{figure1a}}
\subfigure[$\mu = 0.05, \sigma_M = 0.05, b = 0.5, \delta = 0, \lambda = 1$]{\includegraphics[width=0.45\textwidth]{figure1b}}
\subfigure[$\mu = 0.05, \sigma_M = 0.05, b = 0.5, \delta = 0.2, \lambda = 1$]{\includegraphics[width=0.45\textwidth]{figure1c}}
\subfigure[$\mu = 0.05, \sigma_M = 0.03, b = 0.8, \delta = 0.1, \lambda = 0.1$]{\includegraphics[width=0.45\textwidth]{figure1d}}
\caption{Histograms of simulated stockprices $S$ at time $T = 1$, starting at $S_0 = 100$, using the assumptions of Merton’s mixed jump diffusion process for various values of the average jump size $b$, the standard deviation of the associated normal distribution of the jump size $\delta$ and the jump intensity $\lambda$. (# samples $N = 50000$)}
\end{figure}

It is clear that the price process in the Merton model is very different from the Black-Scholes model. However we do wish to compare these processes and their volatilities in particular. It is possible to do this by calculating the variance of the returns of the process. This is done in Navas (2003) for jumps with lognormally distributed sizes. From now on we use only the lognormal distribution for the jump sizes $Y$. We already saw that this gives a nice, closed-form solution for the option price formula. So far, we also discussed $Y$ to be deterministic. This is of course a sub-class of the lognormal distribution, where the standard deviation of $Y$ equals zero. Furthermore, since we have two parameters which we can choose freely, we have enough freedom to fit the distribution of $Y$ properly.

**Theorem 3.3.1 (Navas)** Corresponding to the process considered in Section 3.2.1, consider the price
process $S(t)$ represented by a geometric Brownian motion $z(t)$ with standard deviation $\sigma_M$ and a independent Poisson process $q(t)$ with constant intensity $\lambda$ and random jump size $Y$. Assume that jumps sizes are lognormally distributed with parameters $\gamma$ and $\delta$, where $\gamma$ and $\delta$ are the mean and standard deviation of the associated normal distribution. Then, the total variance of the natural logarithm of the stock price under a jump-diffusion process is given by

$$\text{Var}\{\log \frac{S(t)}{S(0)}\} = \left(\sigma_M^2 + \lambda(\gamma^2 + \delta^2)\right)t.$$  \hfill (3.44)

Following from Theorem 3.3.1 and (3.2) we have the following result:

**Corollary 3.3.2** Given a Jump-Diffusion process as described in Theorem 3.3.1, we have that a price process following a geometric Brownian motion as in the Black-Scholes model, with volatility parameter $\hat{\sigma}_{BS} = \sqrt{\sigma_M^2 + \lambda(\gamma^2 + \delta^2)}$ (3.45)

has the same standard deviation as the Jump-Diffusion process.

So far we did not use the parameters $\gamma$ and $\delta$ (the parameters of the normal distribution corresponding to $Y$) to characterize $Y$ but rather the parameters $b$ and $\delta$, where $b$ is the expected value of $Y$ and $\delta$ the standard deviation of the associated normal distribution. Now these parameters are related as follows:

$$b = \mathbb{E}[Y] = e^{\gamma + \delta^2/2},$$ \hfill (3.46)

$$v = \text{Var}(Y) = e^{2\gamma + \delta^2}(e^{\delta^2} - 1),$$ \hfill (3.47)

$$\gamma = \log\left(\frac{b^2}{\sqrt{v + b^2}}\right),$$ \hfill (3.48)

$$\delta = \sqrt{\log(v/b^2 + 1)}.$$ \hfill (3.49)

We wish to still characterize $Y$ with $b$ and $\delta$, for which we need to express $v$ and $\gamma$ in $b$ and $\delta$.

**Lemma 3.3.3** Let $Y$ be a random variable with a lognormal distribution. Let $\mathbb{E}[Y] = b$ and let $\delta$ be the standard deviation of the associated normal distribution. Then we can calculate $v^2 = \text{Var}(Y)$ and $\gamma$, the mean of the associated normal distribution as follows:

$$\gamma = \log(b) - \frac{1}{2}\log\left(\frac{1}{2}(1 + \sqrt{1 + 4q})\right)$$ \hfill (3.50)

$$v = \frac{b^2(\sqrt{1 + 4q} - 1)}{2}.$$ \hfill (3.51)

where $q$ is defined as

$$q = e^{\delta^2}(e^{\delta^2} - 1).$$ \hfill (3.52)

**Proof.** Substitution of (3.52) and (3.48) into (3.47) yields for $v$

$$v = e^{2\gamma} \cdot q,$$

$$= \left(\frac{b^2}{\sqrt{v + b^2}}\right)^2 \cdot q,$$

$$= \frac{b^4}{v + b^2} \cdot q,$$

which gives us the quadratic equation

$$0 = v^2 + vb^2 - b^4 q.$$
We choose the solution where \( v > 0 \), which gives (3.51).

We can now substitute (3.51) into (3.48):

\[
\gamma = \log \left( \frac{b^2}{\sqrt{\frac{b^2(\sqrt{1+4q-1}-1)}{2} + b^2}} \right),
\]

\[
= \log(b) - \log \left( \sqrt{\frac{1+4q-1}{2} + 1} \right),
\]

\[
= \log(b) - \log \left( \frac{1}{2} (\sqrt{1+4q+1}) \right),
\]

which yields (3.50).

We are now able to compare the two models with similar volatility by choosing the Merton parameters and calculating first \( q \) and \( \gamma \) and then \( \hat{\sigma}_{BS} \).

### 3.3.2 Comparison of Black-Scholes and Merton option prices

In the previous section we found a way to synchronize the volatility of the two models. In this section we will give an analysis of the behavior of the two prices as a function of the parameters in the Jump-Diffusion process. We are particularly interested in which model gives the higher price, how large the difference in price is and how these characteristics depend on the parameters. All option prices have the following characteristics: \( S_0 = 100, K = 100, T = 1 \) year and \( \sigma_M = 0.10 \). We will vary \( \lambda, \delta \) and \( b \), starting with \( \lambda \) in Figure 3.12.

Figure 3.12 shows increasing option prices in \( \lambda \). For the Merton model we already saw this behavior in Figures 3.8 and 3.10 and the increase for the Black-Scholes model is also clear from the increase of \( \hat{\sigma}_{BS} \) in (3.45) and the positive effect this has on the option prices (Figure 3.6).

It turns out that the Black-Scholes option is somewhat higher than the Merton price. This depends on the choice of \( b \), which we will show later on. Furthermore we find that the difference between the two remains roughly the same as we increase \( \lambda \). Hence the choice of \( \lambda \) has an increasing effect on both option prices but the effect on the difference is very small.

A very similar behavior is seen in Figure 3.13 for various \( \delta \). Apart from the increasing behavior which we already saw in Section 3.2.3 and from the increase in variance from (3.45) we again see that the difference between the option prices remains rather constant. Moreover we are typically interested in smaller values of \( \delta \). We would not expect to find a value of \( \delta \) higher than say 0.40. But in this region, \( 0 < \delta < 0.40 \), we find that the increase of the option price is rather small. We thus find that the estimate of \( \delta \) is of relatively small impact on the option price.

The influence of \( b = \mathbb{E}[Y] \) on the option prices is shown in Figures 3.14 and 3.15 for \( \delta = 0.1 \) and \( \delta = 0 \) respectively. We can see that the minimum is attained at \( b = 1 \). This is expected, since the impact of the jumps is also minimal at this point. Furthermore, we have that the Black-Scholes price is higher for negative jumps and the Merton price is generally higher for positive jumps. This means that there is a point of intersection at some value of \( b \). We call this point of intersection \( b^* \). Now if \( \delta = 0 \), the deterministic case, we find that \( b^* = 1 \). Deterministic jumps of size 1 have no influence on the price process, hence in this case the Merton model is equal to the Black-Scholes model and so is its option price. Of course we also find from (3.45) that \( \hat{\sigma}_{BS} = \sigma_M \) in this case.

Now for \( b = 1 \) and \( \delta > 0 \) we have from (3.45) that \( \hat{\sigma}_{BS} > \sigma_M \). This is the result of the jumps being non-deterministic. The average jumps size may be 1 but the deviation of the jumps causes
higher variance of the entire process, and thus higher prices. It turns out that in this case at $b = 1$ the Black-Scholes price is still higher than the Merton price. So we must have $b^* > 1$. In Figure 3.15 the point of intersection is shown when $\delta = 0.1$. We find that $b^* \approx 1.125$. To investigate the points of intersection we plotted $b^*$ as a function of $\delta$ in Figure 3.16.

The point $b^*$ divides the graph into two regions. First the region $b < b^*$ where the Black-Scholes price is higher than the Merton price and the second region, $b > b^*$ where the Merton price is higher. In the first region we can explain the fact that the Black-Scholes price is higher in the following way: While we have drawn a fair comparison in the sense that both price processes have the same variance, the intensity of the jumps (abrupt changes) decreases the price of the option. The jumps that are taken into account with Merton in this region generally have a negative impact on the price of call options because they model movements of the process away from the strike price. If $\delta > 0$ and thus $b^* > 1$ we have for the values of $1 < b < b^*$ that upward effect on the Black-Scholes prices, caused by the high values $\hat{\sigma}_{BS}$ resulting from positive $\delta$, is stronger than the upward effect on the Merton prices caused by the moderately positive jump effects.

As for the second region we have a similar explanation. In this case, the jumps model a change towards the strike price, so this has a positive impact on the call option price. This in contrast to the Black-Scholes model where the high volatility $\hat{\sigma}_{BS}$ corresponds to both negative and positive changes.

So far we discussed the differences between Black-Scholes and Merton’s option prices. To do this we looked at the overall influence of the Merton parameters on the prices and saw that $b$ had the largest impact on the behavior of the prices. Furthermore, increase in both $\lambda$ and $\delta$ caused higher option prices, but these parameters did not have a significant effect on the difference between Black-Scholes and Merton prices.

### 3.4 Option pricing in practice

The previous section gave us insight in the difference between option prices when using Black-Scholes and Merton’s model. We also discussed the influence of the parameters on this. Now if we wish to use these models for option prices using real data, we need some guidelines to estimate parameters.

Let us first consider Figure 3.17, in which a stylized example of a Jump-Diffusion process is given over a time span of 100 years. The index is shown in Figure 3.17(a). The index consists of two compo-
CHAPTER 3. PRICING MODELS

Figure 3.13: Comparison of option prices as a function of the standard deviation of the associated normal distribution $\delta$, where the average (multiplicative) jump size $b = 0.5$, and the jump intensity $\lambda = 1$.

Figure 3.14: Comparison of option prices as a function of the average (multiplicative) jump size $b$, where the jump intensity $\lambda = 1$ and $\delta = 0$.

...components, being a geometric Brownian motion and a jump process, shown in Figures 3.17(b) and 3.17(c) respectively. Now as discussed in Navas (2003) we have that the rates of returns of these components are independent. This led to dividing the variance of the total process into the same two components, as shown in (3.45), where $\gamma$ follows from (3.50) and (3.52) as a function of $b$ and $\delta$. All and all we find that (3.45) holds all variables required for Black-Scholes and Merton’s option price formulas. These variables are shown in Table 3.4, along with their description and the component of the process from which they can be estimated. Note here that the decision of the user for datapoints in Figure 3.17(a) to belong to either Figure 3.17(b) or 3.17(c) is crucial. In this case the decision is clear, but real data may be ambiguous.

We again emphasize that the influence of $\delta$ on the option prices is low in the region $0 < \delta < 0.40$. Therefore we start by simplifying (3.45), by setting $\delta = 0$. In that case, from (3.50) and (3.52) we have that $\gamma = \log(b)$, hence

$$\dot{\sigma}_{BS} = \sqrt{\sigma_M^2 + \lambda \log(b)^2}. \quad (3.53)$$
3.4. OPTION PRICING IN PRACTICE

Figure 3.15: Comparison of option prices as a function of the average (multiplicative) jump size $b$, where the jump intensity $\lambda = 1$, and the standard deviation of the associated normal distribution $\delta = 0.1$.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
<th>Can be estimated from</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{BS}$</td>
<td>standard deviation of total process</td>
<td>Figure 3.17(a)</td>
</tr>
<tr>
<td>$\sigma_M$</td>
<td>standard deviation of Brownian motion</td>
<td>Figure 3.17(b)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>frequency of jumps</td>
<td>Figure 3.17(c)</td>
</tr>
<tr>
<td>$b$</td>
<td>mean impact of jump ($E[Y]$)</td>
<td>Figure 3.17(c)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>uncertainty of jump size (standard deviation of log($Y$))</td>
<td>Figure 3.17(c)</td>
</tr>
</tbody>
</table>

Table 3.4: Variables for option pricing

This leaves us with four parameters to be estimated. Now if we estimate three of them, we can use (3.53) to calculate an implied fourth. This idea gives us the Procedure 3.4.1 for estimating parameters and calculating option prices.

A Matlab function to carry out the procedure can be found in C.3.1.

As an example of Procedure 3.4.1 we will use the data shown in Figure 3.17. The data is already divided into the Brownian motion and the jump part, so we can immediately go to step 2.

In step 2 of the procedure we should choose which value should be implied. Now as we have a rather large time span and since Figure 3.17(b) shows low volatility, we should definitely estimate $\sigma_M$. As for the jumps, Figure 3.17(c) shows little variation in jump sizes, which means that we can probably make a good estimation for $b$. Of course the assumption of $\delta = 0$ is not correct since we do have some uncertainty about $b$. But as mentioned before small changes in $\delta$ are of little influence on option prices, so we can move on. The only variables left are $\sigma_{BS}$ and $\lambda$. Since there are great differences in the interarrival times (ranging from 7 to 35 years) and the sample size is only four, we choose to estimate $\sigma_{BS}$ and use an implied $\lambda$. Hence we go to step III.

The estimated parameters turn out as follows: $\sigma_{BS} = 0.067$, $\sigma_M = 0.028$ and $b = 0.785$. Equation (3.53) now gives us $\lambda = 0.064 \approx 1/16$.

Now step 3 gives us a Black-Scholes option price of €3.76 and a Merton price of €3.30.
Figure 3.16: Intersection point $b^*$ as a function of standard deviation $\delta$, with volatility $\sigma_M = 0.1$ of the Brownian motion and jump intensity $\lambda = 1$.

(a) Merton’s jump-diffusion process

(b) Brownian motion component

(c) jump component

Figure 3.17: Example for estimation of parameters Merton process.
3.4. OPTION PRICING IN PRACTICE

Procedure 3.4.1 (A procedure for option pricing for real data (assuming \( \delta = 0 \).))

1. Take a data set of the underlying process and divide it into two components, being the geometric Brownian motion and the jump component.

2. Decide which parameter should be the one to be estimated ("implied"). Base the decision on the amount of information you have on each parameter. The one with the least information should be implied by the estimates of the other parameters. Now following your decision, let your next step be either I, II, III or IV.

I: (Implied \( \hat{\sigma}_{BS} \))

(a) Take the geometric Brownian motion component of the process. Divide it into subintervals of length \( D \). Calculate the continuously compounded rates of return over these intervals and fit the normal distribution to calculate \( \hat{\sigma}_M \), (B.27).

(b) Take the jump component of the index. Set \( b \) to be the mean value of the jumps. Calculate the inter arrival times of the jumps. Denote the mean interarrival time to be \( \vartheta \) and let \( \lambda = 1/\vartheta \).

(c) Determine the implied \( \hat{\sigma}_{BS} \) from (3.53).

II: (Implied \( \sigma_M \))

(a) Take the data set of the underlying process. Divide it into subintervals of length \( D \). Calculate the continuously compounded rates of return over these intervals and fit the normal distribution to calculate \( \hat{\sigma}_{BS} \), (B.27).

(b) Take the jump component of the index. Set \( b \) to be the mean value of the jumps. Calculate the interarrival times of the jumps. Denote the mean interarrival time to be \( \vartheta \) and let \( \lambda = 1/\vartheta \).

(c) Determine the implied \( \sigma_M \) from (3.53).

III: (Implied \( \lambda \))

(a) Take the data set of the underlying process. Divide it into subintervals of length \( D \). Calculate the continuously compounded rates of return over these intervals and fit the normal distribution to calculate \( \hat{\sigma}_{BS} \), (B.27).

(b) Take the geometric Brownian motion component of the process. Divide it into subintervals of length \( D \). Calculate the continuously compounded rates of return over these intervals and fit the normal distribution to calculate \( \sigma_M \), (B.27).

(c) Take the jump component of the index. Set \( b \) to be the mean value of the jumps.

(d) Determine the implied \( \lambda \) from (3.53).

IV: (Implied \( b \))

(a) Take the data set of the underlying process. Divide it into subintervals of length \( D \). Calculate the continuously compounded rates of return over these intervals and fit the normal distribution to calculate \( \hat{\sigma}_{BS} \), (B.27).

(b) Take the geometric Brownian motion component of the process. Divide it into subintervals of length \( D \). Calculate the continuously compounded rates of return over these intervals and fit the normal distribution to calculate \( \sigma_M \), (B.27).

(c) Take the jump component of the index. Calculate the inter arrival times of the jumps. Denote the mean interarrival time to be \( \vartheta \) and let \( \lambda = 1/\vartheta \).

(d) Determine the implied \( b \) from (3.53).

3. Calculate the option prices using the estimated and implied parameters.
Table 3.5: Example of Procedure 3.4.1.

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Estimated</th>
<th>Implied</th>
<th>BS Price</th>
<th>M Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.4.1 I</td>
<td>$\sigma_M = 0.028, \lambda = 0.046, b = 0.785$</td>
<td>$\hat{\sigma}_{BS} = 0.059$</td>
<td>€3.47</td>
<td>€3.05</td>
</tr>
<tr>
<td>3.4.1 II</td>
<td>$\hat{\sigma}_{BS} = 0.067, \sigma_M = 0.046, b = 0.785$</td>
<td>$\sigma_M = 0.042$</td>
<td>€3.76</td>
<td>€3.42</td>
</tr>
<tr>
<td>3.4.1 III</td>
<td>$\hat{\sigma}_{BS} = 0.067, \sigma_M = 0.028, b = 0.785$</td>
<td>$\lambda = 0.064$</td>
<td>€3.76</td>
<td>€3.30</td>
</tr>
<tr>
<td>3.4.1 IV</td>
<td>$\hat{\sigma}_{BS} = 0.067, \sigma_M = 0.028, \lambda = 0.046$</td>
<td>$b = 0.754$</td>
<td>€3.76</td>
<td>€3.17</td>
</tr>
</tbody>
</table>

To give an idea of the impact of the choice made in step 2, we executed steps I, II and IV as well. Estimated and implied variables are shown in Table 3.5 along with the option prices obtained when using them. Since the Black-Scholes price only depends on $\hat{\sigma}_{BS}$ we find that they are equal when other variables are implied. As for the Merton prices, we again find that they are higher which is consistent with $b < 1$. Furthermore we have that they lie within the interval $[3.05, 3.42]$. So the decision of which procedure to use results in a price difference of about 10%.

### 3.4.1 An example of option pricing using the NVM house price index

In the previous section we used a theoretical sample to demonstrate Procedure 3.4.1. In that sample we had sufficient data to estimate the parameters. In this section we will examine a real data sample, being the NVM house price index. We will see that for this sample, the amount of data is very limited.

The NVM index is shown in Figure 3.18(a). We extrapolated the index with information about the mortgage portfolio of the Rabobank. Since the NVM index starts in 1985, the only jump included in the index is the one that started in 2008. We did not yet see the end of the downfall, so there is only information about the beginning of a jump. The extrapolated data shows a second jump in the early 1980’s.

We start at step I of the procedure, namely dividing the data into two components. Before we do this, we adjust it to having instantaneous jumps, by replacing periods of downfall by discontinuities. This adjusted process is shown in Figure 3.18(b). The two components are now clearly separated and shown in Figures 3.18(c) and 3.18(d) for the Brownian motion and the jumps respectively. With these data sets, we can now go to step II. Similar as in the previous example we have a rather long time span for the Brownian motion component. This gives us an estimate for $\sigma_M$ being 0.069. Since we have very little information on the jumps we choose to estimate $\hat{\sigma}_{BS}$ as well, which gives us a value of 0.102. We now have to estimate either $\lambda$ or $b$ for the jumps. Unfortunately, we cannot get a good estimate for either of the parameters.

As for $b$ we have that the first jump has an impact of $-30\%$. The second jump only shows a decrease of 10%, but since we use yearly data the figure does not show us the decrease we have seen recently in the year 2009. So the total jump size will be bigger than 10%. This leads us to setting $b = 0.75$. Hence we will use the implied $\lambda$ as described in step III. We already gave the estimated variables, so we can move on to calculate the implied $\lambda$. Equation (3.53) gives us $\lambda = 0.068 \approx 1/15$, that is on average one jump every fifteen years. Now the implied $\lambda$ is much larger than we would expect from Figure 3.18(d), which shows an interarrival time of almost 30 years. This is simply due to not having enough information.

In any case we did find values for the parameters, so we can now give option prices. Black-Scholes gives an option price of €5.09 and Merton’s model yields €4.66. We again emphasize that the quality of the parameter estimates is poor. Hence, we must be careful when interpreting or using the option prices, for poorly estimated parameters will inevitably lead to poor option prices.
3.4. OPTION PRICING IN PRACTICE

(a) NVM index (1985=100) extrapolated with returns from Rabo portfolio

(b) NVM index altered to sudden declines

(c) Brownian motion component

(d) Jump component

Figure 3.18: Splitting the NVM index into a Brownian motion and jump components
3.5 Discussion and conclusions

Black-Scholes is by far the most widely applied method for option pricing. However, suggestions have been made to investigate other models, including jumps, such as Merton’s model. In this chapter, we discussed both methods in theory and in practise. To draw conclusions, we must first go back to the assumptions we made and check to which extend they may cause problems for the housing market.

3.5.1 Discussion of theoretical assumptions

Apart from the price process of the underlying asset we saw that Black-Scholes and Merton’s model have the same assumptions, so we start with discussing the credibility of those assumptions.

There are no arbitrage opportunities. Arbitrage opportunities would imply a risk free profit. Difficulties will then arise for calculating the return of a riskless portfolio, and the construction of a risk free portfolio is the basis in both pricing methods. In Chapter 2 we already showed that house price indices have seasonal effects. Myer et al. (1997) argues that the phenomenon of cycles in real estate prices indicates that returns are not independently distributed. This means that future index returns are partly predictable. To establish a derivatives market, there must be sufficient buyers and sellers at any time. It would thus be problematic if there would be consensus among market players on the future behavior of the index. Syz (2008) argues that autocorrelation and cyclic behavior do not necessarily lead to the so called “free lunch” since investments in physical real estate are time consuming and cause high transaction costs.

Naked short selling of securities is permitted. Simply due to the fact that we cannot sell a house which is not in our possession, naked short selling is impossible in the housing market. This prevents us from holding a risk free portfolio.

There are no transaction costs or taxes. It is evident that this assumption does not hold for the housing market. Transactions of real estate are typically associated with high costs. These transaction costs make it impossible to hold a riskless portfolio, since the composition of the assets in such a portfolio changes through time. Buying and selling the underlying to keep the portfolio riskless would simply be too expensive.

There are no dividends during the life of the derivative. Real estate does not yield dividends in the traditional way as for stocks, but we can interpret for instance rental incomes as dividend. Now for Black-Scholes it is possible to include dividend payments as is shown in e.g. Hull (2008).

Security trading is continuous. Again, the assumption of continuous trading is necessary since we must be able to hold a risk free portfolio at all times. Instantaneous property investments are not possible, for these kind of transactions cost a lot of time. We may achieve continuous trading in property derivatives, although some problems arise here as well. Suppose we have a property derivative on an index. Usually, new index values are published quarterly. It is likely that trading on the index derivative concentrates in a short period before and right after the publication of new information.

The risk-free interest rate is constant and the same for all maturities. Such a risk-free rate does not exist, for a riskless asset is purely hypothetical. Effects of different levels of interest rates are shown in Figure 3.5(b). We refer to Daigler (1994), Chapter 4 for more information on variable interest rates. Daigler (1994) argues that the impact of non constant interest rates on the option price is relatively small.

The underlying asset is tradeable. We defined the house price index to be the underlying asset for the options. We already discussed high transaction costs and times for real estate investments. We now arrive at another problem, for a single house does not have the same price development as a house price index. Since we cannot buy an index and if we wish to hold an asset which has exactly the same price process as the index, we need to take on a position on the Dutch housing market in total. This could be done by buying houses of several types in several regions so that
the total price change for all these houses equals the price change in the index. This would of course be very time consuming and very expensive. Moreover, if an investor would have to decrease the amount of the underlying asset to still hold a riskless portfolio he would have to sell ‘parts’ of the houses in all regions. It is clear that the assumption of a tradable underlying asset does not hold.

We have seen that there are some assumptions that cause problems, particularly with the construction of a riskless portfolio. Now some of these problems, like the constant riskless rate assumption also exist for stock options. Others are caused by the particular characteristics of the housing market. The most problematic assumptions are the assumption that there are no arbitrage opportunities and that the underlying asset is tradable. The question rises what the impact on the option price is when these assumption do not hold. Further research is needed here. It may be possible to accept the inconsistencies when the impact is small. If we cannot accept this, other constructions can be investigated, like options on futures. If we would have a liquid futures market on the house price index, both the first and the last assumption are less problematic, for the higher liquidity would lead to less arbitrage opportunities and we would be able to trade the underlying, in this case the futures. We will briefly discuss this in Chapter 4.

The only assumption left to discuss is the price process of the underlying asset, in our case the house price index. The use of a geometric Brownian motion implies a constant volatility. A major problem with nonconstant variance is that the hedge ratio does not provide a risk-free portfolio. A nonconstant variance reduces the accuracy of the option price even if the true average future variance is known (Daigler (1994), p.108). The limited amount of data shows no great changes in volatility over time, e.g. we do not find that the volatility is about 4% is one year and 10% in the next, but we find it to be at a more constant level. However, we cannot draw solid conclusions. Again, further investigation is recommended.

Figure 3.19 shows the histogram of the yearly returns of the NVM HPI and distribution fits. We use the yearly returns to cancel the seasonal effects. The normal distribution fit suggested by the Black-Scholes model is rather poor. The histogram implies that there should be a higher peak and thinner tails. The fat tails that we do see are of course caused by the ‘outlier’ on the left-hand side. Now Merton’s model suggests that these ‘outliers’ can be expected and should be included in the model as jumps. This would lead to a distribution fit as shown in Figure 3.19(b). Clearly in this regard, Merton’s model seems to be much more realistic for the HPI price process.

We find that for both models, assumptions have to made which are questionable. As for the underlying index, it seems plausible to use Merton’s model which includes the jumps.

3.5.2 Discussion of practical application

If we wish to use either Black-Scholes or Merton’s model for option pricing on the house price index, we need to be able to give accurate parameter estimates. There is limited data available of house price indices. There seems to be sufficient data to estimate total volatility, which we can use for Black-Scholes option pricing. However, if we would like to include jumps, we need data of at least three or four jumps to give reasonable estimates. And since jumps occur rarely, this would mean that we would need data over a range of about a hundred years.

The true frequency and impact of jumps is thus not clear, as we saw in the example given in Section 3.4.1. Further research is recommended to give better and more reliable estimates for the jump parameters. Possibilities are to compare the Dutch HPI’s with indices from abroad or with indices of commercial real estate. Finding very similar behavior there may give some more information. On the positive side, expert opinions on the parameters could be investigated, and the models that were discussed in this chapter can be used for scenario analysis.
3.5.3 Conclusions

We discussed the theory of Black-Scholes and Merton’s Jump-Diffusion model and concluded that they are very similar except for the jump component which is added in Merton’s model. The jump component is the reason why the theory of Merton’s model is more suitable for house prices. As for the Black-Scholes model we saw that price changes due to estimation errors of the parameters are relatively small. It is likely that we can give reasonable estimates for $\sigma_{BS}$ and $\gamma$. For Merton’s model we found that $\delta$ has very little impact, which lead to the choice of setting $\delta = 0$. The impact of jump sizes and frequencies was larger and we must thus be careful when estimating them. In addition, since the jumps occur rarely, we have very limited data on jumps making it even more problematic to give good estimates for the parameters.

All in all Merton’s model is theoretically the best choice, but it yields more problems with parameter estimation than the Black-Scholes approach. Furthermore, Black-Scholes is a widely used and respected method for option prices, and the general public may accept these prices more easily. In general, Black-Scholes is easier to use. Nevertheless, more methods for estimating jump parameters must be explored, for it is desirable to use a model closer to the true behavior of house prices.
Chapter 4

Points of attention when introducing HPI derivatives

In this chapter we discuss the difficulties that arise when we wish to introduce a market in House Price Index (HPI) linked products. It consists of three parts.

The first part considers pricing problems. In Chapter 2 we concluded that option pricing for the house price index is not straightforward. The occurrence of sudden downfalls lead us to not only look at Black-Scholes‘ model but also at Merton’s mixed jump-diffusion model for option pricing in Chapter 3. It is clear that some assumptions have to be made which are questionable for the housing market. In Section 4.1 we discuss these problems and Section 4.1.2 we introduce a method which may solve them.

In the second part, Section 4.2, we give pricing methods for three HPI-linked products: a bond, a mortgage and an insurance policy. When wishing to introduce house price derivatives, one should consider which financial product is the best first step. Our objective in this section is not to make this decision, but to describe some possibilities and corresponding pricing methods.

In the third part, we discuss potential hurdles when introducing a market in property derivatives apart from the pricing problems. In Chapter 2 we discussed the developments on property derivatives and found that the introduction of property derivatives in general (so including commercial real estate as well) was anything but easy. For instance, we have seen an initiative fail because there was a lack of liquidity. The potential obstacles that we discuss must be considered as points of attention for further development of house price derivatives.

4.1 Difficulties and possible solutions

In Section 3.5 we discussed the problems with respect to option pricing with the Black-Scholes and Merton’s model. We found that theoretically Merton’s model was more suitable for the housing market, since it includes jumps. On the other hand we experienced problems with estimating the jump-parameters from the available data. Parameter estimation is easier for Black-Scholes, simply because there are less. Furthermore, Black-Scholes is a well known and accepted method for option pricing. This makes Black-Scholes more easy to use. Both models require assumptions which cannot be met when using house price indices as underlying assets. More importantly, we found that the construction of risk free portfolios (which lie at the heart of both models) is impossible.

In Chapter 1 we discussed the effect of expectation on the development of house prices. In this section we focus on the effects that market expectations have on the practise of option pricing.
CHAPTER 4. POINTS OF ATTENTION WHEN INTRODUCING HPI DERIVATIVES

4.1.1 The effect of expectation

For both Black-Scholes and Merton’s model we assumed that the price processes have zero autocorrelation, so that one cannot in any way predict future returns. This lies in the assumption of the Brownian motion (and the jumps corresponding to non-systematic risk) and in the assumption of no arbitrage opportunities.

We already discussed that the effect of expectation results in a slow market, which is typically shown in autocorrelated returns. For the yearly returns of the NVM index, we find an autocorrelation of 0.32. For the quarterly returns this a bit higher, being 0.36. Both values represent the lag 1 autocorrelations. Syz (2008) p.98 gives the annual autocorrelation for various residential indices. These range from 0.33 for the Swiss ZWEX index to 0.69 for the USA S&P/Case-Shiller National Composite index.

We already discussed that autocorrelation leads to partly predictable returns and that this caused consensus between market players about future returns (Section 1.1.1). A negative expectation results in little demand and thus in low prices. This is nonconsistent with the theory behind either Black-Scholes and Merton’s model, for negative expectations (or positive) are always canceled within the risk free portfolio which always results in a return equal to the riskfree rate. So the price movements we see in practise, changing when the general expectation shifts is not consistent with the theoretical prices, which should only react to volatility changes.

The question rises why the market reacts to these expectations. We assumed that it is possible to hold a riskless portfolio. It is however close to impossible to take on a position on the Dutch housing market. It would be extremely expensive and time consuming. If we cannot hold the riskless portfolio, the expected returns (or $\mu$) will not be canceled by the returns of other assets and thus the expected return will influence the derivative price.

Another, more intuitive way of seeing this is that a house is a unique asset. If one wishes to have a position on this house, one has to invest time in it and spend high transaction costs. Moreover, the current owner must be willing to sell. As a result, not everyone is able to invest in this house at any time. This means that the riskless portfolio is not accessible for anyone. This makes it valuable, hence giving higher returns than the riskless rate.

In the next section we will explain how the effect of expectation may be included in the price formula.

4.1.2 Using a risk premium

In e.g. Langens (2008) is shown that the inclusion of expected index movements in derivatives prices is necessary (p.54). A theoretical basis is given in Lord (2000) (p.112). In this section we will give the basic conclusions and pricing formulas.

Expectation pricing consists basically of the use of a risk premium to include market expectations in the option price. Recall that we assumed the riskless rate to be constant and equal for all maturities and we denoted it by $r$. Now suppose we make an estimation of the future (continuously compounded) rate of return for the period from the moment of purchase $t$ until maturity at time $T$ and denote the estimate by $\hat{\mu}_{t,T}$. Than define the risk premium $p$ to be $p = \hat{\mu}_{t,T} - r$. In times of downfall we may have a negative risk premium. It indicates that we expect the index to do worse than the riskless rate. Note that we only include the expectation effect in the risk premium. It is very well possible to modify the premium motivated by liquidity issues or transaction costs. For now however, we choose to ignore these effects and thus assume that there are no such market frictions which have a significant influence on the derivatives prices.

Now in the Black-Scholes pricing formula (3.15) we can use $r + p$ instead of $r$ and this gives us the expectation based option price. Hence the call option price is given by

$$c = S_0 N(d_1) - Ke^{-(r+p)T} N(d_2),$$  (4.1)
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where

\[
\begin{align*}
    d_1 &= \ln \left( \frac{S_0}{K} \right) + \frac{(r + p + \sigma^2/2)T}{\sigma \sqrt{T}}, \quad (4.2) \\
    d_2 &= \ln \left( \frac{S_0}{K} \right) + \frac{(r + p - \sigma^2/2)T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}. \quad (4.3)
\end{align*}
\]

4.1.3 Options on futures

The use of a risk premium is necessary if there are risks not incorporated in the pricing model or in case of a lack of replication possibilities. We saw that the illiquid nature of the housing market causes such a lack. However, the market for forward contracts may become liquid. To obtain this liquidity, more parties must be ‘encouraged’ to participate in the market so that there are always enough risk sellers and risk takers (even in times of downfall). These forward contracts can then be used to create the risk free portfolio for options on house price indices. Let’s for now assume that such a liquid market in house price futures exists. Now to use the futures as underlying for the options we must make the Black-Scholes assumptions as discussed in Section 3.1.1 but then for the futures market instead of the house price index. So we assume that the futures prices follow a geometric Brownian motion.

The futures of course must also be priced, now with the underlying again the house price index. For futures pricing under Black-Scholes assumptions, we would have the futures price \( F_0 \) to be \( F_0 = S_0 e^{rT} \), with \( S_0 \) the current index value, \( r \) the riskless rate and \( T \) the time until maturity.

We have that the Black-Scholes price for call options on a forward contract reads

\[
c_F = e^{-rT} [F_0 N(d_1) - K N(d_2)],
\]

as first described by Black (1976).

The house price index is not replicable and acts as the underlying for the futures contracts, which makes the use of a risk premium for the futures price necessary. Hence we have

\[
F_0 = S_0 e^{(r+p)T},
\]

with \( p \) the risk premium as discussed in the previous section.

As for the options, we have that the underlying is no longer given by \( S \), the index value, but by \( F \), the futures price. So substitution of \( (4.5) \) in \( (4.4) \) gives us for the option price with the future as hedge medium

\[
c = S_0 e^{pT} N(d_1) - K e^{T} N(d_2),
\]

with \( d_1 \) and \( d_2 \) as in \( (4.2) \) and \( (4.3) \).

Now since we do not yet have a liquid futures market, we will not go further into this method for option pricing. More information can be found in Syz (2008).

We choose to use the risk premium directly in the option price, as in \( (4.1) \). If we wish to use the risk premium, we need a method to estimate \( p \).

4.1.4 Estimating risk premia from derivatives prices

To estimate the risk premium \( p \), we need real data for derivatives prices. For Black-Scholes pricing with a risk premium, we have the following variables, \( r \) the risk free rate, \( p \) the risk premium, \( T \) the time until maturity, \( S_0 \) the current stock price, \( K \) the strike price and \( \sigma \) the volatility of the underlying asset. Of course when looking at some derivative we have that the value of the underlying is known, as are the derivative basic characteristics, \( K \) and \( T \). Now if we estimate \( \sigma \) and \( r \), the only unknown parameter is \( p \). But if the derivatives price is known we can of course determine \( p \) as the only unknown parameter in the pricing formula.
Unfortunately, we do not yet have derivatives data for the Dutch housing market. Hence we show the methods for two types of derivatives, firstly the futures on the US house prices and secondly the swaps on the IPD all property index for the UK.

Risk premia for the US housing market

We take a look at the futures market on the S&P/Case-Shiller house price indices. These futures are available for 10 US cities and for the 10-city composite index. We wish the estimate to be robust for local price changes, so we choose to look at the future prices for the composite.

![Figure 4.1: S&P/Case-Shiller 10-city composite HPI and futures prices for Maturity November 2009.](image)

Figure 4.1 shows us the index value and the futures prices for maturity November 2009. We immediately see the decline for both processes. Of course, theoretically for \( p = 0 \) the price movements should be perfectly correlated and with equal volatility. Clearly, this is not the case. The futures price seems to have a higher volatility than the spot price.

To calculate the actual risk premia for the futures, we use the price formula (4.5). We can of course solve this for \( p \). We do not base it on only the initial price \( F_0 \), but at every time \( t \in [0, T] \), which gives us

\[
p_t = \frac{1}{T-t} \log \left( \frac{F_t}{S_t} \right) - r_t,
\]

where \( p_t \) is the implied risk premium at time \( t \) when \( T - t \) is the remaining time until maturity, \( F_t \) and \( S_t \) the futures and index price at time \( t \) and \( r_t \) the riskless rate at time \( t \). We assumed \( r \) to be constant, but now we estimate \( r \) at any point in time, and then assume it to be constant from that point on. Hence \( r_t \) is a ‘variable’. We set the riskless rate \( r_t \) equal to the returns on a US Treasury bill at time \( t \). Since all other parameters are known, we can now calculate \( p \). In Figure 4.2 the risk premia are given for futures on the composite index with maturities in November 2009, 2010, 2011 and 2012.

A few things are immediately apparent from Figure 4.2. Firstly, there is a clear difference in the levels of the premia for the different maturities. Secondly, we find that the process is rather volatile, especially for maturity November 09. Thirdly we find that the the trendlines are very similar, i.e. first downward and then upward. We will discuss these observations in the same order.

**Risk premia for different maturity dates:** In general, we find that the behavior of the risk premia over time for the different maturities is quite similar. The main difference lies in the level of the premia. We find that although all risk premia are of a generally low level, closer maturity dates correspond to much lower risk premia.
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A negative risk premium corresponds to an expectation of index decrease (compared to the riskless rate). A positive risk premium suggests an index increase. The low level of the red line (e.g. in May 2009) thus corresponds to a futures price with maturity November 2009 which is lower than the level of the index in May 2009. The index level was 151 and the futures price was 144. So the market expectations were low for the short term, i.e. a decrease of \(\frac{151 - 144}{151} = 8\%\) in eight months. For comparison, the index showed a decrease of about 13% in the eight months before that. In general, we find that the lower risk premia for closer maturities suggest that the market expectations are more negative for the upcoming period compared to the market expectations in the long run.

**Ups and downs:** Figure 4.2 shows the same kind of behavior for all maturities, i.e. the risk premia see a slight increase in the end of 2007. Around February 2008 they start to decline. The low point is reached in June 2008. The risk premia for maturity November 2009 are the most volatile. Sudden changes in the risk premia are probably caused by the arrival of new information. For instance, in April 2008 we see no great changes in the index, but the futures price decreases rapidly (Figure 4.1). This results is low risk premia. Similarly, in March 2009 we again see a rapid decrease for maturity November 2009, whereas the other maturities only show a slight decrease. The index itself does not explain this behavior, but when for instance experts state that they expect an index decline before November 2009, the investors respond to this and the prices go down. They probably expect the index to recover before the other maturity dates, which explains the lesser effect for those risk premia. So the investors react more strongly to new information for nearer maturity dates. This could explain the higher volatility of the risk premia corresponding to maturity November 2009.

**Trendline behavior:** Since the peaks are probably due to the arrival of new information, we look at the general trend to discuss the ‘long-term’ behavior. More specifically, we are very interested in the impact of the credit crunch on the risk premia. The credit crunch appeared in the Netherlands in the late summer of 2008. Of course, the credit crunch must have changed the general opinion on future developments of house prices. We would expect to see this in decreasing risk premia. However, looking at the index in Figure 4.3, we find that the house prices in the US started to decrease much earlier, that is in the fall of 2007. So the problem now rises that we are not able to compare the risk premia in a period of downfall with risk premia in more prosperous times.

![Figure 4.2: Risk premia for Case-Shiller HPI based futures.](image)
We can draw some conclusions about the general trend however. Firstly, it suggests that the market expectations for house prices reached its low point in the summer of 2008. We find that the general trend of the risk premia in the period from September 2007 until May 2009 shows first a decrease followed by a period of recovery. In Figure 4.3 we see this same behavior in the trend line for the same period. This could indicate that the behavior of the risk premium is similar to the behavior of the index returns. Again, a longer time span for the futures data is necessary. Especially for the period April 2005 to August 2006 would be interesting, for in this period we find that the index returns are decreasing whereas the index itself is still increasing. If the risk premia would show a decrease in the same period, it would be a strong inclination for this dependency. As for this period, we cannot draw solid conclusions.

![Figure 4.3: S&P/Case-Shiller 10-city composite HPI and its returns.](image)

**Estimating risk premia from swap contracts**

Similar to the futures data we can also get information about the risk premia from swap contracts. Swaps are basically instruments that allow periodic payments to be swapped between two counterparties. Typically, one party receives a floating rate (e.g. HPI returns) from and pays a fixed rate to the other swap party for a certain period of time. They are traded mostly on the OTC market. So if for instance an English insurance company wishes to hedge parts of its risk that lies in real estate, it can use a swap contract on the UK IPD All Property Index. The insurance company will receive the 3 month LIBOR rate plus some fixed spread and it pays the floating rates of return on the index. Keeping in mind that the underlying asset is not tradable, we must have a reasonable contract. This means that the expected return from the index must equal the LIBOR rate plus the fixed spread. Since in the UK the LIBOR rate is considered as the riskless rate, we can thus translate this to $\tilde{\mu} = r + p$ with $p$ now the spread. But of course the immediate result is that this spread is a measure for the risk premium.

The use of risk premia in swap contracts is discussed in e.g. IPF (2008). They discuss the factors that influence the risk premium for property swaps. These factors are uncertainty about future returns, an illiquidity premium since the swap market is not yet very liquid and the reflection of market expectations. Their general conclusion is that the risk premium is highly dependent on one’s interpretation of the forward curve. We will see this in the swap rates for the IPD all property index.

Figure 4.4 shows swap rates for the IPD all property index with maturities of December 2008, 2009
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Figure 4.4: Swap prices All Property Index (src: Langens (2008), property.gfigroup.com).

and 2010. This data covers a much longer period than the futures data, so we can now look at the impact of a downfall. We see in Figure 4.5 that the behavior of the all property index is comparable to the Case-Shiller index as shown in Figure 4.3 with respect to the general trend. We are particularly interested in the period where returns are decreasing, but the index is still increasing. For the all property index this is the period between April 2006 and June 2007. Furthermore we are interested in the general difference between the period before downfall and the period afterwards.

Note that the risk premium equals the swap price minus the LIBOR rate. We find a great difference in risk premia between the prosperous period before 2007 and the period of downfall in 2008. We find that the risk premium in the first case is around 5% (100 basis points correspond to 1%). In January 2007 the prices drop to a risk premium around zero. Then, during the summer of 2007 the risk premium decreases to negative values. For the first maturity date it even reaches -15%.

These three phases in the process can be interpreted as follows. First, up until January 2007 the high risk premia are an indication for positive market expectations. In such times it is interesting for investors to buy derivatives instead of buying property, for the systematic risk seems to be low and they avoid the idiosyncratic risks of individual properties. Furthermore, they avoid transaction and management costs.

For the second phase, the risk premia around zero suggest that the market expects the index to remain at a constant level (hence behaving similar to the risk free rate). The turning point to downfall appears in June 2007, at the same moment that the index starts decreasing. The systematic risk at this time is then high, which makes possible investors reluctant. The negative market expectations are clearly visible in the risk premia.

These observations reinforce the idea of partly predictable future returns. The swap rates stagnated around zero, since investors believed the index to be at its peak. And the index decrease could have been only over a small period in time followed by a quick recovery. In retrospect however, we find that investors were right to believe in a longer period of downfall, which was reflected in the negative risk premia.
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Comparison to the futures based risk premia

Both derivatives show negative risk premia in times of downfall. Another similarity is in the difference between the maturity dates. The further away maturity, the higher the risk premium. The behavior of the risk premia in both figures can be explained when looking at the sentiment under the market participants. Hence it reinforces that the effect of expectation cannot be ignored when pricing HPI derivatives.

The main difference between the risk premia we obtained from future and swap prices is that the swap rates show more dramatic risk premia. For the contracts with maturities in November and December 2009 the futures and swaps, we find in April 2008 risk premia of -4% and -17%. This can simply be explained by looking at the index returns. In that same month the composite index showed a return of -1% and the all property index had a return of -5%. So when the index shows more extreme returns, the risk premia (which represent expectations of future returns) are more extreme as well.

Remarks on dealing with risk premia

So far we considered estimating the risk premium based on derivative prices. A different way of dealing with the risk premium is to base it not on data but on expert opinions. This means that experts give their view on future returns for the period until maturity. This would give an estimate for $\hat{\mu}$. Simply subtracting the riskless rate $r$ would then give the estimate for $p$. In the UK, the Investment Property Forum publishes the IPF consensus forecast. This is a compilation of the opinions of 20 to 30 forecasters. In cases where there is no liquid derivatives market, such as the Dutch case, expert opinions could be the solution for estimating the risk premium. It must be noted however, that these experts must have great credibility in the market and that there must be a certain level of consensus between these experts.

Furthermore, it must be noted that we discussed the use of a risk premium in Black-Scholes option pricing, so that the assumption of replicability of the underlying asset can be dropped. We must stress the fact that the replacement of $r$ by $(r + p)$ in the pricing formula may not be applicable for Merton’s model. This is caused by the fact that creating the riskfree portfolio is a bit more elaborate when there are jumps to be considered. Since the theory of jumps in Merton’s models seems to be a good fit for the housing market, possibilities for including a risk premium is a very interesting subject for further research.
4.1.5 Implications for the Dutch housing market

The Dutch housing market is of course very different from the US housing market. This means that the risk premia that we found for the futures give us no information about the risk premia we should use for the Dutch housing market. However, we did show a method for estimating such risk premia. This means that at the time that we have access to derivatives prices with the Dutch house price indices as underlying we are able to estimate the risk premia using this method. Furthermore, since both housing markets are of an illiquid nature we expect to find that the use of a risk premium will be necessary in the Netherlands as well.

4.2 Pricing HPI-linked products

So far in this chapter we discussed pricing issues. We found that market expectations had a great impact on derivative prices which led to the introduction of a risk premium. In this section we will leave these problems and possible solutions behind us and go into methods to price more complicated HPI-linked products. The ‘basic’ derivatives such as futures, swaps and options are suitable for institutional investors, but not for home owners. Two interesting products for the average household could be the index-linked mortgage and a house price insurance. Sections 4.2.2 and 4.2.3 will deal with pricing these products.

Another interesting product could be the index-linked bond. Because of the illiquid nature of the housing market and until now of the house price derivatives market as well, a bond is a very interesting product for hedging house price risk, for the necessity of liquidity is not as high since they are typically OTC traded and the price is thus negotiated instead of determined by the market. We already saw this in the UK where the PIC (a type of bond based on the IPD all property index) was successful while other derivatives were not. So bonds could be a very good first step when introducing HPI-linked derivatives in the Netherlands.

In this section we will use both Merton’s model and various levels of risk premia for option pricing. In Appendix C we give Matlab functions to price these products.

4.2.1 Pricing HPI-linked bonds

It is possible to calculate bond prices based on known option prices. In this section we will describe how to decompose the bond into risk free savings and call options. Then, we can use the option prices to determine the bond price.

Of course to be able to determine the price of the bonds, we must first decide the form of the bonds. A bond will be defined as follows: the principle is always guaranteed and every period the bond holder receives an interest rate on the principle depending on the performance of the underlying asset, in our case the house price index. The bond holder receives the interest rate equal to the performance of the index, only when the performance lies within bounds. For instance, the bounds can be 0% and 10%. Then when the index decreased, the bond holder will not receive interest, but if it increased with more than 10%, the bond holder will only receive 10%.

To illustrate the decomposition, we will first give a simple example. Consider a bank that gives out a zero coupon bond with a maturity of five years. The principal equals \( \mathbf{100} \) and is guaranteed. At maturity the bond holder receives an interest rate equal to the performance of the HPI over the five years, bounded by 0 and 15%. We wish to determine the price \( \mathbf{P} \) of the bond.

The bank can look at the problem in the following way. He knows that he must pay the bond holder \( \mathbf{100} \) at maturity. At time zero this is worth \( \mathbf{100} \cdot e^{-rT} \). Setting the risk free rate \( r = 0.02 \), this comes down to \( \mathbf{90.48} \) which we call \( P \) for principal. As for the interest there are three options. Suppose the index increased with \( x \% \). The first case of \( x < 0 \) means that he does not have to pay any interest at maturity. If \( 0 < x < 15 \), he pays \( x\% \) and if \( x > 15 \) he must pay 15%.

Now suppose that the bank goes long on a call option with a maturity of 5 years. The option is at
the money, which means that the strike price equals the index value at time 0. For simplicity, we take 
\( S_0 = K = €100 \). He has to pay \( €C_{atm} \) for this option. Furthermore, the bank of the bond goes short on a call option, again with a maturity of 5 years. This option has a strike price of 115%, so we have 
\( K = €115 \). He receives \( €C_{15} \) for this option. We again consider the three cases of \( x \). First, \( x < 0 \).
In this case, no option will be exercised and no interest has to be payed to the bond holder. Next, consider \( 0 < x < 15 \). Now, the bank will exercise the atm long option, so that he can buy for \( €100 \) and sell for \( €100 + x \), making a profit of \( €x \). He must also pay the bond holder \( €x \) as interest. Lastly, we consider \( x > 15 \). Now again the bank will exercise its option and the bank buys for \( €100 \). But the option that the bank went short will also be exercised, so the bank has to sell for \( €115 \), thus making a profit of \( €15 \), which is again exactly the same as the rental income of the bond holder. The options payoff and thus of the interest at maturity is shown in Figure 4.6 as a function of the index value at maturity.

![Figure 4.6: Two options payoff.](image)

We thus find that when the bank holds these options, and gives out the bond, it has a riskless portfolio. This means that the bond price must equal 
\( B = P + C_{atm} - C_{15} \). Option prices can be determined as described before. Using the parameters as in the example in Section 3.4.1 we obtain 
\( C_{atm} = €14.19 \)
and 
\( C_{15} = €6.66 \). Hence we have 
\( B = 90.48 + 14.19 - 6.66 = 98.00 \) euros.

In the example we saw that interest payments with bounds can be set off with a long and a short position on call options with strike prices on the lower an upper bound respectively. The same can be done for the (yearly) coupons. We again use an example to illustrate this. Again we have a guaranteed principle of \( €100 \) at the time of maturity of five years, but now we have yearly coupons. They have upper and lower bounds of 0% and 5%. Starting with the guaranteed principle, we again have 
\( P = €90.48 \).

As for the coupons, we speak of yearly returns, so the bank has to take on the long and short position with strike prices of 100% and 105% with a maturity of one year at the beginning of every year. But as the strike prices and terms of the options are the same for each period and the parameters of the index process are assumed constant, the option prices will not change over time. We can thus calculate the prices and obtain 
\( C_{atm} = €4.65 \)
and 
\( C_{05} = €2.11 \). Of course the options will be bought a year before we exercise them, so we need discounting to get the current price. Hence the bond price will equal 
\( B = P + \sum_{i=0}^{4} e^{-ir}C_{atm} - \sum_{i=0}^{4} e^{-ir}C_{05} = 102.69 \) when using a riskless rate of 2%.

Another possibility is making the coupons variable through time, e.g. the first four coupons will be bounded by 0% and 5% but the fifth is bounded from above by 10%. The call option with strike price 110% and a maturity of one year costs \( €0.75 \), hence we now have for the bond price 
\( B = 90.84 + \sum_{i=0}^{4} e^{-ir}4.65 - \sum_{i=1}^{3} e^{-ir}2.11 - e^{-4r}0.75 = 103.94 \), which is of course somewhat more expensive than the former.
4.2. **PRICING HPI-LINKED PRODUCTS**

In Appendix C.4 a Matlab function is given to calculate the price for bonds of this type.

Note that we have used Merton’s model to calculate the prices. Black-Scholes prices for the last type of bond are shown in Table 4.1 for various levels of the risk premium $p$, ranging from $-2\%$ to $3\%$. We find that the price using Merton’s model is close to the zero risk premium case, which is of course as expected since that is the original Black-Scholes price.

Since we have no derivative prices for the Dutch housing market, we cannot give an estimate for $p$ based on real data. Expert opinions could give us a good indication for the value of $p$.

Figure 4.7 shows the NVM index over the period 2001 to 2009. The dotted lines correspond to the exponential growth with $\mu$ ranging from $0\%$ to $5\%$. Assuming $r = 2\%$ for the riskless rate, we have that these lines give the ‘expected’ growth of the index when setting the risk premia $-2\%$ to $3\%$. Now suppose one finds the green line, corresponding to the risk premium of $1\%$ the most probable scenario for the upcoming 5 years, Table 4.1 gives us a bond price of €98.91.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2%$</td>
<td>€110.78</td>
</tr>
<tr>
<td>$-1%$</td>
<td>€106.62</td>
</tr>
<tr>
<td>$0%$</td>
<td>€102.67</td>
</tr>
<tr>
<td>$1%$</td>
<td>€98.91</td>
</tr>
<tr>
<td>$2%$</td>
<td>€95.32</td>
</tr>
<tr>
<td>$3%$</td>
<td>€91.99</td>
</tr>
</tbody>
</table>

Table 4.1: Bond prices for various risk premia.

Figure 4.7: The NVM index and scenarios for future returns for various risk premia.

### 4.2.2 Pricing HPI-linked mortgages

Syz et al. (2008) discusses the use and pricing of index-linked mortgages. They propose an index-linked mortgage to unload housing risk for the average household. The payments of such a mortgage depends on the corresponding house price index. This should have the effect that a homeowner’s wealth is more stable. More particularly, the mortgage could depend on the house price index through variable payments and/or through a variable principal value.

In this study we focus on the design in which the principal is linked to the index. Since the index-linked mortgage has the objective to reduce risk and not additional gain, an asymmetric payoff profile
is proposed, which is shown in Figure 4.8.

![Figure 4.8: Payoff structures for an asset, a put option on the asset and for the combination.](image)

It shows us the payoff at maturity of a portfolio consisting of a certain asset (a house) and a put option on the asset. We find that this portfolio hedges the risk of decreasing asset value. For the homeowners, this means that the combination of having a house and a put with strike price equal to the initial house price protects him from house price declines. Of course, the house is leveraged with a mortgage and the put option costs money. These two can be combined in one mortgage. Mortgage interest rates usually lie around 5%. A put option would require extra costs. To determine the mortgage costs when it includes a put option, we must first determine the put prices. Suppose that the put option costs \( C \) for \( S_0 = 100 \), strike price \( K = 100 \) and a maturity of \( T = 5 \) years. Now we wish to spread the option costs over five years. The annual payments \( x \) must of course be discounted, which gives us

\[
C = \sum_{i=0}^{4} e^{-ir}x.
\]

Hence we can determine the annual interest rate to be

\[
x = \frac{C}{\sum_{i=0}^{4} e^{-ir}}.
\]

In this study we mainly discussed the call option, but of course we can calculate the put option price directly from the call option price by using the put-call parity as discussed in Chapter 3.1.2, equation (3.19). Matlab functions are shown in Appendix C.5.

In the Matlab function we always refer to Merton’s option prices, since setting \( \lambda = 0 \) immediately gives the Black-Scholes price. Moreover, if we set \( \lambda = 0 \) and use \( r + p \) instead of the riskless rate, we obtain the Black-Scholes price when using a risk premium.

So far we discussed how to calculate the additional mortgage rate for the put option. We now wish to examine the usability of the put option. For instance, if the regular mortgage costs 5%, but the put option requires an additional 5%, the homeowner will never choose for the put option, because it is simply too expensive. We choose to follow the methodology in Syz et al. (2008) and set \( T = 5 \) years. As for the option parameters, we first set them equal to the parameters found in Section 3.4.1, as we did for the index-linked bond.

These parameters give us an additional annual rate of 0.97%. This is close to the price for the Swiss equivalent as discussed in Syz et al. (2008), keeping in mind that they used a rather prosperous period for estimating their parameters. They arrived at 0.70%. It is now interesting to examine some variants on the mortgage. Firstly, we choose a maturity of 10 years instead of 5, which could be interesting for homeowners determined to stay in their home for a longer period.

Secondly, we assumed a consumer risk of 5%. This means that if one bought a house leveraged by a mortgage with principal \( €100,000 \) and after \( T \) years the house price index decreased with 10%, the principal reduces to \( €95,000 \) instead of \( €90,000 \). This variant could be interesting for households who have a relatively low amortization rate, so that their own equity invested in the house can bear a part of the risk, in this case at most \( €5,000 \). The fraction of the principal that is covered by the put option is denoted by \( Q \), so \( 0 \leq Q \leq 1 \).
4.2. PRICING HPI-LINKED PRODUCTS

Table 4.2 shows the additional mortgage rates for the variants as discussed above. We find that both modifications result in lower prices, which is the expected result, for either lowering the strike price or increasing the maturity gives a lower probability of exercising the put option.

<table>
<thead>
<tr>
<th>T</th>
<th>Q</th>
<th>p</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>-2%</td>
<td>1.89%</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>-1%</td>
<td>1.38%</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0%</td>
<td>0.98%</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1%</td>
<td>0.67%</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2%</td>
<td>0.45%</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>3%</td>
<td>0.29%</td>
</tr>
</tbody>
</table>

Table 4.2: Additional mortgage rates \( x \) for put options (Merton).

The rate for a maturity of 5 years and a 100% coverage is priced with Merton’s model. We now wish to compare this price (0.97% per annum) with prices that we obtain when using a risk premium for option pricing.

<table>
<thead>
<tr>
<th>T</th>
<th>Q</th>
<th>p</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>-2%</td>
<td>1.89%</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>-1%</td>
<td>1.38%</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0%</td>
<td>0.98%</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1%</td>
<td>0.67%</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2%</td>
<td>0.45%</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>3%</td>
<td>0.29%</td>
</tr>
</tbody>
</table>

Table 4.3: Additional mortgage rates \( x \) for put options using various risk premia.

Similar to bond pricing we need an estimate for the risk premium \( p \). We again give the additional mortgage rates for six levels of \( p \) ranging from −2% to 3%. These prices are shown in Table 4.3. We find that they range from 0.29% pa. for rather positive future expectations to 1.89% for very negative expectations. In general, we find that the risk premia are rather small, especially in prosperous times, when \( p \) is estimated to be above 1%. It is thus very interesting to further investigate the use of index-linked mortgages in the Netherlands. There are many possible variants on the mortgage structure. Monthly payments can be varied as well and a combination can be made. Furthermore, it is interesting to take a closer look at the times till maturity and the level of consumer risk \( Q \). This could differ for individual households in order to fit their personal needs.

4.2.3 Pricing house price insurances

House price insurances are proposed in e.g. Shiller and Weiss (1999). Again several forms of insurances are discussed. In this study we will go into the life-event-triggered insurance policy. In such a policy, the homeowner will receive a payment from the insurance company only when there is a loss experienced by the homeowner. The loss is defined as a situation in which a life event causes the homeowner to suffer from declining house prices. If they continue to live in the same home, or move within the area, house price declines do not have a direct effect on the households wealth. But in this life event, e.g. moving to another state, the homeowner may be forced to sell its house at a loss. This is the loss that is insured.

To the homeowner, such a policy equals a put option with maturity contingent to the life event (compare to Figure 4.8). The required conditions for calling an event a life event can be varied. For simplicity, we assume a move between communities to be a sufficient condition to call it a life event.

Similar to other insurances, we wish to define a fixed annual claim which the homeowner must pay to the insurance company. The homeowner is free to cancel the insurance at any time. The policy has a deductible, which defines a floor below which the policy starts to pay out (compare to \( Q \) for index-linked mortgages). The deductible is thus the consumer risk. If the life event takes place, the insurance company pays the loss in case of a decrease below the floor and the policy is automatically canceled.
Shiller and Weiss (1999) state that since there is no liquid put option market, a ‘break-even’ premium has to be calculated. To get an indication of the premium \( y \), we need to assume that the cost of providing the policies for the insurance company is given by the price of the portfolio of put options. Suppose that a fixed proportion \( \alpha \) of all policies is canceled by the policy holders each period. Now assume that a fixed proportion \( \beta \) of all policy holders at a given time become eligible for a claim each year (so they experience a life event). They receive a claim if the price index has fallen enough to indicate that their home value is less than the floor, and cancel their policies.

We can find the total value of all puts by creating a weighted sum of the put prices. The weights are calculated as follows. Each year a fraction \( \beta \) customers are eligible for a claim. Furthermore, we have that each year \( \alpha \) customers cancel. So after \( n \) year we would expect \( \beta(1-\alpha)^n \) possible claims, since we have \( (1-\alpha)^n \) customers of which a fraction \( \beta \) experience a life event. For each claim, we would like to have a put option. The value of the portfolio of put options can thus be given by

\[
\Pi = \sum_{t=1}^{\infty} \beta(1-\alpha)^t \varphi_t, \tag{4.7}
\]

where \( \varphi_t \) is the put option price with maturity \( t \). We are interested in the yearly premium \( y \), equal for each insurance holder. We assume that the insurance company invests all policy premia in a riskless asset that pays the interest rate \( r \). Now the total income of the insurance company is given by

\[
I = \sum_{t=0}^{\infty} (1-\alpha)^t ye^{-rt} = y \sum_{t=0}^{\infty} ((1-\alpha)e^{-r})^t = \frac{y}{1 - (1-\alpha)e^{-r}}.
\]

For a break even policy premium, we have that \( I = \Pi \), hence we obtain for \( y \)

\[
y = \Pi \left( 1 - (1-\alpha)e^{-r} \right) \tag{4.8}
\]

This leaves us with estimating \( \alpha \) and \( \beta \). For \( \alpha \) we follow the result in Shiller and Weiss (1999) and take 9%. As for \( \beta \) we look at the number of moves between communities. CBS (2009) gives us that 650,000 persons moved from one community to another. The average household consists of 2.3 persons and there are around 7,000,000 households in the Netherlands. This gives us an estimate for \( \beta \) being 4%.

A Matlab function to calculate the premium is given in Appendix C.6. Using the parameter estimates for Merton’s model in Section 3.4.1, we obtain the premia shown in Table 4.4 for various values of \( Q \).

<table>
<thead>
<tr>
<th>( Q )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.18%</td>
</tr>
<tr>
<td>0.99</td>
<td>0.17%</td>
</tr>
<tr>
<td>0.95</td>
<td>0.13%</td>
</tr>
<tr>
<td>0.9</td>
<td>0.10%</td>
</tr>
</tbody>
</table>

Table 4.4: Insurance premia \( y \) (Merton).

We find that if we wish to insure a house bought for €100,000 with a floor of €99,000 so that we have a consumer risk of €1,000 it costs us €170 per annum. In case of a consumer risk of €5,000 the premium is €130. This corresponds to around €11 each month. This seems plausible and it agrees with the premia found in Shiller and Weiss (1999).

We would like to compare these rates to the insurance premia when using a risk premium in Black-Scholes. We again use the risk premia ranging from −2% to 3%. Insurance premia for this range are given in Table 4.5. Similar as for the mortgage rates we find that the rate following from Merton’s price formula is closest to the rate when \( p = 0 \). Of course we have that the insurance for house price declines is more expensive when the market expectations are negative. In case of very positive expectations, we find that the insurance is rather cheap, being only €30 per year for a house worth €100,000.
4.3. DIFFICULTIES FOR INTRODUCING HPI DERIVATIVES

<table>
<thead>
<tr>
<th>( Q )</th>
<th>( p )</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>-2%</td>
<td>0.35%</td>
</tr>
<tr>
<td>0.95</td>
<td>-1%</td>
<td>0.21%</td>
</tr>
<tr>
<td>0.95</td>
<td>0%</td>
<td>0.13%</td>
</tr>
<tr>
<td>0.95</td>
<td>1%</td>
<td>0.08%</td>
</tr>
<tr>
<td>0.95</td>
<td>2%</td>
<td>0.04%</td>
</tr>
<tr>
<td>0.95</td>
<td>3%</td>
<td>0.03%</td>
</tr>
</tbody>
</table>

Table 4.5: Insurance premia \( y \) using various risk premia.

The insurance premia are cheaper than the additional mortgage premia. We can explain this as follows. For the index-linked mortgage, after a period 5 years, the principal of the mortgage will always be reduced when the house value decreased. For the insurance, the policy holder will only receive a compensation when they fulfill certain conditions. That means that when house prices decreased a relatively small amount of insurance holders will receive a claim. For the mortgage, in times of price declines all mortgage holders that arrive at the end of their term (5 years) will automatically receive compensation, whether they wish to move or not.

We must be careful when using the insurance premia. Assumptions are made to arrive at these prices which cannot be true. Most important is the constant behavior of the insurance holders through time. Typically, in times of price decreases the trading volumes on the housing market decrease as well, for less households are willing to sell at a loss. In times of prosperity on the housing market, households have a larger incentive to buy. Of course the introduction of house price insurances may lessen this effect, because the impact of selling at a loss is lower.

Another problem with this price method is the high correlation in the housing market. Insurance premia are typically based on the law of large numbers, meaning that each year approximately the same amount of houses are sold at a loss. Clearly, this is not the case for the housing market. For the insurer this means that one year there may be no claims at all, while in the next year almost every selling household claims a loss. The insurer must thus have very high reserves to prevent a bankruptcy in times of downfall. We believe that further research of pricing methods is necessary before introducing an insurance policy of this kind.

4.3 Difficulties for introducing HPI derivatives

In this section we will give an overview of the difficulties that may arise during the startup process of property derivatives. For the interested reader we strongly recommend Syz (2008) for further information, since we found this to be the most complete and up-to-date source of information on property derivatives.

We start by discussing liquidity. In a survey undertaken by Hermes in May 2006, 27% of the respondents indicate insufficient liquidity to be the main obstacle to trading property derivatives. Liquidity is closely related to the supply and demand balance. Supply and demand is therefore our second point of discussion. Thirdly, we briefly discuss the influence of an untradable underlying asset on market development. We find that this may not lead to difficulties when there is a reliable index. This leads us to discussing index requirements and the available Dutch indices. Lastly, we consider the feedback effects that the introduction of house price derivatives may have on the spot market (i.e. the underlying housing market).

4.3.1 Low liquidity

Hirshleifer (1968) defines liquidity to be “an asset’s capability over time of being realized in the form of funds available for immediate consumption or reinvestment - proximately in the form of money” (p.1). A necessary condition for liquidity is that there are buyers and sellers at any time. It must
be possible at any time for an asset to be traded. We discuss the liquidity of both the spot and the derivatives market.

As for the spot market, it is obvious that the housing market is not a liquid market. This is mostly due to large transaction times and costs. This low liquidity in the spot market makes a liquid derivatives market very interesting for investors, since they would then have an easy access to an (indirect) position on the housing market. Syz and Vanini (2009) describe that in times of a sharp downturn, liquidity evaporates for heterogeneous physical property markets. For example, the time-on-market for housing in the UK more than doubled in the second quarter of 2008 compared to the year before. Furthermore, he states that the mortgage backed securities market globally collapsed in 2008. In this same period, the property derivatives market reached record trading volumes (Syz and Vanini (2009) p.4).

Record trading volumes in the derivatives market as described by Syz may suggest a sufficient liquidity in the derivatives market but we must note that this market is still growing and not yet as liquid as should be for continuous trading (see e.g. www.ipd.com). Moreover, this liquidity grew very slowly. The first attempt for a derivatives market (London Fox Experiment as discussed in Chapter 2) failed due to insufficient liquidity. The fact that the growth in trading volumes continues in times of downturns however is a good sign for further development of liquidity.

4.3.2 Supply and demand

Low liquidity as we have seen in the derivatives markets was not caused by transaction times or costs, but by discrepancies between the buyer and seller sides. That is, the supply and demand are not balanced at any time. For a proper balance between supply and demand, parties are needed to take on both sides of the risk. But in the property world, there seems to be some sort of consensus between parties, i.e. all parties have a very similar view on how the indices will behave in the future. Note that this is mostly caused by the effect of expectation on house prices, which results in autocorrelation of the index and thus partly predictable future returns, as discussed in Chapter 1.1.1. We find that in periods of downfall there are very little parties that are willing to buy the derivatives, resulting in decreasing trading volumes and relatively low derivative prices.

4.3.3 Underlying asset not tradable

We discussed the implications of not having replication possibilities for the house price index on derivative pricing in Section 4.1. It has an impact on the introduction of derivatives in general as well. Namely, in the absence of perfect replicating strategies, a hedging error exists. To reduce this error, the bank needs to find two parties for every deal. It thus acts as an intermediary. We have seen for e.g. weather derivatives, that this may not be problematic. But then it is necessary that we have good, reliable and independent sources for information about the movements of the underlying. For HPI derivatives, this means that we must be very cautious when deciding which index to use as underlying for the derivatives.

4.3.4 Indices

The existence of a reliable index is key to the development of property derivatives. An index is constructed from information about single properties. However, the property market is very heterogeneous, so simply averaging transaction prices or valuations lead to a poor-quality index, Syz (2008). Syz gives a list of basic criteria which should be fulfilled by an index to achieve a high accuracy and to earn trustworthiness (p.53).

**Representativeness:** The index must represent the entire housing market. Hence it must have a strongly reduced idiosyncratic risk by including information about a large number of houses.

**Transparency:** The construction method of the index must be publicly available and understandable. This will lead to a higher trustworthiness of the index.
4.3. DIFFICULTIES FOR INTRODUCING HPI DERIVATIVES

**Track record:** Again to improve trustworthiness, a long track record gives information to the public about the index response to past economic circumstances. Potential investors can use this to judge the representativeness of the index and its robustness.

**Objectivity and minimization of potential fraud:** To prevent manipulation with the figures, we can for instance require a large number of independent data providers. Furthermore, the institution to publish the index should be independent and trustworthy.

**Actuality and high frequency:** High frequency indices require a lot of data. The data must be actual and representative by the time it is used in the index. The frequency of publishing new index information must be high enough to give an accurate view of the market. The time in between publications must however be large enough to gather sufficient new information. In general we have that the higher the number of observations, the higher the possible frequency.

As for the Dutch housing market we have three major parties to publish house price indices, being Kadaster, NVM and the WOX. The NVM index has the longest track record, going back to the 1980s. We will first discuss the differences and similarities between these two indices in terms of the number of registrations, the moment of registration and the calculation method for the index. We follow the analysis of De Jong-Tennekes (2009).

NVM is a brokers organization. They register transactions intermediated by one of their members. Houses sold by auction for instance are thus not represented in the index. The index is based on approximately 70% to 75% of all transactions. Kadaster registers all transactions. We thus have that the representativeness of the Kadaster index is higher than for the NVM index.

As for the moment of registration, we have that the NVM registers the transaction at the time of signing the contract. Kadaster registers when the notary makes up the transaction papers, which is of course somewhat later than the NVM registration. Moreover, NVM shifts the date of registration back with an additional half month. On average, we can say that the time difference between registrations is about three months. So for a fair comparison between the two indices, we should take the NVM index of some quarter and the Kadaster index of the next quarter.

The third difference between the indices is the calculation method. NVM uses the median of all transaction prices for the index. A disadvantage of this method is that it is based on only the transactions that have taken place in the previous period. For instance when in a certain period more houses are sold in the lower segment than in the higher segment in comparison to the last period, both the mean and the median move downwards, whereas the value of houses did not decrease.

The Kadaster index does not have this disadvantage, for it is based on the (theoretical) value of all houses in the Netherlands. The basis of their calculations is that every house has a WOZ value. The WOZ value represents the value of the house without any extras, that is the property itself and not the value of e.g. the kitchen that is included. Basically, the index shows the change in the relation between the average transaction price and the average WOZ value.

The WOX index is based on the same information as the Kadaster index, but the method of calculation is different. The house prices following from transactions are investigated as ‘functions’ of the house characteristics, such as region and type (e.g. apartment, terraced house, detached house). So all of these characteristics have a certain value. Eventually, the value of some house can be determined based on the value of its characteristics. This way, the index is based not only on the sold houses but on all houses. This is, similar to the Kadaster index a strong advantage over the NVM index. Disadvantages of both this method and the method of Kadaster are the lessened transparency and review difficulties. Nevertheless, researchers concluded that the WOX index is an advanced and trustworthy index (Di Bucchianico and Kuhnt (2008)).

The three indices are shown in Figure 4.9. The behavior of the indices is rather similar. Whether they are suitable as underlying for property derivatives is questionable. Especially the usage of the mean and median for the Kadaster and NVM index makes those indices poorer representations of the housing market then the WOX index. So the first requirement from the list is met best by the WOX index. However, the second requirement, transparency, is much better for both the NVM and
CHAPTER 4. POINTS OF ATTENTION WHEN INTRODUCING HPI DERIVATIVES

Kadaster. We find Kadaster even more transparent, for they do not modify the data as much as NVM in terms of registration times. Also, Kadaster uses data from all transactions, which is not the case for the NVM index. The third requirement is best met by the NVM index for it is published since 1986, whereas both the Kadaster index and the WOX index go back to 1995. As for the objectivity requirement, we did not find any information to suggest that there may be issues with either index. The last requirement concerns actuality and frequency. The Kadaster index is published monthly and both the NVM and the WOX index is published quarterly. So Kadaster provides the highest frequency. NVM is however more actual, since they register the transaction costs soonest. The question rises why the decision is made to publish the WOX quarterly instead of monthly as the Kadaster index, since they are based on the same data source. It could suggest that the amount of information is insufficient for a monthly publication of good quality. Further research is necessary in this area.

![Figure 4.9: Dutch house price indices.](image)

4.3.5 Possible Feedback effects

A derivatives market may have feedback effects on the underlying, in this case the housing market in general and/or its index. This is mostly caused by the extra information that becomes available through the derivatives market. It could lead to a higher level of efficiency on the housing market. More specifically, the derivatives market gives information about the expected price development of the housing market. As there is more information about the expectation, we may see a more rapid adaption of the market to this expectation.

Another feedback effect could be on the practise of appraisal for individual properties. Appraisals are now mostly based on the prices of comparable properties. The illiquid nature of the housing market means that transactions are only rarely observed. So the appraisals are based on little and obsolete information. A liquid derivatives market, especially for regional indices, could be very useful for appraisers for they could give extra and more up-to-date information.

The third feedback effect we will discuss is the impact on trading volumes. At the moment one who wishes to invest in the housing market must buy an actual house. For households who wish to invest in housing, a derivatives market would introduce the possibility of renting a house and invest their savings in HPI-linked derivatives. Not only do they then invest in the housing market, they also avoid the idiosyncratic risks of owning a house. If households would go for this scenario, the derivatives market would thus have a negative effect on the transaction volumes in the spot market.
Nevertheless, we do not expect this to be of great impact, for we find it unlikely that many households will go for this option. We believe that their focus is mostly on beating the market through picking suitable houses and actively managing them instead of reducing the idiosyncratic risk and actively taking on positions on derivatives markets.

4.4 Conclusions

Corresponding to the structure of this chapter, we will here summarize the most important conclusions of the three sections. As for the first section, we found that the effect of expectation cannot be ignored when pricing property derivatives. This caused by the lack of replicability of the housing market. We found the use of a risk premium in the Black-Scholes formula to be a solution for this problem. The risk premium can be estimated from derivatives data with the same underlying index or it can be determined through expert opinions. A downside to this solution is that it may not be possible to combine it with Merton’s model. As we are reluctant to ignore sudden jumps in our model, we strongly suggest further research on the possibilities to integrate a risk premium in Merton’s model.

The second part of this chapter consisted of pricing methods for HPI-linked products. We discussed the index-linked bond, which could be an appropriate first step in the development of a HPI derivatives market in the Netherlands. Since we wish to give households the possibility of relaxing their house price risk, we also discuss the HPI-linked mortgage and an insurance policy. We found that the mortgage could be rather costly, but the costs depend highly on the construction of the mortgage. For instance, if a household is willing to keep a small portion of the risk, additional mortgage rates will be strongly reduced.

As for the insurance policy, we are more cautious. First, the pricing method is not very appropriate for the housing market. Second, insurance policies are based on the law of large numbers. But since the high correlation between price movements of individual houses we cannot use this law. For instance, a company that insures cars will receive roughly the same amount of claims per year. For the housing market however, we find that in times of recession almost every house will be sold at a loss, whereas the year before almost every house was sold at a profit. So the insurance company should have very high back-up resources to be able to pay out the losses in times of recession.

In the last part of this chapter we discussed potential hurdles for the introduction of housing derivatives apart from pricing issues. We found that attention should be paid to liquidity. The property derivatives market started out very illiquid, but over the years liquidity slowly grew to a higher level. Important for liquidity is the balance between supply and demand. We find that many investors have the same ideas about future movement of the index which can disturb this balance. In times of recession, this could lead to low trading volumes and strongly decreasing derivatives prices.

Apart from liquidity, we found that not every index is suitable for property derivatives. We considered some characteristics which should be met by house price indices to be suitable. Very important is that the index represents the entire market and of course that it is trustworthy. Three indices for the Dutch housing market are discussed, namely the Kadaster, NVM and WOX index. We found that all indices have some advantages and some disadvantages. The WOX index seems to be the best representation of the Dutch housing market. A drawback is the limited period for which the index is available. This makes it hard to draw conclusions about for instance its response to certain economic circumstances.
Chapter 5

Conclusions & recommendations

In this study we described the possibilities that house price derivatives could yield for institutional investors, mortgage lenders and home owners. Our main focus however was pricing these derivatives, in particular options. Furthermore we described the possible difficulties that may arise when introducing a market in such derivatives. First we will discuss the main conclusions. In Section 5.2 we will give some suggestions for further research.

5.1 Conclusions

We found that the possibilities of a market in house price derivatives are great. Institutional investors could use this market to diversify their portfolio with real estate while avoiding the high transaction and maintenance costs which come with direct investments. Whereas these investors are willing to take on the house price risk for diversification, mortgage providers are good for the other side of the deal, selling the risk and thus reducing the house price risk on their balance sheet. Traditional financial products such as futures and options are very suitable for them. However, home owners are much less likely to take on a position on such markets. We found that households have a housing dominated portfolio. Analysis shows us that with a more diversified portfolio could give similar returns at a significantly lower risk level. Furthermore we find additional risks since most houses are leveraged by a mortgage. Index-linked mortgages or house price insurances could provide a significant decrease of these risks.

The advantages of house price derivatives are thus clear. Nonetheless, there is not yet a liquid market in the Netherlands for such derivatives. In our study of the possible difficulties when introducing the market, we found that autocorrelation in house price movements plays a big role. Autocorrelation causes partly predictable future house price movements and thus some sort of consensus among investors. That is, investors seem to have the same market expectations. This disturbs the balance between supply and demand and leads to liquidity problems. Nevertheless, we have seen in e.g. the UK that liquidity does grow on house price derivatives markets although slowly. In general we find that there are some hurdles to be taken, but we believe it to be possible to achieve a mature market.

As for the pricing issues we found that it is not a straightforward exercise. Figure 5.1 shows the models that are discussed in this study, starting with the Black-Scholes model which is the most used and known model for option pricing. We find that there are some problems with the use of Black-Scholes for pricing options with the underlying being a house price index. These problems can be split in two parts. First, we have that the assumptions of the index following a geometric Brownian motion is incorrect. Namely, the index shows typically a low risk profile, interrupted by sudden downfalls. The second problem lies in the assumption of the underlying asset being tradable. We find that this is hardly the case for the Dutch housing market, which leads to difficulties with constructing a riskless...
CHAPTER 5. CONCLUSIONS & RECOMMENDATIONS

Figure 5.1: Overview of relations between the pricing models as discussed in this study and their corresponding difficulties.

portfolio which lies at the basis of the Black-Scholes pricing formula. These two problems lead to two extensions of the Black-Scholes model, being Merton’s mixed jump-diffusion model and expectation pricing. We will discuss the extensions and their problems according to the numbering of the arrows in Figure 5.1.

1. The extension of the Black-Scholes model to Merton’s model concerns the addition of a jump process to the geometric Brownian motion. Jumps occur according to a Poisson process and the impact of the jump is either deterministic or lognormally distributed. Looking at the histograms of Dutch house price returns we find that the process as described by Merton is more realistic than the geometric Brownian motion. This also agrees with the characteristics as discussed in Chapter 1 which state that external factors cause sudden price movements of great impact.

2. A problem with Merton’s model is parameter estimation. Apart from only the volatility of the process as in Black-Scholes model, we now have to estimate the jump frequency and the parameters of the jump impact (mean and variance). Since jumps are typically rare for house prices, data of jumps is sparse and it is thus impossible to give good estimates for the jump parameters. An analysis of robustness led us to setting the variance of the jump size to zero. This reduces the number of variables to be estimated. Now, Procedure 3.4.1 can be followed to estimate the remaining parameters. We can of course use this procedure for parameter estimation, but must keep in mind that these parameters are still based on little data.

3. The extension of the Black-Scholes model to expectation pricing can be characterized by adding the expected return ($\mu$) as a variable in the price model. This is done by the use of a risk premium. We simply replace the riskless rate $r$ in Black-Scholes model by $r + p$, where $p$ is the risk premium and $r + p = \mu$. So the only extra parameter to be estimated is the risk premium. It can be done by analyzing derivatives prices with the same underlying index.

4. Again, the problem of this model is parameter estimation. To estimate it from data, we need derivatives prices with the same underlying index. And these derivatives do not yet exists. In this case we can use expert opinions of future market developments to estimate $\mu$ and thus $p$. 
5.2 Recommendations for further research

In this study, we have described and analyzed a lot of aspects of the introduction of house price derivatives. There are numerous possibilities for further research, regarding both pricing methods and practical issues. We will give some suggestions here.

- It would be interesting to do market research on two levels. Firstly among home owners to see whether they are aware of their high house price risk and if they were willing to spend money on protection from this risk. Secondly it would be interesting to interview possible market players (both buying and selling side) on their willingness to participate on a house price derivatives market and on their views of future price movements to gain insight in how high the level of consensus between these investors is in the Netherlands.

- At this time, we are slowly recovering from the credit crunch. It will be interesting to take a closer look at the property derivatives which were available over a longer period, covering the credit crunch to see what the effect was on both prices and trading volumes. And how long will it take to recover from it? Since there is not yet a very liquid market, such characteristics could give a good indication of the influence of dramatic price changes in a premature market in the Netherlands.

- For expectation pricing in this study, we considered the riskfree rate $r$, the risk premium $p$ and the expected return $\mu$ to be related as follows: $\mu = r + p$. This means that the risk premium can be determined when we have $\mu$ (we assume $r$ to be known). However, the risk premium may be influenced by other factors as well, for instance by transaction costs and liquidity issues. It would be very interesting to further investigate the factors that influence the risk premium.

- As for pricing the options, we found that the main difficulties lie with parameter estimation. We suggest more research on that point for both models. Furthermore, since both models solve distinct problems of the Black-Scholes assumptions, it would be very interesting to explore the possibilities of combining the two. That is, to use a model that includes both jumps and market expectations.
Appendix A

Markowitz for household portfolios

In this appendix we discuss the average Dutch household portfolio. We wish to show that the average household has an ‘overinvestment’ in housing. This means that housing is a larger portion of their wealth than optimal. To decide which portfolio is optimal we use Markowitz portfolio optimization. In Section A.1 we briefly discuss the theory behind Markowitz portfolio optimization theory and discuss its practicalities and limitations. Subsequently in Section A.2 we cover the characteristics of the average Dutch household’s portfolio. Finally in Section A.3 we apply Markowitz to the household’s portfolio.

A.1 Markowitz portfolio optimization

Markowitz portfolio optimization is based on the idea that the relevant information about securities can be summarized by three figures: the mean return, the standard deviation of the returns and the correlation with other assets’ returns. This means that one doesn’t need to know any information about the firm one wishes to invest in, such as dividend policy, earnings, market share, strategy, quality of management etcetera. Furthermore it says that any person or business should hold the same portfolio.

Markowitz assumes asset’s return to be normally distributed random variable. Furthermore it defines risk as the standard deviation of return. He then models a portfolio as a weighted combination of assets so that the return of a portfolio is the weighted combination of the assets’ returns.

A.1.1 Expected returns-Variance of return (E-V) rule

Let the random variable $R_i$, $i = 1, 2, \ldots, n$, be the return on the $i^{th}$ security. Furthermore we have that $E[R_i] = \mu_i$ and $\sigma_{ij} = E[(R_i - E[R_i])(R_j - E[R_j])]$ the covariance of the securities. We let $X_i$ be the fraction of security $i$ in the investment portfolio, excluding short sales, so $0 \leq X_i \leq 1 \forall i \in \{1, 2, \ldots, n\}$ and of course $\sum_{i=1}^{n} X_i = 1$.

Now we can define the random variable $R := \sum_{i=1}^{n} R_i X_i$ to be the yield on the portfolio as a whole.

We now simply obtain the expected value and the variance to be

$$E[R] = \sum_{i=1}^{n} X_i E[R_i] = \sum_{i=1}^{n} X_i \mu_i,$$  \hspace{1cm} (A.1)

$$\text{Var}[R] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} X_i X_j.$$ \hspace{1cm} (A.2)
We wish to arrive at a formulation of the portfolio optimization problem. Above we saw that the expectation and variance of the yield of a portfolio can be calculated when we know the expected values, the variances and covariances of the asset weights. We now wish to describe this idea in vector notation and as a function of the asset weights.

We define the vectors \( \mu = (\mu_1, \ldots, \mu_n) \), \( X = (X_1, \ldots, X_n)^T \) and the correlation matrix

\[
\Sigma = \begin{pmatrix}
\sigma_{11} & \cdots & \sigma_{1n} \\
\vdots & \ddots & \vdots \\
\sigma_{n1} & \cdots & \sigma_{nn}
\end{pmatrix}
\] (A.3)

We have the set \( X := \{ X \in [0,1]^n | \sum_{i=1}^n X_i = 1 \} \) to be all possible portfolios. In vector notation we have for the expected value and variance of the portfolio yield as functions of the weights

\[
E(X) = \mu^T X,
\]

\[
V(X) = X^T \Sigma X.
\] (A.5)

Now we can define the set \( S \) of all possible outcomes to be

\[
S := \{ (E,V) \in \mathbb{R}_+^2 | \exists X \in X : E = E(X), V = V(X) \}.
\] (A.6)

We suppose this to be the feasible set in which we wish to solve either optimization problem

1. \( \min(V) \), such that \( E \geq e \) for some \( e \in \mathbb{R}_+ \), or
2. \( \max(E) \), such that \( V \leq v \) for some \( v \in \mathbb{R}_+ \).

We look at the first optimization problem. Now according to the definition of \( E \) and \( V \), this problem is a quadratic programming problem, i.e. having quadratic objective function and linear constraints. For positive definite matrices \( \Sigma \), this optimization problem has a unique solution (see Fletcher (1981)), which can be calculated as in e.g. Goldfarb and Idnani (1983).

The assumption of \( \Sigma \) being positive definite is violated either if a security can be obtained as a linear combination of other securities (so no full row/column rank), or if there is a riskless security. A riskless security would correspond to a zero-row and column in the matrix. This is easily seen, since this would lead to the determinant being zero and of course the requirement of positive determinants for all principal minors (upper left square sub-matrices) is equivalent to the matrix being positive definite. In this study we can easily rule out the possible violations as we consider at most five assets, so we can safely assume that \( \Sigma \) is positive definite.

The second optimization problem has linear objective function and a quadratic constraint. This cannot be solved in a direct manner, as we can for the first. But the existence of the relation between the two formulations is clear and will be formalized in the following proposition.

**Proposition A.1.1** Let \( \Sigma \) be positive definite. Assume in the following parts that \( e \) and \( v \) lie within the feasible region. We have for the optimization problems 1. and 2. with feasible regions given by (A.4), (A.5) and (A.6).

a) If \( X^* \) is the portfolio corresponding to the optimal solution \( (E_1, V_1) \) in problem 1 and furthermore we have that \( E_1 = e = \mu^T X^* \), then \( X^* \) also solves problem 2 provided that we choose \( v = V_1 = X^T \Sigma X^* \).

b) If \( X^* \) is the portfolio corresponding to the optimal solution \( (E_2, V_2) \) in problem 2 and furthermore we have that \( V_2 = v = X^T \Sigma X^* \), then \( X^* \) also solves problem 1 provided that we choose \( e = E_2 = \mu^T X^* \).

The proof is straightforward, using reductio ad absurdum and can be found in Korn (1997).
A.2. The average household portfolio

If we wish to optimize the average household’s portfolio, we must first gain insight in the current situation. Table A.1 gives the contribution of several asset- and liability types for the average household in the Netherlands. As we can see, real housing is by far the biggest contribution to a household’s wealth with a contribution of 50%. We must emphasize here that these numbers correspond to the average household and only 52% of the households actually own a house. So we can safely assume that the contribution of a real estate, given that the family owns one, is even larger than 50%. Furthermore, since a house is a large investment, a mortgage is necessary to finance it. The size of liabilities is almost completely determined by a mortgage (96%). So in both assets and in liabilities housing is the major factor. The majority of the financial assets lies in pension schemes and life insurance, with contributions of 21% and 10% respectively. A minor factor in the financial assets are stocks and bonds. Stocks represent about 1.3% and bonds (both long term and short term) represent only 1.1%. As an alternative to stocks and bonds, 2.2% of the assets is allocated to collective investment schemes. These schemes participate in bond and stock markets as well as in real estate and other securities. The diverse nature of the collective investment schemes brings us to leave them out of further consideration.

Our basic interest lies in the distribution of saving accounts, bonds, stocks and housing in the asset portfolios of households. We wish to see the portfolio as an investment portfolio and we cannot consider all durable goods to be investments. The same holds for pension schemes and life insurances. We assume that a household does not wish to switch these assets for other assets which may lead to a better investment portfolio. We leave out the checking accounts for similar reasons.

Of course, the arguments for not viewing these assets as investments can also hold for housing. One could say that a house is the ultimate durable consumer good, held to live in, not to gain a profit. These are all valid reasons, but as we look at the domination of housing in portfolios we see that studying the effect of considering housing as a pure investment may lead to interesting changes in how we should build up our portfolios.

Now when we restrict ourselves to savings, stocks, bonds and housing, we obtain Table A.2, which shows the distribution of just these six types of assets. The total of the investment assets corresponds to €203,000, which is a little over 60% of the total asset value. Figure A.1 shows us the asset weights as in Table A.2. Again we can see that the portfolio is very unbalanced, even more now, since we do not include pension schemes in the investment portfolio.
### APPENDIX A. MARKOWITZ FOR HOUSEHOLD PORTFOLIOS

#### Assets

**Deposits**
- Checking account: 6,947
- Saving accounts: 34,488

**Total**
- Total deposits: 41,434

**Securities**
- Stocks: 4,293
  - Dutch: 3,133
  - International: 1,161
- Investment funds: 7,315
  - Dutch: 3,967
  - International: 3,348
- Long term bonds (> 1 year): 3,829
  - Dutch: 2,783
  - International: 1,046
- Short term bonds (< 1 year): 21
  - Dutch: 19
  - International: 2

**Total**
- Total securities: 15,458

**Pension**
- Pension: 68,660

**Life insurance**
- Life insurance: 34,016

**Non financial assets**
- Housing: 160,839
- Other durable goods: 83,252

**Total**
- Total non financial assets: 172,494

**Assets total**
- Total assets: 332,062

#### Liabilities

**Mortgages**
- Mortgages: 78,202

**Consumer credit**
- Consumer credit: 3,282

**Liabilities total**
- Total liabilities: 81,484

**NETT EQUITY**
- Nett equity: 250,578

Table A.1: Balance sheet for the average Dutch household (x 1000€)

<table>
<thead>
<tr>
<th>Asset type</th>
<th>Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>17.0%</td>
</tr>
<tr>
<td>Dutch stocks</td>
<td>1.5%</td>
</tr>
<tr>
<td>International stocks</td>
<td>0.6%</td>
</tr>
<tr>
<td>Bonds</td>
<td>1.9%</td>
</tr>
<tr>
<td>Housing</td>
<td>79.0%</td>
</tr>
</tbody>
</table>

Table A.2: Asset allocation for the average Dutch household investment portfolio.
A.3. OPTIMAL HOUSEHOLD PORTFOLIOS

A.3 Data evaluation

We examine a simplified portfolio consisting of stocks, bonds (both international and Dutch), cash (Dutch) and housing (Dutch). The yearly rates of return are shown in Figure A.2. Clearly, stocks give the highest return but even more obvious is the risk (standard deviation) which is significantly larger than the risks associated with the other investment classes. The mean rate of return and the standard deviation are shown in Table A.3. As one would expect, higher returns entail higher risks and the highest risk lies in stock investments. Housing and bond investments show much lower risk and the foremost safest investment is of course cash. Besides the risk and return of the assets we also need information about the correlation between these return rates. Table A.4 shows the correlation between the return rates of the asset classes. Housing is negatively correlated with bonds and interest rates, but these correlations are very close to zero so we cannot draw solid conclusions from these results. The highest correlation is found between bond returns and interest rates. This is probably caused by the fact that if the interest rates are high, coupon payments for new bonds will be high, for if not, one is better with saving. On the other hand, the market

<table>
<thead>
<tr>
<th>Asset</th>
<th>Return</th>
<th>Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>10.22%</td>
<td>18.90%</td>
</tr>
<tr>
<td>AEX</td>
<td>6.84%</td>
<td>26.26%</td>
</tr>
<tr>
<td>Euribor 3 m</td>
<td>4.73%</td>
<td>2.18%</td>
</tr>
<tr>
<td>NVM HPI</td>
<td>6.26%</td>
<td>4.31%</td>
</tr>
<tr>
<td>Lehman agg. BI</td>
<td>7.02%</td>
<td>5.16%</td>
</tr>
</tbody>
</table>

Table A.3: Return & risk for four asset classes

Figure A.2: Yearly rates of return (1986-2008)
value of existing bonds will decrease, since their coupon payments are getting relatively low and this makes investing in them less profitable. The latter must be dominant, since we found positive correlation.

### A.3.2 The efficient frontier for household portfolios

Using the data characteristics found in Section A.3.1, we let Matlab perform its ‘portopt’ function to optimize portfolios consisting of these four assets for various risk-levels. Figure A.3 shows us the maximal rate of return as a function of risk. This is done for portfolios including and excluding housing. For the including housing portfolio, we find for instance that if we wish to have a risk level of 3.6%, we could obtain an expected return close to 7% if we were to keep an optimal portfolio. The average household clearly does not hold an optimal portfolio. The average household clearly does not hold an optimal portfolio, for their return is about 6% for a risk level also close to 3.6%. The expected return of an optimal portfolio without housing for this same risk level is somewhat higher then for the average household, about 6.3%. This means that a household renting a house, and keeping an optimized portfolio (excluding housing of course) with the same risk level as an average household, can increase its expected return with more than 0.5% by taking a position on the housing market.

As we have an overview of which returns we can obtain by optimizing portfolios, we now wish to know what this optimal portfolio looks like. We suspect that for lower risk levels the amount of cash held should be large, whereas for high risk we expect a stock-dominated portfolio. Figure A.4 shows the weights for the four asset classes in an optimal portfolio as a function of the level of risk. In other words, the maximal return we find in Figure A.3 for some risk level is obtained by holding a portfolio in which each asset has the fraction shown in in Figure A.4 for that same risk level. So if we wish to optimize our portfolio and our risk level should be 3.6%, an optimal portfolio gives us an expected return of about 7% and this portfolio is characterized by holding zero cash and investments of about 10% in stocks, 40% in housing and 50% in bonds. As we expected, low risk levels are characterized by the Euribor rates. But although intuitively we would maybe expect the portfolio for lowest risk level to contain only cash investments, we can see that adding bonds and stocks to our portfolio reduces instead of increases the risk. This is the result of the effects of inter correlations and diversification.
A.3. OPTIMAL HOUSEHOLD PORTFOLIOS

A.3.3 A brief note on the robustness of Markowitz portfolio optimization

We examined the robustness of portfolio optimization by varying the source data and the sample periods of the returns of assets. We investigated the changes in efficient frontier and in the behavior of the asset weights in the optimal portfolios. From these results we can draw the following conclusions. First, if the expected rate of return or risk changes but the correlation matrix remains approximately the same, the efficient frontier will barely move. The weights will be very similar and the expected return rate does not increase much because of diversification of the portfolio. Secondly, if the rate of return and the risk remain the same but the correlation matrix shifts at some points, the effect on the efficient frontier is there, although not large, as are the effects on the asset weights. Thirdly, if the expected risk, the return and the correlation matrix are all different for one in four asset classes, the efficient frontier will be quite different, but the weights will show similar behavior since three out of four asset classes still have the same characteristics and correlations. Finally if all asset classes have a slightly changed rate of return and risk and moreover the correlation matrix is different, the effects on efficient frontier can be quite large as can be the effects on asset weights. Changing the sampling period of asset returns will cause this to happen. Therefore we should be careful by choosing this period: it should not be too small and one must be able to argue that this period is typical for, or as close as can be to what will happen in the future.
A.4 Conclusions

We can draw the following conclusions. The average household suffers from ‘overinvestment’ in housing. This is clear from Figure A.1. We wished to show that this high level of housing in their portfolio is not optimal. This is definitely the case. Figure A.3 shows that the rate of return could be increased by about 1% for the same risk level, if we have greater freedom in choosing the asset weights in our portfolios. The maximal contribution of housing to an optimal portfolio for whatever risk level is about 50%. It also showed that households having a portfolio excluding housing, i.e. households renting their homes could gain about 0.5% by taking a position in the housing market.

As we examined the robustness of Markowitz portfolio optimization, we found that we should be careful in selecting the data. Especially differences in the correlation matrix will lead to shifts in the asset weights and the efficient frontier. But we can safely conclude that a housing component of about 80% is far too large, and diversification will lead to much higher returns at a given risk level.
Appendix B

Preliminaries for option pricing

This appendix considers preliminaries for option pricing. We discuss different definitions for the rate of return and continuous compounding, Brownian motions, Itô’s lemma and the lognormal distribution.

B.1 Rate of return & continuous compounding

There are two widespread definitions for the rate of return ($\mu_i$) of an asset $i$: arithmetic and geometric. Formally we have for a single period the rate of return defined as $\frac{p_t - p_{t-1}}{p_{t-1}}$, where $p_t$ is the asset value at time $t$. Now for longer time spans the difference between arithmetic and geometric rates become clear. Consider for example an asset which is at time $t = 0$ worth 100 euro’s. At time $t = 1$ its value has dropped to 50 euro’s, but at time $t = 2$ the asset reached its initial value of a hundred euro’s again. Now, the two single periods give us returns of respectively $-50\%$ and $+100\%$. The average and thus the arithmetic rate of return given by $\frac{-50 + 100}{2} = 25\%$. The geometric rate of return though is 0\%, since we started as well as ended with an asset worth 100 euro’s.

The expected rate of return will be estimated by the arithmetic average of past values. The use of the arithmetic rate of return means that we expect that investors focus on single period returns. Both the arithmetic and geometric rate of returns are dependent on the length of the time interval. This makes it difficult to compare returns. In this section we will introduce the continuously compounded rate of return, which is a variant of the geometric rate of return and suitable for comparing returns over different assets and time intervals.

Suppose we have an interest rate of 12\% per annum. If the interest rate is measured with annual compounding, €100 will grow to €100 \cdot 1.12 = €112 in one year. This corresponds to an annual geometric rate of return.

But if we now have semiannual compounding, we receive €100 \cdot 1.06^2 = €112.36. Similar for monthly compounding, we obtain €100 \cdot 1.01^{12} = €112.68. In general we have that if we invest an amount $A$ for $n$ years with an interest rate $R$ compounded $m$ times per annum, the final value will be

$$A \left(1 + \frac{R}{m}\right)^{mn}.$$  \hspace{1cm} (B.1)

It is easy to see that if we let $m$ tend to infinitely, the final value will be

$$\lim_{m \to \infty} A \left(1 + \frac{R}{m}\right)^{mn} = Ae^{Rn}.$$  \hspace{1cm} (B.2)
This principle is called continuous compounding, since the situation of \( m \to \infty \) approaches the situation with continuous returns. If we wish to calculate the continuously compounded rate \( R_c \) corresponding to a rate \( R_m \) compounded \( m \) times per year, we simply obtain

\[
A e^{R_c n} = A \left(1 + \frac{R_m}{m}\right)^{mn},
\]

\[
A e^{R_c} = A \left(1 + \frac{R_m}{m}\right)^{m},
\]

\[
R_c = m \log \left(1 + \frac{R_m}{m}\right).
\]

(B.3)

**B.2 Brownian motion**

A Brownian motion is a special type of continuous Markov process. To get an idea of what a Brownian motion is, we first look at a symmetric random walk which take a unit step in each time interval to either the left or the right. We now alter the process by taking smaller steps in smaller time intervals. We now let both the time intervals \( \Delta t \) as the step sizes \( \Delta x \) go to zero with \( \Delta x = \sqrt{\Delta t} \). The limiting process is a Brownian motion.

The formal definition for a Brownian motion is given e.g. in Ross (1996).

**Definition B.2.1** A stochastic process \([z(t), t \geq 0]\) is said to be a Brownian motion if:

1. \( z(0) = 0 \),
2. \( z \) has stationary independent increments,
3. for every \( t > 0 \), \( z(t) \) is normally distributed with mean 0 and variance \( t \).

The mean slope of a stochastic process over time is known as the drift rate of the process. The variance per unit time is called the variance rate. Now for a basic Brownian motion, we have drift rate of zero, and a variance of one, where one unit of time corresponds to a year. The zero drift rate is a necessary condition for being a stationary process, but the independency of distinct parts of time makes that it is a stochastic process. We will now consider the generalized Brownian motion, which is still a stochastic process, but no longer a martingale, as we include a drift.

**B.2.1 Generalized Brownian motion**

**Definition B.2.2** We define a generalized Brownian motion in variable \( x \) in terms of \( dz \)

\[
dx = a \, dt + b \, dz,
\]

(B.4)

where \( a \) and \( b \) are constants.

We can interpret the two terms on the right-hand-side of equation (B.4) separately. The \( a \, dt \) term shows that differentiation to \( t \) yields \( a \), hence \( x \) has a drift rate of \( a \). The \( b \, dz \) terms can be seen as an addition of noise to the process. Since the process is continuous, we cannot add independent normally distributed noise points as is common in discrete processes. Instead
B.3. ITÔ’S LEMMA AND THE LOGNORMAL DISTRIBUTION

the noise is to be $b$ times a Brownian motion $z$. A standard Brownian motion has a standard deviation of 1.0, hence $b$ times this motion has a standard deviation of $b$. Now we have for the characteristics of the generalized Brownian motion that for some time interval of $T$ years the mean change of $x$ equals $aT$ and the variation of change equals $b^2T$. So the generalized Brownian motion has a drift rate of $a$ and a variance rate of $b^2$.

B.2.2 Geometric Brownian motion

As we saw in the previous section, we would like to see a process in which the rate of return remains constant instead of the drift rate. We obtain such a process by multiplying the drift rate with the stock price. That is, if $S$ is the stock price at time $t$, the expected drift rate should be $\mu S$, where $\mu$ represents the constant expected rate of return for a very small time interval. Were there no volatility, the model would be $dS = \mu S \, dt$, or $\frac{dS}{S} = \mu \, dt$. Furthermore, the stock prices are then deterministic, so we obtain by integration the stock price at time $T$, $S_T = S_0 e^{\mu T}$.

We now wish to add a noise factor such that we again have volatility. As we discussed above we wish to have a variable standard deviation for this noise, more specifically, we would like to have a standard deviation which grows linearly with the stock prices. In that case we obtain that the uncertainty about the percentage return is stable, regardless the stock price. This leads to the following noise term, $\sigma S \, dz$, where the constant $\sigma$ is the volatility of the process. We now have for the model the following equation

$$dS = \mu \, dt + \sigma \, dz.$$ (B.5)

Equation (B.5) is the most widely used model for stock price behavior. $\mu$ and $\sigma$ represent the expected rate of return and the volatility of the process.

Discrete-time model

The discrete-time model corresponding to the model in (B.5) is

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}.$$ (B.6)

$\Delta S$ is the change in stock price over the time interval $\Delta t$, where of course $\Delta t$ should be small. As before $\mu$ and $\sigma$ correspond to the expected rate of return and volatility of the process, furthermore we denote by $\epsilon$ a random variable with standard normal distribution. The left hand side of (B.6) is the return of the stock over the period $\Delta t$. Note that this is not the same as the return rate $\mu$. As $\mu$ is the expected return per unit time, we do have that $E[\frac{\Delta S}{S}] = \mu \Delta t$. The term $\sigma \epsilon \sqrt{\Delta t}$ corresponds to the stochastic part of this return and is of course normally distributed with mean 0 and variance $\sigma^2 \Delta t$. So we have for the distribution of the return over interval $\Delta t$

$$\frac{\Delta S}{S} \sim \mathcal{N}(\mu \Delta t, \sigma^2 \Delta t).$$ (B.7)

B.3 Itô’s lemma and the lognormal distribution

Definition B.3.1 We define $y$ to follow an Itô process, if it can be written as

$$dy = a(y,t) \, dt + b(y,t) \, dz,$$ (B.8)
where \( a, b \) are functions of \( y \) and \( t \) and \( z \) is a brownian motion.

**Theorem B.3.2** Suppose \( y \) follows an Itô process. Now let \( G \) be a twice differentiable function of \( y \) and \( t \). Then \( G \) follows the following process

\[
\begin{align*}
    dG &= \left( \frac{\partial G}{\partial y} a(y,t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} b(y,t)^2 \right) dt + \frac{\partial G}{\partial y} b(y,t) \, dz, \\
\end{align*}
\]

where \( z \) is a brownian motion.

We will not give the formal proof, but restrict ourselves to the outline. Extensive calculus is omitted. Detailed information can be found in Itô (1951). We start by considering the taylor expansion of the function \( \Delta G \) in two variables, \( y, t \).

\[
\Delta G = \frac{\partial G}{\partial y} \Delta y + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} \Delta y^2 + \frac{\partial^2 G}{\partial y \partial t} \Delta y \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} \Delta t^2 + \ldots. 
\]  

(B.10)

We are eventually interested in the limit where \( \Delta y \) and \( \Delta t \) tend to zero.

First we wish to be able to evaluate the function \( G \) of \( t \) and a variable \( y \) that follows an Itô process \( dy = a(y,t) dt + b(y,t) \, dz \). In discrete time this corresponds to

\[
\Delta y = a(y,t) \Delta t + b(y,t) \epsilon \sqrt{\Delta t}, 
\]

(B.11)

with \( \epsilon \) a standard normally distributed random variable.

Second order terms in equation (B.10) will be ignored in the limit to zero. But the term

\[
\Delta y^2 = b^2 \epsilon^2 \Delta t + \text{terms of higher order of } \Delta t 
\]

(B.12)

has a term of first order \( \Delta t \) and will thus be included. This yields

\[
\Delta G = \frac{\partial G}{\partial y} \Delta y + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} \Delta y^2. 
\]

(B.13)

We now wish to evaluate \( \epsilon^2 \Delta t \). Since \( \text{Var}(\epsilon) = 1 = \mathbb{E}[\epsilon^2] - \mathbb{E}[\epsilon]^2 \) and \( \mathbb{E}[\epsilon] = 0 \), we have that \( \mathbb{E}[\epsilon^2] = 1 \) and \( \mathbb{E}[\epsilon^2] \Delta t = \Delta t \). It can be shown that \( \text{Var}(\epsilon^2 \Delta t) \) is of order \( \Delta t^2 \), so we treat the random variable as being deterministic and thus equal to its expected value.

Substituting \( \Delta y \) and \( \Delta y^2 \) in (B.13), we obtain

\[
\Delta G = \frac{\partial G}{\partial y} (a(y,t) \Delta t + b(y,t) \epsilon \sqrt{\Delta t}) + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} b^2(y,t) \Delta t. 
\]

(B.14)

Now, taking the limit of \( \Delta t \) to zero, we have

\[
dG = \left( \frac{\partial G}{\partial y} a(y,t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} b(y,t)^2 \right) dt + \frac{\partial G}{\partial y} b(y,t) \, dz. 
\]

(B.15)

This completes the sketch of the proof.

The following lemma is analogous to Itô’s lemma, but for Poisson processes. We will not go into the theory, but only state it as it is given in Merton (1971). The proof can be found in Kushner (1967).
Theorem B.3.3 Let \( q(t) \) be an independent Poisson process. Let the event be that a state variable \( x(t) \) has a jump in amplitude of size \( J \), where \( J \) is a random variable whose probability measure has compact support. Then, a Poisson differential equation for \( x(t) \) can be written as

\[
\frac{dx}{dt} = f(x, t) dt + g(x, t) dq
\]  

(B.16)

and the corresponding differential generator \( \mathcal{L}_x \) is defined by

\[
\mathcal{L}_x := h_t + f(x, t) h_x + \mathbb{E}_t[\lambda(h(x + Jg, t) - h(x, t))],
\]

(B.17)

where \( \mathbb{E}_t \) is the conditional expectation over the random variable \( J \), conditional on knowing \( x(t) = x \), and where \( h(x, t) \) is a \( C^1 \) function of \( x \) and \( t \).

B.3.1 Lognormality of stock prices

We can use Itô’s lemma to evaluate the function \( G(S, t) = \log(S) \) where \( S \) is as in (B.5),

\[
\frac{dS}{S} = \mu dt + \sigma dz.
\]  

(B.18)

We have

\[
\frac{\partial G}{\partial S} = \frac{1}{S}, \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \quad \frac{\partial G}{\partial t} = 0,
\]  

(B.19)

so it directly follows from Itô’s lemma that

\[
dG = \left(\frac{\mu S}{S} - \frac{\sigma^2 S^2}{2}ight) dt + \frac{\sigma S}{S} dz = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz,
\]  

(B.20)

which is of course again a generalized Brownian motion, which implies that the change in \( G(S) = \log(S) \) over a time period \( T \) is normally distributed. That is

\[
\log(S_T) - \log(S_0) = \log\left(\frac{S_T}{S_0}\right) \sim \mathcal{N}\left((\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right)
\]  

(B.21)

and

\[
\log(S_T) \sim \mathcal{N}\left(\log(S_0) + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right).
\]  

(B.22)

Now since the log of \( S_T \) is normally distributed, we have that \( S_T \) has a lognormal distribution. A lognormally distributed random variable takes values in \( \mathbb{R}^+ \), which corresponds to non-negative prices. Furthermore, we have for the expected value

\[
\mathbb{E}[S_T] = S_0 e^{\mu T}
\]

(B.23)

and this fits with the definition of \( \mu \) as the rate of return. For the variation we have

\[
\text{Var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1).
\]  

(B.24)
B.3.2 Estimating the rate of return and the volatility

We use the model shown above for price development. This model is determined by the parameters $\mu$ and $\sigma$ where $\mu \Delta t$ is the mean return in time $\Delta t$ and its standard deviation is $\sigma \sqrt{\Delta t}$. In short

$$\frac{\Delta S}{S} \sim \mathcal{N}(\mu \Delta t, \sigma^2 \Delta t).$$  \hspace{1cm} (B.25)

We saw that this model implies lognormality of $S_T$, equation (B.22). The model-assumptions and hence the lognormality can be used to estimate the rate of return and the volatility of the price development. Let us first look at the distribution of the continuously compounded rate of return as a random variable $r$. We already saw in equation (B.2) that $S_T = S_0 e^{rT}$. This gives for $r$

$$r = \frac{1}{T} \log\left(\frac{S_T}{S_0}\right)$$  \hspace{1cm} (B.26)

and using (B.22) we obtain

$$r \sim \mathcal{N}(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}),$$  \hspace{1cm} (B.27)

so the continuously compounded rate of return per annum is normally distributed with mean $\mu - \sigma^2/2$ and standard deviation $\sigma/T$. Of course we have a smaller standard deviation as $T$ increases. Note that the expected value of the continuously compounded rate of return is not equal to $\mu$ but somewhat smaller. This is caused by uncertainty. To illustrate this, suppose we have for some asset a rate of return with mean zero and zero variance. Then the expected rate of return is of course always zero. But if we still have a mean zero rate of return, but high variance we could have that one year we have a return of 10% and the other a return of −10%. The arithmetic mean return is of course still zero, but the actual return will be $-1\%$ for $100 + 0.10 \cdot 100 - 0.10 \cdot 110 = 99$. This is of course a very simple example, but in general we can say that higher uncertainty lead to a negative shift in mean return.

Now as for the volatility of the price developments we define this to be the standard deviation of the continuously compounded rate of return. Define

$n + 1$: number of observations

$S_i$: price level ultimo interval $i$, with $i = 0, 1, \ldots, n$

$\tau$: length of intervals.

Define the variables $\mu_i = \log\left(\frac{S_i}{S_{i-1}}\right)$, for $i = 1, 2, \ldots, n$. The standard deviation $s$ of the $\mu_i$’s is given by

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (\mu_i - \bar{\mu})^2},$$  \hspace{1cm} (B.28)

where $\bar{\mu}$ is the mean of $\mu_i$. Equation (B.21) shows that the standard deviation of $\mu_i$ equals $\sigma \sqrt{\tau}$. This means that the variable $s$ is an estimate of $\sigma \sqrt{\tau}$. Hence we have for the estimate of $\sigma$

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}.$$  \hspace{1cm} (B.29)
Appendix C

Matlab functions

This appendix contains the Matlab functions which we use in this study. They are organized as follows. We start with the option price formulas for Black-Scholes model and Merton’s jump-diffusion model. Secondly we show functions with data series as input which will give parameter estimates and functions to calculate implied parameters. These functions are then used in procedure 3.4.1, shown in Section C.3. The last three sections cover pricing formulas for index-linked bonds, mortgages and a house price insurance.

C.1 Option price formulas

This function will give the Black-Scholes price for a call option, corresponding with equation (3.15). Note that we use the error function (erf) to calculate the cumulative normal distribution function $N(\cdot)$. Both functions are related as $N(x) = \frac{1}{2}(1 + \text{erf}(\frac{x}{\sqrt{2}}))$.

Matlab Function C.1.1 (Black-Scholes option price formula)

```matlab
% Matlab code for Black-Scholes option pricing (Call).
% INPUT:
% S0: stock price at time t
% t: current time
% K: strike price
% r: riskfree rate
% sigma: standard deviation of Brownian
% T: time until maturity (years)
function C = bsf(S0,t,K,r,sigma,T)

tau = T-t;
if tau > 0
    d1 = (log(S0/K) + (r + 0.5*sigma^2)*tau)/(sigma*sqrt(tau));
    d2 = d1 - sigma*sqrt(tau);
    Nd1 = 0.5*(1 + erf(d1/sqrt(2)));
    Nd2 = 0.5*(1 + erf(d2/sqrt(2)));
    C = S0*Nd1 - K*exp(-r*tau)*Nd2;
else
    C = max(S0-K,0);
end
```
The following function will give Merton’s option price for a call option, assuming the jump sizes to be deterministic. It corresponds to equation (3.41).

**Matlab Function C.1.2 (Merton’s option price formula (deterministic jumps))**

```matlab
function C = mertondet(S0, t, K, r, sigma, T, b, lambda)
    tau = T - t;
    c = zeros(1, 21);
    p = zeros(1, 21);
    hc = zeros(1, 21);
    for n = 0:20
        c(n + 1) = bsf(S0 * (b^n) * exp(-lambda * (b - 1) * tau), t, K, r, sigma, T);
        p(n + 1) = exp(-lambda * tau) * (lambda * tau)^n / prod(1:n);
        hc(n + 1) = c(n + 1) * p(n + 1);
    end
    C = sum(hc);
end
```

This function gives the option price according to Merton’s model with jump sizes lognormally distributed as we have seen in equation (3.43).

**Matlab Function C.1.3 (Merton’s option price formula (lognormal jumps))**

```matlab
function C = merton(S0, t, K, r, sigma, T, k, delta, l)
end
```

% Matlab code for option pricing (Call)
% using Merton's Jump Diffusion model with lognormal jump sizes.
% INPUT:
% S0: current stock price
% t: current time
% K: Strike price
% r: riskfree rate
% sigma: volatility brownian
% T: time of maturity
% k: Expected jump size (= b−1)
% ∆: standard deviation log(Y)
% l: Poisson rate lambda

This function gives the option price according to Merton’s model with jump sizes lognormally distributed as we have seen in equation (3.43).
C.2. PARAMETER ESTIMATES

```
hc = zeros(1,21);
for ind=0:20
    c = bsf(S0,t,K,((r-l*k+ ind*log(1+k)/T),(sqrt(sigma^2+ind*Delta^2/T)),T));
    hc(ind+1) = c*exp(-l*(1+k)*T)*(l*(1+k)*T)^ind/prod(1:ind);
end
C = sum(hc);
```

C.2 Parameter estimates

In this section we will give functions to estimate parameters and calculate implied variables from equation (3.53) as described in Procedure 3.4.1. We will assume that we have three data sets as in Figure 3.17, being the total price process, the Brownian motion component and the jump component.

Matlab Function C.2.1 (Parameter estimates for real data)

```
function sBS = getSigmaBS(Dt,P)
    N = length(P);
    ret = [];
    for i=2:N
        ret = [ret; log(P(i)/P(i-1))];
    end
    sBS = sqrt(var(ret)/Dt);
end
```

```
function sM = getSigmaM(Dt,M)
    N = length(M);
    ret = [];
    for i=2:N
        ret = [ret; log(M(i)/M(i-1))];
    end
    sM = sqrt(var(ret)/Dt);
end
```
To determine the implied parameters based on the estimated parameters, we use the result given in (3.45). We first give a function to calculate the parameters of the lognormal distribution as a function of $b$ and $\gamma$. For $\gamma$ this is shown in (3.50).

Matlab Function C.2.2 (Calculate parameters of lognormal distribution as a function of $b$ and $\gamma$)

```matlab
function [b,v,lambda,delta] = getLogparameters(b,delta)
    q = exp(delta^2)*(exp(delta^2)-1);
    gamma = log(b/sqrt((sqrt(1+4*q)+1)/2));
    v = b^2*(sqrt(1+4*q)-1)/2;
end
```

We can now give the functions for the implied parameters.
Matlab Function C.2.3 (Calculate implied parameters)

% Matlab code to estimate sigma_Bs
% INPUT:
% sM: estimated sigma_M
% lambda: estimated lambda
% b: estimated b
% Δ: estimated Δ (usually zero)

function sBShat = getImpliedSigmaBS(sM,lambda,b,Δ)
    [b,v,gamma,Δ] = getLogparameters(b,Δ);
    sBShat = sqrt(sM^2 + lambda*(gamma^2+Δ^2));
end

% Matlab code to estimate sigma_M
% INPUT:
% sBS: estimated sigma_Bs
% lambda: estimated lambda
% b: estimated b
% Δ: estimated Δ (usually zero)

function sMhat = getImpliedSigmaM(sBS,lambda,b,Δ)
    [b,v,gamma,Δ] = getLogparameters(b,Δ);
    sMhat = sqrt(sBS^2 - lambda*(gamma^2+Δ^2));
end

% Matlab code to estimate lambda
% INPUT:
% sBS: estimated sigma_Bs
% sM: estimated sigma_M
% b: estimated b
% Δ: estimated Δ (usually zero)

function lambdahat = getImpliedLambda(sBS,sM,b,Δ)
    if(Δ ≠ 0)
        lambdahat = 'Δ must equal zero';
    end
    if(Δ == 0)
        gamma = log(b);
        lambdahat = (sBS^2 - sM^2)/gamma^2;
    end
end

% Matlab code to estimate lambda
% INPUT:
% sBS: estimated sigma_Bs
% sM: estimated sigma_M
% lambda: estimated lambda
% Δ: estimated Δ (usually zero)

function bhat = getImpliedB(sBS,sM,lambda,Δ)
    if(Δ ≠ 0)
        bhat = 'Δ must equal zero';
    end
    if(Δ == 0)
C.3 Procedure 3.4.1

We will use one function to do all possible steps in Procedure 3.4.1. This means that we will have a lot of in- and output variables. We assume that δ equals zero and that we have the components of the price process.

Matlab Function C.3.1 (Perform procedure 3.4.1)

```matlab
function [sigmaBS, sigmaM, lambda, b, PBS, PM] = procedure(S0, t, K, r, T, D, P, M, tj, J, I)

if I == 1
    sigmaM = getSigmaM(D, M);
    lambda = getLambda(tj);
    b = getB(J);
    sigmaBS = getImpliedSigmaBS(sigmaM, lambda, b, 0);
end

if I == 2
    sigmaBS = getSigmaBS(D, P);
    lambda = getLambda(tj);
    b = getB(J);
    sigmaM = getImpliedSigmaM(sBS, lambda, b, 0);
end

if I == 3
```

```matlab
    sigmaM = getSigmaM(D, M);
    lambda = getLambda(tj);
    b = getB(J);
    sigmaBS = getImpliedSigmaBS(sigmaM, lambda, b, 0);
end
end
```
C.4 Pricing bonds

The following function can be used to price bonds of the type described in Section 4.2.1. This means that we have two arrays of lower and upper bounds for coupon returns and that the principal is always guaranteed at maturity. This function gives the bond price based on Merton’s model for option pricing, so the same process variables are needed as for the option price function. These variables can of course follow from Matlab Function C.3.1 Now if one needs the Black-Scholes price, one can put $\sigma_M = \sigma_{BS}$ and either $\lambda = 0$, or $b = 1$ and $\delta = 0$. Of course, for expectation pricing we can use the Black-Scholes price and use $r + p$ instead of $r$.

Matlab Function C.4.1 (Determine bond price)

```matlab
function Price = getBondprice(P,lb,ub,int,r,sigmaM,lambda,b,D)

k = b-1;
[b,v, gamma, D] = getLogparameters(b,D);

Lib = length(lb);
Lub = length(ub);

if Lib ~= Lub
    Price = 'choose vectors of same length';

end
```

```matlab
db = getSigmaBS(P,D);
dma = getSigmaM(D,M);
b = getB(J);
lambda = getImpliedLambda(sBS,sM,b,0);
end
if I == 4
db = getSigmaBS(D,P);
dma = getSigmaM(D,M);
lambda = getLambda(tj);
getImpliedB(sBS,sM,lambda,0);
end
PBS = bsf(S0,t,K,r,sigmaBS,T);
PM = merton(S0,t,K,r,sigmaM,T,b,0,lambda);
end
```

```matlab
sigmaBS = getSigmaBS(D,P);
sigmaM = getSigmaM(D,M);
b = getB(J);
lambda = getImpliedLambda(sBS,sM,b,0);
end
```

C.4 Pricing bonds

The following function can be used to price bonds of the type described in Section 4.2.1. This means that we have two arrays of lower and upper bounds for coupon returns and that the principal is always guaranteed at maturity. This function gives the bond price based on Merton’s model for option pricing, so the same process variables are needed as for the option price function. These variables can of course follow from Matlab Function C.3.1 Now if one needs the Black-Scholes price, one can put $\sigma_M = \sigma_{BS}$ and either $\lambda = 0$, or $b = 1$ and $\delta = 0$. Of course, for expectation pricing we can use the Black-Scholes price and use $r + p$ instead of $r$.

Matlab Function C.4.1 (Determine bond price)

```matlab
function Price = getBondprice(P,lb,ub,int,r,sigmaM,lambda,b,D)

k = b-1;
[b,v, gamma, D] = getLogparameters(b,D);

Lib = length(lb);
Lub = length(ub);

if Lib ~= Lub
    Price = 'choose vectors of same length';

end
```

```matlab
db = getSigmaBS(P,D);
dma = getSigmaM(D,M);
b = getB(J);
lambda = getImpliedLambda(sBS,sM,b,0);
end
if I == 4
db = getSigmaBS(D,P);
dma = getSigmaM(D,M);
lambda = getLambda(tj);
getImpliedB(sBS,sM,lambda,0);
end
PBS = bsf(S0,t,K,r,sigmaBS,T);
PM = merton(S0,t,K,r,sigmaM,T,b,0,lambda);
end
```

C.4. PRICING BONDS

```matlab
sigmaBS = getSigmaBS(D,P);
sigmaM = getSigmaM(D,M);
b = getB(J);
lambda = getImpliedLambda(sBS,sM,b,0);
end
if I == 4
    sigmaBS = getSigmaBS(D,P);
sigmaM = getSigmaM(D,M);
lambda = getLambda(tj);
getImpliedB(sBS,sM,lambda,0);
end
PBS = bsf(S0,t,K,r,sigmaBS,T);
PM = merton(S0,t,K,r,sigmaM,T,b-1,0,lambda);
```
end

if Llb == Lub
    L = Llb;
    Long = zeros(1,L);
    Short = zeros(1,L);

    MaturityBond = L*int;
    PPrincipal = P * exp(-r*MaturityBond);

    for i = 1:L
        MaturityOption = int;
        KLong = (1+(lb(i)/100)) * P;
        KShort = (1+(ub(i)/100)) * P;

        Long(i) = exp(-r*(i-1))*merton(P,0,KLong,r,sigmaM,MaturityOption,k,\Delta,lambda);
        Short(i) = exp(-r*(i-1))*merton(P,0,KShort,r,sigmaM,MaturityOption,k,\Delta,lambda);
    end

    PLong = sum(Long);
    PShort = sum(Short);

    PAll = PPrincipal + PLong - PShort;
    Price = PAll;
end

C.5 Pricing index-linked mortgages

For index-linked mortgages, we need to use put prices. The following function gives the put price when the call price is known, corresponding to the put-call parity as shown in (3.19).

Matlab Function C.5.1 (Determine put price from call price)

```matlab
% C: call price
% S0: initial stock price
% K: strike price
% r: riskfree rate
% T: maturity

function P = getPut(C,S0,K,r,T)
    P = C+K*exp(-r*T)-S0;
end
```

As we can now determine the put prices, the following function can be used to calculate additional mortgage rates. Similar to as described in Section C.4 it is possible to use this function for Black-Scholes prices and expectation pricing.

Matlab Function C.5.2 (Determine additional interest rate for put option)

```matlab
% Q: fraction of the house value to be insured (1-house owners risk)
% T: maturity (years)
```
% r: riskfree rate
% sigma: volatility index
% b: average jump size (multiplicative)
% ∆: uncertainty jumpsize
% lamda: jump rate

function x = getPutcosts(Q,T,r,sigma,b,Δ,lambda)
S0 = 100;
K = Q*S0;
c = merton(S0,0,K,r,sigma,T,b−1,Δ,lambda);
p = getPut(c,S0,K,r,T);
som =0;
for j=0:(T−1)
    som = som + exp(−j*r);
end
x = p/som;
end

C.6 Pricing house price insurances

The following function gives annual insurance premia for a house price insurance. Again, we can also use this function for Black-Scholes option pricing by setting b = 1 and/or λ = 0. The expectation price is then the Black-Scholes price with substitution of r by r + p.

Matlab Function C.6.1 (Determine insurance premium)
% Q: fraction of the house value to be insured (1−house owners risk)
% alpha: fraction of cancelations pa
% beta: fraction of insurance holders eligible for claim pa
% r: riskfree rate
% sigma: volatility index
% b: average jump size (multiplicative)
% ∆: uncertainty jumpsize
% lamda: jump rate

function y = getInsuranceP(Q,alpha,beta,r,sigma,b,Δ,lambda)
S0 = 100;
K = Q*S0;
t = 1;

c = merton(S0,0,K,r,sigma,t,b−1,Δ,lambda);
p = getPut(c,S0,K,r,t);
z = beta*(1-alpha)*p;
Z=z;
Som = z;
while z>2/1000
    t = t+1;
    c = merton(S0,0,K,r,sigma,t,b−1,Δ,lambda);
p = getPut(c,S0,K,r,t);
z = beta*(1-alpha)^t)*p;
    Som = Som+z;
end
y = Som*(1-(1-alpha)*exp(−r));
end
Bibliography


