Generating all possible permutations with a minimal fixed restriction of any multiset by adjacent interchanges

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Bachelor project

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This paper is dedicated to my advisors.

Abstract. In this paper a conjecture is given on how to generate all permutations of a multiset where two subsequent permutations only differ by an interchange of two adjacent objects. This is not always possible as shown in [17] by C.W. KO and F. Ruskey. With a fixed minimal restriction on the permutations it may be possible to generate all permutations of the multiset. By removing the stutter permutations from the set of all permutations we may achieve this minimal restriction. This is also proven to be the minimal amount of permutations that have to be removed to be able to generate the permutations by adjacent interchanges. This paper partly proves and therefore strengthens the conjecture given by D. H. Lehmer [8].

1. Introduction

In D. H. Lehmer’s paper [8] he states:

Many problems of optimization involve the examination of permutations of distinct or possible nondistinctive objects. Often the number of these permutations becomes too great to deal with effectively even at high speeds. In other cases the number is modest enough to contemplate an exhaustive examination. In such cases one can consider optimizing the procedures used.

Although there are a lot of articles about combinatorial generation, not many of them discuss the use of a minimal change algorithm. In ‘Loopless generation of multiset permutations by prefix shifts’[19] A. Williams gives some reasons why the area of combinatorial generation has its importance.

The area of combinatorial generation research is so important to computer science that Knuth has dedicated over 400 pages to the subject in his volume 4 A of The Art of Computer Programming [13, 14]. The research area is applicable whenever it is necessary to efficiently consider every possible object of a particular type, such as binary strings of length n, permutations of \{1, 2, ..., n\}, binary trees with n nodes, linear extensions of a partially ordered set, spanning trees of a directed graph, or perfect elimination orders of a chordal graph.

The most useful results in combinatorial generation tend to have a mathematical aspect and an algorithmic aspect. For example, two of the most well-known results in combinatorial generation are the binary reflected Gray code [16] and the de Bruijn cycle [15]. Both results provide a clever order for the binary strings of length n. The binary reflected Gray code provides an order in which each successive string can be obtained from the previous by changing the value of a single bit, while de Bruijn cycles provide an order in which each successive string can be obtained from the previous by removing the rightmost bit and inserting a new leftmost bit.

In general, the mathematical aspect of combinatorial generation involves the discovery of a minimal-change order. A minimal-change order is an order in which each successive object can be obtained from the previous by making one small modification of a certain type. The existence or non-existence of minimal-change orders depend upon the type of object and the type of modification. New results in this area are often quite difficult to find, but the results that are found tend to be elegant and simple.

Key words and phrases. Stutter permutations, Discrete mathematics, Combinatorica, Combinatorial generation.
In this paper the modification is used that D. H. Lehmer provided in his paper about minimal-change order [8]. If we can generate all possible permutations of a multiset where two subsequent permutations only differ by an interchange of two adjacent objects, then the calculations made on two subsequent permutations may be much the same since they only differ by an interchange of two adjacent objects, therefore possibly reduce the number of calculations made in total.

1.1. Applications [19]. Efficient algorithms for generating multiset permutations have a number of applications. If the multiset is simply a set, then applications include communication in point-to-point multiprocessor networks [9]. If the multisets corresponding set contains only two elements, then applications include cryptography (where orders have been implemented in hardware at NSA), genetic algorithms, software and hardware testing, statistical computation (e.g., for the bootstrap, and Diaconis and Holmes [10]).

Minimal-change orders also tend to have diverse applications. For example, the binary reflected Gray code was designed at Bell Labs for telephone systems, but has since found applications in information and communication technology, analog-to-digital conversion, error correction, and decreased power consumption in hand-held devices. It has also been used in the CODACON spectrometer, and appears in research titles ranging from measurement and instrumentation [11] to quantum chemistry [12].

There are also playful ways to use these generating methods. In the article 'Combinatorial Choreography' [20] T. Verhoeff describes a situation where the permutations of the multiset are used for positioning dancers on a line. The fixed restriction is also a restriction on the choreography. This way the problems are mathematically the same, but describing the problem gets more intuitive.

2. Notation and Conventions

The first question related to making an algorithm for generating multiset permutations is knowing if it is possible with the used modification. The goal is to generate, without duplicates, all possible permutations of a multiset where two subsequent permutations only differ by an interchange of two adjacent objects. For this paper we use the following notation.

Let \( \vec{n} = (n_0, n_1, \ldots, n_t) \) denote a multiset with \( n_i \)'s. Throughout the paper we will assume that \( n = n_0 + n_1 + \ldots + n_t \). Let \( S(\vec{n}) \) be the set of all permutations of the multiset \( \vec{n} \). A neighbor swap is when you swap two adjacent numbers (neighbors) in a permutation. Interchange digits are digits \( k \in \{1, 2, \ldots, n-1\} \) that indicate a swap of the elements \( x_k \) and \( x_{k+1} \) of a permutation \( x \in S(\vec{n}) \). Note that under a neighbor swap the new permutation is still an element of \( S(\vec{n}) \).

\( I(S) \) denotes a neighbor-swap graph over the elements of the set \( S \). This is a graph where the nodes of the graph are the elements of \( S \) and two nodes are connected if they differ by a single neighbor swap. For the notation of a neighbor-swap graph on the set of all permutations of a multiset we use \( I(\vec{n}) \).

The existence of an algorithm to create all the permutations of \( S(\vec{n}) \) without duplicates with only the use of neighbor swaps is the same as finding a Hamilton path in the neighbor-swap graph.

In the subject of adjacent interchange algorithms we can categorize the previous results into three different subjects. Permutations, combinations and multiset permutations. In that order I will summarize what is known.

3. Permutations

Permutations is the subset of multiset permutations, in which case each number can be in the multiset only once. In other words, for \( \vec{n} : n_i \in \{0, 1\} \ \forall i \in \mathbb{N}^+ \)

3.1. Steinhaus-Johnson-Trotter algorithm. The Steinhaus-Johnson-Trotter algorithm [7], referred to as the SJT algorithm, is an algorithm that generates all of the permutations of \( n \) elements. Each permutation in the sequence that it generates differs from the previous permutation by a neighbor swap. This algorithm finds a Hamiltonian path in a neighbor-swap graph of all permutations of \( n \) different elements.
It is a simple algorithm and computationally efficient. It has the advantage that subsequent computations on the permutations may be sped up because subsequent permutations are very similar to each other.

3.2. Recursive Structure. The sequence of permutations for a given number \( n \) can be formed from the sequence of permutations for \( n - 1 \) by placing the number \( n \) into each of the possible \( n \) positions in each of the shorter permutations. When the permutation on \( n - 1 \) items is an even permutation then the number \( n \) is placed in all possible positions in descending order, from \( n \) down to 1. When the permutation on \( n - 1 \) items is odd, the number \( n \) is placed in all the possible positions in ascending order.

In this way, each permutation differs from the previous one either by the single-position-at-a-time motion of \( n \), or by a change of two smaller numbers inherited from the previous sequence of shorter permutations. In either case, this difference is just the transposition of two adjacent elements. When \( n > 1 \) the first and final elements of the sequence, also differ in only two adjacent elements (the positions of the numbers 1 and 2), as may easily be shown by induction. Therefore this is a Hamilton Cycle on all permutations.

4. Combinations

The subset with only combinations of the multisets can be defined by all the multisets that are constructed by only using 0’s and 1’s. In other words, all the multisets \( \vec{n} \) where \( n_i = 0 \), for all \( i > 2 \).

It is straightforward to develop an algorithm for generating combinations in lexicographic order. Bitner, Ehrlich, and Reingold [1] developed an algorithm for generating combinations where each successive combination differs from its predecessor by the interchange of a 0 and a 1 somewhere in the bit string. It was independently discovered by Buck and Wiedemann [2], Eades, Hickey, and Read [3], and Ruskey and Miller [5] that if each successive combination differs by the interchange of an adjacent 01 or 10 pair, that the generation is possible if and only if \( n_0 \) and \( n_1 \) are odd (except for the trivial cases \( n_1 = 0, 1, n - 1, n \)).

**Theorem 4.1.** The neighbor-swap graph \( I(\vec{n}) \) for \( n = (n_0, n_1) \) has a Hamilton path if and only if (i) \( n_0 = 0, 1, n, n - 1, \) or (ii) both \( n_0 \) and \( n_0 \) are odd.

4.1. Combinations unable to have a Hamilton path. To prove it is not always possible to generate all the different combinations through adjacent interchange, Buck and Wiedemann [2], Eades, Hickey, and Read [3], and Ruskey and Miller [5] all used the bipartite property of the graph.

**Lemma 4.2.** The adjacent interchange graph is bipartite

**Proof.** We argue that the graph has no odd cycles. Consider a vertex \( v \) on a cycle in the graph. As the cycle is traversed we eventually return to \( v \) via a sequence of interchanges. For each interchange there must be a corresponding reversed interchange, and so there are an even number of vertices on the cycle.

In other words we can classify permutations as odd and even permutations. The canonical sequence, \( \text{Can}(\vec{n}) \), is defined to be \( 0^{n_0}1^{n_1}...l^{n_t} \). Let \( D(\vec{n}) \) denote the parity difference, which is the difference between the number of sequences that are an even number of adjacent interchanges from \( \text{Can}(\vec{n}) \) (even permutations), and those who are an odd number of adjacent interchanges from \( \text{Can}(\vec{n}) \) (odd permutations). The canonical sequence itself is therefore an even permutation.

For a Hamilton path to exist, \( D(\vec{n}) \) has to be less than or equal to 1. Using induction, it was shown in [2, 3, 5] that the difference in the number of vertices in the two bipartitions is zero if both \( n_0 \) and \( n_1 \) are odd and otherwise

\[
\begin{pmatrix}
\left\lfloor \frac{n}{2} \right\rfloor \\
\left\lfloor \frac{k}{2} \right\rfloor
\end{pmatrix}
\]

(4.1)
Expression 4.1 is greater than one if \( k \neq 0,1,n-1,n \). Therefore the neighbor-swap graph \( I(\vec{n}) \) for \( n = (n_0, n_1) \) has no Hamilton path if \( (i) n_0 \neq 0,1,n-1,n \), or \( (ii) n_0 \) or \( n_1 \) is even.

\[ 4.2. \text{Proof for existing Hamilton path.} \] A Hamilton cycle does not exist, because \( \text{Can}(\vec{n}) \) has only one neighbor. The non-trivial part of theorem 4.1 is proven in [2, 3] through an existential approach. The theorem is proven using induction. \( I(\vec{n}) \) is split into four parts depending on what the last two elements of the permutation are. Each can be 0 or 1. These subgraphs are isomorphic to smaller graphs \( I(\vec{m}) \) with \( m = n-2 \). Known paths and cycles are then surgically glued to create a path for the neighbor-swap graph on \( S(\vec{n}) \).

Where Eades, Hickey and Read [3], and Buck and Wiedemann [2] only prove Theorem 4.1 through an existential approach, Ruskey [7] found a concrete adjacent interchange algorithm. His proof is a huge construction of the Hamilton path that can be found on the graph \( I(\vec{n}) \) with \( n = (n_0, n_1) \). But because this is a constructive method, the proof itself can be used to generate all possible combinations with only adjacent interchanges.

5. Multiset Permutations

This the final and most complex part which considers all possible \( \vec{n} \).

\[ 5.1. \text{Multiset permutations unable to have a Hamilton path.} \] In lemma 4.2 is already proven that a neighbor-swap graph is bipartite. Therefore, a Hamilton path only exists if the parity difference is 0 or 1.

It will be useful to have a compact notation for multinomial coefficients.

\[
(n; \vec{n}) = (n; n_o, n_1, ..., n_t) = \binom{n}{n_o, n_1, ..., n_t} = \frac{n!}{n_o! n_1! ... n_t!}
\]

The following theorem is proven by C.W. Ko and F. Ruskey [17]. Let \( \text{Odd}(\vec{n}) \) denote the number of \( n_i \) that are odd, and let \( m = \lfloor \frac{1}{2}n \rfloor \).

THEOREM 5.1. \( D(n) \) is as follows.

\[ D(n) = \begin{cases} 
(m; \vec{m}) & \text{if } \text{Odd}(\vec{n}) \leq 1 \\
0 & \text{otherwise.}
\end{cases} \]

This theorem proves that there exists no Hamilton paths if \( \text{Odd}(\vec{n}) \leq 1 \). But also gives rise to the following conjecture.

CONJECTURE 5.1. A Hamilton path exists in the adjacent interchange graph \( I(\vec{n}) \) if \( \text{Odd}(\vec{n}) \leq 1 \).

\[ 5.2. \text{Proof for existence of a Hamilton path.} \] In 1992 G. Stachowiak [18] proved the conjecture 5.1 given in [17]. Instead of operating on permutations and multisets, he converted the idea to linear extensions for unions of posets. His proof also cuts a neighbor-swap graph into smaller graphs and then combines the Hamilton cycles found on the smaller graphs to a Hamilton cycle on the bigger graph.

An extra finding in his research is that if \( \{n_0, n_1, n_2\} = \{1,1,\text{even}\} \) there is only a Hamilton path and no Hamilton cycle. But in all other cases, where \( \text{Odd}(\vec{n}) \leq 1 \), there is a Hamilton cycle over the neighbor-swap graph.

This proof is still an existential proof and not a constructive proof. This means no program or method to generate all possible multiset permutations is given explicitly. The only thing proven here is that it is possible to generate all possible multiset permutations.
6. Conjecture of Lehmer

The conjecture given by D. H. Lehmer [8] is a bit different from the theorems proven above. D. H. Lehmer not only states that it may be possible to generate all permutations of a multiset by interchanging digits for the multisets with no parity difference. But he also declares that it may be possible to generate all permutations of a multiset with parity difference with a minimal amount of spurs.

**Definition 6.1.** A spur is a neighborswap you directly reverse to the original.

Visually a spur is a line segment of length 1 that connects one isolated node to a Hamilton path or cycle on the rest of the nodes.

By saying all permutations can be generated with a minimal amount of spurs, Lehmer suggests a method with which we may be able to generate all permutations of a multiset with minimal change.

**Conjecture 6.1.** (Lehmer 1965) A Hamilton path with $D(I(\vec{n})) - 1$ spurs exists in the adjacent interchange graph $I(\vec{n})$

Hence, if we remove the nodes that are visited via a spur, the newly found graph will have a Hamilton path.

It is easy to see that we need at least $D(I(\vec{n})) - 1$ spurs, since this follows from the bipartite nature of the neighbor-swap graph. At least $D(I(\vec{n})) - 1$ even permutations have to be removed from $S(\vec{n})$ in order for the remaining graph to have a Hamilton path. For the rest of the article we will strengthen this conjecture 6.1 to this conjecture.

**Conjecture 6.2.** A Hamilton cycle with $D(I(\vec{n}))$ spurs exists in the adjacent interchange graph $I(\vec{n})$

(It is trivial that from conjecture 6.2 follows conjecture 6.1.)

7. Stutter Permutations

For the remainder of this article we define a new term. This term is introduced by T. Verhoeff in Personal Communication, Oct. 2012. In this section I will give some of his results.

**Definition 7.1.** A stutter permutation is a permutation with $x_i = x_{i+1}$ for all $i$ odd.

A stutter permutation has several properties. For once, it is an even permutation. From the canonical sequence $Can(\vec{n})$ an even number of neighbor swaps is used to get to a stutter permutation.

With $m = \lfloor \frac{1}{2} n \rfloor$, a multiset has $(m; \vec{m})$ stutter permutations if $Odd(\vec{n}) \leq 1$.

We can divide a permutation into pairs $x_i, x_{i+1}$ for all $i$ odd. This way the first two elements of the permutation are paired up, the second two are paired up and so on. $|x_1x_2|x_3x_4|...$

If $x_i = x_{i+1}$ for $i$ is odd we call it a stutter pair. Interchanging the first non-stutter pair will get you a new permutation with a different parity. These permutations can be paired. This is the case for every non-stutter permutation. A stutter permutation does not have a non-stutter pair, therefore these are the only permutations that can not be paired in this way with a permutation of a different parity.

Every permutation, except for the stutter permutations, has a counterpart with a different parity. Therefore the parity difference of the whole graph $D(I(\vec{n}))$ is equal to the number of stutter permutations. This proves theorem 5.1 in a more intuitive and elegant way. This proof is also by T. Verhoeff in personal communication.

Let $NS(\vec{n})$ be the set of all non-stutter permutations of $\vec{n}$. Because all the properties of the stutter permutations we put forth a new conjecture.

**Conjecture 7.1.** A Hamilton path exists on the adjacent interchange graph $I(NS(\vec{n}))$
Without stutter permutations every permutation in the neighbor-swap graph can be paired with a counterpart with a different parity. Maybe the ones without a counterpart were the ones creating all the problems in the first place.

Some small multiset cases were solved by T. Verhoeff via a Hamilton path finder. (see table 1 for the results) These cases only contradict conjecture 7.1 in some special cases. The exceptions are the multisets \( M(n_0, n_1) \) with \( n_0 \) or \( n_1 \) odd and \( M(1, 1, n_2) \) with \( n_2 \) even. In the other cases it didn’t contradict the conjecture and therefore makes the conjecture more plausible on all other cases. Proving the theorem by calculation via computer is not done because of the computational complexity \( \text{NP} \) for finding Hamilton paths. But if we prove the theorem via a constructive method to make the Hamilton path on a \( I(\vec{u}) \), then we can also create an algorithm to generate all possible permutations via neighbor swaps.
8. Binary cases $n = (n_0, n_1)$

It is already known that for $n_0$ and $n_1$ odd there is a Hamilton path on all permutations. So just the prove remains for the existence of a Hamilton path or cycle on $I(\vec{n})$ when $n_0$ or $n_1$ is even. The prove is by T. Verhoeff in personal communication, Oct. 2012. We will split into two cases. One case where both $n_0$ and $n_1$ are even and a second where only one $n_0$ or $n_1$ is even. For proving the second case we consider the case where $n_0$ is odd and $n_1$ is even and prove there is a Hamilton path on $I(\vec{n})$.

**Theorem 8.1.** When $\vec{n}$ is a vector to give multisets $M(\vec{n})$ with $n_i = 0$ for all $i > 2$, then the following is true.

1. For both $n_0$ and $n_1$ odd, there is a Hamilton path on $I(\vec{n})$.
2. For only $n_0$ is odd, there is a Hamilton path on $I(\vec{n})$.
3. For both $n_0$ and $n_1$ even, there is a Hamilton cycle on $I(\vec{n})$.

Case (1) is already proven before. Note that in the Hamilton path for case (1) the Hamilton path contains the following edges:
- $0^{n_0}1^{n_1} \leftrightarrow 0^{n_0-1}101^{n_1-1}$
- $1^{n_0}0^{n_1} \leftrightarrow 1^{n_0-1}010^{n_1-1}$

We will prove case (3) of the theorem also inductively, the proof will be supported by visualization through the example $\vec{n} = (4, 2)$.

**Proof.** For the proof we will divide the graph $I(\vec{n})$ into subgraphs depending on the last two elements of every vertex. For case (2) the set of all permutations $S(\vec{n})$ is broken down into four separate groups

$$
A(n_0, n_1) = \{(x_1, x_2, ..., x_n) \in S(\vec{n}) : x_n = x_{n-1} = 1\}
$$
$$
B(n_0, n_1) = \{(x_1, x_2, ..., x_n) \in S(\vec{n}) : x_n = x_{n-1} = 0\}
$$
$$
C(n_0, n_1) = \{(x_1, x_2, ..., x_n) \in S(\vec{n}) : x_n = 1, x_{n-1} = 0\}
$$
$$
D(n_0, n_1) = \{(x_1, x_2, ..., x_n) \in S(\vec{n}) : x_n = 0, x_{n-1} = 1\}
$$

(8.1)

![Figure 1](image.png)

**Figure 1.** Different subgraphs on $I(4, 2)$

Then $I(A(\vec{n}))$, $I(B(\vec{n}))$, $I(C(\vec{n}))$ and $I(D(\vec{n}))$ are subgraphs of $I(\vec{n})$ which contain all the vertices. The following isomorphic relations hold.

- $I(A(n_0, n_1))$ is isomorphic to $I(M(n_0, n_1 - 2))$
- $I(B(n_0, n_1))$ is isomorphic to $I(M(n_0 - 2, n_1))$
- $I(C(n_0, n_1))$ is isomorphic to $I(M(n_0 - 1, n_1 - 1))$
- $I(D(n_0, n_1))$ is isomorphic to $I(M(n_0 - 1, n_1 - 1))$

Both $C(\vec{n})$ and $D(\vec{n})$ are isomorphic to graphs that have a Hamilton path. By interchanging the last two elements of a permutation we can switch from $I(C(\vec{n}))$ to $I(D(\vec{n}))$ and back. This way we can combine the Hamilton paths on the two segments into a Hamilton cycle on both.
All the stutter permutations of $S(\vec{n})$ are contained in $A(\vec{n})$ and $B(\vec{n})$. The stutter permutations in $M(n_0, n_1 - 2)$ are the same as in $A(n_0, n_1)$ and the stutter permutations in $M(n_0 - 2, n_1)$ are the same as in $B(n_0, n_1)$. By induction we can now say that $A(\vec{n})$ and $B(\vec{n})$ have a cycle on their non-stutter permutations.

To conclude that there is a cycle on $I(NS(\vec{n}))$ we need to combine the three cycles found. To do so we strengthen the theorem. Without loss of generality we assume $n_0$ and $n_1$ are greater than 2.

**Lemma 8.2.** When $\vec{n}$ is a vector to give multisets $M(\vec{n}) : n_i = 0$ for all $i > 2$, the following is true. For both $n_0$ and $n_1$ even, there is a Hamilton cycle on $I(NS(\vec{n}))$ that includes the following edges. $0^{n_0-1}1^{n_1}0 \leftrightarrow 0^{n_0-2}101^{n_1-1}0$ and $1^{n_1-1}0^{n_0-1}1 \leftrightarrow 1^{n_1-2}0^{n_0-1}0$.

This theorem is true for $n_0 = n_1 = 2$, hence the cycle $0110 \leftrightarrow 1010 \leftrightarrow 1001 \leftrightarrow 0101 \leftrightarrow 0110$. 

**Figure 5.** Cycle on non stutter permutations in $I(4,2)$
Via induction we know that:

- \( I(NS(A(n_0, n_1))) \) has a Hamilton cycle with edges:
  - (A1) \( 0^n_0 - 1^{n_1} 1^n_0 - 1^{n_1} 11 \leftrightarrow 0^n_0 - 2^{n_1} 1^n_0 - 3^{n_1} 10 11 \)
  - (A2) \( 1^{n_1} - 3^{n_0} 1^n_1 - 1^{n_0} 11 \leftrightarrow 1^{n_1} - 4^{n_0} 1^n_1 - 1^{n_0} 11 \)

- \( I(NS(B(n_0, n_1))) \) has a Hamilton cycle with edges:
  - (B1) \( 0^{n_0} - 2^{n_1} 1^n_0 - 0^{n_0} 00 \leftrightarrow 0^n_0 - 4^{n_1} 1^n_0 - 1^{n_0} 00 \)
  - (B2) \( 1^{n_1} - 0^{n_0} - 2^{n_1} 1^n_0 - 2^{n_0} 00 \leftrightarrow 1^{n_1} - 1^{n_0} - 1^{n_0} 00 \)

- \( I(C(n_0, n_1)) \) has a Hamilton path with edges:
  - (C1) \( 0^{n_0} - 1^{n_1} 1^n_0 - 1^{n_1} 01 \leftrightarrow 0^n_0 - 2^{n_1} 1^n_0 - 1^{n_1} 01 \)
  - (C2) \( 1^{n_1} - 0^{n_0} - 1^{n_1} 01 \leftrightarrow 1^{n_1} - 2^{n_0} 01^{n_0} - 2^{n_0} 01 \)

- \( I(D(n_0, n_1)) \) has a Hamilton path with edges:
  - (D1) \( 0^{n_0} - 1^{n_1} 1^n_0 - 1^{n_1} 10 \leftrightarrow 0^n_0 - 2^{n_1} 1^n_0 - 2^{n_1} 10 \)
  - (D2) \( 1^{n_1} - 0^{n_0} - 1^{n_1} 10 \leftrightarrow 1^{n_1} - 2^{n_0} 01^{n_0} - 2^{n_0} 10 \)

We can connect the path from \( I(C(\vec{n})) \) with the path from \( I(D(\vec{n})) \) by adding the edges \( 0^n_0 - 1^{n_1} 1^n_0 - 1^{n_1} 10 \leftrightarrow 0^n_0 - 1^{n_1} 1^n_0 - 1^{n_1} 01 \) and \( 1^{n_1} - 0^{n_0} - 1^{n_1} 10 \leftrightarrow 1^{n_1} - 0^{n_0} - 1^{n_1} 01 \). Now the two paths form a Hamilton cycle on the combined neighbor-swap graph. See figure 4 for these paths on \( I(4, 2) \). The edges (A1) and (C1) are parallel, therefore we can combine the two cycles into one cycle that covers the vertices of both graphs. Then we can combine the edges (D2) and (B2) are parallel, therefore we can combine the cycle in \( I(NS(B(n_0, n_1))) \) with the other cycle. The newly constructed cycle through all \( NS(\vec{n}) \) has the desired edges for the induction. These are the edges (D1) and (C2).

**Figure 6. Stutter on \( I(4, 2) \)**

**Figure 7. Path on \( B(4, 2) \)**

**Figure 8. Cycle on non stutter permutations in \( I(4, 2) \)**

Hence, with the lemma 8.2 the theorem 8.1 is also proven. □

We will prove case (2) of the theorem inductively, the proof will again be supported by visualization through an example \( \vec{n} = (3, 2) \).

**Proof.** Because \( n_0 \) is odd, \( 0^n_0 1^{n_1} \) is not a stutter permutation. This permutation only has one neighbor in \( I(NS(\vec{n})) \) so it has to be an endpoint of the Hamilton path. For the proof of case (2) we will devise the set of all permutations \( S(\vec{n}) \) into two separate groups.

\[
\begin{align*}
P(n_0, n_1) &= \{(x_1, x_2, ..., x_n) \in S(\vec{n}) : x_n = 1\} \\
Q(n_0, n_1) &= \{(x_1, x_2, ..., x_n) \in S(\vec{n}) : x_n = 0\}
\end{align*}
\]  

Then \( I(P(\vec{n})) \) and \( I(Q(\vec{n})) \) are subgraphs of \( I(\vec{n}) \) which contain all the vertices. Note that \( Q(n_0, n_1) \) yields all the stutter permutations of \( S(\vec{n}) \) because \( n_0 \) is the only one that is odd. In order
to be a stutter permutation the last element has to be a 0. The following isomorphic relations hold.

- $I(P(n_0, n_1))$ is isomorphic to $I(M(n_0, n_1 - 1))$
- $I(Q(n_0, n_1))$ is isomorphic to $I(M(n_0 - 1, n_1))$

Because of this isomorphic relation $I(P(\bar{n}))$ has a Hamilton path from $0^{n_0}1^{n_1 - 1}1$ to $1^{n_1 - 1}0^{n_0}1$. It is easy to see that $NS(Q(n_0, n_1)) = NS(M(n_0 - 1, n_1))$ with lemma 8.2 we know via the isomorphic relation that $I(NS(Q(n_0, n_1)))$ has a Hamilton cycle on all non-stutter permutations. The cycle contains the vertex $1^{n_1 - 1}0^{n_0 - 1}1$, thus we can open the cycle at this point and connect it with the path over $I(P(\bar{n}))$ in order to get a Hamilton path over $I(NS(\bar{n}))$.

In total this is a proof for Theorem 8.1. So for Binary cases the conjecture 7.1 is true.
Multisets $M(1, 2, n_2)$ with $n_2 = \text{even}$

We now define $M(1, 2, \text{even})$ as the infinite group of multisets $M(\vec{n})$ with $\vec{n} : n_0 = 1; n_1 = 2; n_2 = \text{even}$ with $n_i = 0$ for all $i > 2$ that have stutter permutations. The proof and findings in this section are my own.

Lemma 8.3. $I(\text{NS}(M(1, 2, \text{even})))$ has a Hamilton cycle.

This lemma will be proven via a constructive proof. For the proof we look at the structure of the graph $I(\text{NS}(M(1, 2, \text{even})))$. For visual support of the proof we use an example $S(1, 2, 4)$.

8.1. Layers in $I(1, 2, \text{even})$. The graph can be built out of layers, where each layer indicates the position of the 0 in the permutation.

In the first layer the 0 will be at position $x_1$ and in the second layer the 0 will be at position $x_2$ of a permutation. In total, in the $i$-th layer the 0 will be at position $x_i$ of a permutation. In each layer the position of the 0 is static so only the 1’s and 2’s can move around. From every permutation you can shift one layer up or down through a neighbor swap which switches the position of the 0.

The bottom layer is isomorphic to $I(n_1, n_2)$ this is a graph that shapes like a triangle. The top of the triangle is the permutation 012$^{n_2}$1 going one step left means shifting the right 1 with the 2 on its left-hand side. Going one step right means shifting the left 1 with the 2 on its right-hand side. See figure 17 for an example.

The $i$th layer of the graph $I(1, 2, 4))$ is almost the same, except for the fact that in this layer the 0 prevents some of the interchanges of 2’s and 1’s. See Figure 15 for the fourth layer of the graph $I(1, 2, 4)$.

Every layer has at most three parts. First there is a part where the two 1’s come before the 0 (painted red), second there is a part where the 0 is in between the two 1’s (painted blue) and there is a part where the 1’s come after the 0 (painted black).

The top layer has all the stutter permutations and is isomorphic to $I(\text{NS}(2, 4))$. For this layer we already know it has a Hamilton path on all non-stutter permutations, see theorem 8.1.

8.2. Paths on different sections. In total the graph $I(\text{NS}(1, 2, 4))$ will look like in Figure 16. The proof that this graph has a Hamilton cycle is constructive and will give you an explicit Hamilton cycle.
For the proof we will deliberately separate the top layer from the graph. If we find a Hamilton cycle on both parts we can connect them into a Hamilton cycle on the full graph. We already know we can find a cycle on the top layer by theorem 8.1. For the rest of the graph we will divide the graph into separate parts, find paths on those parts and connect them to form a cycle.

We will divide the graph into bottom, top and middle parts.
8.2.1. Paths on bottom part of $I(1,2, even)$. The bottom part broadly is characterized by the fact that it consists of the permutation where the 0 stands left of the two 1’s. The starting points of the bottom paths will be $02^k12^{n_2-k}1$. From the starting point we will start shifting the second 1 to the left till it reaches the first 1 to $02^k112^{n_2-k}$. After this we will move up a layer by shifting the 0 to the right. Then we shift the second 1 all the way to the right and after that again go up a layer. We repeat this process until we reach $2^k1012^{n_2-k}$. In figure 17 is shown which paths we get on $I(NS(1,2,4))$.

8.2.2. Paths on top part of $I(1,2, even)$. The top part of the graph consists of all the permutations with the two 1’s left of the zero. The permutations with the 0 on the end are excluded from this part.

For the top paths we will use the starting points $12^k12^{n_2-k-1}02$ and $12^{n_2}01$. From these starting points we will do almost the same as for the bottom paths, but now we will shift the left 1 from left to right in the permutations and we will move down a layer each time. If we repeat this process we eventually get to $2^k1012^{n_2-k}$ and $2^{n_2}101$. These paths end in the same points as the bottom paths. Hence, we can connect them to get paths from $02^k12^{n_2-k-1}1$ to $12^k12^{n_2-k-1}02$ for $k < n_2$ and a path from $02^{n_2}11$ to $12^{n_2}01$. The paths partly go through the middle part.

Add the edges $02^k12^{n_2-k-1}1 \leftrightarrow 02^{k+1}12^{n_2-k+1}$ for $k$ odd and the edges $12^k12^{n_2-k-1}02 \leftrightarrow 12^{k+1}12^{n_2-k-2}02$ for $k$ even. Note that these are indeed neighbor swaps and with these edges the separate paths form one path from $012^{n_2}1$ to $12^{n_2}01$. This path is shown in figure 18 for $I(NS(1,2,4))$.

8.2.3. Paths on middle part of $I(1,2, even)$. The middle part consists of all the permutations where the 0 is between the 1’s. Some of the permutations are already included in the path given in figure 18. For the middle path we want it to start at $012^{n_2}1$ and go to $12^{n_2}01$ while connecting all the remaining permutations in the middle. This way we have a Hamilton cycle that contains all permutations except for those in the top layer.

For the middle paths we divide the path into $\frac{n_2}{2}$ different paths. With each starting point $12^{2k}02^{n_2-2k}1$ with $k \in \{0,1, \ldots, \frac{n_2}{2} - 1\}$. From this point we start to shift the right 1 to the left, until we reach $12^{2k}0212^{n_2-2k-1}$. Then we shift the left 1 one space to the right and then move the right 1 all the way to the right. Repeat this zigzag process until you reach $2^{2k}10212^{n_2-2k-1}$.
At this permutation move the 0 to the right to go up one layer, and mirror the path made on the previous layer. The path ends at $12^{2k+1}012^{n_2-2k-1}$ and contains all the remaining permutations on these layers.

If we add the edges $12^{2k+1}02^{n_2-2k-1}1 \leftrightarrow 12^{2k+2}02^{n_2-2k-2}1$ and the edge $012^{n_2}1 \leftrightarrow 012^{n_2}1$, we create a path from $012^{n_2}1$ to $12^{n_2}01$ that covers all the remaining permutations. The two paths combined give a cycle on all vertices of $I(NS(1, 2, 4))$ except for those in the top layer. This cycle is shown in figure 19 for $I(NS(1, 2, 4))$.

8.2.4. Combining separate cycles. We already know that the top layer is isomorphic to $I(n_1, n_2)$ which is a binary case. For this binary case we know that the cycle on the non stutter permutations has the edge $12^{n_2}10 \leftrightarrow 212^{n_2-1}10$ following from lemma 8.2. This edge is parallel to the edge $12^{n_2}01 \leftrightarrow 212^{n_2-1}01$, which is included in the graph on the rest of the layers. We can now connect these two cycles via the parallel edges to get a Hamilton cycle on $I(NS(1, 2, 4))$.
This method of construction works for all $n_2$ even. Therefore, the conjecture 7.1 is true for this infinite group of multisets. The Hamilton cycle found via this algorithm on $I(\text{NS}(1, 2, 4))$ is shown in figure 20. For $I(\text{NS}(1, 2, 6))$ figure 21 gives you the path on the bottom and top part and figure 22 gives you the Hamilton cycle over all vertices.
Figure 21. Top and bottom paths on $I(1, 2, 4))$

Figure 22. Hamilton cycle on $I(1, 2, 6))$
Conclusion

The conjecture given isn’t totally true. There are some cases where a Hamilton cycle is not possible on all NS($\vec{n}$). All binary cases where at least one of $n_0$ and $n_1$ is odd do not have a Hamilton cycle on all NS($n_0, n_1$). They only have a Hamilton path on those permutations. Also, Stachowiak proved in [18] that the case $M(1,1,n_2)$ with $n_2$ even only has a Hamilton path on all permutations. Since this multiset does not have any stutter permutations this also is an exception on the conjecture 7.1. Other than these few exceptions we only found new multisets that support the conjecture. Plus, in these exceptions we can still generate all possible permutations of a multiset Also we have proven that $M(1,2,n_2)$ with even $n_2$ is an infinite family of multisets that support conjecture 7.1.

The proofs and findings in this paper do support Lehmer’s conjecture 6.1. Only for the binary case with $n_0$ or $n_1$ odd there isn’t yet a proof that a Hamilton path with $D(I(\vec{n}))−1$ spurs exists. For this proof we will have to include one of the stutter permutations in the path. This is not trivial for cases with $n_0 \geq 5$ because a path from $0^{n_0}1^{n_1}$ to $1^{n_1−1}0^{n_0}1$ containing all vertices ending on a 1 is connected to a cycle containing all vertices ending on a 0. The cycle will be broken near $1^{n_1−1}0^{n_0}1$ and near this point a stutter permutation has to be added, but the nearest stutter permutation is five neighbor swaps away.

The stutter permutations are the permutations that destroy the symmetry of the neighbor-swap graphs. Interchanges by odd interchange digits give the same permutation for stutter permutations. Via the stutter permutation the proof for the parity difference of a neighbor-swap graph is much more elegant and more intuitive. In conclusion all results strengthen the conjecture 7.1. For the remaining multisets, all that lacks is a proof.
References


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