Controller synthesis for $\mathcal{L}_2$ behaviors using rational kernel representations

by

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Chapter 1

Introduction

The analysis of system interconnections is at the heart of many problems in modeling, simulation and control. Indeed, when focusing on control, the controller synthesis question amounts to finding a dynamical system (a controller) that, after interconnection with a given plant, results in a controlled system that is supposed to perform a certain task in a more desirable manner than the plant without controller. Usually the control synthesis problem is formulated as a feedback optimisation problem in which the plant and controller interact through a number of distinguished channels that have been divided in input and output variables.

The behavioral theory of dynamical systems has been advocated as a conceptual framework in which especially interconnection structures of dynamical system can be studied in an input-output independent setting. There are many conceptual, pedagogic and practical reasons for doing so. Please see [14, 16] for a detailed account on this matter.

One key problem concerning the interconnection of dynamical systems involves the question when a given dynamical system $K$ can be implemented (or realised) as the interconnection of a dynamical system $P$, that is supposed to be given, and a second dynamical system $C$, that is supposed to be designed. With the interpretation that $P$ and $K$ denote plant and controlled system, this question is therefore equivalent to a synthesis question for the controller $C$.

Within the behavioral framework this question received a very complete and elegant answer for the class of linear time-invariant systems that admit representations in terms of polynomial difference or polynomial differential operators [9, 11]. A rather complete theory has been developed for such representations that covers, among other things, $H_\infty$, LQ and $H_2$ optimal control.

It is the purpose of this report to reconsider the controller synthesis question for specific classes of linear and time-invariant $L_2$ systems that admit representations in terms of rational functions. In doing so, we depart from the setting proposed in [18] of considering infinitely smooth trajectories of $C^\infty$ as solutions to “rational” differential equations. Instead, rational functions in $H_\infty$ are viewed as multiplicative operators on $L_2$ functions to define $L_2$ behaviors or, using Parseval, $L_2$ behaviors in the time domain. In particular, this gives a frequency domain treatise on a very general synthesis problem for interconnected systems.

The first chapter of this report contains a summary of a literature study about the behavioral approach as it is introduced by [15]. Also the interconnection methods that result in the controller synthesis problems are discussed here. Chapter 3 starts with notational remarks on spaces and operators that are required for the introduction of the $L_2$ behaviors. Also the controller synthesis problems and the resulting algorithms that solve those problems are given in a $L_2$ behavioral framework. Now the algorithms are known, it is possible to use them in an example, which is the LQ problem in Chapter 4. Not only the usage of the algorithms is given but also a method of model reduction is applied. Afterwards, the results of the work are discussed and some recommendations for further research on $L_2$ behaviors are given.
Chapter 2

Literature

As mentioned in the introduction of this report, this chapter contains an introduction about modeling and control in the “classical” behavioral context. The information in this chapter is a short summary of earlier published work [2, 3, 9, 14, 15].

First of all, the concept of a behavior is introduced, which needs some introduction of systems described by (kernels of) polynomial matrices. Some useful properties of behaviors and polynomial matrices are summarised. Because we are interested in control using a behavioral approach, also properties like controllability and observability of behaviors are discussed.

In the second section of this chapter, the interconnection of dynamical systems is an issue. In fact, control is just an interconnection of a so called plant behavior with a controller behavior! Two different interconnection types, namely the full- and partial-interconnection, are introduced, which yield in the two different controller synthesis problems that are solved in a $\mathcal{L}_2$ behavioral framework in the next chapter.

2.1 Behaviors

2.1.1 Dynamical systems in a behavioral form

A dynamical system can be described as a triple:

\[ \Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{B}) \], where

\( \mathbb{T} \subseteq \mathbb{R} \) (or \( \mathbb{T} \subseteq \mathbb{C} \)) is the time- (or frequency) axis,
\( \mathbb{W} \subseteq \mathbb{R}^w \) is the signal space, where the variables take their values and
\( \mathbb{B} \subseteq \mathbb{W}^T \) is the behavior, which are the admissible trajectories.

A trajectory \( \mathbb{w} \) is defined as a mapping \( \mathbb{w} : \mathbb{T} \rightarrow \mathbb{W} \).

One can interpret:

\( \mathbb{w} \in \mathbb{B} \): the model allows (or explains) the trajectory \( \mathbb{w} \),
\( \mathbb{w} \notin \mathbb{B} \): the model forbids (or rejects) the trajectory \( \mathbb{w} \).

The above introduced dynamical system is a so called manifest system, where the signal space \( \mathbb{W} \) contains the “manifest” variables. This are the variables the model aims at. It is also possible that a system contains auxiliary modeling variables, which are introduced as latent variables. The system description then changes in:

\[ \Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathbb{B}_{\text{full}}) \], where
L is the latent variable space and \( B_{\text{full}} \subseteq (\mathcal{W} \times L)^\mathbb{T} \) is the full behavior.

This latent variable system \( \Sigma_L = (\mathbb{T}, \mathcal{W}, L, B_{\text{full}}) \) induces a manifest system \( \Sigma = (\mathbb{T}, \mathcal{W}, B) \), where the manifest behavior is given by:

\[
B = \{ w : \mathbb{T} \to \mathcal{W} \mid \exists l : \mathbb{T} \to L \text{ such that } (w, l) \in B_{\text{full}} \}.
\]

**Example 2.1.** An input / state / output system can be described with the following set of (differential) equations:

\[
\begin{align*}
\frac{d}{dt} w(t) &= f(w(t), u(t)); \\
y(t) &= h(w(t), u(t)),
\end{align*}
\]

where \( w(t) \in \mathcal{W} \), \( u(t) \in U \), \( y(t) \in Y \), \( t \in \mathbb{T} = \mathbb{R} \).

This can be seen as a latent variable system \( \Sigma_L \), given by the introduced quadruple. In this description, the signal spaces \( \mathcal{W} \) and \( L \) are described by: \( \mathcal{W} = U \times Y \) and \( L = \mathcal{X} \). The full behavior related with this system is given by:

\[
B_{\text{full}} = \{ \text{all } (u, y, x) : \mathbb{R} \to U \times Y \times \mathcal{X} \text{ that satisfy the system equations} \}.
\]

It is also possible to obtain the induced manifest system \( \Sigma \), where the (manifest) behavior in the triple is defined as: \( B = \text{all } (\text{input} / \text{output}-\text{pairs}) \).

**Properties of dynamical systems**

The dynamical systems discussed in this report have the following three properties:

- **Linear:** \((w_1, w_2 \in B) \land (\alpha, \beta \in \mathbb{R}) \Rightarrow (\alpha w_1 + \beta w_2 \in B)\).
- **Time-invariant:** \((w \in B) \land (t \in \mathbb{R}) \Rightarrow (\sigma^t w \in B)\), where \( \sigma^t \) denotes the \( t \)-shift, \( \sigma^t f(t') := f(t' + t) \).
- **Differential:** meaning \( B \) consists of the solutions of a system of differential equations.

### 2.1.2 Kernel representations of linear differential systems

The dynamical systems in this chapter are assumed to be linear and are described using transfer functions, state space realisations and kernel representations of polynomial matrices. Because earlier done research is using polynomial matrices, this type of representations (and its properties) is introduced shortly.

**Polynomial matrices**

Because of the “differential” property of a dynamical system, the following formulation of a set of \( g \) \( (n \) times) differential equations for the manifest variables \( w_1, w_2, \ldots, w_w \), is introduced:

\[
\begin{align*}
\sum_{j=1}^{w} R_{1,j}^0 w_j + \sum_{j=1}^{w} R_{1,j}^1 \frac{d}{dt} w_j + \cdots + \sum_{j=1}^{w} R_{1,j}^n \frac{d^n}{dt^n} w_j &= 0 \\
\sum_{j=1}^{w} R_{2,j}^0 w_j + \sum_{j=1}^{w} R_{2,j}^1 \frac{d}{dt} w_j + \cdots + \sum_{j=1}^{w} R_{2,j}^n \frac{d^n}{dt^n} w_j &= 0 \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \\
\sum_{j=1}^{w} R_{g,j}^0 w_j + \sum_{j=1}^{w} R_{g,j}^1 \frac{d}{dt} w_j + \cdots + \sum_{j=1}^{w} R_{g,j}^n \frac{d^n}{dt^n} w_j &= 0.
\end{align*}
\]

The coefficients in these equations, \( R_{i,j}^k \), do have 3 indices:

\( i = 1, \ldots, g \): for the \( i \)-th differential equation,
2.1. BEHAVIORS

For the variable $w_j$ involved,
k = 1, \ldots, n: for the order $\frac{d^k}{dt^k}$ of differentiation.

In vector/matrix notation this is given as:

$$
w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_w \end{bmatrix}, \quad R_k = \begin{bmatrix} R_{1,1}^k & R_{1,2}^k & \cdots & R_{1,w}^k \\ R_{2,1}^k & R_{2,2}^k & \cdots & R_{2,w}^k \\ \vdots & \vdots & \ddots & \vdots \\ R_{g,1}^k & R_{g,2}^k & \cdots & R_{g,w}^k \end{bmatrix}
$$

yields

$$R_0w + R_1 \frac{d}{dt}w + \cdots + R_w \frac{d^w}{dt^w}w = 0,$$

with $R_0, \ldots, R_w \in \mathbb{R}^{w \times w}$.

This notation can be rewritten as a polynomial matrix, which is a polynomial with matrix coefficients:

$$\hat{R}(\xi) = \hat{R}_0 + \hat{R}_1 \xi + \cdots + \hat{R}_n \xi^n,$$

with $\hat{R}_0, \ldots, \hat{R}_n \in \mathbb{R}^{n \times n}$. These polynomials can be embedded in the matrix itself, which results in the polynomial matrix $\hat{R}(\xi)$:

$$\hat{R}(\xi) = \begin{bmatrix} \hat{R}_{1,1}(\xi) & \hat{R}_{1,2}(\xi) & \cdots & \hat{R}_{1,n}(\xi) \\ \hat{R}_{2,1}(\xi) & \hat{R}_{2,2}(\xi) & \cdots & \hat{R}_{2,n}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{R}_{n,1}(\xi) & \hat{R}_{n,2}(\xi) & \cdots & \hat{R}_{n,n}(\xi) \end{bmatrix} \in \mathbb{R}^{n \times n}[\xi],$$

where $\hat{R}_{i,j}$ are polynomials (with real coefficients).

To find a solution $w(t) \in \mathbb{R}^w$ for this set of differential equations, one has to solve:

$$\hat{R} \left( \frac{d}{dt} \right) w = 0,$$

with $\hat{R}(\xi)$ a polynomial matrix as introduced.

**Important note:**

In this literature chapter of this report, it is assumed that $w$ is infinitely differentiable, which is denoted by $w \in C^\infty(\mathbb{R}, \mathbb{R}^w)$.

So, $\hat{R} \left( \frac{d}{dt} \right)$ is associated with a mapping $C^\infty(\mathbb{R}, \mathbb{R}^w) \to C^\infty(\mathbb{R}, \mathbb{R}^w)$. A solution $w(t)$ is then an element in $C^\infty(\mathbb{R}, \mathbb{R}^w)$ or $B = \{ w \in C^\infty(\mathbb{R}, \mathbb{R}^w) \mid \hat{R} \left( \frac{d}{dt} \right) w = 0 \}$:

$$w(t) \in \ker \hat{R} \left( \frac{d}{dt} \right).$$

When also latent variables, $l(t) \in \mathbb{R}^l$, are included in the system, this results in an extra polynomial matrix $\hat{M} \left( \frac{d}{dt} \right)$ that puts constraints on those variables:

$$\hat{R} \left( \frac{d}{dt} \right) w = \hat{M} \left( \frac{d}{dt} \right) l, \text{ with } \hat{R} \in \mathbb{R}^{w \times w} [\xi] \text{ and } \hat{M} \in \mathbb{R}^{w \times l} [\xi].
$$

**Unimodular polynomial matrix:**

A common used polynomial matrix is the so called unimodular matrix, which is defined as:

**Definition 2.2.** Let $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$. Then $U(\xi)$ is said to be a unimodular matrix if there exists a polynomial matrix $V(\xi) \in \mathbb{R}^{g \times g}[\xi]$ such that $V(\xi)U(\xi) = I$. Equivalently, $\det U(\xi)$ is equal to a nonzero constant.

A very nice aspect of unimodular polynomial matrices is described in the following lemma (using [9, Thm. 2.5.4] and the previous definition):

**Lemma 2.3.** Let $R(\xi) \in \mathbb{R}^{g \times g}$ and $U(\xi) \in \mathbb{R}^{g \times g}[\xi]$. Define $R'(\xi) := U(\xi)R(\xi)$. Denote the behaviors corresponding to $R(\xi)$ and $R'(\xi)$ by $B$ and $B'$ respectively. Then:
1. $B \subseteq B'$,
2. If in addition, $U^{-1}(\xi) \in \mathbb{R}^{r \times g}$ exists and if $U^{-1}(\xi) \in \mathbb{R}^{\times g}$, then $B = B'$. This is the case if $U(\xi)$ is a unimodular matrix!

So, this results in a very nice theorem that is often used later on:

**Theorem 2.4.** Let $B$ be a behavior described by the kernel of the polynomial matrix $R(\xi) \in \mathbb{R}^{* \times w}[\xi]$ (such that $R(\frac{d}{dt})w = 0$ with $w(t) \in \mathbb{R}^{w}$). Let $U(\xi) \in \mathbb{R}^{w \times w}[\xi]$ be a unimodular matrix. Then, the behavior $B' = \ker(UR) = \ker(R) = B$. So, the possible trajectories of the kernel $UR$ are the same as those of the kernel $R$.

**Conversion of latent representations into manifest representations:**

It is possible to convert systems, which contain latent variables (like represented in Eq. (2.1)), into an equivalent system description, that only contains manifest variables. The manifest variables in this system will have the same behavior! Consider the full behavior $B_{\text{full}}$ associated with (2.1) and let $B$ be its corresponding manifest behavior. Lemma 2.3 says the this polynomial representation (2.1) is equivalent to

$$U(\frac{d}{dt})R(\frac{d}{dt})w = U(\frac{d}{dt})M(\frac{d}{dt})l,$$

where $U(\xi) \in \mathbb{R}^{* \times g}[\xi]$ is a unimodular matrix. With such a unimodular matrix, it is possible to row-reduce the matrix $M(\xi)$, such that:

$$U(\xi)M(\xi) = \begin{bmatrix} M_1(\xi) \\ O_{r \times 1} \end{bmatrix},$$

with $M_1(\xi)$ a (to be estimated) polynomial matrix, $O_{r \times 1}$ a zero matrix with dimensions $r \times 1$ with $r$ a new introduced dimension.

When this unimodular matrix is found, it also has to be pre-multiplied with $R(\xi)$, which results in a partition of the two polynomial matrices:

$$U(\frac{d}{dt})R(\frac{d}{dt})w = U(\frac{d}{dt})M(\frac{d}{dt})l \iff \begin{bmatrix} R_1(\frac{d}{dt}) \\ R_2(\frac{d}{dt}) \end{bmatrix}w = \begin{bmatrix} M_1(\frac{d}{dt}) \\ O_{r \times 1} \end{bmatrix}l,$$

where $R_1(\xi) \in \mathbb{R}^{(s-r) \times w}[\xi]$, $R_2(\xi) \in \mathbb{R}^{r \times w}[\xi]$ and $M_1(\xi) \in \mathbb{R}^{(s-r) \times \times g}[\xi]$.

Using [9, Thm. 6.2.6], one can say that the behavior of the (latent) representation of the system in (2.1) is equivalent to the one described by the following (manifest) representation:

$$R_2(\frac{d}{dt})w = 0. \quad (2.2)$$

So, by pre-multiplication with a unimodular matrix it is possible to represent a system with latent variables in an equivalent system with only manifest variables. That is, $B = \ker R_2(\frac{d}{dt})$.

**Linear differential systems**

A linear differential system, e.g. described by a polynomial matrix as above, with a finite number of variables is denoted by: $\mathcal{L}^*$, where $*$ is the number of variables in the system.

The behavior $B$, which is the set of trajectories $w$ that are in the kernel of the operator $R(\frac{d}{dt})$. Therefore, $R(\frac{d}{dt})w = 0$ is called a kernel representation of the system with a behavior: $B = \ker R(\frac{d}{dt})$.

$R \in \mathbb{R}^{p \times w}[\xi]$ induces a minimal kernel representation of $B$, when there only exist polynomial matrices $R' \in \mathbb{R}^{g \times w}[\xi]$ that induce a kernel representation of the same behavior $B$ where $p \leq g$, i.e. rowdim($R$) $\leq$ rowdim($R'$).
Some properties of behaviors that are described by linear differential equations $\mathcal{L}^*$ are:

Intersection: $(B_1, B_2 \in \mathcal{L}^*) \Rightarrow (B_1 \cap B_2 \in \mathcal{L}^*)$

Addition: $(B_1, B_2 \in \mathcal{L}^*) \Rightarrow (B_1 + B_2 \in \mathcal{L}^*)$

Projection: $(B \in \mathcal{L}^{w_1+w_2}) \Rightarrow (\Pi_{w_1} B \in \mathcal{L}^{w_1})$

Action of a linear differential operator: $(B \in \mathcal{L}^{w_1}, P \in \mathbb{R}^{w_2 \times w_1}[\xi]) \Rightarrow (P(\frac{\partial}{\partial \xi})B \in \mathcal{L}^{w_2})$

Inverse image of a linear differential operator: $(B \in \mathcal{L}^{w_2}, P \in \mathbb{R}^{w_2 \times w_1}[\xi]) \Rightarrow ((P(\frac{\partial}{\partial \xi}))^{-1} B \in \mathcal{L}^{w_1})$

Another useful aspect that has to be known is when a behavior is included in another behavior. This is described in the following proposition [2, Prop. 2.5.1]:

**Proposition 2.5.** Let $B^1, B^2 \in \mathcal{L}^w$ be represented by kernel representations $R_1(\frac{\partial}{\partial \xi})w = 0$ and $R_2(\frac{\partial}{\partial \xi})w = 0$, respectively. Then $B^1 \subseteq B^2$ if and only if there exists an $F \in \mathbb{R}^{\times \times [\xi]}$ such that $FR_1 = R_2$.

### 2.1.3 Partitions of variables and kernels

As in “classical” system theory, plants are described as input-output systems. In the behavioral approach, only trajectories of (manifest) variables $w$ are used. The nice thing about this behavioral framework is that no separation in variables has to be made for using the theories associated with the framework. It can be seen as a “direction-less” representation of systems. However, it is possible to partition this variable into parts that can be seen as inputs and outputs of the system as it can be in physical systems. This also results in a partition of the kernel itself, such that one part is related with the inputs and the other part with the outputs.

#### Inputs

The **inputs** of the system are the arbitrary functions in $w$. In other words: the requirement $w \in \mathcal{B}$ leaves some of the components $\{w_1, w_2, \ldots, w_w\}$ unconstrained, which are named the inputs.

These inputs can be found by searching for free variables in the behavior $\mathcal{B}$, defined as [3]:

Let $\mathcal{B} \in \mathcal{L}^w$, $w = (w_1, w_2, \ldots, w_w)$ and $I = \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, w\}$. The functions $w^I = (w_{i_1}, w_{i_2}, \ldots, w_{i_k})$ are obtained by selecting from $w$ only the components which are in the index set $I$. This elimination can be described with the projection $P_I(\mathcal{B})$ and results in a (new) linear differential system:

The set of variables $\{w_{i_1}, \ldots, w_{i_k}\}$ is called **free** in $\mathcal{B}$ if $P_I(\mathcal{B}) = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{|I|})$, where $|I| = m(\mathcal{B})$, the input cardinality of the set $I$. So, no constraints are applied on this set $\{w_{i_1}, \ldots, w_{i_k}\}$.

There exist a various number of sets that contain free variables, therefore the so called **maximally free set** is introduced:

**Definition 2.6.** A set of variables $w^I$ is called maximally free in $\mathcal{B}$ if it is free and if for any $I' \subseteq \{1, 2, \ldots, w\}$ such that $I \subseteq I'$ we have $P_{I'}(\mathcal{B}) \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{|I'|})$. So, if $w^I$ is maximally free, then any set of variables obtained by adding a variable $w_i$ (which is not in $w^I$ already of course) to it, results in a set that is not free anymore.

#### Outputs

With this definition of maximally free variables, the total set of (manifest) variables $w$ can be partitioned in a so called input/output partitioning. When the inputs are known, it can easily be shown that the resulting variables are outputs. This is given in the following definition:

**Definition 2.7.** Let $\mathcal{B} \in \mathcal{L}^w$, $w = (w_1, w_2, \ldots, w_w) = (w^{(1)}, w^{(2)})$, with $w^{(1)} = (w_1, w_2, \ldots, w_m)$ and $w^{(2)} = (w_{m+1}, w_{m+2}, \ldots, w_w)$. If $w^{(1)}$ is maximally free, $w^{(1)} = u$ is called an input in $\mathcal{B}$ and $w^{(2)} = y$ is called an output in $\mathcal{B}$.

Here, the dimension of $w^{(2)}$ is named the **output cardinality** $p(\mathcal{B})$ of the behavior.
Cardinalities

Now, two cardinalities, which are used to estimate the number of input- and output components, are introduced. These two integer invariants and also the number of components in an element \( w \) and the McMillan Degree (see appendix A.1) associated with a behavior \( \mathcal{B} \in \mathcal{L}^w \) are summarised because they are used later on:

\[
\begin{align*}
\dim(w) &= q, \\
\max\{k \in \mathbb{N} \mid \{w_{i_1}, w_{i_2}, \ldots, w_{i_k}\} \text{ is free in } \mathcal{B}\} &= m(B), \\
\dim(w) - m(B) &= p(B), \\
\text{the McMillan degree of } B &= n(B).
\end{align*}
\]

Partition of kernel

The partitioning of the variables in \( w \) implies that it should also be possible to split the kernel, that describes the complete trajectories, into two separate parts. This is described in the following proposition \cite[Prop. 2.9.4]{9}:

**Proposition 2.8.** Let \( R \in \mathbb{R}^{\mathbf{8} \times \mathbf{w} }[\xi] \) induce a kernel representation of \( \mathcal{B} \). Let \( w = (u, y) \) be a partition of \( w \) and let \( R = [Q P] \) be a corresponding partition of \( R \). Then,

1. \( u \) is \( \mathcal{C}^\infty \)-free if and only if \( \text{rank}(R) = \text{rank}(Q P) = \text{rank}(P) \),
2. once \( u \) is fixed, \( y \) is bound if and only if \( P \) has full column rank, i.e. \( \text{rank}(P) = \dim(y) \),
3. \( w = (u, y) \) is an i/o partition if and only if \( \text{rank}(R) = \text{rank}(P) = \text{coldim}(P) \).

If \( R(\frac{d}{dt})w = 0 \) is a minimal kernel representation, then \( w = (u, y) \) is an i/o partition if and only if \( P \) is square and nonsingular (so, \( \det(P) \neq 0 \)).

The input/output representation of the behavior \( \mathcal{B} \in \mathcal{L}^w \), with the partitioned manifest variable \( w = (u, y) \) and polynomial matrices \( P \in \mathbb{R}^{y \times y}[\xi] \), \( \det(P) \neq 0 \), \( Q \in \mathbb{R}^{y \times u}[\xi] \) is then given by:

\[
\mathcal{B} = \{(u, y) \mid P \left( \frac{d}{dt} \right) y = -Q \left( \frac{d}{dt} \right) u \},
\]

where \( -P^{-1}Q \) is defined as the transfer matrix of \( \mathcal{B} \) w.r.t. the given input/output partition.

**2.1.4 Controllable, observable, autonomous and partly controllable behaviors**

Because control in a behavioral context is one of the main issues in this report, one has to look whether the systems are controllable or observable. This is relevant if one wants to apply control on the system, which is for example not possible if a plant is autonomous. In this section, properties like observability and controllability are summarised in a behavioral context.

**Controllable behaviors**

\( \mathcal{B} \in \mathcal{L}^w \) is called **controllable** if for all \( w_1, w_2 \in \mathcal{B} \) there exists a \( w \in \mathcal{B} \) and \( T \geq 0 \) such that

\[
w(t) = \begin{cases} w_1(t), & t < 0, \\ w_2(t), & t > T. \end{cases}
\]

Using kernel representations of polynomial matrices (\( \mathcal{B} \in \mathcal{L}^w, R \in \mathbb{R}^{\mathbf{8} \times \mathbf{w}} \), with \( R(\frac{d}{dt})w = 0 \)) controllability of a behavior \( \mathcal{B} \) only holds if

\[
\text{rank}(R(\lambda)) = r \text{ for all } \lambda \in \mathbb{C} \quad \text{\cite[Thm. 5.2.5]{9}}.
\]

So, \( \mathcal{B} \) is controllable if the rank equals \( r \) for all possible \( \lambda \in \mathbb{C} \).
A weaker form of controllability is stabilizability, which enables to steer towards a desired trajectory asymptotically. When considering linear systems, it would be possible to steer a trajectory to zero. This can be defined as: \( B \in \mathcal{L}^w \) is called stabilizable if for each \( w \in B \), there exists a \( w' \in B \) such that

\[
\begin{align*}
w'(t) &= w(t), & \text{for } t < 0, \\
w'(t) &\to 0, & \text{as } t \to \infty.
\end{align*}
\]

Stabilizability of a behavior \( B (B \in \mathcal{L}^w, R \in \mathbb{R}^{\times \times w}[\xi]) \), with representation \( R(\frac{d}{dt}) w = 0 \) only holds if:

\[
\text{rank}(R(\lambda)) = r \text{ for all } \lambda \in \mathbb{C}^\ast \quad [9, \text{Thm. 5.2.30}].
\]

Observable behaviors

Not only controllability is important for (linear) systems, but also observability is an issue. When a system \( \Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, B) \) contains two variables \((w_1 \text{ and } w_2)\), \( w_2 \) is said to be observable from \( w_1 \) if

\[
(w_1, w_2'), (w_1, w_2'') \in B \Rightarrow w_2' = w_2''.
\]

This is equivalent to the statement that there exists a map \( f : \mathbb{W}_1^\ast \to \mathbb{W}_2^\ast \) such that \( w_2 = f(w_1) \) for all \((w_1, w_2) \in B\).

In terms of the behavioral framework, observability can be described with the following proposition:

**Proposition 2.9.** Let \( B \in \mathcal{L}^{w_1 + w_2} \) be represented by \( R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2 \). Then \( w_2 \) is observable from \( w_1 \) if and only if \( R_2(\lambda) \) has full column rank for all \( \lambda \in \mathbb{C} \) [9, Thm. 5.3.3].

Also a weaker form of observability is known, which is the detectability. Let \( B \in \mathcal{L}^{w_1 + w_2} \) again. \( w_2 \) is said to be detectable from \( w_1 \) if

\[
(w_1, w_2'), (w_1, w_2'') \in B \Rightarrow \lim_{t \to \infty} w_2'(t) - w_2''(t) = 0.
\]

Detectability can also be described in terms of behaviors:

Let \( B \in \mathcal{L}^{w_1 + w_2} \) be represented by \( R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2 \). Then \( w_2 \) is detectable from \( w_1 \) if and only if \( R_2(\lambda) \) has full column rank for all \( \lambda \in \mathbb{C}^\ast \) [9, Thm. 5.3.17].

Autonomous behaviors

Controllable behaviors are discussed in the previous subsection, however the complete opposite type of behaviors is not discussed yet. This is the so called autonomous behavior, in which it is not possible to move from a given trajectory to another trajectory:

A time-invariant dynamical system \( \Sigma = (\mathbb{R}, \mathbb{W}, B) \) is called autonomous if for all \( w_1, w_2 \in B \),

\[
w_1(t) = w_2(t) \text{ for } t \leq 0 \quad \Rightarrow \quad w_1 = w_2.
\]

This can also be described (\( B \in \mathcal{L}^w \) and \( R \in \mathbb{R}^{\times \times w}[\xi] \), inducing the kernel representation \( R(\frac{d}{dt}) w = 0 \) as: \( B \) is autonomous if \( \text{rank}(R) = w \) or \( B \) is a finite dimensional vector space.

Partly controllable behaviors

Behaviors that are neither autonomous or controllable have a controllable sub-behavior within them. A controllable sub-behavior within every behavior is the zero behavior (\( B = 0 \)). However, the largest controllable sub-behavior is most interesting:

**Definition 2.10.** Therefore, the controllable part of a behavior \( B \in \mathcal{L}^w \) is defined as \( B_{\text{cont}} \in \mathcal{L}^w \) satisfying:

1. \( B_{\text{cont}} \) is controllable,
2. $\mathcal{B}_{\text{cont}} \subseteq \mathcal{B}$ and
3. if $\mathcal{B}' \in \mathcal{L}^w$ is controllable and if $\mathcal{B}' \subseteq \mathcal{B}$ then $\mathcal{B}' \subseteq \mathcal{B}_{\text{cont}}$.

When the largest sub-behavior is defined, one can distinguished two different parts in the complete behavior, namely a controllable- and an autonomous part: $\mathcal{B} = \mathcal{B}_{\text{cont}} \oplus \mathcal{B}_{\text{aut}}$.

The second item in the definition above allows us to use Lemma 2.3, which makes it possible to estimate the kernel representation of the controllable behavior. So, if $R\left(\frac{d}{dt}\right)w = 0$ is a kernel representation of $\mathcal{B}$, then there exist a kernel $R' \in \mathbb{R}^{* \times w}[\xi]$ and a polynomial matrix $F \in \mathbb{R}^{* \times *}[\xi]$ such that $R'\left(\frac{d}{dt}\right)w = 0$ is a kernel representation for $\mathcal{B}_{\text{cont}}$ and $FR' = R$. In the case if $R\left(\frac{d}{dt}\right)w = 0$ is a minimal kernel representation of $\mathcal{B} \in \mathcal{L}^w$, $R'\left(\frac{d}{dt}\right)w = 0$ is also minimal and $F$ is (square and) nonsingular. Because $F$ is nonsingular and $R$ and $R'$ have the same rank we note that $\mathcal{B}$ and $\mathcal{B}_{\text{cont}}$ have the same input cardinalities ($p(\mathcal{B}) = p(\mathcal{B}_{\text{cont}})$).

The construction of $F$ and $R'$ from a behavior induced by the kernel $R$, which is in Smith form, is explained in the following example:

**Example 2.11.** Let $R\left(\frac{d}{dt}\right)w = 0$ be a minimal kernel representation of $\mathcal{B} \in \mathcal{L}^w$. Let $U, V$ be unimodular matrices that bring $R$ to its Smith form (explained in appendix A.1):

$$URV = \begin{bmatrix} S & 0 \end{bmatrix},$$

where $S$ has dimension $p(\mathcal{B})$, is nonsingular and diagonal. Define $R' := [I_{p} 0]V^{-1}$, with $I_{p}$ the identity matrix of dimension $p(\mathcal{B})$. This yields to a minimal kernel representation $R'\left(\frac{d}{dt}\right)w = 0$ of $\mathcal{B}_{\text{cont}}$. Correspondingly, $F$ is equal to $U^{-1}S$. Moreover, each $\mathcal{B}_{\text{aut}}$ has $\det(S)$ as its characteristic polynomial.

With this decomposition, it is possible to define an observable image representation for $\mathcal{B}_{\text{cont}}$ using $V$. We define $M \in \mathbb{R}^{w \times w}[\xi]$ as the last $m(\mathcal{B})$ columns of $V$, i.e.

$$M := V \begin{bmatrix} 0 \\ I_{m} \end{bmatrix},$$

where $I_{m}$ is the identity matrix of dimension $m(\mathcal{B})$. Then, $w = M\left(\frac{d}{dt}\right)w = l$ is an observable image representation of $\mathcal{B}_{\text{cont}}$.

## 2.2 Interconnection

In this section, control can be viewed as an interconnection of a plant and a controller, which will be done using the behavioral approach. In the first part of this section, a normal interconnection between two dynamical systems will be introduced. Later on, one of those dynamical systems will be replaced by a controller, which is also a dynamical system. This results in some nice theorems that provide information about implementable controllers for a certain desired (closed loop) behavior.

### 2.2.1 Interconnection of dynamical systems

![Figure 2.1: Interconnection of dynamical systems.](image)

In this section, two dynamical systems are interconnected with each other. Those systems can be described with the following triple representations:

$$\Sigma_1 = (T, \mathcal{W}_1 \times \mathcal{W}_2, \mathcal{B}_1), \quad \Sigma_2 = (T, \mathcal{W}_2 \times \mathcal{W}_3, \mathcal{B}_2).$$
2.2. INTERCONNECTION

These dynamical systems have a common time axis $T$ and a common factor $W_2$ in the signal space $W$. An interconnection of two dynamical systems, $\Sigma_1$ and $\Sigma_2$, through a common variable, $w_2$, is denoted as $\Sigma_1 \wedge w_2 \Sigma_2$ and is defined as [14]:

$$\Sigma_1 \wedge w_2 \Sigma_2 := (T, W_1 \times W_2 \times W_3, B)$$

with

$$B = \{(w_1, w_2, w_3) : T \rightarrow W_1 \times W_2 \times W_3 \mid (w_1, w_2) \in B_1 \text{ and } (w_2, w_3) \in B_2\}.$$ 

The interconnected situation is illustrated in Figure 2.1. The (common) variable $w_2$ is called the interpolation variable.

$\Sigma_1 \wedge w_2 \Sigma_2$ is called a regular interconnection if the output cardinalities of $\Sigma_1$ and $\Sigma_2$ add up to that of the interconnection:

$$p(B_1 \cap B_2) = p(B_1) + p(B_2). \tag{2.3}$$

Using the definitions of the previous sections, the interconnected situation can also be represented as kernel representation. For example, if the dynamical systems $\Sigma_1$ and $\Sigma_2$ are represented as:

- kernel representation of $\Sigma_1$: $R_{11}(\frac{d}{dt})w_1 + R_{12}(\frac{d}{dt})w_2 = 0$,
- kernel representation of $\Sigma_2$: $R_{22}(\frac{d}{dt})w_2 + R_{23}(\frac{d}{dt})w_3 = 0$,

this results in the representation for the interconnected dynamical system:

$$\begin{bmatrix} R_{11}(\frac{d}{dt}) & R_{12}(\frac{d}{dt}) & 0 \\ 0 & R_{22}(\frac{d}{dt}) & R_{23}(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = 0.$$

In many interconnections it is natural to suppress the interpolation variable after the interconnection, such that:

$$\Sigma_1 \wedge \Sigma_2 = (T, W_1 \times W_2, B')$$

with

$$B' = \{(w_1, w_3) : T \rightarrow W_1 \times W_3 \mid \exists w_2 : T \rightarrow W_2 \text{ such that } (w_1, w_2) \in B_1 \text{ and } (w_2, w_3) \in B_2\}.$$ 

This suppression is depicted in Figure 2.2. One is able to suppress variables using the elimination technique that is introduced in the end of section 2.1.2.

2.2.2 Feedback interconnection

In "classical" control, a feedback controller is a common used control strategy, which can be described as follows.

An interconnection is called a feedback interconnection if, after permutation of components, there exists a partition of $w_1$, $w_2$ and $w_3$ into $w_1 = (v_1, z_1)$, $w_2 = (u, y_1, y_2)$ and $w_3 = (v_2, z_2)$ such that the following conditions hold:
1. in the system $\Sigma_1$, $(v_1, y_2, u)$ is input and $(z_1, y_1)$ is output, and the transfer matrix from $(v_1, y_2, u)$ to $(z_1, y_1)$ is proper.
2. in the system $\Sigma_2$, $(v_2, y_1, u)$ is input and $(z_2, y_2)$ is output, and the transfer matrix from $(v_2, y_1, u)$ to $(z_2, y_2)$ is proper.
3. in the system $\Sigma_1 \land w_2 \Sigma_2$, $(v_1, v_2, u)$ is input and $(z_1, z_2, y_1, y_2)$ is output, and the transfer matrix from $(v_1, v_2, u)$ to $(z_1, z_2, y_1, y_2)$ is proper.

![Figure 2.3: “Classical” feedback interconnection.](image)

This feedback interconnection of two behaviors is depicted in Figure 2.3. Also is mentioned in [10] that for a regular feedback interconnection must hold that the output cardinalities of the controlled behavior is the sum of the output cardinalities of the to be interconnected behaviors (2.3) but there also is such a relation for the McMillan degrees:

$$n(B_1 \cap B_2) = n(B_1) + n(B_2).$$

Because we are using the behavioral framework, there is no need to separate variables into inputs and outputs. So, no directions have to be specified in the description of the system. Then, the “classical” case, which is depicted in Figure 2.3, is in the behavioral context depicted as in Figure 2.1.

### 2.2.3 Control as interconnection

Before we can discuss about control, we have to introduce the plant that needs to be controlled of course. This plant can be depicted with the block schematic depicted in Figure 2.4(a). It contains two kind of variables:

1. variables to be controlled $w$ (taking values in $\mathcal{W}$),
2. control variables $c$ (taking values in $\mathcal{W}_c$).

The control variables are those variables in the plant that we are allowed to put constraints on (e.g. in the form $C(\frac{\partial}{\partial t})c = 0$). Knowing those two sets of variables, the to be controlled plant can be described by the triple:

$$\Sigma_p = (T, \mathcal{W} \times \mathcal{W}_c, \mathcal{P}_{\text{full}}),$$

with the full plant behavior

$$\mathcal{P}_{\text{full}} := \{(w, c) \in C^\infty(\mathbb{R}, \mathbb{R}^{w+c}) | (w, c) \text{ satisfies the plant equations}\}.$$

We assume that the to be controlled plant (and the controller) are linear differential systems (so in $\mathcal{L}^\ast$). Thus, we assume $\mathcal{P}_{\text{full}} \in \mathcal{L}^{w+c}$. Also the manifest behavior for the plant, $\mathcal{P} \in \mathcal{L}^w$, can be defined, which eliminates $c$ from the variable space:

$$\mathcal{P} = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^w) | \exists c \text{ such that } (w, c) \in \mathcal{P}_{\text{full}}\}.$$  

As just mentioned, the control variables put constraints on the behavior of the system. This can be done...
by introducing a controller, which is a dynamical system that can be described with the following triple:

\[ \Sigma_c = (T, W_c, C). \]

The trajectories of those variables \( c \) are described in the controller behavior \( C \in \mathcal{P}^c \):

\[ C = \{ c \in \infty (\mathbb{R}, \mathbb{R}^c) \mid \text{c satisfies the controller equations} \}. \]

To obtain a controlled system, an interconnection of those two dynamical systems, through \( c \), has to be made. This means that \( c \) has to satisfy the laws for both \( P_{\text{full}} \) as well as for \( C \). This can be described as:

\[ \Sigma_p \wedge \Sigma_c = (T, W \times W_c, K_{\text{full}}), \]

with the full controlled behavior:

\[ K_{\text{full}} = \{ (w, c) \mid (w, c) \in P_{\text{full}} \text{ and } c \in C \}. \]

**Example 2.12.** Let a full plant behavior \( P_{\text{full}} \) be described with the kernel representation:

\[ R_1(\frac{d}{dt})w + R_2(\frac{d}{dt})c = 0, \]

and a controller behavior \( C \) be described by:

\[ C(\frac{d}{dt})c = 0. \]

Then, the full controlled behavior \( K_{\text{full}} \) can be described by:

\[ \begin{bmatrix} R_1(\frac{d}{dt}) & R_2(\frac{d}{dt}) \\ 0 & C(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0. \]

"Outside" the interconnected system, one is mostly interested in the trajectories of the variables \( w \). For that reason, the manifest controlled behavior \( K \) is introduced, which eliminates the control variables from \( K_{\text{full}} \):

\[ K := \{ w \mid \exists c \text{ such that } (w, c) \in K_{\text{full}} \}. \]

This is equivalent to:

\[ K = \{ w \mid \exists c \in C \text{ such that } (w, c) \in P_{\text{full}} \}. \]

Because of this formulation, we say that \( C \) implements the behavior \( K \). This behavior is depicted in the block schematic in Figure 2.5 (but later on is shown that in the full interconnection case, \( c \) is part of the controlled behavior).

**Full- and partial interconnection:**

In this case, \( C \) implements \( K \) by partial interconnection with \( P \), because only the variables \( c \) are used to interconnect with. It can also be possible that no separation can be made between control- and to-be-controlled variables. In these situations, the complete variable \( w \) is used to interconnect with between the controller and the plant. This is defined as control using a full interconnection and is depicted by the block schematic in Figure 2.6.
2.3 Implementability of a controller

Now one knows the definition and the most important properties of a behavior and also knows how interconnections of behaviors work, the question is whether it is possible to find a controller $C$ that implements a desired controlled behavior $K$ for a system, described by a plant behavior $P$. This issue is separated into two possible situations in this section:

1. **Control using full interconnection:**
   One uses all (manifest) variables $w$ for the interconnection between the plant and controller.

2. **Control using partial interconnection:**
   It is possible to separate the variables of the plant such that only a part of it is used for the interconnection with the controller. This is depicted in Figure 2.5.

Those two different situations result in different conditions that have to hold for the controller-existence questions. So, in the first subsection these conditions are given for the full interconnection case, and in the second one it is done when a part of the variables is used in the interconnection.

### 2.3.1 Full interconnection

First of all, the case that all manifest variables $w$ are used to interconnect the plant with the controller. In this subsection, the plant behavior $P \in \mathcal{L}^w$ is described using the polynomial matrix $R(\xi) \in \mathbb{R}^{\ast \times w}[\xi]$ such that $P = \ker(R)$. Because of the full interconnection, the controller behavior can be written as $C \in \mathcal{L}^w$ with also a polynomial matrix $C(\xi) \in \mathbb{R}^{\ast \times w}[\xi]$ such that $C = \ker(C)$. Interconnection of those two systems will result in the “desired” or controlled behavior $K \in \mathcal{L}^w$.

With this information about the plant-, the ("desired") controlled- and (to be estimated) controller behavior, the following question has to be answered:

Which manifest controlled behaviors $K \in \mathcal{L}^w$ are regularly implementable by full interconnection, i.e. for which $K \in \mathcal{L}^w$ there exists a controller $C \in \mathcal{L}^w$ such that $K = P \cap C$ holds?

The answer of this question can be given, when behaviors are described by kernel representations with polynomial matrices, using the following theorem [3]:

---

**Figure 2.5:** Interconnected plant and controller with manifest controlled behavior $K$.

**Figure 2.6:** Full interconnection.
Theorem 2.13. Let \( \mathcal{P}, \mathcal{K} \in \mathcal{L}^w \). Let \( R(\frac{d}{dt})w = 0 \) and \( K(\frac{d}{dt})w = 0 \) be minimal kernel representations of \( \mathcal{P} \) and \( \mathcal{K} \), respectively. Then the following statements are equivalent:

1. \( \mathcal{K} \) is regularly implementable w.r.t. \( \mathcal{P} \) by full interconnection,
2. there exists a polynomial matrix \( F \) with \( F(\lambda) \) full row rank for all \( \lambda \in \mathbb{C} \), such that \( R = FK \),
3. \( \mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P} \).

Proof. \((1 \iff 3)\) Proven in [3, Lemma 7]. Only the equivalence of 1 and 2 will be proved here.

\((1 \Rightarrow 2)\) Let \( C \) be such that \( \left( \frac{d}{dt} \right)w = 0 \) is a minimal representation of \( \mathcal{K} \). Then there exists a unimodular \( U \) such that \( \text{col}(R, C) = UK \). This implies \( R = FK \), with \( F \) consisting of the upper rows of \( U \).

\((2 \Rightarrow 1)\) Assume \( R = FK \). Let \( V \) be such that \( \text{col}(F, V) \) is unimodular. Define \( C = VK \). Then \( \left( \frac{d}{dt} \right)w = 0 \) is a minimal representation of \( \mathcal{K} \), and thus \( \mathcal{K} \) is regularly implemented by the controller \( C = \ker(C) \).

In the next chapter of this report, an algorithm for finding a (rational representation of a) controller is introduced.

2.3.2 Partial interconnection

Definition 2.14. A given \( \mathcal{K} \in \mathcal{L}^w \) is called implementable with respect to \( \mathcal{P}_{\text{full}} \) by interconnection through \( c \), if there exists a controller \( C \in \mathcal{L}^e \) such that \( C \) implements \( \mathcal{K} \) through \( c \).

A very important question is:

Which manifest controlled behaviors \( \mathcal{K} \in \mathcal{L}^w \) are implementable, i.e. for which \( \mathcal{K} \in \mathcal{L}^w \) there exists a controller \( C \in \mathcal{L}^e \) such that (2.4) holds?

To be able to answer this question, the so called hidden behavior needs to be introduced. The hidden behavior is the set of trajectories that \( w \) can have even after nullifying the control variables:

Definition 2.15. Let \( \mathcal{P}_{\text{full}} \in \mathcal{L}^{w+e} \). The hidden behavior \( \mathcal{N} \in \mathcal{L}^w \) is the behavior consisting of the to-be-controlled variable trajectories that can occur when the control variables are restricted to be zero:

\[ \mathcal{N} := \{ w \in C^\infty(R, R^w) | (w, 0) \in \mathcal{P}_{\text{full}} \} . \]

This behavior is also named “hidden” behavior because it is not possible to estimate \( \mathcal{N} \) from the controller’s view. There is only control on the variables \( c \), which are distinguished controller variables that will, in general, be independent of \( w \). This hidden behavior is depicted in Figure 2.7.

Knowing the definition of the hidden behavior, the “most beautiful theorem” from Willems and Trentelman can be introduced: the Controller Implementability Theorem

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\(^1\)This statement is equivalent with \( \mathcal{K} \subseteq \mathcal{P} \)

\(^2\)\( \text{col}(A, B) = [A^T, B^T]^T \)
Theorem 2.16. (Controller Implementability Theorem)
Let \( \mathcal{P}_{\text{full}} \in \mathcal{L}^{w+c} \) be the full plant behavior, \( \mathcal{P} \in \mathcal{L}^w \) be the manifest plant behavior, and \( \mathcal{N} \in \mathcal{L}^w \) be the hidden behavior. Then \( \mathcal{K} \in \mathcal{L}^w \) is implementable by a controller \( \mathcal{C} \in \mathcal{L}^c \) acting on the control variables if and only if \( \mathcal{N} \subseteq \mathcal{K} \subset \mathcal{P} \).

Proof. The proof can be separated in different parts. First the given condition \( \mathcal{N} \subset \mathcal{K} \subset \mathcal{P} \) will be proved. After that, the “if” and “only if” parts of the theorem will be proved [17].

\( \mathcal{N} \subset \mathcal{K} \subset \mathcal{P} \):
Theorem 2.16 shows that \( \mathcal{K} \) can be any behavior that is wedged in between the given behaviors \( \mathcal{N} \) and \( \mathcal{P} \). This condition is quite intuitive:
\( \mathcal{K} \subset \mathcal{P} \): The controlled behavior has to be a part of the plant behavior. This is logical, because the controller implemented puts a restriction on the behavior of the plant.
\( \mathcal{K} \supset \mathcal{N} \): The behavior \( \mathcal{N} \) must remain possible whatever be the controller. This is also logical, because a controller can have an output \( c = 0 \), which results in the hidden behavior.

Only if...
Let \( \mathcal{P}_{\text{full}} \in \mathcal{L}^{w+c} \) be the full behavior of the plant. Assume that \( \mathcal{K} \in \mathcal{L}^w \) is implemented by \( \mathcal{C} \in \mathcal{L}^c \). Then
\[
\mathcal{P} = \{ w \in C^\infty(\mathbb{R}, \mathbb{R}^w) \mid \exists c \in C^\infty(\mathbb{R}, \mathbb{R}^c) \text{ such that } (w, c) \in \mathcal{P}_{\text{full}} \},
\]
\[
\mathcal{K} = \{ w \in C^\infty(\mathbb{R}, \mathbb{R}^w) \mid \exists c \in \mathcal{C} \text{ such that } (w, c) \in \mathcal{P}_{\text{full}} \},
\]
\[
\mathcal{N} = \{ w \in C^\infty(\mathbb{R}, \mathbb{R}^w) \mid (w, 0) \in \mathcal{P}_{\text{full}} \}.
\]
Clearly, \( \mathcal{N} \subset \mathcal{K} \subset \mathcal{P} \), as claimed.

if...
This part uses kernel representations and makes use of the earlier mentioned proposition 2.5. Assume that \( \mathcal{N} \subset \mathcal{K} \subset \mathcal{P} \). The full plant behavior \( \mathcal{P}_{\text{full}} \) can be described with the kernel representation:
\[
R(\frac{d}{dt})w = M(\frac{d}{dt})c.
\]

Then the hidden behavior \( \mathcal{N} \) can be described by the kernel representation \( R(\frac{d}{dt})w = 0 \).

Using the proposition, we conclude that \( \mathcal{K} \) admits a kernel representation of the form
\[
F(\frac{d}{dt})R(\frac{d}{dt})w = 0 \quad \text{with} \quad F(\frac{d}{dt}) \in \mathbb{R}^{\infty*}[\mathcal{K}].
\]
We now have to prove that the controller \( F(\frac{d}{dt})M(\frac{d}{dt})c = 0 \) implements \( \mathcal{K} \).

In other words:
\( \mathcal{K}' \) (the manifest behavior of \( R(\frac{d}{dt})w = M(\frac{d}{dt})c \)) is described by \( F(\frac{d}{dt})M(\frac{d}{dt})c = 0 \) and equals the behavior \( \mathcal{K} \).

Let \( w' \in \mathcal{K}' \), then \( w' \) satisfies \( R(\frac{d}{dt})w' = M(\frac{d}{dt})c \) for some \( c \) such that \( F(\frac{d}{dt})M(\frac{d}{dt})c = 0 \). This implies that:
\[
F(\frac{d}{dt})R(\frac{d}{dt})w' = 0, \ \text{whence} \ w' \in \mathcal{K} \text{ and } \mathcal{K}' \subset \mathcal{K}.
\]
Conversely, let \( w \in \mathcal{K} \). Then \( F(\frac{d}{dt})R(\frac{d}{dt})w = 0 \) and there exists a \( c \) such that \( R(\frac{d}{dt})w = M(\frac{d}{dt})c \). This \( c \) hence satisfies also \( F(\frac{d}{dt})M(\frac{d}{dt})c = 0 \). Hence \( w \) is such that
\[
R(\frac{d}{dt})w = M(\frac{d}{dt})c \text{ for some } c \text{ that satisfies } F(\frac{d}{dt})M(\frac{d}{dt})c = 0.
\]
Consequently, \( w \in \mathcal{K}' \) and \( \mathcal{K} \subset \mathcal{K}' \).

If \( \mathcal{K} \in \mathcal{L}_\text{cont}^w \), then the controller implements it can also be taken to be controllable. For if \( \mathcal{C} \) implements \( \mathcal{K} \), then its controllable part also implements \( \mathcal{K} \). \( \Box \)

In additions to implementability issues, the hidden behavior \( \mathcal{N} \) plays a role in the properties of observability and detectability within the full plant behavior \( \mathcal{P}_{\text{full}} \). The following lemma (Lemma 3.13 of [2]), which will be used later on, formalises the statement that \( w \) is observable or detectable from \( c \) in some cases:

Lemma 2.17. Let \( \mathcal{P}_{\text{full}} \in \mathcal{L}^{w+c} \) and let \( \mathcal{N} \) be the hidden behavior as defined in Definition 2.2. Then we have
1. in \( \mathcal{P}_{\text{full}} \), \( w \) is observable from \( c \) if and only if \( \mathcal{N} = 0 \), and
2. in \( \mathcal{P}_{\text{full}} \), \( w \) is detectable from \( c \) if and only if \( \mathcal{N} \) is autonomous and stable.
Chapter 3

Controller synthesis: an $L_2$ behavioral approach

Now the idea of a behavior is introduced in the Literature study, algorithms for controller synthesis using an $L_2$ behavioral approach are introduced in this chapter. This will be done for both interconnection problems that are given in Chapter 2. First of all, some spaces and operators have to be introduced, which is done in the first section of this chapter.

When those notations are defined, the $L_2$ behavior can be introduced. This type of behavior differs from the one introduced by [15]. Now, we don’t require the trajectories $w$ to be infinite smooth, but they have to be elements of $L_2$ in the time domain, or using Parseval, elements of $L_2$ in the frequency domain.

These $L_2$-behaviors are used to solve the full- and partial-interconnection problems, that are already introduced in Chapter 1, but also recapped in the last section of this chapter using a $L_2$ behavioral approach.

3.1 Notation

3.1.1 Dynamical systems

Recapping, following the behavioral formalism, a dynamical system [2] is a triple:

$$
\Sigma = (T, W, B),
$$

where $T \subseteq \mathbb{R}$ is the time-axis, $W$ is the variable signal space, which typically contains inputs and outputs and will be taken to be a finite dimensional vector space throughout, and $B \subseteq W^T$ is the behavior.

3.1.2 Hilbert spaces

The $L_2$, $L_2^+$ and $L_2^-$ (Hilbert) spaces are defined as:

$$
\begin{align*}
L_2 &= L_2(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R}^q \mid \| f \|_2 < \infty \}, \\
L_2^+ &= L_2(\mathbb{R}_+) := \{ f : \mathbb{R}_+ \to \mathbb{R}^q \mid \| f \|_2 < \infty \}, \\
L_2^- &= L_2(\mathbb{R}_-) := \{ f : \mathbb{R}_- \to \mathbb{R}^q \mid \| f \|_2 < \infty \}.
\end{align*}$$

(3.1a)

where the norms are defined as:

$$
\| f \|_2 = \sqrt{\langle f, f \rangle} = \left( \int_T |f(\mu)|^2 \, d\mu \right)^{\frac{1}{2}},
$$

(3.1b)
and the inner products

\[ \langle f, g \rangle = \int_{\mathbb{T}} f(\mu)^T g(\mu) d\mu, \]  

(3.1c)

with \( \mathbb{T} \) equal to \( \mathbb{R} \), \( \mathbb{R}_+ = [0, \infty) \) or \( \mathbb{R}_- = (-\infty, 0) \), respectively. So, in short we define \( L_2(\mathbb{T}) \) with \( \mathbb{T} \subseteq \mathbb{R} \). The superscript \( q \) means that it is possible to map to multi-variable systems. By definition it follows that any \( w \in L_2 \) can be uniquely decomposed as \( w = w_+ + w_- \), with \( w_+ \in L_2^+ \) and \( w_- \in L_2^- \), where \( \| w \|_2 = \| w_+ \|_2 + \| w_- \|_2 \).

This can be written as a direct sum decomposition:

\[ L_2(\mathbb{R}) = L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_-). \]

For completeness, when using discrete time, we use the \( \ell_2^+ \) and \( \ell_2^- \) spaces:

\[
\ell_2^+ = \ell_2^+(\mathbb{Z}) := \{ h : \mathbb{Z} \to \mathbb{R}^q \mid \| h \|_2 < \infty \}, \\
\ell_2^- = \ell_2^-(\mathbb{Z}) := \{ h : \mathbb{Z} \to \mathbb{R}^q \mid \| h \|_2 < \infty \},
\]

with the norm

\[ \| h \|_2 = \sqrt{\langle h, h \rangle} = \left( \sum_{\mu \in \mathbb{T}} |h(\mu)|^2 \right)^{\frac{1}{2}}, \]

and inner product

\[ \langle h, k \rangle = \sum_{\mu \in \mathbb{T}} h(\mu)^T k(\mu), \]

where \( \mathbb{T} \) is equal to \( \mathbb{Z}, \mathbb{Z}_+ \) and \( \mathbb{Z}_- \), respectively. Similarly, \( \ell_2 = \ell_2^+ \oplus \ell_2^- \).

### 3.1.3 Hardy spaces

Also the normed Hardy spaces \( \mathcal{H}_p^+ \) and \( \mathcal{H}_p^- \) have to be introduced, where \( p = 1, 2, \ldots, \infty \):

\[
\mathcal{H}_p^+ := \{ f : \mathbb{C}^+ \to \mathbb{C}^q \mid \| f \|_{\mathcal{H}_p^+} < \infty \}, \\
\mathcal{H}_p^- := \{ f : \mathbb{C}^- \to \mathbb{C}^q \mid \| f \|_{\mathcal{H}_p^-} < \infty \},
\]

(3.2a)

where \( \mathbb{C}^+ := \text{Re } s > 0 \) and \( \mathbb{C}^- := \text{Re } s < 0 \), with \( s = \sigma + j\omega \). Note that \( \mathcal{H}_p^+ \) is analytic\(^3\) on \( \{ \mathbb{C}^+ \cup \infty \} \) and that \( \mathcal{H}_p^- \) is analytic on \( \{ \mathbb{C}^- \cup -\infty \} \). The \( \mathcal{H}_p^+ \) spaces are the classical Hardy spaces.

The norms for those spaces are defined as

\[
\| f \|_{\mathcal{H}_p^+} = \begin{cases} 
\sup_{\sigma > 0} \left( \int_{-\infty}^{\infty} |f(\sigma + j\omega)|^p d\omega \right)^{\frac{1}{p}}, & 0 < p < \infty, \\
\sup_{\sigma > 0} \sup_{\omega \in \mathbb{R}} |f(\sigma + j\omega)|, & p = \infty,
\end{cases}
\]

(3.2b)

and

\[
\| f \|_{\mathcal{H}_p^-} = \begin{cases} 
\sup_{\sigma < 0} \left( \int_{-\infty}^{\infty} |f(\sigma + j\omega)|^p d\omega \right)^{\frac{1}{p}}, & 0 < p < \infty, \\
\sup_{\sigma < 0} \sup_{\omega \in \mathbb{R}} |f(\sigma + j\omega)|, & p = \infty.
\end{cases}
\]

(3.2c)

\(^3\)A function is analytic if it is complex differentiable.
Note: To make the Hardy spaces well defined normed spaces on the closure \(\overline{\mathbb{C}^+}\), also \(\mathbb{C}^0 = j\mathbb{R}\) has to be included in the domain of definition. Here, this is explained specifically for \(f \in \mathcal{H}^+_p\). Define \(\mathcal{F} : \mathbb{C}^+ \cup \mathbb{C}^0 \to \mathbb{C}^q\) by setting
\[
\mathcal{F}(s) := \begin{cases} f(s), & s \in \mathbb{C}^+, \\ \lim_{\sigma \to 0} f(\sigma + j\omega), & s = \sigma + j\omega \in \mathbb{C}^0. \end{cases}
\]
Define:
\[
\|f\|_p := \begin{cases} \infty \int_{-\infty}^{\infty} |\mathcal{F}(j\omega)|^p d\omega, & 0 < p < \infty, \\ \sup_{\omega \in \mathbb{R}} |\mathcal{F}(j\omega)|, & p = \infty. \end{cases}
\]
(3.3)
Then, \(\|f\|_p = \|f\|_{\mathcal{H}^+_p}\) so that we generally can compute the \(\mathcal{H}^+_p\) norm of \(f \in \mathcal{H}^+_p\) on the imaginary axis by the expression (3.3). This is similar for any \(f \in \mathcal{H}^-_p\).

3.1.4 Rational functions and units

Another issue that is required for the algorithm later on, is the introduction of (normed) spaces that consist of rational functions. We need them, because we are interested in algorithms that are using state space representations, which can be written as rational functions (e.g. \(f = C(sI - A)^{-1}B + D\)).

The rational functions in a space are denoted using the prefix \(R\). In this report, the rational Hardy spaces \(RH^+_p\) and \(RH^-_p\) are used in algorithms and definitions:
\[
RH^+_p := \{ f \in \mathcal{H}^+_p \mid f \text{ is rational} \}, \quad RH^-_p := \{ f \in \mathcal{H}^-_p \mid f \text{ is rational} \}. \quad (3.4)
\]
Rational functions whose inverses belong to the same class are called units and denoted with the prefix \(U\). That is:
\[
UH^+_\infty := \{ U \in RH^+_\infty \mid U^{-1} \in RH^+_\infty \}, \quad UH^-_\infty := \{ U \in RH^-_\infty \mid U^{-1} \in RH^-_\infty \}.
\]
Note that units are necessarily square rational matrices.

3.1.5 Laplace transformation

A very commonly used transformation is the Laplace operation. The Laplace transform: \(\mathcal{L} : L_2(\mathbb{R}, \mathbb{R}^q) \to L_2(\mathbb{C}, \mathbb{C}^q)\) defines an isometry between \(L_2\) and \(\mathcal{L}_2\) and is defined by:
\[
\mathcal{L}(f) = F(s) := \int_{-\infty}^{\infty} f(t)e^{-st} dt.
\]
Its inverse \(\mathcal{L}^{-1} : L_2 \to L_2\) is given by:
\[
\mathcal{L}^{-1}(F) := \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)e^{st} ds.
\]
Because this transformation is isomorph, the operation is norm- and inner-product preserving. That is \((f, g) = (\mathcal{L}(f), \mathcal{L}(g))\).
This transformation can be applied to the (complete) \( L_2 \) space, as defined in (3.1), which results in a new inner-product space \( L_2 \):

\[
L_2 := \{ f : \mathbb{C} \rightarrow \mathbb{C}^q \mid \|f\|_2 < \infty \},
\]

which inherits the following norm:

\[
\|f\|_2^2 = \int_{-\infty}^{\infty} |f(j\omega)|^2 d\omega,
\]

and the inner product:

\[
\langle f, g \rangle = \int_{-\infty}^{\infty} f(j\omega) H g(j\omega) d\omega.
\]

We then have that \( L_2 \) admits the orthogonal decomposition

\[
L_2 = \mathcal{H}_2^+ \oplus \mathcal{H}_2^-,
\]

so that any element \( w \in L_2 \) can be uniquely decomposed as \( w = w_+ + w_- \) where

\[
w_+ := \Pi_+ w, \quad \text{with} \quad \Pi_+ : L_2 \rightarrow \mathcal{H}_2^+,
\]

\[
w_- := \Pi_- w, \quad \text{with} \quad \Pi_- : L_2 \rightarrow \mathcal{H}_2^-.
\]

Here, \( \Pi_+ \) and \( \Pi_- \) denote the canonical projections from \( L_2 \) onto \( \mathcal{H}_2^+ \) and \( \mathcal{H}_2^- \), respectively. With this knowledge, it is possible to extend the previous introduced \( \mathcal{H}_2^+ \) and \( \mathcal{H}_2^- \) spaces into inner-product spaces. Summarizing:

\[
L : L_2(\mathbb{R}) \rightarrow L_2 = \mathcal{H}_2^+ \oplus \mathcal{H}_2^-,
\]

\[
L : L_2(\mathbb{R}^+) \rightarrow \mathcal{H}_2^+,
\]

\[
L : L_2(\mathbb{R}^-) \rightarrow \mathcal{H}_2^-.
\]

### 3.1.6 Stable and anti-stable representations

The plants and controlled descriptions in this report are represented by elements in either of the Hardy spaces \( \mathcal{RH}^\infty_+ \) and \( \mathcal{RH}^\infty_- \) that are defined in (3.4). In those definitions is given, that the norm on \( \mathbb{C}^+ \) or \( \mathbb{C}^- \) respectively should be finite. This means that the “poles” used in the representations have to be in the left- or right-half plane, because integrating over a pole results in a norm going to infinity. So recapping, these spaces can also be named as:

\[
\mathcal{RH}^\infty_{\text{stable}} := \mathcal{RH}^\infty_+ \quad \text{and} \quad \mathcal{RH}^\infty_{\text{anti-stable}} := \mathcal{RH}^\infty_-.
\]

### 3.1.7 Mappings in \( \mathcal{RH}^\infty_+ \) and \( \mathcal{RH}^\infty_- \)

Now the Hardy spaces are introduced, we can use them to define the behavior of the plants and controllers. In this section, representations using elements of two different spaces are used, namely: \( \mathcal{RH}^\infty_+ \) (stable representations) and \( \mathcal{RH}^\infty_- \) (anti-stable representations).

First of all, the stable ones are discussed, let \( \Theta \in \mathcal{RH}^\infty_+ \). Then we associate with \( \Theta \) a mapping

\[
\Theta : L_2 \rightarrow L_2, \quad \Theta : \mathcal{H}_2^+ \rightarrow \mathcal{H}_2^+,
\]

by defining

\[
(\Theta w)(s) := \Theta(s)w(s), \quad \text{where} \ w \in \{L_2, \mathcal{H}_2^+, \mathcal{H}_2^-\},
\]

which is the usual “multiplication” operator in the frequency domain.
Conversion to the time domain:

To make those mappings more clear, we interpret (3.7) in the time domain as follows:

$$\theta(t) := L^{-1}(\Theta(s)) \in L_1(\mathbb{R})$$

which means that the impulse θ is absolute integrable. The multiplication (3.7) in the frequency domain is equivalent to a convolution in time domain. Therefore, we define the mapping \( \theta : L_2 \to L_2 \) as:

$$\theta w(t) := (\theta * w)(t) := \int_{-\infty}^{\infty} \theta(t-\tau)w(\tau) d\tau.$$  

Note that whenever \( \Theta \in \mathcal{R}\mathcal{H}^\pm_{\infty} \) we have that \( \theta(t) = 0 \) for all \( t < 0 \), which results in an impulse response \( \theta(t) \) that is zero for \( t < 0 \).

So, when using three elements \( w \) from the three different spaces \( (L_2, L_2^+, L_2^-) \) and taking the convolution with a (stable) impulse response \( \theta \), this results in Figure 3.1. The results of these convolutions are in the spaces \( L_2, L_2^+ \) and \( L_2^- \) respectively, which can be converted into the frequency space using the Laplace operator.

So, to summarize the mappings in \( \mathcal{R}\mathcal{H}^\pm_{\infty} \) the following theorem is introduced:

**Theorem 3.1.** Using \( \Theta(s) \in \mathcal{R}\mathcal{H}^\pm_{\infty} \) (so \( \theta(t) \in L_1(\mathbb{R}) \), where \( \theta(t) = 0 \) for all \( t < 0 \)), with possible arguments \( L_2, \mathcal{H}^+_{2}, \mathcal{H}^-_{2} \) (or \( L_2, L_2^+ \) and \( L_2^- \)), this results in the mappings:

$$\Theta : L_2 \to L_2, \quad \theta : L_2 \to L_2,$$

$$\Theta : \mathcal{H}^+_{2} \to \mathcal{H}^+_{2}, \quad \theta : L_2^+ \to L_2^+,$$

$$\Theta : \mathcal{H}^-_{2} \to L_2, \quad \theta : L_2^- \to L_2.$$  

As mentioned, we can also define similar mappings with anti-stable representations \( \Psi \in \mathcal{R}\mathcal{H}^\pm_{\infty} \), according to:

$$\Psi w(s) := \Psi(s)w(s), \quad \text{where } w \in \{ L_2, \mathcal{H}^+_{2}, \mathcal{H}^-_{2} \}.$$  

This also results in three possible mappings, which are also depicted in the time domain. Therefore, the inverse Laplace transformation has to be applied to \( \Psi \), such that

$$\psi(t) := L^{-1}(\Psi(s)), \quad \text{which defines the convolution } (\psi w)(t) := (\psi * w)(t).$$
In continuous time, we introduce the left- and right-shift operators,

3.1.8 Left- and right-shift invariant systems

so, with these results in the time domain, the mappings in the frequency domain can be summarized in the following theorem.

Theorem 3.2. Using \( \Psi(s) \in \mathcal{RH}_\infty \) (so \( \psi(t) \in L_1(\mathbb{R}) \), where \( \psi(t) = 0 \) for all \( t > 0 \)), with possible arguments \( L_2, \mathcal{H}_2^+ \) and \( \mathcal{H}_2^- \) (or \( L_2, L_2^+ \) and \( L_2^- \)), this results in the mappings:

\[
\begin{align*}
\Psi &: L_2 \to L_2, \\
\psi &: L_2 \to L_2, \\
\Psi &: \mathcal{H}_2^+ \to L_2, \\
\psi &: L_2^+ \to L_2, \\
\Psi &: \mathcal{H}_2^- \to \mathcal{H}_2^-, \\
\psi &: L_2^- \to L_2. 
\end{align*}
\]  

(3.9)

3.1.8 Left- and right-shift invariant systems

In continuous time, we introduce the left- and right-shift operators, \( \hat{\sigma}_L \) and \( \hat{\sigma}_R \) respectively, which refer to shifts of signals with respect of the time axis. Those operations are defined as:

\[
\begin{align*}
\hat{\sigma}_L &: L_2 \to L_2, \\
\hat{\sigma}_R &: L_2 \\
\hat{\sigma}_L &: L_2^+ \to L_2^+, \\
\hat{\sigma}_R &: L_2^+ \to L_2^+, \\
\hat{\sigma}_L &: L_2^- \to L_2^-, \\
\hat{\sigma}_R &: L_2^- \to L_2^-,
\end{align*}
\]

which are defined by:

\[
\begin{align*}
(\hat{\sigma}_L \hat{\omega})(t) &= \hat{\omega}(t + \tau), \\
(\hat{\sigma}_R \hat{\omega})(t) &= \hat{\omega}(t - \tau),
\end{align*}
\]

with \( \hat{\omega}(t) \) an element of \( L_2, H_2^+ \) or \( H_2^- \). Because we are using rational mappings, we transform these operators to the Laplace domains, so the new operators act on \( L_2 \) and the Hardy spaces \( \mathcal{H}_2^+ \) and \( \mathcal{H}_2^- \):

\[
\begin{align*}
\sigma_L &: L_2 \to L_2, \\
\sigma_L &: \mathcal{H}_2^+ \to \mathcal{H}_2^+, \\
\sigma_L &: \mathcal{H}_2^- \to \mathcal{H}_2^-, \\
\sigma_R &: L_2 \to L_2, \\
\sigma_R &: \mathcal{H}_2^+ \to \mathcal{H}_2^+, \\
\sigma_R &: \mathcal{H}_2^- \to \mathcal{H}_2^-,
\end{align*}
\]

\[
\begin{align*}
(\sigma_L w)(s) &= e^{s\tau} w(s), \\
(\sigma_L w)(s) &= e^{s\tau} (w(s) - \int_0^\tau \hat{\omega}(t)e^{-st} dt), \\
(\sigma_L w)(s) &= e^{s\tau} w(s), \\
(\sigma_R w)(s) &= e^{-s\tau} w(s), \\
(\sigma_R w)(s) &= e^{-s\tau} (w(s) - \int_{-\tau}^0 \hat{\omega}(t)e^{-st} dt). 
\end{align*}
\]
These mappings in time domain are depicted in Figure 3.3 for a signal \( \hat{w}(t) \in H^+_2 \). Note that \( \sigma_L \) and \( \sigma_R \) do not define isometries when applied to \( H^+_2 \) and \( H^-_2 \) respectively. When applied to \( L^2 \) they do.

In the beginning of this chapter, a dynamical system is introduced as a triple \( \Sigma = (T, W, B) \), where \( B \) is the behavior of the system. This behavior describes the trajectories of the variables in the time domain. Using the two introduced operators, one can define the following properties of the system:

**Definition 3.3.** A dynamical system \( \Sigma = (T, W, B) \) is said to be left-shift invariant if \( \sigma_L B \subseteq B \) and right-shift invariant if \( \sigma_R B \subseteq B \).

### 3.1.9 Inner- and outer representations

A nice property of a rational mapping \( \Theta \in RH^+_{\infty} \) or \( \Psi \in RH^-_{\infty} \) is that an inner-outer decomposition can be made for them [12]. This has some nice properties, which are required in an algorithm later on. First, the definitions for inner- and outer rational matrices have to be given.

**Definition 3.4.** Define a \( T \in RH^+_{\infty} \) which describes the transfer function \( y(s) = T(s)u(s) \), where \( \{y, u\} \in L^2 \). This \( T \) is called inner if holds that:

\[
\|Tu\|_2 = \|u\|_2, \quad \text{for all } u \in L^2.
\]

So, one has to verify if:

\[
\int_{-\infty}^{\infty} u(s)^* T(s)^* T(s) u(s) d\omega = \int_{-\infty}^{\infty} u(s)^* u(s) d\omega,
\]

where * denotes the conjugate transposed. As can be seen, \( T \) is inner if \( T(s)^* T(s) = I \) for all \( s \in \mathbb{C}^0 \). Knowing this, a nice property of an inner matrix can be obtained by observing eigenvalues:

\[
eig(T(s)^* T(s)) = \eig(I) = 1,
\]

which is equal to:

\[
\sqrt{\eig(T(s)^* T(s))} = \sigma(T(s)) = 1.
\]

So, in the case if \( T \) is inner, the singular values for all frequencies \( \omega \) are equal to 1.
For SISO systems $T \in \mathcal{RH}_\infty^+$ is inner if it is a finite Blaschke product multiplied by a complex number $p$ of unit modulus ($|p|^2 = 1$). That is
\[
T(s) = p\prod_{i=1}^{n} \frac{|a_i|}{a_i} \frac{s + a_i}{s - a_i},
\]
where $\text{Re}(a_i) > 0$. One can show that in this situation, the Bode plot (or singular value plot) is equal to 1 for all possible frequencies $\omega$.

Now inner is known, also outer has to be defined [7, 12]:

**Definition 3.5.** An operator $T : \mathcal{L}_2 \rightarrow \mathcal{L}_2$ as multiplication, where $T \in \mathcal{RH}_\infty^+$, is outer if $\text{im} T$ is dense in $\mathcal{L}_2$.

This definition results in the following theorem:

**Theorem 3.6.** Let $T : \mathcal{L}_2 \rightarrow \mathcal{L}_2$, $T \in \mathcal{RH}_\infty^+$, be as defined by multiplication. Then, the following three items are equivalent:

1. $T$ is outer,
2. The determinental rank of $T$, which is largest integer $r$ for which there exists a non-zero $r \times r$ minor of $T$, is equal to the “normal” matrix rank evaluated at a certain $\lambda \in \mathbb{C}^+$, so:
\[
\text{rank}_{\text{det}}(T(s)) = \text{rank}_{\lambda} T(\lambda), \quad \forall \lambda \in \mathbb{C}^+.
\]
3. When $T(s)$ is written in Smith-McMillan form (see Appendix A.1), there are no zeros in the right half plane.

**Note:** If $T$ is also a square matrix, one can notice that $T$ is a unit $\in \mathcal{UH}_\infty^+$.

**Theorem 3.7.** As defined in [12], every matrix $T \in \mathcal{RH}_\infty^+$ of dimension $n \times m$ containing rational elements, $T \in \mathcal{RH}_\infty^+$, can be decomposed as:
\[
T = T_i T_o,
\]
where $T_i$ is inner and $T_o$ is outer. If $n \geq m$, then $T_o$ is square, whereas if $n \leq m$, then $T_i$ is square.

With appropriate adaptations on definitions, those theorems and definitions can also be applied using $T \in \mathcal{RH}_\infty^-$.

**Example 3.8.** This inner-outer decomposition will be illustrated using a very simple example. In this case, the function that has to be decomposed is defined as:
\[
g(s) = \frac{s + 3}{s - 4} \in \mathcal{RH}_\infty^+.
\]
This function can be decomposed as:
\[
g(s) = g_i(s)g_o(s) = \frac{s + 3}{s - 4} = \frac{s + 4}{s - 4} \frac{s + 3}{s + 4},
\]
inner \hspace{1cm} outer
3.2 Rational representations of behaviors

Using the notational remarks of the previous section and the definition of the infinite smooth behaviors in the Literature chapter, it is possible to introduce the $L_2$ behaviors. In the previous section, a stable-and an anti-stable-mapping are introduced. They can both be used to represent $L_2$ behaviors. Using those mappings, we have to take into account that the manifest variables $w$, which are now in frequency domain, can be elements of the following three spaces: $L_2$, $H_2^+$ and $H_2^-$.

In this section, the $L_2$ behavior of a plant $P$ is defined but it is of course similar for those that describe the controller- and controlled behavior $C$ and $K$.

First, we associate $L_2$ systems with an arbitrary rational element $P \in RH_\infty^+$. This $P$ is similar to the already introduced $\Theta$ in (3.8). To preserve the usage of kernels later on in the algorithms, the behavior of a plant $P$ is introduced as follows for the manifest variables $w \in (L_2, H_2^+, H_2^-)$:

\[
P := P(P) := \{ w \in L_2 \mid P(s)w(s) = 0 \} = \ker P \subset L_2,
\]

\[
P_+ := P_+P := \{ w \in H_2^+ \mid P(s)w(s) = 0 \} = \ker P \subset H_2^+,
\]

\[
P_- := P_-P := \{ w \in H_2^- \mid P(s)w(s) \in H_2^+ \} = \ker \Pi_-P \subset H_2^-,
\]

where $P \in RH_\infty^+$ and $\Pi_-$ is a canonical projection that is introduced before.

We have that:

**Lemma 3.9.** For $P \in RH_\infty^+$, the systems $P(P)$, $P_+(P)$ and $P_-(P)$ are linear and right-shift invariant subsets of $L_2$, $H_2^+$ and $H_2^-$, respectively.

We will denote by $M, M_+$ and $M_-$ the classes of all linear and right-shift invariant systems in $L_2$, $H_2^+$ and $H_2^-$ that admit representations as the kernel of a rational element $P \in RH_\infty^+$.

Similarly, for any $\hat{P} \in RH_\infty^-$ we introduce the three dynamical systems ($C, C^\dagger, P_{(\pm)}$):

\[
P := \hat{P}(P) := \{ w \in L_2 \mid \hat{P}(s)w(s) = 0 \} = \ker \hat{P} \subset L_2,
\]

\[
P_+ := \hat{P}_+P := \{ w \in H_2^+ \mid \hat{P}(s)w(s) \in H_2^+ \} = \ker \Pi_+\hat{P} \subset H_2^+,
\]

\[
P_- := \hat{P}_-P := \{ w \in H_2^- \mid \hat{P}(s)w(s) = 0 \} = \ker \hat{P} \subset H_2^-,
\]

where $\hat{P} \in RH_\infty^-$ and $\Pi_+$ is the canonical projection $L_2 \rightarrow H_2^+$.

**Lemma 3.10.** For $\hat{P} \in RH_\infty^-$ the systems $\hat{P}(P)$, $\hat{P}_+(P)$ and $\hat{P}_-(P)$ are linear and left-shift invariant subsets of $L_2$, $H_2^+$ and $H_2^-$, respectively.

We will denote by $L, L_+$ and $L_-$ the classes of all linear and left-shift invariant systems in $L_2$, $H_2^+$ and $H_2^-$ that admit representations as the kernel of a rational element $\hat{P} \in RH_\infty^-$.

3.3 Controller synthesis - full interconnection

In this section we consider the full interconnection controller synthesis problem that can be formally stated as follows:

**3.3.1 Problem description**

**Problem 3.11.** Given two linear left-shift invariant systems $P$ and $K$ in the class $L$ (or $L_+$ or $L_-$), construct, if it exists, a linear left-shift-invariant system $C$ such that $P \cap C = K$. Any such system is said to implement $K$ for $P$ by full interconnection. A similar problem formulation applies to the model classes $M, M_+$ and $M_-$. 
CHAPTER 3. CONTROLLER SYNTHESIS: AN $\mathcal{L}_2$ BEHAVIORAL APPROACH

Figure 3.4(a) gives an illustration of the problem treated in this section, namely the existence of a controller $C$ that after intersection with the plant $P$ results in the controlled behavior $K$. In this section, systems are interconnected through all variables $w$. That is, we consider full-interconnections as depicted in Figure 3.4(b).

Our synthesis algorithm is inspired by the polynomial analog that has been treated in [9, 11]. Our algorithm leads to explicit rational representations of all systems $C$ that implement $K$ for $P$, whenever they exist.

**Theorem 3.12.** Given the systems $P, K \in \mathcal{L}_2(\pm)$ (or $\mathcal{M}(\pm)$). There exists a controller $C \in \mathcal{L}_2(\pm)$ (or $\mathcal{M}(\pm)$), that implements $K$ for $P$ by full interconnection if and only if $K \subset P$.

Our proof of this theorem is similar to the one in [9], which gives a comparable result for behaviors that are defined as the continuous and infinitely often differentiable functions in the kernel of a polynomial differential operator.

The proof of Theorem 3.12 will result from the following theorems and algorithms:

**Theorem 3.13.** Let $K, P \in \mathcal{R}\mathcal{H}_\infty^+$ and let $P(\pm) = P(\pm)(P)$ and $K(\pm) = K(\pm)(K)$ be their associated behaviors as defined in (3.10) (so, $P, K \in \mathcal{M}(\pm)$).

Equivalent are:

1. $K \subset P$,
2. $K_+ \subset P_+$,
3. $K_- \subset P_-$,
4. $\exists F \in \mathcal{R}\mathcal{H}_\infty^+$ such that $P = FK$.

Moreover, $K = P \iff K_+ = P_+ \iff K_- = P_- \iff \exists U \in \mathcal{U}\mathcal{H}_\infty^+$ such that $P = UK$.

**Proof.**

(iiv $\Rightarrow$ {i, ii, iii}):  

- iiv $\Rightarrow$ i: $K, P \subset \mathcal{L}_2$.

Suppose $P = FK$. Take $w \in K \subset \mathcal{L}_2$. Then, $v = Kw = 0$, so also $Pw = FKw = Fv = 0$. This implies that $P(s)w(s) = 0$, so $w \in \mathcal{P}$. Since $w \in K$ is arbitrary, it follows that $K \subset P$.

- iiv $\Rightarrow$ ii: $K, P \subset \mathcal{H}_\infty^+$, so $w \in \mathcal{H}_\infty^+$:

This proof is identical to the case when $K, P \subset \mathcal{L}_2$.  

Figure 3.4: Full interconnection problem.
3.3. CONTROLLER SYNTHESIS - FULL INTERCONNECTION

- iv $\Rightarrow$ iii:
  \( \mathcal{K}, \mathcal{P} \subseteq \mathcal{H}_{\infty}^{+} \), so \( w \in \mathcal{H}_{\infty}^{+} \):
  Again, suppose \( P = FK \) and take \( w \in \mathcal{K} \). Define \( v = Kw \). Then \( v \in \mathcal{H}_{\infty}^{+} \) and hence \( Pw = FKw = Fv \). Now, \( P \in \mathcal{R}\mathcal{H}_{\infty}^{+} \) and \( v \in \mathcal{H}_{\infty}^{+} \). Consequently, \( Fv \in \mathcal{H}_{\infty}^{+} \) and it follows that \( Pw \in \mathcal{H}_{\infty}^{+} \). Using (3.10) this gives that \( w \in \mathcal{P} \). Since \( w \in \mathcal{K} \) is arbitrary, \( w \in \mathcal{P} \) which gives the result.

\( (iv \Leftarrow \{i,ii,iii\}) : \)

- iv $\Leftarrow$ i:
  \( \mathcal{K}, \mathcal{P} \subseteq \mathcal{L}_{2} \), so \( w \in \mathcal{L}_{2} \):
  Using the definition of \( \mathcal{K} \), one can write:
  \[
  \mathcal{K} = \{ w \in \mathcal{L}_{2} \mid (Kw,v)_{\mathcal{L}_{2}} = 0 \forall v \in \mathcal{L}_{2} \} = \{ w \in \mathcal{L}_{2} \mid (w,K^{*}v)_{\mathcal{L}_{2}} = 0 \forall v \in \mathcal{L}_{2} \} = (\text{im } K^{*})^{\perp}
  \]
  where \( K^{*} : \mathcal{L}_{2} \to \mathcal{L}_{2} \) is the dual- or adjoint operator in \( \mathcal{R}\mathcal{H}_{\infty}^{+} \) defined by \( K^{*}(s) = K^{T}(s^{-1}) \).
  Something similar can be applied to the plant behavior. So, \( \mathcal{K} \subseteq \mathcal{P} \) implies that \( \mathcal{P}^{\perp} \subseteq \mathcal{K}^{\perp} \) and using the previous definition of \( \mathcal{K} \), this results in
  \[
  \overline{\text{im } P^{*}} \subseteq (\text{im } K^{*})^{\perp},
  \]
  where the bar denotes the closure in \( \mathcal{L}_{2} \). For rational operators the latter implies that:
  \[
  \text{im } P^{*} \subset (\text{im } K^{*})^{\perp},
  \]
  because in that case the images are closed.
  Then we can say that for some \( e_{i} \), \( P^{*}e_{i} \in \text{im } K^{*} \), so there exists a \( v_{i} \) such that:
  \( P^{*}e_{i} = K^{*}v_{i} \).
  This can be extended to a set of \( v_{i} \)'s, such that:
  \( P^{*} = K^{*}X \) with \( X = (v_{1}, \ldots, v_{p}) \in \mathcal{R}\mathcal{H}_{\infty}^{+} \subset \mathcal{R}\mathcal{H}_{\infty}^{+} \).
  Then, we can rewrite this to \( P = X^{*}K \), where \( F \) is equal to the dual operator \( X^{*} \in \mathcal{R}\mathcal{H}_{\infty}^{+} \).

- iv $\Leftarrow$ ii:
  \( \mathcal{K}, \mathcal{P} \subseteq \mathcal{H}_{\infty}^{+} \), so \( w \in \mathcal{H}_{\infty}^{+} \):
  This proof is similar to the one in the previous item, except that now the \( \mathcal{H}_{\infty}^{+} \) inner product is used. However, \( \mathcal{H}_{\infty}^{+} \) inherited this inner product from \( \mathcal{L}_{2} \).

- iv $\Leftarrow$ iii:
  \( \mathcal{K}, \mathcal{P} \subseteq \mathcal{H}_{\infty}^{+} \), so \( w \in \mathcal{H}_{\infty}^{+} \):\( \text{Now, } \mathcal{K} \) can be written as:
  \[
  \mathcal{K} = \{ w \in \mathcal{H}_{\infty}^{+} \mid \langle \Pi_{-}Kw, v \rangle_{\mathcal{H}_{\infty}^{+}} = 0 \forall v \in \mathcal{H}_{\infty}^{+} \}
  = \{ w \in \mathcal{H}_{\infty}^{+} \mid \langle w, K^{*}\Pi_{-}v \rangle_{\mathcal{H}_{\infty}^{+}} = 0 \forall v \in \mathcal{H}_{\infty}^{+} \}
  = (\text{im } K^{*}\Pi_{-})^{\perp},
  \]
  where \( K^{*} \) and \( \Pi_{-} \) are adjoint operators. This can also be done for the plant behavior \( \mathcal{P} \). As in item (iv $\Leftarrow$ i), \( \mathcal{P}^{\perp} \subseteq \mathcal{K}^{\perp} \), so: \( (\text{im } P^{*}\Pi_{-})^{\perp} \subset (\text{im } K^{*}\Pi_{-})^{\perp} \). Then there exists a \( X \in \mathcal{R}\mathcal{H}_{\infty}^{+} \) such that \( P^{*}\Pi_{-} = K^{*}\Pi_{-}X \).
  So, one can say that \( \Pi_{-}P = X^{*}\Pi_{-}K \), where \( F = X^{*} \).

Equality condition:
Using the previous items, one can say that \( \mathcal{P} = \mathcal{K} \) if and only if \( P = U_{1}K \) and \( K = U_{2}P \) with both \( U_{1} \) and \( U_{2} \) in \( \mathcal{R}\mathcal{H}_{\infty}^{+} \). Moreover, if \( U_{1} \) and \( U_{2} \) satisfy these conditions, then \( P = U_{1}U_{2}K \) and \( K = U_{2}U_{1}P \). If \( P \) and \( K \) are full rank, we find that \( U_{1} = U_{2}^{-1} \), which completes the proof.

\[^{2}e_{i} = [0, \ldots, 1, \ldots, 0] \text{, with the 1 on the } i^{th} \text{ position.}\]
Also anti-stable mappings can be used in the representations, which yields the following theorem:

**Theorem 3.14.** Let \( \hat{K}, \hat{P} \in \mathcal{RH}_\infty \) and let \( \mathcal{P}(\pm) = \mathcal{P}(\hat{P}) \) and \( \mathcal{K}(\pm) = \mathcal{K}(\hat{K}) \) as in (3.11) (so, \( \mathcal{P}, \mathcal{K} \in \mathcal{L}(\pm) \)). Equivalent are:

- \( \mathcal{K} \subset \mathcal{P} \)
- \( \mathcal{K}^+ \subset \mathcal{P}^+ \)
- \( \mathcal{K}^- \subset \mathcal{P}^- \)
- \( \exists \hat{F} \in \mathcal{RH}_\infty \) such that \( \hat{P} = \hat{F} \hat{K} \).

Moreover, \( \mathcal{K} = \mathcal{P} \iff \mathcal{K}^+ = \mathcal{P}^+ \iff \mathcal{K}^- = \mathcal{P}^- \iff \exists \hat{U} \in \mathcal{RH}_\infty \) such that \( \hat{P} = \hat{U} \hat{K} \).

The proof of this theorem is similar to the one of Theorem 3.13 and therefore is not included in this report. Those theorems are used in the following algorithm.

### 3.3.2 Algorithm

With the preceding theorems, it is possible to define the algorithm that is used to find controllers that result in the desired controlled behavior (by full interconnection). In this case, this will be done using trajectories \( w \in \mathcal{L}_2 \) and anti-stable mappings in \( \mathcal{RH}_\infty \). The algorithm will result in one controller \( \mathcal{C} \in \mathcal{L} \).

We emphasize that the construction is similar for controllers \( \mathcal{C} \) in the classes \( \mathcal{L}(\pm) \) and \( \mathcal{M}(\pm) \).

**Algorithm 3.1.** Given \( \mathcal{P}, \mathcal{K} \in \mathcal{RH}_\infty \) that define the systems \( \mathcal{P} \) and \( \mathcal{K} \):

- \( \mathcal{P} = \{ w \in \mathcal{L}_2 \mid P(s)w(s) = 0 \} = \ker P \)
- \( \mathcal{K} = \{ w \in \mathcal{L}_2 \mid K(s)w(s) = 0 \} = \ker K \)

**Aim:** Find \( \mathcal{C} \in \mathcal{RH}_\infty \) that defines the system \( \mathcal{C} := \mathcal{C}(C) = \{ w \in \mathcal{L}_2 \mid C(s)w(s) = 0 \} = \ker C \in \mathcal{L} \), such that \( \mathcal{C} \) implements \( \mathcal{K} \) for \( \mathcal{P} \) in the sense that \( \mathcal{P} \cap \mathcal{C} = \mathcal{K} \) by full interconnection.

**Step 1:**
Verify whether \( \mathcal{K} \subset \mathcal{P} \). If so, there must exists a mapping \( F \in \mathcal{RH}_\infty \) such that \( P = FK \). If not, the algorithm ends and no controller exists that implements \( \mathcal{K} \) for \( \mathcal{P} \).

**Step 2:**
Introduce a unit \( U \in \mathcal{UH}_\infty \) such that the obtained \( F \) matrix can be brought into column reduced form, so:

\[
\mathcal{T} = FU = [F_1, 0],
\]

where \( F_1 \in \mathcal{RH}_\infty \) is square and full rank.

**Step 3:**
Extend this matrix \( \mathcal{T} \) using \( W = [0, I] \) such that:

\[
\mathcal{X} = \begin{bmatrix} \mathcal{T} \\ W \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \\ I \end{bmatrix},
\]

is square and a unit. Define \( W = WU^{-1} \). Then \( W \in \mathcal{RH}_\infty \).

**Step 4:**
Now, we have to introduce the behavior of the (to be estimated) controller:

\[
\mathcal{C} = \{ w \in \mathcal{L}_2 \mid C(s)w(s) = 0 \} = \ker C,
\]

where

\[
C = WK \in \mathcal{RH}_\infty.
\]
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Proof. In the controlled behavior $\mathcal{K}$, the restrictions of the plant as well as the restrictions applied by the controller have to be satisfied:

$$\mathcal{K} = \ker \begin{bmatrix} P \\ C \end{bmatrix} = \ker K = \ker(\Lambda K),$$

where $\Lambda \in \mathcal{UH}_\infty$ is a unit. Because the unit $\overline{\Lambda}$ is a multiplication of a unit $U$ with a certain matrix $\Lambda = \text{col}(F, W)$, also $\Lambda$ has to be a unit. Therefore:

$$\mathcal{K} = \ker \begin{bmatrix} P \\ C \end{bmatrix} = \ker \begin{bmatrix} F \\ W \end{bmatrix} K,$$

so a possible controller is described as the kernel of the rational function

$$C = WK$$

that belongs to $\mathcal{RH}_\infty$.

Step 5:
This controller is not the only one that realizes the desired controlled behavior $\mathcal{K}$, because another multiplication with a unit $Q \in \mathcal{UH}_\infty$ is possible:

$$\mathcal{K} = \ker \begin{bmatrix} \begin{bmatrix} I \\ Q_2 \end{bmatrix} Q_1 \\ P \end{bmatrix} = \ker \begin{bmatrix} Q_1 P + Q_2 W K \end{bmatrix},$$

where $Q_1, Q_2 \in \mathcal{RH}_\infty$ and $Q_2$ is full rank.

Result:
With this pre-multiplication, all possible controllers are given by the parametrization:

$$\mathcal{C}_{\text{par}} = \{ Q_1 P + Q_2 W K \mid Q_1, Q_2 \in \mathcal{RH}_\infty, Q_2 \text{ full rank} \},$$

such that the controller behavior $\mathcal{C} = \ker C$ with $C \in \mathcal{C}_{\text{par}}$.

Now this algorithm is known, the proof of Theorem 3.12 can be given:

Proof.
($\Rightarrow$): This is trivial.
($\Leftarrow$): If $\mathcal{K} \subset \mathcal{P}$, then there exists a $F$ as in Theorem 3.13 or 3.14, which can be used in Algorithm 3.1 that estimates a controller.

3.4 Controller synthesis - partial interconnection

Now the full interconnection situation is studied, an extension to the partial interconnection problem can be made. This makes it possible to take aspects like disturbances as noise into account. This expands the scope of applications to e.g. robust control problems.

In this partial interconnection case, depicted in Fig. 3.5(a), the behavior $\mathcal{P}_{\text{full}}$ consists of trajectories $(w, c) \in \mathcal{L}_2$ in which $w$ is interpreted as the set of manifest variables and $c$ as the set of “controller” variables. Refer to $\mathcal{P}_{\text{full}} \subseteq (W \times C)^2$ as the full plant behavior (here $C$ is a variable signal space). Any such system defines a manifest plant behavior:

$$\mathcal{P}_{\text{manifest}} = \{ w \in \mathcal{L}_2 \mid \exists c \in \mathcal{L}_2 : (w, c) \in \mathcal{P}_{\text{full}} \}.$$ 

If $\mathcal{P}_{\text{full}}$ is in $\mathcal{L}(\pm)$ (or in $\mathcal{M}(\pm)$) then it is immediate that $\mathcal{P}_{\text{manifest}}$ also belongs to $\mathcal{L}(\pm)$ (or to $\mathcal{M}(\pm)$).
Problem 3.15. Given two linear left-shift invariant systems $P_{\text{full}}$ and $K$ in the class $L$ (or $L_+$ or $L_-$), where $P_{\text{full}} \subseteq (\mathcal{W} \times \mathcal{L})^T$ and $K \subseteq \mathcal{W}^T$. Find, if it exists, a linear left-shift invariant system $C$ in the class $L$ (or $L_+$ or $L_-$) such that:

$$K = \{ w \in \mathcal{L}_2 \mid \exists c \in \mathcal{L}_2 : (w, c) \in P_{\text{full}}, c \in C \}.$$

Any such system is said to implement $K$ for $P_{\text{full}}$ by partial interconnection.

A similar problem formulation applies for the model classes $M_-$, $M_+$ and $M_-$ and for $(w, c) \in H^+_2$ or $(w, c) \in H^-_2$.

Note: As shown in Fig. 3.5(a), the system $C$ restricts $c$ only (and not $w$), the full plant behavior $P_{\text{full}}$ restricts $w$ as well as $c$ and the controlled behavior $K$ restricts $w$ only.

Now, the full plant behavior $P_{\text{full}}$ has to be introduced using stable- and anti-stable rational representations, as done for behaviors in the previous sections:

$$P_{\text{full}} := P_{\text{full}}(P) := \{(w, c) \in \mathcal{L}_2 \mid P(s) \begin{bmatrix} w(s) \\ c(s) \end{bmatrix} = 0 \} = \ker P \subset \mathcal{L}_2,$$

$$P_{\text{full},+} := P_{\text{full},+}(P) := \{(w, c) \in H^+_2 \mid P(s) \begin{bmatrix} w(s) \\ c(s) \end{bmatrix} = 0 \} = \ker P \subset H^+_2, \quad (3.12)$$

$$P_{\text{full},-} := P_{\text{full},-}(P) := \{(w, c) \in H^-_2 \mid P(s) \begin{bmatrix} w(s) \\ c(s) \end{bmatrix} \in H^+_2 \} = \ker \Pi_- P \subset H^-_2,$$

where $P = [P_1 \quad P_2] \in \mathcal{RH}^+_\infty$ and $\Pi_-$ is a canonical projection that is introduced before. The separation of $P$ in $P_1$ and $P_2$ is such that:

$$P(s) \begin{bmatrix} w(s) \\ c(s) \end{bmatrix} = \begin{bmatrix} P_1(s) & P_2(s) \end{bmatrix} \begin{bmatrix} w(s) \\ c(s) \end{bmatrix} = P_1(s)w(s) + P_2(s)c(s).$$

A similar definition can be made using anti-stable representations $\hat{P} = [\hat{P}_1 \quad \hat{P}_2] \in \mathcal{RH}^-\infty$:

$$P_{\text{full},+} := P_{\text{full},+}(\hat{P}) := \{(w, c) \in \mathcal{L}_2 \mid \hat{P}(s) \begin{bmatrix} w(s) \\ c(s) \end{bmatrix} = 0 \} = \ker \hat{P} \subset \mathcal{L}_2,$$

$$P_{\text{full},+} := P_{\text{full},+}(\hat{P}) := \{(w, c) \in H^+_2 \mid \hat{P}(s) \begin{bmatrix} w(s) \\ c(s) \end{bmatrix} \in H^-_2 \} = \ker \Pi_+ \hat{P} \subset H^+_2 \quad (3.13)$$

Later on, the in the previous chapter introduced “hidden behavior” $N$, as introduced in the Literature chapter, is required in the algorithm. In the case we use the so called $L_2$ behaviors, this $N$ has to be defined using the just introduced full behaviors (with stable- as well anti-stable representations) as:

$$N = N(P) := \{ w \in \mathcal{L}_2 \mid (w, 0) \in P_{\text{full}}(P) \} = \{ w \in \mathcal{L}_2 \mid P_1(s)w(s) = 0 \} = \ker P_1 \subset \mathcal{L}_2,$$

$$N_+ = N_+(P) := \{ w \in H^+_2 \mid (w, 0) \in P_{\text{full},+}(P) \} = \{ w \in H^+_2 \mid P_1(s)w(s) = 0 \} = \ker P_1 \subset H^+_2,$$

$$N_- = N_-(P) := \{ w \in H^-_2 \mid (w, 0) \in P_{\text{full},-}(P) \} = \{ w \in H^-_2 \mid P_1(s)w(s) \in H^+_2 \} = \ker \Pi_- P_1 \subset H^-_2,$$

where $P = [P_1 \quad P_2] \in \mathcal{RH}^+_\infty$ and $N_+$, $N_-$ are the “full” $N$ obtained from $N$.

Figure 3.5: Partial interconnection problem and the “hidden” behavior $N$. 
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where \( \hat{P} = [\hat{P}_1 \ P_2] \in \mathcal{RH}_\infty \).

Now we can use the theorem that is introduced in the literature chapter of this report, namely:

**Theorem 3.16.** Given the systems \( \mathcal{K}, \mathcal{N}, \mathcal{P}_{\text{manifest}} \in \mathbb{L}_{(\pm)} \) (or \( \mathbb{M}_{(\pm)} \)). There exists a controller \( \mathcal{C} \in \mathbb{L}_{(\pm)} \) (or \( \mathbb{M}_{(\pm)} \)), that implements \( \mathcal{K} \) for \( \mathcal{P}_{\text{full}} \) by partial interconnection if and only if \( \mathcal{N} \subset \mathcal{K} \subset \mathcal{P}_{\text{manifest}} \).

To prove this theorem, the theorems used in the previous subsection, the elimination-theorem for obtaining the manifest behavior and the algorithm, that is to be introduced in the next section, are used.

First, an elimination theorem will be given (and proved), which makes it possible to obtain a manifest behavior, which only contains trajectories \( w \), that fulfills the full behavior:

**Theorem 3.17.** Suppose that \( \mathcal{P}_{\text{full}} = \ker \ P \) with \( P = [P_1 \ P_2] \in \mathcal{RH}_\infty^+ \). If there exists a unit \( U(s) \in \mathcal{UH}_\infty^+ \) such that:

\[
U(s)P_2(s) = \begin{bmatrix} 0 \\ P_2^0(s) \end{bmatrix},
\]

where \( P_2^0 \in \mathcal{RH}_\infty^+ \) is outer and the zero block has \( p \) rows, then the manifest behavior is given by:

\[
\mathcal{P}_{\text{manifest}} = \{ w \in \mathcal{L}_2 \mid P_1^0(s)w(s) = 0 \} = \ker P_1^0(s),
\]

where \( P_1^0 \in \mathcal{RH}_\infty^+ \) consists of the first \( p \) rows in:

\[
U(s)P_1(s) = \begin{bmatrix} P_1^0(s) \\ P_1^0(s) \end{bmatrix}.
\]

This also holds for \( w \in \mathcal{H}_\infty^+ \) and \( w \in \mathcal{H}_2^+ \), which results in the the manifest plant behaviors described by:

\[
\mathcal{P}_{\text{manifest}} = \{ w \in \mathcal{L}_2 \mid P_1^0(s)w(s) = 0 \} = \ker P_1^0, \\
\mathcal{P}_{\text{manifest},+} = \{ w \in \mathcal{H}_2^+ \mid P_1^0(s)w(s) = 0 \} = \ker P_1^0, \\
\mathcal{P}_{\text{manifest},-} = \{ w \in \mathcal{H}_2^- \mid P_1^0(s)w(s) \in \mathcal{H}_2^+ \} = \ker \Pi_1 - P_1^0,
\]

where \( \mathcal{P}_{\text{manifest}(\pm)} \in \mathbb{M} \).

**Proof.** Partitioning using the unit gives us the following equivalent description for the full behavior \( \mathcal{P}_{\text{full}} \):

\[
P_1^0(s)w(s) = 0, \quad (3.14a)
\]
\[
P_1^0(s)w(s) = -P_2^0(s)c(s). \quad (3.14b)
\]

Equation (3.14a) introduces a constraint on the manifest variable \( w(s) \).

So, if \( w(s) \) is such that \( (w(s), c(s)) \in \mathcal{P}_{\text{full}} \) for some \( c(s) \), then \( w(s) \) has to satisfy (3.14a). The proof is complete if one shows that (3.14b) is redundant in the sense that \( w \in \mathcal{L}_2 \) is not constrained by (3.14b). That is,

\[
\forall w(s) \exists c(s) \text{ such that } P_1^0(s)w(s) = -P_2^0(s)c(s). \quad (3.15)
\]

The sufficient condition for this is that \( P_2^0(s) \) is an outer matrix. This matrix can be square, but then one knows that \( P_2^0(s) \) is a unit which is invertible, which results in:

\[
c(s) = P_2^{0-1}(s)P_1^0(s)w(s).
\]

This completes the proof. However, if \( P_2^0(s) \) is not square (but “wide”), it is possible to rewrite \( P_2^0(s)c(s) \) as:

\[
P_2^0(s)c(s) = [P_2^0(s), P_2^{0''}(s)] \begin{bmatrix} c(s) \\ c''(s) \end{bmatrix},
\]

where \( P_2^{0''}(s) \) is a square outer matrix (and so a unit). Then it is possible to find a \( c(s) \) that holds (3.15) by setting \( c''(s) = 0 \).\qed
An equivalent theorem has to be introduced for behaviors that are described using anti-stable rational mappings:

**Theorem 3.18.** Suppose that $\mathcal{P}_{\text{full}} = \ker P$ with $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \in \mathcal{RH}_\infty$. If there exists a unit $U(s) \in U\mathcal{H}_\infty$ such that:

$$U(s)P_2(s) = \begin{bmatrix} 0 \\ P_2'(s) \end{bmatrix},$$

where $P_2'' \in \mathcal{RH}_\infty$ is outer and the zero block has $p$ rows, then the manifest behavior is given by:

$$\mathcal{P}_{\text{manifest}} = \{ w \in \mathcal{L}_2 \mid P_1'(s)w(s) = 0 \} = \ker P_1',$$

where $P_1' \in \mathcal{RH}_\infty$ consists of the first $p$ rows in:

$$U(s)P_1(s) = \begin{bmatrix} P_1'(s) \\ P_1'(s) \end{bmatrix}.$$

This also holds for $w \in \mathcal{H}_2^+$ and $w \in \mathcal{H}_2^-$, which results in the the manifest plant behaviors described by:

$$\mathcal{P}_{\text{manifest}} = \{ w \in \mathcal{L}_2 \mid P_1'(s)w(s) = 0 \} = \ker P_1,'$$

$$\mathcal{P}_{\text{manifest, +}} = \{ w \in \mathcal{H}_2^+ \mid P_1'(s)w(s) \in \mathcal{H}_2^- \} = \ker P_1,'$$

$$\mathcal{P}_{\text{manifest, -}} = \{ w \in \mathcal{H}_2^- \mid P_1'(s)w(s) = 0 \} = \ker P_1,'$$

where $\mathcal{P}_{\text{manifest, (±)}} \in \mathcal{L}_2$.

The proof of this theorem is similar to the one of Theorem 3.17, so it’s not useful to include it here in this report.

### 3.4.1 Algorithm

Now the algorithm for deriving controller representations can be given. This is, as in the full interconnection case, done using trajectories $(w, c) \in \mathcal{L}_2$ and rational anti-stable mappings in $\mathcal{RH}_\infty$:

**Algorithm 3.2.** Given $P, K \in \mathcal{RH}_\infty$ that define the systems $\mathcal{P}_{\text{full}}$ and $\mathcal{K}$:

$$\mathcal{P}_{\text{full}} = \{ (w, c) \in \mathcal{L}_2 \mid P(s) \begin{bmatrix} w(s) \\ c(s) \end{bmatrix} = 0 \} = \ker P,$$

$$\mathcal{K} = \{ w \in \mathcal{L}_2 \mid K(s)w(s) = 0 \} = \ker K,$$

where $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$.

**Aim:** Find $C \in \mathcal{RH}_\infty$ that defines the system:

$$C := C(C) := \{ c \in \mathcal{L}_2 \mid C(s)c(s) = 0 \} = \ker C,$$

such that $C$ implements $K$ for $\mathcal{P}_{\text{full}}$ by partial interconnection through $c$.

**Step 1:**

Represent the manifest plant behavior $\mathcal{P}_{\text{manifest}}$ by applying the elimination theorems (Theorems 3.17 and 3.18) on the full plant behavior $\mathcal{P}_{\text{full}}$. So, find $P_1' \in \mathcal{RH}_\infty$ such that:

$$\mathcal{P}_{\text{manifest}} = \{ w \in \mathcal{L}_2 \mid P_1'(s)w(s) = 0 \} = \ker P_1'.$$

**Step 2:**

Verify whether there holds that the “hidden behavior” $\mathcal{N}$ is a subset of the controlled behavior $\mathcal{K}$ (so, $\mathcal{N} \subset \mathcal{K}$). If so, there should exist a mapping $X \in \mathcal{RH}_\infty$ such that $K = XP_1$ (See Theorems 3.13 and 3.14). If not, the algorithm stops and no controller can be found.
Step 3:
Now one has to verify if $\mathcal{K} \subset \mathcal{P}_{\text{manifest}}$, so there should exist a $Y \in \mathcal{RH}_\infty^-$ such that $P'_1 = YK$. If not, the algorithm stops here.

Step 4:
Compress the number of columns of the found matrix $Y$ by post-multiplying with a unit $U \in \mathcal{UH}_\infty^-$ as done in step 2 of Algorithm 3.1:

$$\bar{Y} = YU = [Y_1, 0],$$

where $Y_1 \in \mathcal{RH}_\infty^-$ and full rank.

Step 5:
Extend this matrix $\bar{Y}$ using $\bar{W} = [0 \ 1]$ such that:

$$\bar{X} = \begin{bmatrix} \bar{Y} \\ \bar{W} \end{bmatrix} = \begin{bmatrix} Y_1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $\bar{W} = W$, so $W = \bar{W}U^{-1}$ and $\bar{X} \in \mathcal{UH}_\infty^-$ is a unit.

Step 6:
Compute a possible controller, also using a similar step as in Algorithm 3.1, such that:

$$\bar{C}(s)w(s) = 0 \quad \Rightarrow \quad \bar{C} = WP_1,$$

and because an interconnection is made between the controller and the plant using the interconnect variables $c$, one has to use:

$$P_1w = -P_2c,$$

which results in the mapping $C(s)c(s) = 0$, where:

$$C = WP_2 \in \mathcal{RH}_\infty^-.$$

Now this algorithm is known, the proof of Theorem 3.16 can be given:

Proof.
$$(\Rightarrow):$ This is trivial.
$$(\Leftarrow):$ If $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}_{\text{manifest}}$, then it is possible to obtain the desired controlled behavior by partial interconnection with a controller using the introduced algorithm. \qed
Chapter 4

Example: LQ optimal control problem

The construction of system representations that define a desired controlled behavior is one of the research topics in our Control Systems group. In earlier publications [4, 6] has shown that it is possible to represent an optimisation problem as a dynamical system that has to be solved. One of those examples is a linear quadratic cost function, which is known as the LQR problem. In this chapter, a more detailed view on this problem is given. The resulted controlled representation is used to verify the full interconnection algorithm using the $L_2$ behavioral approach. Also a method to reduce this controlled system description is introduced in this chapter.

Note:
In this chapter, functions related with time, so for example $x(t)$ and $u(t)$, are shortened as $x$ and $u$ to make the formulae more clear. The dot above functions, e.g. $\dot{x}$ or $\dot{\lambda}$, mention the derivative with respect to time.

4.1 Optimisation problem

Define a plant $\Sigma_G$, whose behavior is described by the following state space representation:

$$\Sigma_G : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \\ x(0) = x_0. \end{cases}$$

where $x : \mathbb{R} \rightarrow \mathbb{R}^n$, $y : \mathbb{R} \rightarrow \mathbb{R}^y$ and $u : \mathbb{R} \rightarrow \mathbb{R}^u$. So, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times u}$, $C \in \mathbb{R}^{y \times n}$ and $D \in \mathbb{R}^{y \times u}$.

An optimal controller has to minimise the following cost $J$:

$$J(x, u) = \frac{1}{2} \int_{t_0}^{t_f} x^T Q x + u^T R u dt + \frac{1}{2} x^T(t_f) E x(t_f) = \int_{t_0}^{t_f} F(x, u) dt + \Phi(x) \bigg|_{t=t_f},$$

where $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{u \times u}$ and $E \in \mathbb{R}^{n \times n}$.

The system description and the cost can be summarised in the following optimisation problem that has to be solved.

$$P_{opt} = \min_{x, u} J(x, u) = \min_{x, u} \int_{t_0}^{t_f} F(x, u) dt + \Phi(x) \bigg|_{t=t_f} \quad (4.1a)$$

subject to:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad (4.1b)$$
CHAPTER 4. EXAMPLE: LQ OPTIMAL CONTROL PROBLEM

where

\[ F(x, u) = \frac{1}{2} \left( x^T Q x + u^T R u \right), \]
\[ \Phi(x) = \frac{1}{2} x^T E x. \]

The optimisation problem can be rewritten in the (dual) Lagrangian form:

\[
\begin{aligned}
\ell(\lambda) &= \min_{x, u} L(x, u, \lambda) = \min_{x, u} \int_{t_0}^{t_f} \left[ F(x, u) + \lambda^T (Ax + Bu - \dot{x}) \right] dt + \Phi(x) |_{t=t_f} \\
&= \min_{x, u} \left( 1, F(x, u) + \lambda^T (Ax + Bu) \right) - \langle \lambda, \dot{x} \rangle + \Phi(x) |_{t=t_f} \\
&= \min_{x, u} \left( 1, H(x, u, \lambda) \right) + \langle \lambda, x \rangle + \langle \langle 1, [\lambda] \rangle \rangle_{t=0} + \langle \langle 1 - \lambda, [\Phi(x)] \rangle \rangle_{t=t_f}
\end{aligned}
\]

where \(H(x, u, \lambda)\) is the Hamiltonian function, which represents an energy function, and \(\ell(\lambda)\) is named the dual cost.

\[
\begin{aligned}
\langle \cdot, \cdot \rangle_2 \text{ is the } L_2 \text{ type inner product } (\langle x, y \rangle = \int_{t_0}^{t_f} x^T y \, dt) \text{ and } \langle \cdot, \cdot \rangle \text{ is the Euclidean type inner product.}
\end{aligned}
\]

\(L(x, u, \lambda)\) is called the Lagrangian, \(\lambda : \mathbb{R} \to \mathbb{R}^n\) are the Lagrange multipliers (or also named co-states), \(H(x, u, \lambda)\) the Hamiltonian function, which represents an energy function, and \(\ell(\lambda)\) is named the dual cost.

The optimisation problem has an optimal solution \(p^* = (x^*, u^*, \lambda^*)\) if all partial derivatives of the Lagrangian are equal to zero:

\[
\nabla L(x^*, u^*, \lambda^*) = 0.
\]

This results in the three K.K.T. conditions that have to hold:

\[
\begin{aligned}
\frac{\partial L}{\partial x} |_{p^*} &= 0 \quad \rightarrow \quad Ax^* + Bu^* = \dot{x}^*, \\
\frac{\partial L}{\partial u} |_{p^*} &= 0 \quad \rightarrow \quad Qx^* + \lambda^T A = -\dot{\lambda}^*, \\
\frac{\partial L}{\partial u} |_{p^*} &= 0 \quad \rightarrow \quad Ru^* + B^T \lambda^* = 0.
\end{aligned}
\]

Also the final penalty in the cost function results in a condition for the final value of the Lagrange multipliers, namely:

\[
\lambda^*(t_f) = \nabla_x \Phi(x^*(t_f)) = Ex^*(t_f).
\]

These conditions can be summarized in the following system \(\Sigma_H:\)

\[
\Sigma_H : \begin{bmatrix} \dot{x}^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix},
\]

with

\[
u^* = -R^{-1}B^T \lambda^* \quad \text{and} \quad x(0) = x_0 \quad \text{and} \quad \lambda^*(t_f) = Ex^*(t_f).
\]

Here, \(H\) is a Hamiltonian matrix that contains a stable- and anti-stable part. (4.2a) is an autonomous system, whose general solution needs some more introduction before it can be written as a rational kernel representation.
4.2 General solution of an autonomous system

Before a rational kernel representation (in frequency domain) of an autonomous system can be defined, one has to verify if there exists an $L_2$ solution for the set of differential equations (4.2a).

First, only $L^+_2$ trajectories are taken into account, so the integral bounds in the optimisation problem (4.1) are positive ($t_0, t_f \in \mathbb{R}^+$). The solution of an autonomous system, in the time domain $t \in [0, \infty)$ (using differential operators), is given by:

$$(\frac{d}{dt} - a)x(t) = 0, \quad x(0) = x_0 \iff x(t) = e^{at} x_0 \in L^+_2, \quad a < 0, \quad t \geq 0.$$ 

One can denote that the value of $a$ should be smaller than zero, because otherwise the solution $x(t)$ is unstable, and not an element of $L^+_2$ anymore.

Now we can transform this solution to the frequency domain using the Laplace operator:

$$X(s) = \mathcal{L}(x(t)) = \int_0^\infty e^{at} e^{-st} dt \quad x_0 = \int_0^\infty e^{(a-s)t} dt \quad x_0 = \frac{1}{a-s} e^{(a-s)t} \bigg|_{t=0}^{t=\infty} x_0.$$ 

Because $x \in L^+_2$, we have that $X \in \mathcal{H}^+_2$.

This is the case when Re($a-s$) < 0, so Re $a -$ Re $s < 0$. In this case, we can state that:

$$X(s) = \frac{1}{s-a} x_0 \in \mathcal{H}^+_2, \quad \text{where } a < 0 \text{ and ROC: } \{ s \in \mathbb{C} \mid \text{Re } s > \text{Re } a \}.$$ 

Similar for the case that $\dot{x} \in L^-_2$ (and so $\dot{X} \in \mathcal{H}^-_2$):

$$(\frac{d}{dt} - b)\dot{x}(t) = 0, \quad \dot{x}(0) = \dot{x}_0 \iff \dot{x}(t) = e^{bt} \dot{x}_0 \in L^-_2, \quad b > 0, \quad t \leq 0.$$ 

In this case, $b$ has to be positive, because the time $t < 0$ and the solution has to be stable (so in $L^-_2$). This can also be converted into the frequency domain using the Laplace operator, namely:

$$\dot{X}(s) = \mathcal{L}(\dot{x}(t)) = -\frac{1}{s-b} \dot{x}_0 \in \mathcal{H}^-_2, \quad \text{where ROC: } \{ s \in \mathbb{C} \mid \text{Re } s < \text{Re } b \}.$$ 

Now, we are interested in a possible $L_2$ solution, which doesn’t seem to be possible! There is only the trivial solution if $x_0$ is zero, because then the solution is zero over all time (which is an element of $L_2$).

In the situation of the previous section, the autonomous system (4.2a) has a “stable”- and “anti-stable” part, which is equivalent to the both situations depicted in this section. One part results in a solution in $L_2^+$ and the other part in the solution in $L_2^-$.

So, this dynamical system has to be separated. One way to do this is the “classical” state transformation using the solution of the algebraic Riccati equation, which only results in the stable part (which is a solution in $L^+_2$). This is done in [6], and is recapped in the next section.

It should also be possible to split the dynamical system into a “stable”- and “anti-stable” system which can be solved separately and generate the $L^+_2$ and $L^-_2$ solution. This is done in Section 4.4.

4.3 Separation of Hamiltonian system: using Riccati transformation

As mentioned in Section 4.1, the Hamiltonian system contains a “stable”- and a “anti-stable” part. However, this can be reduced to a stable system, because this representation is not minimal! A minimal realization can be obtained by applying the following state transformation [6]:

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} I & 0 \\ S & I \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} \iff \begin{bmatrix} x \\ \sigma \end{bmatrix} = \begin{bmatrix} I & 0 \\ -S & I \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}, \quad (4.3)$$
and
\[
\frac{d}{dt} \begin{bmatrix} x \\ \sigma \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} I & 0 \\ -S & I \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \iff \begin{bmatrix} \dot{x} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -S & I \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\dot{S} & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}.
\]

So, one can state that
\[
\dot{\sigma} = -S\dot{x} + \dot{\lambda} - \dot{S}x.
\]

This results in the following dynamical system:
\[
\begin{bmatrix} \dot{x} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\dot{S} & 0 \end{bmatrix} + \begin{bmatrix} I & 0 \\ -S & I \end{bmatrix} \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I & 0 \\ S & I \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix},
\]

which can be rewritten as:
\[
\begin{bmatrix} \dot{x} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} A - BR^{-1}B^TS \\ -BR^{-1}B^T \end{bmatrix} \chi + \begin{bmatrix} A - BR^{-1}B^TS \\ -BR^{-1}B^T \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix},
\]

where \( \chi = A^T S + SA - SB^T SR^{-1} + Q + \dot{S} \).

To make the state variable \( x \) independent of the "co-state" variable \( \sigma \), there has to hold that \( \chi \) and the \( \sigma(t_f) \) are equal to zero. In this situation, \( \sigma \) will be zero for all \( t \), because if \( \chi = 0 \) the differential equation for \( \sigma \) becomes:
\[
\dot{\sigma} = -(A - BR^{-1}B^T S)^T \sigma,
\]

which is "anti-stable". If the final value of \( \sigma \) is zero, it has to be zero for all time \( t \). If \( \sigma(t_f) \) has to be zero, \(-S(t_f)x(t_f) + \lambda(t_f)\) has to be zero (see (4.3)). Known is that \( \lambda(t_f) = Ex(t_f) \) (4.2b), so there has to hold that \( S(t_f) = E \).

So, summarised:
\[
\chi = A^T S + SA - SB^T SR^{-1} + Q + \dot{S} = 0,
\]

\[
S(t_f) = E.
\]

\( \chi \) is known as the differential Ricatti equation, which in the steady state situation \( (t_f \to \infty) \) reduces to the algebraic Ricatti equation:
\[
\chi_{ss} = A^T S + SA - SB^T SR^{-1} + Q = 0.
\]

So, if we use the (stable) solution of this ARE for \( S \), then \( \sigma \) is an uncontrollable state, so that (4.2a)-(4.2b) results in the equivalent \( L^+_2 \) system:
\[
\begin{align*}
\dot{x} &= (A - BR^{-1}B^T S)x, \\
u &= -R^{-1}B^T Sx.
\end{align*}
\]

### 4.4 Separation of Hamiltonian system: eigenvalue decomposition

In this section, a start is made for a possible other method of obtaining the stable- and unstable part of a Hamiltonian matrix in (4.2a) separately. To start the separation, an eigenvalue decomposition is calculated:
\[
HV = V\Lambda \iff H = VAV^{-1},
\]

where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n) \) with \( \lambda_i > 0 \) and \( V = [V_{\text{anti-stable}}, V_{\text{stable}}] \) are the eigenvectors of the Hamiltonian matrix \( H \). Remark that this Hamiltonian matrix has rank \( 2n \).

As can be seen, there is a separation between the "stable"- and "anti-stable" eigenvalues and eigenvectors, which results in two "solutions". Mentioned in Section 4.2 is that this makes it possible to calculate
4.4. SEPARATION OF HAMILTONIAN SYSTEM: EIGENVALUE DECOMPOSITION

Simulation results using separate parts $H_{\text{stable}}$ and $H_{\text{anti-stable}}$.

Figure 4.1: Simulation of $H_{\text{stable}}$ and $H_{\text{anti-stable}}$ separately.

A $L_2$ "solution", using projections $\Pi_+$ and $\Pi_-$, for the total Hamiltonian system, because the stable part results in a $L_2^+$ solution and the anti-stable part in a $L_2^-$ solution. The value at time $t = 0$ is the initial value of the state vector $[\dot{x} \lambda]$. The idea of the separation is that the autonomous part of the optimal controlled system can be written as:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = H_{\text{stable}} \begin{bmatrix} x \\ \lambda \end{bmatrix} + H_{\text{anti-stable}} \begin{bmatrix} x \\ \lambda \end{bmatrix} = H \begin{bmatrix} x \\ \lambda \end{bmatrix}.$$  

Here, the matrices $H_{\text{stable}}$ and $H_{\text{anti-stable}}$ are defined as:

$$H_{\text{stable}} = V_{\text{stable}} \cdot [\text{diag}(-\lambda_1, \ldots, -\lambda_n), 0] \cdot V^{-1},$$

$$H_{\text{anti-stable}} = V_{\text{anti-stable}} \cdot [0, \text{diag}(\lambda_1, \ldots, \lambda_n)] \cdot V^{-1},$$

where $H_{\text{stable}}$ and $H_{\text{anti-stable}}$ both have rank $n$, where $[\dot{x} \lambda] = H_{\text{stable}} [\dot{x} \lambda]$ has an $L_2^+$ solution and $[\dot{x} \lambda] = H_{\text{anti-stable}} [\dot{x} \lambda]$ has an $L_2^-$ solution.

**Note** that I use the complete inverse matrix of $V$ because I extend the "eigenvalue"-matrix with zeros. It is also possible to multiply with the $\text{diag}(\pm \lambda_1, \ldots, \pm \lambda_n)$ only and take the upper- or lower part $V^{-1}$.

It is important to stress that the combined solution for positive- and negative time is not a solution for the complete system!

**Example 4.1.** In the introduced LQ optimal control problem, we use the following values:

$$A = \begin{bmatrix} -4 & 2 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad R = 0.75.$$  

This results in the Hamiltonian matrix $H$:

$$H = \begin{bmatrix} -4 & 2 & -1.3333 & 1.3333 \\ 1 & -4 & 1.3333 & -1.3333 \\ -0.1 & 0 & 4 & -1 \\ 0 & -0.1 & -2 & 4 \end{bmatrix}, \quad \text{where} \quad [\dot{x} \lambda] = H [x \lambda].$$

The eigenvalue decomposition of this Hamiltonian results in the eigenvalues and eigenvectors:

$$\Lambda = \text{diag}(5.4390, 2.5854, -5.4390, -2.5854) \quad \text{and} \quad V = \begin{bmatrix} 0.1544 & 0.0349 & 0.8156 & -0.8159 \\ -0.1751 & -0.0425 & -0.5785 & -0.5779 \\ 0.5621 & 0.5780 & 0.0082 & -0.0144 \\ 0.7934 & 0.8142 & -0.0044 & -0.0131 \end{bmatrix}.$$
Using the mentioned equations, one can split this Hamiltonian matrix into:

\[
H_{\text{stable}} = \begin{bmatrix}
-4.0061 & 2.0053 & -0.6667 & 0.7497 \\
1.0068 & -4.0061 & 0.5836 & -0.6667 \\
-0.0500 & 0.0062 & -0.0061 & 0.0068 \\
-0.0062 & -0.0500 & 0.0053 & -0.0061
\end{bmatrix},
\]

\[
H_{\text{anti-stable}} = \begin{bmatrix}
0.0061 & -0.0053 & -0.6667 & 0.5836 \\
0.0068 & 0.0061 & 0.7497 & -0.6667 \\
-0.0500 & -0.0062 & 4.0061 & -1.0068 \\
0.0062 & -0.0500 & -2.0053 & 4.0061
\end{bmatrix}.
\]

It is valid that the sum of these two matrices results in the original Hamiltonian matrix:

\[H = H_{\text{stable}} + H_{\text{anti-stable}}.\]

Also holds that the rank of \(H_{\text{stable}}\) and \(H_{\text{anti-stable}}\) is equal to 2, while the rank of the complete Hamiltonian matrix is equal to 4.

It is possible to simulate the two parts to obtain “projected” \(L_2^-\) and \(L_2^+\) solutions as:

\[
\begin{align*}
\begin{bmatrix}
\hat{x} \\ \lambda
\end{bmatrix} &= H_{\text{anti-stable}} \begin{bmatrix} \hat{x} \\ \lambda
\end{bmatrix}, \quad t < 0, \\
\begin{bmatrix}
\hat{x} \\ \lambda
\end{bmatrix} &= \begin{bmatrix} x(0) \\ \lambda(0)
\end{bmatrix}, \quad t = 0, \\
\begin{bmatrix}
\hat{x} \\ \lambda
\end{bmatrix} &= H_{\text{stable}} \begin{bmatrix} \hat{x} \\ \lambda
\end{bmatrix}, \quad t > 0.
\end{align*}
\]

This is done for the given example using the initial value at time \(t = 0\) as \((x_1, x_2, \lambda_1, \lambda_2) = 20\).

The results are depicted in Figure 4.1, where one can see that the stable “states” are \(x_1\) and \(x_2\) while the anti-stable “states” are \(\lambda_1\) and \(\lambda_2\). This has some relation with the separation done in the previous section, because there also \(x_1\) and \(x_2\) are the stable states for \(t > 0\).

4.5 Model reduction using modal truncation

4.5.1 Approximation problem

It is possible to reduce the controlled behavior description using modal truncation. The first step for this reduction is to calculate an eigenvalue decomposition of the Hamiltonian matrix (4.2a) again:

\[HV = V\Lambda \iff H = V\Lambda V^{-1},\]

where \(\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n)\) and \(V = [V_{\text{anti-stable}}, V_{\text{stable}}]\) are the eigenvalues and eigenvectors of \(H\). To make this section more clear, the autonomous system is described as:

\[\dot{\xi} = H\xi \quad \text{(so } \xi = [\hat{x}] \text{ compared with (4.2a))}.\]

The idea of model reduction is to reduce the states which have less influence on the complete system dynamics. This means that we are truncating the eigenvectors that are less dominant, which implies that the eigenvectors coupled with the smallest eigenvalues will be removed.

Because a Hamiltonian system is used, there are positive- and negative eigenvalues, which result in the “stable”- and “anti-stable” dynamics, that both can be reduced.

In this section, the number of positive- and negative eigenvalues that are truncated will be the same, so the reduced order system will have the following eigenvalues:

\[\Lambda_r = \text{diag}(\lambda_1, \ldots, \lambda_r, -\lambda_1, \ldots, -\lambda_r)\]
which implies the set of eigenvectors:

\[ V_r = [V_r, \text{anti-stable}, V_r, \text{stable}] \]

where \( r < n \). The truncated eigenvalues and eigenvectors are named the complement eigenvalues and eigenvectors, and are given by:

\[ \Lambda_c = \text{diag}(\lambda_{r+1}, \ldots, \lambda_n, -\lambda_{r+1}, \ldots, -\lambda_n) \quad \text{and} \quad V_c = [V_c, \text{anti-stable}, V_c, \text{stable}] \]

### 4.5.2 Applied projection for reduction

To reduce the number of states in the system, a projection has to be applied to the original state vector. This results in a new system description, and so in a new Hamiltonian matrix \( H_{\text{reduced}} \). For this projection, the eigenvectors calculated in the eigenvalue decomposition of the previous subsection are used.

The to be defined projection to the reduced number of states \( \Pi_{V_r} \) is visualized in Figure 4.2. Here, the state vector \( \xi \) is separated into the states that are used in the reduced order model (\( \xi_r \)) and the complement of those states (\( \xi_c \)):

\[ \xi = \xi_r \oplus \xi_c, \quad \text{where} \quad \xi_r := \Pi_{V_r} \xi, \quad \xi_c := \Pi_{V_r^\perp} \xi \quad \text{and} \quad (\xi_r, \xi_c) = 0. \]

The projection \( \Pi_{V_r} \) has to result in a subspace \( V_r \subset V \), where the reduced order space should be spanned by the desired eigenvectors \( V_r \), so \( V_r = \text{im} \ V_r \). Because of some properties that are required for a projection, we define \( \Pi_{V_r} \) and \( \Pi_{V_r^\perp} \) as:

\[ \Pi_{V_r} := V_r (V_r^T V_r)^{-1} V_r^T, \]
\[ \Pi_{V_r^\perp} := I - \Pi_{V_r} = V_r (V_r^T V_r)^{-1} V_r^T = V_c (V_c^T V_c)^{-1} V_c^T. \]

There has to be verified if this projection is a valid one. One property of the projection should be that if it is applied twice, this should be the same as applying it once:

\[ \Pi_{V_r} \Pi_{V_r} = \Pi_{V_r}. \]

Another aspect that needs to be verified is that the introduced projection used should be the “best” one. So using the visualization in Figure 4.2, the distance between \( \xi \) and \( \xi_r \) should be smaller than the distance between \( \xi \) and all possible \( \xi_0 \in V_r \). Summarized, there has to be proven that:

\[ \|\xi - \xi_r\|^2 \leq \|\xi - \xi_0\|^2 \quad \forall \xi_0 \in \text{im} \ V_r. \]  
\[ (4.4) \]
The right hand side of this equation can be rewritten as:

\[ \|\xi - \xi_0\|^2 = \| (\xi - \xi_r) + (\xi_r - \xi_0) \|^2, \]

which can be split in two separate parts:

(A) The first part of this equation is rewritten such that:

\[ \xi - \xi_r = \xi - \Pi_{V_r}\xi = \xi - V_r(V_r^T V_r)^{-1}V_r^T \xi = (I - V_r(V_r^T V_r)^{-1}V_r^T)\xi = \Pi_{\perp_r}\xi. \]

So, this part in the norm is in the complementary space of the \( V_r: (\xi - \xi_r) \in V_r^\perp. \)

(B) In the second part, \( \xi_r \) is an element of \( V_r \) and also \( \xi_0 \) is chosen in this same subspace.

So, summarizing: \( \xi_r, \xi_0 \in V_r \) and also \( (\xi_r - \xi_0) \in V_r. \)

Then known is that

\[ \langle \xi - \xi_r, \xi_r - \xi_0 \rangle = 0 \quad \text{and} \quad \langle \xi_r - \xi_0, \xi - \xi_r \rangle = 0, \]

which can be used in rewriting the right hand side of (4.4) again:

\[ \|\xi - \xi_0\|^2 = \| (\xi - \xi_r) + (\xi_r - \xi_0) \|^2 \]
\[ = \| (\xi - \xi_r) \|^2 + \| (\xi_r - \xi_0) \| + \langle \xi_r - \xi_0, \xi - \xi_r \rangle + \|\xi_r - \xi_0\|^2 \]
\[ = \|\xi - \xi_r\|^2 + \|\xi_r - \xi_0\|^2 \geq \|\xi - \xi_r\|^2. \]

So, proven is that (4.4) holds for all \( \xi_0 \in \text{im} \ V_r = V_r. \)

With those two properties, it is shown that a proper projection to obtain the reduced order states \( \xi_r \in V_r \) is chosen.

4.5.3 Using projection to obtain reduced order model

Now we can apply this projection to the states as described in the Hamiltonian dynamical system (4.2a).

So, applying \( \Pi_{V_r} \) to the original state vector \( \xi \), results in:

\[ \xi_r = \Pi_{V_r}\xi = V_r(V_r^T V_r)^{-1}V_r^T \xi = V_r\alpha_r, \]

where \( \alpha_r = (V_r^T V_r)^{-1}V_r^T \xi \) are the coefficients of \( \xi_r \) in the basis \( V_r \). We are interested in those coefficients, because other wise the dimension of \( \xi_r \) is equal to the dimension of \( \xi \) and the same number of ODEs has to be solved. Using this method, a smaller set of ODEs, for the \( \alpha_r \) coefficients, has to be solved, which saves simulation time!

Combining the coefficients of \( \xi_r \) in the basis \( V_r \) and the dynamical system \( \dot{\xi} = H\xi \) results in:

\[ \dot{\alpha}_r = (V_r^T V_r)^{-1}V_r^T \dot{\xi} = (V_r^T V_r)^{-1}V_r^T H\xi \]
\[ = (V_r^T V_r)^{-1}V_r^T H [\xi_r + \xi_0] \]
\[ = (V_r^T V_r)^{-1}V_r^T [V_r(V_r^T V_r)^{-1}V_r^T \xi + (I - V_r(V_r^T V_r)^{-1}V_r^T)\xi] \]
\[ = (V_r^T V_r)^{-1}V_r^T HV_r(V_r^T V_r)^{-1}V_r^T \xi + (V_r^T V_r)^{-1}V_r^T H(I - V_r(V_r^T V_r)^{-1}V_r^T)\xi \]
\[ = (V_r^T V_r)^{-1}V_r^T HV_r\alpha_r + (V_r^T V_r)^{-1}V_r^T HV_r(V_r^T V_r)^{-1}V_r^T \xi \]
\[ = (V_r^T V_r)^{-1}V_r^T HV_r\alpha_r + (V_r^T V_r)^{-1}V_r^T HV_r(V_r^T V_r)^{-1}V_r^T \xi \] \( \in V_r \)
\[ := H_r\alpha_r, \]

where the weighted inner product is defined as: \( \langle a, b \rangle_H = a^T H b \) and the reduced order Hamiltonian as:

\[ H_r = (V_r^T V_r)^{-1}V_r^T HV_r. \]
Also the equation for the desired control input, \( u = -R^{-1}B^T\lambda = Q\xi \) (4.2b) has to be transformed:
\[
u = Q\xi = Q[\xi_r + \xi_c] = Q[V_r(V_r^TV_r)^{-1}V_r^T\xi + (I - V_r(V_r^TV_r)^{-1}V_r^T)\xi] = QV_r\alpha_r + Q(I - V_r(V_r^TV_r)^{-1}V_r^T)\xi = QV_r\alpha_r + QV_r(V_r^TV_r)^{-1}V_r^T\xi \in V_r^\perp V_r.
\]

Now let \( Q_r = QV_r \) and set \( u_r = Q_r\alpha_r \) as approximation of \( u \).

This results in the following reduced order system that needs to be solved:
\[
\begin{align*}
\dot{\alpha}_r &= (V_r^TV_r)^{-1}V_r^THV_r\alpha_r, \\
u_r &= QV_r\alpha_r.
\end{align*}
\]

### 4.5.4 Verify if \( K_{\text{reduced}} \subseteq K \)

The set of derived equations (4.2a)-(4.2b) describes the “desired” behavior a system should have if it is interconnected with a controller. This is, as mentioned in the previous chapter and in literature, known as the controlled behavior \( K \). When using the behavioral approach theory, it is quite important to know if the, just introduced, reduced order system describes a behavior that is a subset of the original controlled behavior, as described in the full order system.

So one has to verify if \( K_{\text{reduced}} \subseteq K \). If this holds and also \( K \subset P \) is the case for the full interconnected situation (as in Section 3.3), it should be possible to estimate a controller behavior \( C \) using the reduced order controlled behavior \( K_{\text{reduced}} \). In this optimal control example, we are interested in the full interconnection situations, so \( \xi \) is fully observable (which in the state space case means that \( C = I \)).

The verification problem will be solved in two parts, namely first the autonomous part that describes the behavior of the (reduced) states, and then the part that describes the relation between the states and the “controller” output \( u \).

**Autonomous part**

In this chapter, the original controlled behavior has \( 2n \) states that can be described as followed:
\[
\xi = \sum_{k=1}^{2n} v_k\xi_k \in V,
\]
where \( v_k \) are the eigenvectors that can be obtained from the \( V = [v_1, \ldots, v_{2n}] \) matrix of the eigenvalue decomposition of \( H \). This Hamiltonian matrix puts some constraints on those states in (4.2a), which result in the trajectories for \( \xi \), with the corresponding trajectories for \( u \), that fulfill the controlled situation. This means that \( \xi \in K^\xi \) when (4.2a) holds, where \( K^\xi \) is the set of trajectories \( \xi \) that are in \( K \) (which can be defined, because the other part of the behavior \( K \), namely \( u \), is a linear combination of those trajectories \( \xi \)). Also is known that \( \xi \in V \) as defined in the previous section and depicted in Figure 4.1.

In the reduced order situation, the “state” vector \( \alpha_r \) has \( 2r \) states, where \( r \leq n \) and is given by:
\[
\alpha_r = (V_r^TV_r)^{-1}V_r^T\xi.
\]
As one can see, \( \alpha_r \) is an element of a subspace of the complete \( V \) (where \( \xi \) lives in), namely \( V_r \subseteq V \).

Because \( \xi \) is an element of the controlled behavior, also \( \alpha_r \) has to be one. Therefore we can say that:
\[
\alpha_r \in K^\xi \subseteq K^\xi,
\]
which means that the states of the reduced order system are in the controlled behavior described by the full order model.
Relation between states and output $u$

In the full order model, the relation between the controller “output” $u$ and the states is a linear one:

$$u_i = \sum_{j=1}^{2n} q_{ij} \xi_j,$$

with $i$ the number of controller outputs and $q_{ij}$ an element of $Q$.

This controller output trajectory $u$ is, by definition, an element of the controlled behavior $K_u$. In the reduced order situation however, the matrix $Q$ is transformed to a reduced weighting matrix $Q_r$ and post-multiplied by elements $\alpha_{r,j}$ instead of elements $\xi_j$.

This does not mean that the trajectories of $u$ in the reduced case are not an element of the controlled behavior $K_u$ however! The first step that has to be made is that $\alpha_{r,j} \in V_r \subseteq V$, and so is in the controlled behavior $K_u^{\text{reduced}}$. The $Q$ matrix is post-multiplied with eigenvectors of this subspace $V_r$, which means that multiplying with $Q_r$ does not change the subspace of the coefficients $\alpha_{r,j}$ and so $u \in V_r$. Because $V_r \subset V$, one can say that the trajectories of $u$ in the reduced situation are also in $K_u^{\text{reduced}} \subset K_u$.

**Theorem 4.2.** This model reduction accomplishes $K_u^{\text{reduced}} \subseteq K$. So, if holds that $K \subseteq P$, Theorem 3.12 holds and now there also does exist a controller behavior $C$ such that $P \cap C = K_u^{\text{reduced}}$ by full interconnection.

4.5.5 Examples on reduction

The reduction technique will be illustrated with a simple example:

**Simple example: fourth order diagonal Hamiltonian:**

Let the full order Hamiltonian system be described by:

$$\dot{\xi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \xi = H \xi \quad \text{and} \quad u = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xi = Q \xi.

Then the solution for $\xi(t)$ would be:

$$\xi(t) = \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{12t} & 0 & 0 \\ 0 & 0 & e^{-12t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix} \xi(0) = e^{Ht} \xi(0).$$

Because model reduction should be applied, an eigenvalue decomposition ($HV = VA$) is calculated, where (in this simple example) $V = I$ and $A = \text{diag}(1, 12, -12, -1)$.

Only the largest eigenvalues are interesting, so $V$ has to be separated such that:

$V_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, which correspond to the eigenvalues $(12, -12)$ of $H$ and

$V_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$, which correspond to the eigenvalues $(1, -1)$ of $H$.

In this situation, the reduced order Hamiltonian matrix $H_r$ is given by: $H_r = \begin{bmatrix} 12 & 0 \\ 0 & -12 \end{bmatrix}$.

The reduced $Q_r$ matrix is in this example: $Q_r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

4.6 Conversion to rational kernel representations

For the algorithms used in Chapter 3, rational $\mathcal{H}_\infty$ expressions have to be defined to represent the $L_2$ behaviors for the plants and desired “controlled” descriptions. Section 4.2 has already shown that it is not possible to define a rational representation for the behavior of an autonomous dynamical system that contains stable- as well as unstable poles.
4.6. CONVERSION TO RATIONAL KERNEL REPRESENTATIONS

The first part of this section will contain the conversion of a normal input-state-output state space representation of a plant into a rational one. This results in the plant behavior $P \subset \mathcal{L}_2$. In the second part, controlled behaviors $K$ that are given by dynamical systems as in (4.2a)-(4.2b) will be converted into rational kernel representations (taking into account the problem that is mentioned in Section 4.2). When both dynamical systems are converted into rational kernel representations, it is possible to start the algorithm described in Chapter 3.

4.6.1 Conversion of the plant description

In general, a dynamical system of a plant is given by a state space representation as:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du. \quad (4.5)$$

This system has to be first converted into the frequency domain such that a $\mathcal{L}_2$ behavior $P$ can be defined for it. A general representation of i.s.o. state space dynamical systems in frequency domain is given by:

$$y(s) = [C(sI - A)^{-1}B + D]u(s). \quad (4.6a)$$

In the behavioral approach, the “classical” inputs and outputs are combined into a new variable, named $w(s) = \begin{bmatrix} y(s) \\ u(s) \end{bmatrix}$, which results in the following representation for the plant in the frequency domain:

$$P(s)w(s) = \begin{bmatrix} -I & C(sI - A)^{-1}B + D \end{bmatrix}w(s) = 0. \quad (4.6b)$$

One can notice that this representation is already a rational one, however as mentioned in the $\mathcal{L}_2$ behavioral framework, stable- and anti-stable representations ($\mathcal{RH}_\infty^+$ and $\mathcal{RH}_\infty^-$) can be used for the kernels of the behaviors. So, therefore those two cases are treated separately:

Anti-stable representations ($\mathcal{RH}_\infty^-$)

In this subsection, the representation problem is divided into three separate problems, namely the cases when the plant dynamics are unstable, stable or contain positive- as well as negative poles:

Unstable plants ($\text{eig}(A) > 0$):

As defined in Section 3.1.6, the norm of the to be estimated $P(s)$, that is an element of $\mathcal{RH}_\infty^-$, has to be finite on $\mathbb{C}^-$. As can be seen in (4.6b), this is already a good anti-stable representation if the plant is unstable itself:

$$P(s)w(s) = 0 \iff P(s) = \begin{bmatrix} -I & C(sI - A)^{-1}B + D \end{bmatrix} \in \mathcal{RH}_\infty^- \text{ if } \text{eig}(A) > 0.$$

This will be illustrated using a simple SIMO example:

Example 4.3. Let the plant be described by the general i.s.o. state space representation with the matrices:

$A = \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $C = I$, $D = 0$, where $\text{eig}(A) > 0$. This will result in the rational representation $P(s)$:

$$P(s) = \begin{bmatrix} -1 & 0 & \frac{3s + 5}{s - 1} \\ 0 & -1 & \frac{s^2 + 4}{s - 1} \end{bmatrix} \in \mathcal{RH}_\infty^-, \text{ where } P(s)w(s) = 0.$$

Stable plants ($\text{eig}(A) < 0$):

However, it should also be possible to derive an anti-stable rational representation for plants that contain stable eigenvalues (so, $\text{eig}(A) < 0$). Therefore, equation (4.6a) has to be rewritten. To keep the calculations simple, we assume that there is no feed-through (so $D = 0$), but a similar conversion is also possible if
there is a feed through from the input to the output of the system.

\[ y(s) = C(sI - A)^{-1}B u(s) \]

\[ (C^T C)^{-1}C^T y(s) = (C^T C)^{-1}C C(sI - A)^{-1}B u(s) \]

\[ (C^T C)^{-1}C^T y(s) = (sI - A)^{-1}B u(s) \]

\[ (sI - A)(C^T C)^{-1}C^T y(s) = (sI - A)(sI - A)^{-1}B u(s) \]

\[ \frac{(sI - A)(C^T C)^{-1}C^T}{\notin \mathcal{RH}_\infty} y(s) = B u(s) \]

\[ (sI - \Psi)^{-1}(sI - A)(C^T C)^{-1}C^T y(s) = (sI - \Psi)^{-1}B u(s), \]

where \( \text{eig}(\Psi) > 0 \) to make the left- and right hand side of the equation rational anti-stable.

This results in the kernel for the plant’s behavior \( \hat{P}(s) \in \mathcal{RH}_\infty \), namely:

\[ \hat{P}(s) = \left[ -(sI - \Psi)^{-1}(sI - A)(C^T C)^{-1}C^T \quad (sI - \Psi)^{-1}B \right] \in \mathcal{RH}_\infty. \]  \hspace{1cm} (4.7)

Because, in this situation, all eigenvalues of the \( A \)-matrix are negative, it is a good idea to choose the introduced matrix \( \Psi = -A \), because then a part of the operator \( \hat{P}(s) \) is inner [13].

**Plants containing positive- and negative eigenvalues:**

In this situation one can use the same expression as used in the stable case, namely (4.7), however there are some restrictions on the choice of \( \Psi \) now. There should not occur pole-zero cancellation of the “zeros”, which are the original poles of the plant, with the introduced “poles”, which are the eigenvalues of the matrix \( \Psi \) (so, no cancellation in the part \((sI - \Psi)^{-1}(sI - A))\).

This can be preserved if we chose the \( \Psi \) matrix to be a diagonal matrix which contains the negative eigenvalues of the original matrix multiplied with \(-I\).

**Summarized representations using \( P(s) \in \mathcal{RH}_\infty \):**

The eigenvalues of \( A \) are given by:

\[ \text{eig}(A) = \text{diag}(\lambda_1, \ldots, \lambda_n), \quad \text{with} \quad A \in \mathbb{R}^{n \times n}, \]

which results in the following representations \( P(s), \hat{P}(s) \) and \( \tilde{P}(s) \) for the earlier three mentioned cases:

\[ \forall \lambda_i > 0 \quad (\text{unstable plant}) \quad : \quad P(s) = [-I \quad C(sI - A)^{-1}B + D], \]

\[ \forall \lambda_i < 0 \quad (\text{stable plant}) \quad : \quad \hat{P}(s) = [-\frac{(sI - \Psi)^{-1}(sI - A)(C^T C)^{-1}C^T \quad (sI - \Psi)^{-1}B}{\notin \mathcal{RH}_\infty}, \quad \Psi = -A, \]

\[ \exists \lambda_i > 0 \land \exists \lambda_j < 0 \quad : \quad \tilde{P}(s) = [-\frac{(sI - \tilde{\Psi})^{-1}(sI - A)(C^T C)^{-1}C^T \quad (sI - \tilde{\Psi})^{-1}B}{\notin \mathcal{RH}_\infty}, \quad \tilde{\Psi} = -A, \]

where \( i, j \in \{1, \ldots, n\} \) and \( \Psi = -\text{diag} \lambda_{s_1}, \ldots, \lambda_{s_n} \) where \( \lambda_{s_n} \) are all negative eigenvalues in \( A \). For all those representations holds that:

\[ P(s), \hat{P}(s), \tilde{P}(s) \in \mathcal{RH}_\infty. \]

**Stable representations (\( \mathcal{RH}^+_\infty \))**

The stable representations (denoted as \( Q(s), \hat{Q}(s) \) and \( \tilde{Q}(s) \)) can be derived on the same way as done in the previous subsection for the anti-stable representations. Therefore, only a summary will be given:

\[ \forall \lambda_i > 0 \quad (\text{unstable plant}) \quad : \quad Q(s) = [-\frac{(sI - \Psi)^{-1}(sI - A)(C^T C)^{-1}C^T \quad (sI - \Psi)^{-1}B}{\notin \mathcal{RH}_\infty}, \quad \Psi = -A, \]

\[ \forall \lambda_i < 0 \quad (\text{stable plant}) \quad : \quad \hat{Q}(s) = [-I \quad C(sI - A)^{-1}B + D], \]

\[ \exists \lambda_i > 0 \land \exists \lambda_j < 0 \quad : \quad \tilde{Q}(s) = [-\frac{(sI - \tilde{\Psi})^{-1}(sI - A)(C^T C)^{-1}C^T \quad (sI - \tilde{\Psi})^{-1}B}{\notin \mathcal{RH}_\infty}, \quad \tilde{\Psi} = -A, \]
where \( i, j \in \{1, \ldots, n\} \) and \( \tilde{\Psi} = -\text{diag}(\lambda_{u_1}, \ldots, \lambda_{u_k}) \) where \( \lambda_{u_i} \) are all positive eigenvalues in \( A \). Now, there holds that:

\[
Q(s), \quad \dot{Q}(s), \quad \dot{\tilde{Q}}(s) \in \mathcal{RH}_\infty^+.
\]

### 4.6.2 Conversion of the controlled description

As already said before, the controlled description of the system contains an autonomous part, which results in the trajectories for the states, and a linear equation, that results in the “controller-output” trajectories \( u(t) \). The representation of this latter is easy using rational operators, however it is not so simple for the autonomous part. As mentioned in Section 4.2 it is not possible to estimate a \( L_2 \) solution (except the trivial one) for an autonomous system. Therefore a separation has been made into \( L_2^- \) and \( L_2^+ \) solutions, which represent the anti-stable and stable autonomous systems. In this section the same separation will be made:

\[
\dot{y} = Hy, \quad \text{with} \quad \text{eig}(H) = \text{diag}(\lambda_1, \ldots, \lambda_n) \quad \text{and} \quad \forall i, \lambda_i > 0 \vee \forall \lambda_i < 0, \quad \text{where} \quad i \in \{0, \ldots, n\}.
\]

#### Unstable autonomous system

In this situation, it is only possible to define a \( L_2^- \) solution in time domain. Using this time domain, one can write the behavior of the autonomous system as:

\[
B_u = \{ y(t) \in L_2^- \mid y(t) = y(0)e^{Ht}, \quad \text{where} \quad \text{eig}(H) > 0 \}\subset L_2^-.
\]

Because this is an unstable autonomous system with a bounded solution on \( t < 0 \), one can say that a right-shifted version of \( y(t) \) is an element of the behavior. So, \( y_r(t) = y(t - \tau) \in B_u \) in this case. Therefore, an unstable autonomous system results in a right-shift invariant behavior.

The trajectories in time domain, \( y(t) \), are elements of \( L_2^- \), so in frequency domain the behavior will contain “trajectories” \( y(s) \in \mathcal{H}_\infty^- \) (as introduced in Chapter 3). Because there is shown that the behaviors should be right-shift invariant, the restrictions applied to \( y(s) \) will be represented using operators that are in \( \mathcal{RH}_\infty^- \). For this combination, the following behavior using kernels is defined:

\[
B_u = \{ y(s) \in \mathcal{H}_\infty^- \mid K(s)y(s) \in \mathcal{H}_\infty^- \} = \ker \Pi_\infty K(s) \subset \mathcal{H}_\infty^-.
\]

A solution for \( y(s) \) in the case that all eigenvalues of \( H \) are positive is, as derived in Section 4.2, given by:

\[
y(s) = -(sI - H)^{-1}y(0) \in \mathcal{H}_\infty^-.
\]

There has to hold that \( K(s)y(s) \in \mathcal{H}_\infty^+ \), so therefore the operator \( K(s) \) can be defined as:

\[
K(s) = (sI + H)^{-1}(sI - H) \in \mathcal{RH}_\infty^+ \quad \Rightarrow \quad K(s)y(s) = -(sI + H)^{-1}(sI - H)(sI - H)^{-1}y(0) \in \mathcal{H}_\infty^+.
\]

#### Stable autonomous system

For stable autonomous systems it is possible to derive a \( L_2^+ \) solution for the trajectories \( y(t) \) in time domain. So, in this case the behavior \( B_s \), that contains trajectories \( y(t) \) for this type of systems is given by:

\[
B_s = \{ y(t) \in L_2^+ \mid y(t) = y(0)e^{Ht}, \quad \text{where} \quad \text{eig}(H) < 0 \}\subset L_2^+.
\]

Same kind of story…. Left shift invariance required, so \( \mathcal{RH}_\infty^- \) and trajectories \( w(s) \in \mathcal{H}_\infty^+ \)…
4.7 Example controller synthesis: Full interconnection using Riccati

The starting point in this example is the following is the state space MIMO plant given (4.5), where

\[
A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}, \quad C = I, \quad D = 0.
\]

As one can mention, all states \( x \) are used as outputs of the plant. There is also no feed through in this situation. We have to convert this system into a rational kernel representation:

\[
P(s)w(s) = 0, \quad \text{where } P \in \mathcal{RH}_\infty \text{ and } w(s) = \begin{bmatrix} p(s) \\ u(s) \end{bmatrix}.
\]

In this example, we require an anti-stable representation for \( P \), because later on the controlled behavior \( K \) is autonomous and this behavior has to be left shift-invariant (so \( P, K, C \in \mathbb{L}(G) \)). It is possible that some extra “poles” have to be introduced for obtaining an anti-stable representation, however in this example this is not the case:

\[
P(s) = \begin{bmatrix} -1 & 0 & 2 & \frac{4(s-4) + 1}{s-4} \\ 0 & -1 & 0 & 1 \end{bmatrix} \in \mathcal{RH}_\infty.
\]

Here, the controlled system \( K \) has to fulfill the given quadratic optimisation problem (4.1), where \( Q = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, R = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \) and \( E \) is the solution of the ARE, namely: \( E \approx \begin{bmatrix} 4 & -15 \\ -15 & 76 \end{bmatrix} \).

The controlled behavior \( K \) should be represented using an anti-stable rational kernel representation:

\[
K(s)w(s) = 0, \quad \text{where } K \in \mathcal{RH}_\infty. \quad \text{Using the plant dynamics, the weighting matrices } Q \text{ and } R \text{ and the solution of the ARE } S, \text{ the controlled behavior is, using Section 4.6.2, the kernel of:}
\]

\[
K(s) = \begin{bmatrix} \frac{s+6}{s-a} & \frac{1}{s-a} & \frac{4(s-4) + 1}{s-a} \\ \frac{1}{s-a} & 0 & 0 \\ \frac{4(s-4) + 1}{s-a} & 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty.
\]

The “poles” \( a, b > 0, b \neq a \) are introduced to bring \( K(s) \) into a anti-stable rational form.

Now the plant- and controlled behavior, \( P \) and \( K \) respectively, are given using rational kernel representations, the algorithm can be applied. This is done using Matlab, where the rational matrices are implemented using zpk-objects.

Start algorithm for the full interconnection case:

**Step 1:**
Verify if the controlled behavior \( K \) is a subset of the plant behavior \( P \):

\[
P(s) = F(s)K(s) \quad \Rightarrow \quad F(s) = \begin{bmatrix} \frac{s-a}{s-4} & \frac{1}{s-a} & \frac{4(s-4) + 1}{s-a} \\ 0 & 0 & 0 \\ \frac{4(s-4) + 1}{s-a} & 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty.
\]

There does exist a rational matrix \( F(s) \), so we can continue to the next step.

**Step 2:**
Column reduce the estimated \( F(s) \). This is done in two steps: first, transform \( F(s) \) such that there are as many columns non-empty as the rank of \( F(s) \), secondly, put the empty columns on the last entries of the matrix.

\[
\tilde{F}(s) = F(s)\tilde{U}(s) \quad \Rightarrow \quad \tilde{U}(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{s-b}{s-a} \\ \frac{1}{s-a} & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{UH}_\infty, \quad \tilde{F}(s) = \begin{bmatrix} 0 & \frac{s-a}{s-4} & \frac{4(s-4) + 1}{s-4} \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty.
\]
Now, the columns have to be replaced such that:

\[ \overline{F}(s) = F(s)U(s) \implies U(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \frac{s-b}{s-a} \\ 0 & 1 & \frac{1}{2} \frac{s-a}{s-b} & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{UH}_\infty. \]

One has to verify if \( U(s) \) is really a unit, so its inverse has to be estimated for verification:

\[ U^{-1}(s) = \begin{bmatrix} 0 & 1 & 0 & -\frac{s-b}{s-a} \\ -\frac{s-a}{2} & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{UH}_\infty. \]

Concluding this step, the column reduced version of \( \overline{F}(s) \) equals:

\[ \overline{F}(s) = [F_1(s), 0] = \begin{bmatrix} \frac{s-a}{s-b} & 2 \frac{s-b}{s-a} & 0 & 0 \\ -\frac{s-a}{s-b} & 0 & 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty. \]

**Step 3:**

Estimate the \( W(s) \) matrix, which is required to obtain a possible controller:

\[ \overline{W}(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies W(s) = \overline{W}(s)U^{-1}(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

**Step 4:**

A possible controller that, after full interconnection with the plant, results in the controlled behavior \( K \) is:

\[ C_1(s) = W(s)K(s) = \begin{bmatrix} \frac{s-a}{s-b} & \frac{1}{s-b} & 0 & 0 \\ \frac{1}{s-a} & \frac{1}{s-b} & 0 & 1 \end{bmatrix} \in \mathcal{RH}_\infty. \]

As mentioned before, the behavioral framework does not require a separation of the variable \( w \) into inputs and outputs. This can be seen in the result above, because the controller tries to “restrict” the outputs of the plant in the first row when a separation in the variable space is made. So, mathematically the interconnection with this controller results in the desired controlled behavior, but this representation is not directly implementable for a real system, because outputs of a plant can’t be used as inputs. Therefore, another representation for this controller behavior has to be found in the next step.

**Step 5:**

Another controller has to be found using the matrices \( Q_1(s) \) and \( Q_2(s) \) as defined in the algorithm:

\[ C(s) = Q_1(s)P(s) + Q_2(s)C_1(s), \quad \text{where } Q_1, Q_2 \in \mathcal{RH}_\infty \text{ and } Q_2 \text{ full rank.} \]

If we choose \( Q \) such that:

\[ Q_1 = \begin{bmatrix} \frac{1}{2} \frac{s-a}{s-b} & -\frac{1}{2} \frac{1}{s-b} \\ 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty \text{ and } Q_2 = \begin{bmatrix} \frac{1}{2} \frac{s-a}{s-b} & -2 \\ 0 & 1 \end{bmatrix} \in \mathcal{RH}_\infty, \]

then this results in the desired controller:

\[ C(s) = \begin{bmatrix} \frac{1}{2} \frac{s-a}{s-b} & -\frac{15}{s-b} & \frac{1}{s-b} & 0 \\ \frac{5}{s-b} & \frac{1}{s-b} & 0 & \frac{1}{s-b} \end{bmatrix} \in \mathcal{RH}_\infty. \]

This is the “desired” LQR gain which we are looking for!

**Note:** Denote that the values in the Ricatti solution \( S \) and the values in the estimated rational expressions are rounded to integers for simplification.
Chapter 5

Conclusions and recommendations

Conclusions

We considered the problem of controller synthesis for specific classes of $L_2$ functions. Operators in the classes $RH_{\infty}^+$ of stable rational functions and $RH_{\infty}^-$ of anti-stable rational functions define linear right-shift invariant $L_2$ behaviors and linear left-shift invariant $L_2$ behaviors by considering their kernel spaces.

Given two $L_2$ behaviors $P$ and $K$ we solve the question to synthesise a third $L_2$ system $C$ that realises $K$ in the sense that the full- or partial interconnection of $P$ and $C$ satisfies $K = P \cap C$. Necessary and sufficient condition for the existence of a “controller” $C$ are the inclusions $K \subset P$ or $N \subset K \subset P_{manifest}$ respectively.

The general quadratic cost problem is given as example to demonstrate the full-interconnection algorithm for the construction of a rational representation of $C$. As intermezzo, a type of model reduction that can be applied to a Hamiltonian matrix is introduced.

The results of this Master work are not only presented in this report and my graduation report (which is included in the Appendix), but a part of it (the full interconnection situation) is also submitted to the IEEE Conference on Decision and Control 2008 in Mexico [8].

Recommendations

As shown in the example, it is possible to obtain a controller, which is valid in a mathematical manner, but not implementable in a real situation. Therefore, an extension on the algorithm is required such that it is possible to find implementable controllers directly.

Also some more research is required for the partial-interconnection case. It is possible that there are some limitations due to some requirements on the matrices in the elimination theorem. Studies already started for infinite smooth continuous behaviors in [11]. A nice aspect of this partial interconnection case is that disturbances like noise can be taken into account, which may yield in robust control problems.

Because, as in the given example, Hamiltonian functions can be used to describe controlled behaviors $K$, it can be possible to take dissipativity properties into account for the controller synthesis problems, because the Hamiltonian has some specific properties about energies. Also some more research has to be done on the model reduction and the separation of the stable- and unstable parts for this type of autonomous systems.

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Bibliography

Appendix A

System representations

In the appendix, different representations used in the previous chapters are discussed. It is possible to convert a certain system representation into another one. For example, a system described by the kernel of a polynomial matrix can be rewritten into a (output-nulling) state space realisation that results in the same behavior. It is also possible to convert representations into a form that results in easier computing problems. An example is the Smith form used for polynomial matrices. This is discussed in the first section of this appendix. The second section contains some information about different state space representations that can be used.

A.1 Smith- and Smith-McMillan Form

In this appendix the Smith- and Smith-McMillan form of polynomial and rational matrices are discussed [1, 5]. With this latter representation, it is quite easy to estimate the poles and zeros of a MIMO system. In this appendix, the so called Smith form is introduced in the first subsection, which is used to derive the Smith-McMillan form. This transformation results in the McMillan degree, which is an important cardinality of a behavior. When the Smith-McMillan form is derived, the poles and the zeros can be found very easily (Section A.1.3). This will be illustrated using a simple MIMO system.

A.1.1 Smith Form

The system, that is rewritten in the Smith- and Smith-McMillan form in this appendix, is denoted by a \( m \times p \) polynomial or rational matrix \( G(s) \) with rank \( r \) (with \( r \leq \min\{m, p\} \)). This matrix can be written in the following form:

\[
G(s) = \frac{1}{d(s)} \cdot P(s),
\]

where \( d(s) \) is the least common multiple (lcm) of the denominators of all the elements in \( G(s) \). After this division, \( P(s) \) will be transformed to the Smith form. This can be done by the three elementary operations which can be applied to a matrix:

**Definition A.1.** An elementary operation on a polynomial or rational matrix is one of the following three operations:

1. interchange of two rows or two columns,
2. multiplication of one row or one column by a constant,
3. addition of one row (column) to another row (column) times a polynomial or rational.
These operations can be applied by pre-multiplication or post-multiplication with unimodular polynomial matrices, which are known as elementary matrices in the polynomial case (or units when rational matrices are used). Pre-multiplication with such a matrix results in a row operation on the polynomial matrix, and post-multiplication results in a column operation on it.

When applying those multiplications of unimodular matrices (or units) to a polynomial matrix $P(s)$, this results in a new matrix $S(s)$.

**Definition A.2.** Those two matrices, $P(s)$ and $S(s)$, are equivalent matrices, denoted by $P(s) \sim S(s)$, if there exists a set of unimodular polynomial matrices (or units) $L_i$ and $R_i$, such that:

$$S(s) = L_1(s)L_2(s) \cdots L_{n_1}(s)P(s)R_1(s)R_2(s) \cdots R_{n_2}(s).$$

When choosing suitable $L_i$ and $R_i$ matrices, one can transform $P(s)$ into a pseudo-diagonal $m \times p$ matrix $S(s)$, which is named the Smith form, and has the following structure:

$$S(s) = \text{diag} [ \epsilon'_1(s), \epsilon'_2(s), \ldots, \epsilon'_r(s), 0, 0, \ldots, 0 ],$$

where $r$ is the normal rank of $G(s)$. Furthermore $\epsilon'_i(s)$ are the monic polynomials that are known as the invariant factors of $P(s)$. They have the property that

$$\epsilon'_i(s) \mid \epsilon'_{i+1}(s), \quad i \in [1, r - 1],$$

which means that $\epsilon'_i(s)$ is a factor in $\epsilon'_{i+1}(s)$, i.e. $\epsilon'_i(s)$ divides $\epsilon'_{i+1}(s)$.

To obtain those polynomials, the (monic) greatest common divisor (gcd) of all the $i \times i$ minor determinants of $P(s)$ have to be estimated. Those are denoted by $\chi_i(s)$, where $\chi_0(s) = 1$. After estimating those polynomials, the invariant factors $\epsilon'_i(s)$ can be calculated by:

$$\epsilon'_i(s) = \frac{\chi_i(s)}{\chi_{i-1}(s)}.$$

### A.1.2 Smith-McMillan form

When the polynomials $\chi_i(s)$ and $\epsilon'_i(s)$ are determined, it is a short step to the Smith-McMillan form of the transfer matrix. Just as in the Smith form, the transfer matrix $G(s)$ can be transformed to a polynomial matrix $M(s)$ that is an equivalent of $G(s)$: $G(s) \sim M(s)$:

$$M(s) = L(s)G(s)R(s).$$

The matrix $M(s)$ is known as the Smith-McMillan form, which has the following structure:

$$M(s) = \text{diag} \begin{bmatrix} \frac{\epsilon_1(s)}{\psi_1(s)}, \frac{\epsilon_2(s)}{\psi_2(s)}, \ldots, \frac{\epsilon_r(s)}{\psi_r(s)}, 0, 0, \ldots, 0 \end{bmatrix},$$

where

$$\frac{\epsilon_i(s)}{\psi_i(s)} \sim \frac{\epsilon'_i(s)}{d(s)}$$

with all possible cancellations between $\epsilon'_i(s)$ and $d(s)$ being performed. The polynomials $\epsilon_i(s)$ and $\psi(s)$ are coprime, so the gcd is equal to 1, and have the following characteristics:

$$\epsilon_i(s) \mid \epsilon_{i+1}(s), \quad i \in [1, r - 1],$$

$$\psi_{i+1}(s) \mid \psi_i(s), \quad i \in [1, r - 1].$$
A.1.3 Poles and zeros

Now the polynomial matrix is transformed to its Smith-McMillan form, it is easy to estimate the poles of the system. The pole polynomial is defined as

$$p(s) = \prod_{i=1}^{r} \psi_i(s) = \psi_1(s)\psi_2(s) \cdots \psi_r(s).$$

The roots of $p(s) = 0$ are the poles of the transfer matrix $G(s)$. The total number of poles in the system is given by $n = \deg[[p(s)]]$, which is known as the McMillan degree. This is the dimension of the minimal state space description of $G(s)$ (so if the order of the representation of $G(s)$ is larger then the McMillan degree, pole-zero cancellation occurs).

The zeros of the system can be found in a similar way. The zero polynomial is defined as

$$z(s) = \prod_{i=1}^{r} \epsilon_i(s) = \epsilon_1(s)\epsilon_2(s) \cdots \epsilon_r(s).$$

The transmission zeros of the system are equal to the roots of $z(s) = 0$. If there exists a transmission zero, it has to be a factor in (at least) one of the $\epsilon_i(s)$ polynomials. This results that there exists a $\epsilon_i(s) = 0$. So, a transmission zero can be defined to be any value for $s$ for which $G(s)$ loses rank.

**Example A.3.** Consider the following example of a system with $m = 3$ outputs and $p = 2$ inputs. The transfer matrix $G(s)$ is shown below and has a normal rank $r = 2$:

$$G(s) = \begin{bmatrix}
\frac{1}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\
\frac{(s+2)(s+8)}{(s+1)(s+2)} & \frac{2s^2-s-8}{2(s+1)(s+2)} \\
\frac{s^2+s-4}{(s+2)(s+2)} & \frac{2s^2-s-8}{(s+2)(s+2)}
\end{bmatrix}.$$

The first step is to determine $d(s)$, which is the lcm of all denominators in $G(s)$:

$$d(s) = \text{lcm}\left\{[(s+1)(s+2)]^4(s+1)^2\right\} = (s+1)(s+2).$$

The next step is to determine the polynomial matrix $P(s)$:

$$P(s) = \begin{bmatrix}
1 & -1 \\
\frac{s^2+s-4}{s-2} & \frac{2s^2-s-8}{2s-4}
\end{bmatrix}.$$  

Because we want to estimate the Smith form of $G(s)$, the invariant factors $\epsilon_i(s)$ and therefore the polynomials $\chi_i(s)$ have to be determined.

$$\chi_0(s) = 1$$

$$\chi_1(s) = \gcd[1, -1, (s^2+s-4), (2s^2-s-8), (s^2-4), (2s^2-8)] = 1$$

$$\chi_2(s) = \gcd\left[\begin{array}{c}
\frac{1}{s^2+s-4} & -1 \\
\frac{2s^2-s-8}{s^2-4} & \frac{2s^2-s-8}{2s^2} \\
\frac{s^2+s-4}{s+2} & \frac{2s^2-s-8}{s+2}
\end{array}\right] = \gcd[(s+2)(s-2), (s+2)(s-2), s(s+2)(s-2)] = (s+2)(s-2)$$

This results in the invariant factors:

$$\epsilon_1'(s) = \frac{\chi_1(s)}{\chi_0(s)} = \frac{1}{1} = 1$$

$$\epsilon_2'(s) = \frac{\chi_2(s)}{\chi_1(s)} = \frac{(s+2)(s-2)}{1} = (s+2)(s-2)$$

so the Smith form of $P(s)$ is:

$$S(s) = \begin{bmatrix}
1 & 0 \\
0 & (s+2)(s-2)
\end{bmatrix}.$$
Now the transformation to the Smith-McMillan form can be applied:

\[
\begin{align*}
\epsilon_1(s) / \psi_1(s) &= \epsilon'_1(s) / d(s) = \frac{1}{(s+1)(s+2)} \\
\epsilon_2(s) / \psi_2(s) &= \epsilon'_2(s) / d(s) = \frac{(s+2)(s-2)}{(s+1)(s+2)} = \frac{(s-2)}{(s+1)}
\end{align*}
\]

which results in

\[
G(s) \sim M(s) = \begin{bmatrix}
\frac{1}{(s+1)(s+2)} & 0 \\
0 & \frac{(s-2)}{(s+1)}
\end{bmatrix}.
\]

With the previous steps it is possible to derive the zero- and pole polynomials, which results in the following two equations:

\[
\begin{align*}
p(s) &= [(s+1)(s+2)](s+1) = (s+1)^2(s+2) \\
z(s) &= s - 2
\end{align*}
\]

So, the McMillan degree for this system is 3 and there are two poles at \( s = -1 \) and one pole at \( s = -2 \). There is a transmission zero at \( s = 2 \), so if this is the case, the rank of the system drops from 2 to 1.

### A.2 State space representations

![Figure A.1: Different state space realisations.](image)

In this section, three different state space representations are given. Those are depicted in Figure A.1. The purpose of this introduction is to show that those different representations can result in the same behaviors. It is also nice to see that there is a relation between the kernel representations, used in this report, and the output nulling state space representation that is introduced in this section.

First of all, the input/state/output state space representation is introduced as:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du,
\end{align*}
\]

where \( x \) are states, \( u \) inputs and \( y \) outputs. (A.1)

We define the trajectories \( w \) as \( w := (y_u) \), so the behavior of an input/state/output state space representation can be described by:

\[
\mathcal{B}_{i.s.o.}(A, B, C, D) = \{ w \mid \exists x : (A.1) \text{ holds} \}.
\]

The second type of state space representation is the so called output nulling state space representation:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
0 &= Cx + Dw,
\end{align*}
\]

where \( x \) are states and \( w \) trajectories. (A.2)

This form looks more like a kernel representation with a polynomial matrix, where the “input” is \( w \) and the “output” is equal to 0. The behavior of a system described by an output nulling representation is described by:

\[
\mathcal{B}_{o.n.}(A, B, C, D) = \{ w \mid \exists x : (A.2) \text{ holds} \}.
\]
The third and last state space form is the **driving variable** representation:

\[
\begin{align*}
\text{d.v. } \left\{ \begin{array}{l}
x' = Ax + Bu \\
y = Cx + Du
\end{array} \right. \quad & \text{where } x \text{ are states, } v \text{ the driving variable and } w \text{ trajectories,} \\
\end{align*}
\]

(A.3)

with its behavior description:

\[B_{\text{d.v.}}(A, B, C, D) = \{w \mid \exists x, \exists v : \text{(A.3) holds}\}.\]

**Conversions between different state space descriptions:**

It is possible to convert state space descriptions described by one of the three above mentioned forms into each other. In this section, those conversions are given by:

\[\Sigma_{\text{i.s.o.}} \leftrightarrow \Sigma_{\text{o.n.}}:\]

\[
\begin{align*}
\text{i.s.o. } \left\{ \begin{array}{l}
x' = Ax + Bu \\
y = Cx + Du
\end{array} \right. & \iff \text{o.n. } \left\{ \begin{array}{l}
x' = Ax + [0, B] y \\
y = Cx + [-I, D] y
\end{array} \right. \\
\end{align*}
\]

\[\Sigma_{\text{i.s.o.}} \rightarrow \Sigma_{\text{d.v.}}:\]

\[
\begin{align*}
\text{i.s.o. } \left\{ \begin{array}{l}
x' = Ax + Bu \\
y = Cx + Du
\end{array} \right. & \implies \text{d.v. } \left\{ \begin{array}{l}
x' = A x + B v \\
y = C \begin{bmatrix} x & u \end{bmatrix} + D \begin{bmatrix} u & 1 \end{bmatrix} v
\end{array} \right. , \quad \text{(note: } v = u)\.
\end{align*}
\]
Appendix B

Graduation paper

In the following pages, the graduation paper is included.
Controller synthesis for $L_2$ behaviors using rational kernel representations

Mark Mutsaers

Abstract—This paper considers the controller synthesis problem for the class of linear time-invariant $L_2$ behaviors. Classes of LTI $L_2$ systems are introduced whose behavior can be represented as the kernel of a rational operator. Given a plant and a controlled system in this class, an algorithm is developed that produces a rational kernel representation of a controller that, when interconnected with the plant, realizes the controlled system. This result generalizes similar synthesis algorithms in the behavioral framework for infinitely smooth behaviors that allow representation as kernels of polynomial differential operators. A part of this work has been submitted to the IEEE Conference on Decision and Control [6].

I. INTRODUCTION

The analysis of system interconnections is at the heart of many problems in modeling, simulation and control. Indeed, when focusing on control, the controller synthesis question amounts to finding a dynamical system (a controller) that, after interconnection with a given plant, results in a controlled system that is supposed to perform a certain task in a more desirable manner than the plant without controller. Usually the control synthesis problem is formulated as a feedback optimisation problem in which the plant and controller interact through a number of distinguished channels that have been divided in input and output variables.

The behavioral theory of dynamical systems has been advocated as a conceptual framework in which especially interconnection structures of dynamical system can be studied in an input-output independent setting. There are many conceptual, pedagogic and practical reasons for doing so. Please see [12], [13] for a detailed account on this matter. One key problem concerning the interconnection of dynamical systems involves the question when a given dynamical system $K$ can be implemented (or realized) as the interconnection of a dynamical system $P$, that is supposed to be given, and a second dynamical system $C$, that is supposed to be designed. With the interpretation that $P$ and $K$ denote plant and controlled system, this question is therefore equivalent to a synthesis question for the controller $C$.

Within the behavioral framework this question received a very complete and elegant answer for the class of linear time-invariant systems that admit representations in terms of polynomial difference or polynomial differential operators [7], [8]. A rather complete theory has been developed for such representations that covers, among other things, $H_\infty$, LQ and $H_2$ optimal control.

It is the purpose of this paper to reconsider the controller synthesis question for specific classes of linear and time-invariant $L_2$ systems that admit representations in terms of rational functions. In doing so, we depart from the setting proposed in [15] of considering infinitely smooth trajectories of $C^\infty$ as solutions to “rational” differential equations. Instead, rational functions in $H_\infty$ are viewed as multiplicative operators on $L_2$ functions to define $L_2$ behaviors or, using Parseval, $L_2$ behaviors in the time domain. In particular, this gives a frequency domain treatise on a very general synthesis problem for interconnected systems.

The paper is organised as follows. In Section II some notational remarks about spaces and operators, which are required in the algorithm, are introduced. In Section III, the $L_2$ behavior will be introduced, which is used in the full- and partial interconnection problems, and their controller synthesis algorithms, in Section IV. Section V contains an example that illustrates the full-interconnection case, which is discussed in Section VI. In the last section, the results of this paper are discussed and some recommendations for further research on $L_2$ behaviors are given.

II. NOTATION

A. Dynamical systems

Following the behavioral formalism, a dynamical system [2] is a triple:

$$\Sigma = (T, W, B),$$

where $T \subset \mathbb{R}$ or $T \subset C$ is the time- or frequency-axis, $W$ is the variable signal space, which typically contains inputs and outputs and will be taken to be a finite dimensional vector space throughout, and $B \subset W^T$ is the behavior, that will be defined in Section III.

B. Hilbert spaces

The $L_2$, $L_2^+$ and $L_2^-$ (Hilbert) spaces are defined as:

$$L_2 = L_2(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R}^q \mid \| f \|_2 < \infty \},$$

$$L_2^+ = L_2(\mathbb{R}_+) := \{ f : \mathbb{R}_+ \to \mathbb{R}^q \mid \| f \|_2 < \infty \},$$

$$L_2^- = L_2(\mathbb{R}_-) := \{ f : \mathbb{R}_- \to \mathbb{R}^q \mid \| f \|_2 < \infty \}.$$

Here, $\| \cdot \|$ denotes the Euclidean norm and

$$\| f \|_2 = \sqrt{\langle f, f \rangle} = \left( \int_T f(\mu)^H f(\mu) d\mu \right)^{\frac{1}{2}}.$$  (1b)

where $T$ denotes the signal domain $\mathbb{R}$, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$, respectively. These spaces become Hilbert spaces when equipped with the standard inner product

$$\langle f, g \rangle = \int_T f(\mu)^H g(\mu) d\mu.$$  (1c)

Superscripts $q$ indicate the dimension of signals. By definition it follows that any $w \in L_2$ can be uniquely decomposed as $w = w_+ + w_-$, with $w_+ \in L_2^+$ and $w_- \in L_2^-$, where
\[ \|w\|_2 = \|w_+\|_2 + \|w_-\|_2. \]

This can be written as a direct sum decomposition:
\[ L_2(\mathbb{R}) = L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_-). \]  

(1d)

C. Hardy spaces

Now, Hardy spaces are introduced as \( \mathcal{H}_p^+ \) and \( \mathcal{H}_p^- \), where \( p = \{1, 2, \ldots, \infty\} \) by setting:

\[ \mathcal{H}_p^+ := \{ f : \mathbb{C}^+ \rightarrow \mathbb{C}^q \mid \| f \|_{\mathcal{H}_p^+} < \infty \}, \]
\[ \mathcal{H}_p^- := \{ f : \mathbb{C}^- \rightarrow \mathbb{C}^q \mid \| f \|_{\mathcal{H}_p^-} < \infty \}, \]

where \( \mathbb{C}^+ := \text{Re}\{s\} > 0 \) and \( \mathbb{C}^- := \text{Re}\{s\} < 0 \), with \( s = \sigma + j \omega \). So, functions in \( \mathcal{H}_p^+ \) are analytic in \( \mathbb{C}^+ \cup \{\infty\} \) and functions in \( \mathcal{H}_p^- \) are analytic in \( \mathbb{C}^- \cup \{-\infty\} \). The \( \mathcal{H}_p^+ \) spaces are classical Hardy spaces.

The norms of functions in \( \mathcal{H}_p^+ \) and \( \mathcal{H}_p^- \) are defined as

\[ \|f\|_{\mathcal{H}_p^+} = \lim_{\sigma \to \infty} \left( \int_{-\infty}^{\infty} |f(\sigma + j \omega)|^p d\omega \right)^{\frac{1}{p}}, \quad 0 < p < \infty, \]
\[ \lim_{\sigma \to -\infty} \sup_{\omega \in \mathbb{R}} |f(\sigma + j \omega)|, \quad p = \infty, \]

and

\[ \|f\|_{\mathcal{H}_p^-} = \lim_{\sigma \to \infty} \left( \int_{-\infty}^{\infty} |f(\sigma + j \omega)|^p d\omega \right)^{\frac{1}{p}}, \quad 0 < p < \infty, \]
\[ \lim_{\sigma \to \infty} \sup_{\omega \in \mathbb{R}} |f(\sigma + j \omega)|, \quad p = \infty. \]

It is remarked that the tangential limits \( \sigma \to 0 \) in the above expressions exist, which makes the Hardy spaces well defined normed spaces.

D. Rational functions and units

The prefixes \( \mathcal{R} \) and \( \mathcal{U} \) denote, respectively, rational functions and units in the Hardy spaces \( \mathcal{H}_p^+ \) and \( \mathcal{H}_p^- \) as in

\[ \mathcal{R}\mathcal{H}_p^+ := \{ f \in \mathcal{H}_p^+ \mid f \text{ is rational} \}, \]
\[ \mathcal{R}\mathcal{H}_p^- := \{ f \in \mathcal{H}_p^- \mid f \text{ is rational} \}, \]

and

\[ \mathcal{U}\mathcal{H}_p^+ := \{ U \in \mathcal{R}\mathcal{H}_p^+ \mid U^{-1} \in \mathcal{R}\mathcal{H}_p^+ \}, \]
\[ \mathcal{U}\mathcal{H}_p^- := \{ U \in \mathcal{R}\mathcal{H}_p^- \mid U^{-1} \in \mathcal{R}\mathcal{H}_p^- \}. \]

Note that units are necessarily square rational matrices.

E. Laplace transformation

The Laplace transform, \( \mathcal{L} : L_2(\mathbb{R}) \rightarrow L_2(\mathbb{C}) \), defines an isometry between \( L_2 \) and \( L_2 \):

\[ |\mathcal{L}(f)|(s) := \int_{\mathbb{R}} f(t) e^{-st} dt, \quad s \in \mathbb{C}. \]

This transformation can be applied to the (complete) \( L_2 \) space, as defined in (1), which results in a new inner-product space \( L_2 \):

\[ \mathcal{L}_2 := \{ f : \mathbb{C} \rightarrow \mathbb{C}^q \mid \| f \|_2 < \infty \}, \]

which inherits the following norm:

\[ \|f\|_2^2 = \int_{-\infty}^{\infty} |f(j\omega)|^2 d\omega, \]

and the inner product on complex valued functions

\[ \langle f, g \rangle = \int_{-\infty}^{\infty} f(j\omega) g(j\omega) d\omega. \]

Then, as in (1d), \( \mathcal{L}_2 \) admits the orthogonal decomposition

\[ \mathcal{L}_2 = \mathcal{L}_2^+ \oplus \mathcal{L}_2^-, \]

so that any element \( w \in \mathcal{L}_2 \) can be uniquely decomposed as \( w = w_+ + w_- \), where

\[ w_+ := \Pi_+ w, \quad \text{with} \quad \Pi_+ : \mathcal{L}_2 \rightarrow \mathcal{H}_2^+, \]
\[ w_- := \Pi_- w, \quad \text{with} \quad \Pi_- : \mathcal{L}_2 \rightarrow \mathcal{H}_2^- \].

Here, \( \Pi_+ \) and \( \Pi_- \) denote the canonical projections from \( \mathcal{L}_2 \) onto \( \mathcal{H}_2^+ \) and \( \mathcal{H}_2^- \), respectively.

With this knowledge, it is possible to extend the previous introduced \( \mathcal{H}_2^+ \) and \( \mathcal{H}_2^- \) spaces into inner-product spaces.

Summarising:

\[ \mathcal{L} : L_2(\mathbb{R}) \rightarrow \mathcal{L}_2 = \mathcal{H}_2^+ \oplus \mathcal{H}_2^-, \]
\[ \mathcal{L} : L_2(\mathbb{R}_+) \rightarrow \mathcal{H}_2^+, \]
\[ \mathcal{L} : L_2(\mathbb{R}_-) \rightarrow \mathcal{H}_2^- \].

F. Mappings in \( \mathcal{R}\mathcal{H}_2^+ \) and \( \mathcal{R}\mathcal{H}_2^- \)

Elements of \( \mathcal{R}\mathcal{H}_2^+ \) (also known as stable functions \( \mathcal{R}\mathcal{H}_\infty^{\text{stable}} \)) and \( \mathcal{R}\mathcal{H}_2^- \) (known as anti-stable functions \( \mathcal{R}\mathcal{H}_\infty^{\text{anti-stable}} \)) define dynamical systems in the following manner.

Let \( \Theta \in \mathcal{R}\mathcal{H}_2^+ \). Then \( \Theta \) defines a number of mappings

\[ \Theta : \mathcal{L}_2 \rightarrow \mathcal{L}_2, \quad \Theta : \mathcal{H}_2^+ \rightarrow \mathcal{L}_2, \quad \Theta : \mathcal{H}_2^- \rightarrow \mathcal{L}_2, \]

by defining

\[ (\Theta w)(s) := \Theta(s) w(s), \quad \text{where} \quad w \in \{ \mathcal{L}_2, \mathcal{H}_2^+, \mathcal{H}_2^- \}, \]  

which is the usual “multiplication” operator in the frequency domain.

Conversion to the time domain:

To make this mapping more clear, (2) is in the time domain interpreted as follows:

\[ \theta(t) = L^{-1}(\Theta(s)) \in L_1(\mathbb{R}), \]

which means that \( \theta(t) \) is absolute integrable and \( \theta(t) = 0 \) for all \( t < 0 \). The multiplication in the frequency domain (2) is equivalent to a convolution in time domain. Therefore, we define an equivalent mapping \( \hat{\theta} : \mathcal{L}_2 \rightarrow \mathcal{L}_2 \) as:

\[ (\hat{\theta} \hat{w})(t) := (\theta \ast \hat{w})(t) := \int_{-\infty}^{\infty} \theta(t - \tau) \hat{w}(\tau) d\tau. \]
Theorem 2.1: Let \( \theta \in RH_{\infty}^+ \) and \( \Psi \in RH_{\infty}^+ \). Results with different input signals \( \{ L_2, L_2^+, L_2^\ast \} \) depicted in time domain.

When taking elements \( \hat{w} \) from the three different spaces \( L_2, L_2^+ \) and \( L_2^\ast \), convolution with \( \theta \) results in functions that lie in \( L_2, L_2^+ \) and \( L_2^\ast \), respectively. See Fig. 1(a)-(c).

If \( w = \mathcal{L}(\hat{w}) \) is the Laplace transform of \( \hat{w} \in L_2 \) then we have that \( \mathcal{L}(\theta w) = \theta w \).

To summarise the mappings in \( RH_{\infty}^+ \) the following theorem is introduced:

**Theorem 2.2:** Let \( \Theta(s) \in RH_{\infty}^+ \) (so \( \theta = \mathcal{L}^{-1}(\Theta) \in L_1(\mathbb{R}) \) with \( \theta(t) = 0 \) for all \( t < 0 \), with possible arguments \( L_2, H_2^+ \) and \( H_2^\ast \) (or \( L_2, L_2^+ \) and \( L_2^\ast \)). Then:

\[
\begin{align*}
\Theta : L_2 & \rightarrow L_2, \\
\Theta : H_2^+ & \rightarrow H_2^+, \\
\Theta : H_2^\ast & \rightarrow L_2.
\end{align*}
\]

Similar mappings can be defined by elements \( \Psi \in RH_{\infty}^+ \).

Indeed, let

\[
(\Psi w)(s) := \Psi(s)w(s), \quad \text{where } w \in \{ L_2, H_2^+, H_2^\ast \}.
\]

G. Left- and right-\( \tau \)-shift invariant systems

Before introducing shift invariant systems, the left- and right-\( \tau \)-shift operators have to be defined in frequency domain, \( \sigma_L \) and \( \sigma_R \) respectively. The left- and right-\( \tau \)-shifts usually refer to shifts (of \( \tau \)) of signals with respect to the time axis, defined as:

\[
(\sigma_L \hat{w})(t) := \hat{w}(t + \tau),
\]

\[
(\sigma_R \hat{w})(t) := \hat{w}(t - \tau),
\]

with \( \hat{w}(t) \) an element of \( L_2, H_2^+ \) or \( H_2^\ast \). Here, these operators on \( L_2 \) and the Hardy spaces \( H_2^+ \) and \( H_2^\ast \) are introduced as follows:

\[
\begin{align*}
\sigma_L : L_2 & \rightarrow L_2, & (\sigma_L w)(s) &= e^{s\tau}w(s), \\
\sigma_L : H_2^+ & \rightarrow H_2^+, & (\sigma_L w)(s) &= e^{s\tau}w(s) - \int_0^\tau \hat{w}(t)e^{-st}dt, \\
\sigma_L : H_2^\ast & \rightarrow H_2^\ast, & (\sigma_L w)(s) &= e^{s\tau}w(s), \\
\sigma_R : L_2 & \rightarrow L_2, & (\sigma_R w)(s) &= e^{-s\tau}w(s), \\
\sigma_R : H_2^+ & \rightarrow H_2^+, & (\sigma_R w)(s) &= e^{-s\tau}w(s), \\
\sigma_R : H_2^\ast & \rightarrow H_2^\ast, & (\sigma_R w)(s) &= e^{-s\tau}w(s) - \int_0^\tau \hat{w}(t)e^{-st}dt.
\end{align*}
\]

A subset \( P \) of \( L_2 \) is said to be left-shift invariant (or right-shift invariant) whenever \( \sigma_L P \subseteq P \) (or \( \sigma_R P \subseteq P \)). Similar definitions apply for subsets of \( H_2^+ \) and \( H_2^\ast \). Note that \( \sigma_L \) and \( \sigma_R \) do not define isometries when applied to functions in \( H_2^+ \) and \( H_2^\ast \), respectively. When interpreted in the time domain, these mappings are depicted in Fig. 2 for a signal \( \hat{w} \in L_2^\ast \).

H. Inner- and outer representations

A nice property of a rational mapping \( \Theta \in RH_{\infty}^+ \) or \( \Psi \in RH_{\infty}^+ \) is that an inner-outer decomposition can be made for them [9]. This has some nice properties, which are required in an algorithm later on. First, the definitions
for inner- and outer rational matrices have to be given.

**Definition 2.1:** Define a $T \in \mathcal{RH}_n^+$ which describes the transfer function $y(s) = T(s)u(s)$, where $\{y, u\} \in \mathcal{L}_2$. This $T$ is called inner if holds that:

$$\|Tu\|_2 = \|u\|_2,$$

for all $u \in \mathcal{L}_2$.

So, one has to verify if:

$$\int_{-\infty}^{\infty} u(s)^* T(s)^* T(s) u(s) d\omega = \int_{-\infty}^{\infty} u(s)^* u(s) d\omega,$$

where $*$ denotes the conjugate transposed. As can be seen, $T$ is inner if $T(s)^* T(s) = I$ for all $s \in \mathbb{C}^n$. Knowing this, a nice property of an inner matrix can be obtained by observing eigenvalues:

$$\text{eig}(T(s)^* T(s)) = \text{eig}(I) = 1,$$

which is equal to:

$$\sqrt{\text{eig}(T(s)^* T(s))} = \sigma(T(s)) = 1.$$

So, in the case if $T$ is inner, the singular values for all frequencies $\omega$ are equal to 1.

For SISO systems $T \in \mathcal{RH}_n^{+,*}$ is inner if it is a finite Blaschke product multiplied by a complex number $p$ of unit modulus $|p| = 1$. That is

$$T(s) = p \prod_{i=1}^{n} \frac{|a_i|}{a_i} \frac{s + a_i}{s - a_i},$$

where $\text{Re}(a_i) > 0$. One can show that in this situation, the Bode plot (or singular value plot) is equal to 1 for all possible frequencies $\omega$.

Now inner is known, also outer has to be defined [9]:

**Definition 2.2:** An operator $T : \mathcal{L}_2 \to \mathcal{L}_2$ as multiplication, where $T \in \mathcal{RH}_n^{+,\infty}$, is outer if $\text{im} T$ is dense in $\mathcal{L}_2$.

This definition results in the following theorem:

**Theorem 2.3:** Let $T : \mathcal{L}_2 \to \mathcal{L}_2$, $T \in \mathcal{RH}_n^{+,\infty}$, be as defined by multiplication. Then, the following three items are equivalent:

1) $T$ is outer,
2) The determinental rank of $T$, which is largest integer $r$ for which there exists a non-zero $r \times r$ minor of $T$, is equal to the “normal” matrix rank evaluated at a certain $\lambda \in \mathbb{C}^+$, so:

$$\text{rank}_{\mathbb{R}}(T(s)) = \text{rank}_{\mathbb{C}}(T(\lambda)), \quad \forall \lambda \in \mathbb{C}^+,$$

3) When $T(s)$ is written in Smith-McMillan form [1], [5], there are no zeros in the right half plane.

**Note:** If $T$ is also a square matrix, one can notice that $T$ is a unit $\mathcal{UH}_n^{+,*}$.

**Theorem 2.4:** As defined in [9], every matrix $T \in \mathcal{RH}_n^{+,\infty}$ of dimension $n \times m$ containing rational elements, $T \in \mathcal{RH}_n^{+,\infty}$, can be decomposed as:

$$T = T_1 T_2,$$

where $T_1$ is inner and $T_2$ is outer. If $n \geq m$, then $T_2$ is square.

With appropriate adaptations on definitions, those theorems and definitions can also be applied using $T \in \mathcal{RH}_n^{+,\infty}$.

**III. RATIONAL REPRESENTATIONS OF BEHAVIORS**

In the previous section, stable- and anti-stable rational mappings have been introduced on Hilbert spaces. In this section we will associate behaviors in $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$ defined through the null spaces of these mappings. So here, the variables $w$ can be any element of the spaces $\mathcal{L}_2$, $\mathcal{H}_2^+$ or $\mathcal{H}_2^-$.

**Definition of a $\mathcal{L}_2$ behavior $\mathcal{P}$**

First, behaviors associated with mappings $P$ from the space of rational stable Hardy functions are discussed. For any $P \in \mathcal{RH}_n^+$ the three dynamical systems $(\mathbb{C}, \mathbb{C}_q, \mathcal{P}(\pm))$ are defined:

$$\mathcal{P}(P) := \{w \in \mathcal{L}_2 \mid Pw = 0\} = \ker P \subset \mathcal{L}_2,$$

$$\mathcal{P}_+(P) := \{w \in \mathcal{H}_2^+ \mid Pw = 0\} = \ker P \subset \mathcal{H}_2^+, \quad (3)$$

$$\mathcal{P}_-(P) := \{w \in \mathcal{H}_2^- \mid Pw = 0\} = \ker \Pi_- P \subset \mathcal{H}_2^-.$$  

Here, $\Pi_-$ is a canonical projection that is introduced before. We have that:

**Lemma 3.1:** For $P \in \mathcal{RH}_n^+$, the systems $\mathcal{P}(P)$, $\mathcal{P}_+(P)$ and $\mathcal{P}_-(P)$ are linear and right-shift invariant subsets of $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$, respectively.

The classes of all linear and right-shift invariant systems in $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$ that admit representations as the kernel of a rational element $P \in \mathcal{RH}_n^+$ are denoted by $\mathcal{M}$, $\mathcal{M}_+$ and $\mathcal{M}_-$.

Similarly, for any $\tilde{P} \in \mathcal{RH}_n^+$ three dynamical systems $(\mathbb{C}, \mathbb{C}_q, \mathcal{P}(\pm))$ can be introduced:

$$\mathcal{P}(\tilde{P}) := \{w \in \mathcal{L}_2 \mid \tilde{P}w = 0\} = \ker \tilde{P} \subset \mathcal{L}_2,$$

$$\mathcal{P}_+(\tilde{P}) := \{w \in \mathcal{H}_2^+ \mid \tilde{P}w = 0\} = \ker \Pi_+ \tilde{P} \subset \mathcal{H}_2^+, \quad (4)$$

$$\mathcal{P}_-(\tilde{P}) := \{w \in \mathcal{H}_2^- \mid \tilde{P}w = 0\} = \ker \tilde{P} \subset \mathcal{H}_2^-,$$

where $\Pi_+ : \mathcal{L}_2 \to \mathcal{H}_2^+$ is the canonical projection introduced before.

**Lemma 3.2:** For $\tilde{P} \in \mathcal{RH}_n^+$ the systems $\mathcal{P}(\tilde{P})$, $\mathcal{P}_+(\tilde{P})$ and $\mathcal{P}_-(\tilde{P})$ are linear and left-shift invariant subsets of $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$, respectively.
Now, the classes of all linear and left-shift invariant systems in $\mathcal{L}_2$, $\mathcal{H}_x^2$ and $\mathcal{H}_x^{-2}$ that admit representations as the kernel of a rational element $\hat{P} \in \mathcal{RH}_\infty^2$ are denoted by $\mathbb{L}_x, \mathbb{L}_x^+$ and $\mathbb{L}_x^-$.

IV. CONTROLLER SYNTHESIS

In this section of the paper, the introduced $\mathcal{L}_2$ behaviors will be used to solve the synthesis problems for the so called full- and partial interconnection situations that occur in control problems. Those two cases, with their algorithms that solve the synthesis problems, are defined separately:

A. Full Interconnection - Problem

**Problem 4.1:** Given two linear left-shift invariant systems $\mathcal{P}$ and $\mathcal{K}$ in the class $\mathcal{L}_2$ (or $\mathbb{L}_x^+$ or $\mathbb{L}_x^-$), construct, if it exists, a linear left-shift-invariant system $\mathcal{C}$ such that $\mathcal{P} \cap \mathcal{C} = \mathcal{K}$.

Any such system is said to implement $\mathcal{K}$ for $\mathcal{P}$ by full interconnection.

A similar problem formulation applies to the model classes $\mathbb{M}_x, \mathbb{M}_x^+$ and $\mathbb{M}_x^-.

Fig. 3(a) gives an illustration of the problem treated in this section, namely the existence of a controller $\mathbb{C}$ that after intersection with the plant $\mathcal{P}$ results in the controlled behavior $\mathcal{K}$. Here, systems are interconnected through all variables $w$. That is, we consider full-interconnections as depicted in Fig. 3(b).

Our synthesis algorithm is inspired by the polynomial analog that has been treated in [7], [8]. Our algorithm leads to explicit rational representations of all systems $\mathcal{C}$ that implement $\mathcal{K}$ for $\mathcal{P}$, whenever they exist.

**Theorem 4.1:** Given the systems $\mathcal{P}, \mathcal{K} \in \mathbb{L}_x^\pm$ (or $\mathbb{M}_x^\pm$).

There exists a controller $\mathcal{C} \in \mathbb{L}_x^\pm$ (or $\mathbb{M}_x^\pm$), that implements $\mathcal{K}$ for $\mathcal{P}$ by full interconnection if and only if $\mathcal{K} \subset \mathcal{P}$.

Our proof of this theorem is similar to the one in [7], which gives a comparable result for behaviors that are defined as the continuous and infinitely often differentiable functions in the kernel of a polynomial differential operator. The proof of Theorem 4.1 will result from the following theorems and algorithms:

**Theorem 4.2:** Let $\mathcal{K}, \mathcal{P} \in \mathcal{RH}_\infty^2$ and let $\mathcal{P}(\pm) = \mathcal{P}(\pm)(\mathcal{P})$ and $\mathcal{K}(\pm) = \mathcal{K}(\pm)(\mathcal{K})$ be their associated behaviors as defined in (3) (so, $\mathcal{P}, \mathcal{K} \in \mathbb{M}_x^\pm$).

Equivalent are:

i. $\mathcal{K} \subset \mathcal{P}$,

ii. $\mathcal{K}_+ \subset \mathcal{P}_+$,

iii. $\mathcal{K}_- \subset \mathcal{P}_-$,

iv. $\exists F \in \mathbb{RH}_\infty^2$ such that $\mathcal{P} = FK$.

Moreover, $\mathcal{K} = \mathcal{P} \iff \mathcal{K}_+ = \mathcal{P}_+ \iff \mathcal{K}_- = \mathcal{P}_- \iff \exists U \in \mathbb{UX}_\infty^2$ such that $\mathcal{P} = UK$.

The proof of this theorem can be found in the Appendix. Also anti-stable mappings can be used in the representations, which yields the following theorem:

**Theorem 4.3:** Let $\hat{\mathcal{K}}, \hat{\mathcal{P}} \in \mathbb{RH}_\infty^2$ and let $\mathcal{P}(\pm) = \mathcal{P}(\hat{\mathcal{P}})$ and $\mathcal{K}(\pm) = \mathcal{K}(\hat{\mathcal{K}})$ as in (4) (so, $\mathcal{P}, \mathcal{K} \in \mathbb{L}_x^\pm$).

Equivalent are:

i. $\mathcal{K} \subset \mathcal{P}$,

ii. $\mathcal{K}_+ \subset \mathcal{P}_+$,

iii. $\mathcal{K}_- \subset \mathcal{P}_-$,

iv. $\exists F \in \mathbb{RH}_\infty^2$ such that $\mathcal{P} = \hat{F} \mathcal{K}$.

Moreover, $\mathcal{K} = \mathcal{P} \iff \mathcal{K}_+ = \mathcal{P}_+ \iff \mathcal{K}_- = \mathcal{P}_- \iff \exists U \in \mathbb{UX}_\infty^2$ such that $\mathcal{P} = \hat{U} \mathcal{K}$.

The proof of this theorem is similar to the one of Theorem 4.2 and therefore is not included in this paper. Those theorems are used in the following algorithm.

B. Full interconnection - Algorithm

An algorithm (based on [8]) for obtaining $\mathcal{C} \in \mathbb{RH}_\infty^2$ using trajectories $w \in \mathcal{L}_2$ is given. This describes the behavior $\mathcal{C} \in \mathbb{L}_x$, however this algorithm is similar for finding behaviors $\mathcal{C}$ in the classes $\mathbb{M}_x^\pm$ and $\mathbb{M}_x^\pm$.

**Algorithm 1:** Given $P, K \in \mathbb{RH}_\infty^2$ that define the systems $\mathcal{P}$ and $\mathcal{K}$ as in (4).

**Aim:** Find $\mathcal{C} \in \mathcal{RH}_\infty^2$ that defines the system $\mathcal{C} := \mathcal{C}(\mathcal{C}) \in \mathbb{L}_x$ as in (4), such that $\mathcal{C}$ implements $\mathcal{K}$ for $\mathcal{P}$ in the sense that $\mathcal{P} \cap \mathcal{C} = \mathcal{K}$ by full interconnection.

**Step 1:** Verify whether $\mathcal{K} \subset \mathcal{P}$. If so, there exists a mapping $\mathcal{F} \in \mathbb{RH}_\infty^2$ such that $\mathcal{P} = \mathcal{F} \mathcal{K}$. If not, the algorithm ends and no controller exists that implements $\mathcal{K}$ for $\mathcal{P}$.

**Step 2:** Introduce a unit $U \in \mathbb{UX}_\infty^2$. This unit is introduced to bring $\mathcal{F}$ into column reduced form: $\tilde{\mathcal{F}} = \mathcal{F}U = \left[ \begin{array}{c} F_1 \\ 0 \end{array} \right]$, where $F_1 \in \mathbb{RH}_\infty^2$ is square and full rank.

**Step 3:** Extend this matrix $\tilde{\mathcal{F}}$ using $\overline{\mathcal{W}} = [0 \mid I]$ such that:

$$\tilde{\Lambda} = \left[ \begin{array}{c} \tilde{\mathcal{F}} \\ \overline{\mathcal{W}} \end{array} \right] = \left[ \begin{array}{c} F_1 \\ 0 \end{array} \right]$$

where $\overline{\mathcal{W}} = \mathcal{W}U$, so $\mathcal{W} = \mathcal{W}U^{-1}$ and $\tilde{\Lambda} \in \mathbb{UX}_\infty^2$ is a unit.

**Step 4:** In the controlled behavior $\mathcal{K}$, the restrictions of the plant as well as the restrictions applied by the controller have to be satisfied:

$$\mathcal{K} = \ker \left[ \overline{\mathcal{W}}^T \right] = \ker \left[ \mathcal{W}^T \right] \ker \left( \overline{\mathcal{W}}^T \right) \ker \left( \mathcal{W}^T \right),$$

where $\Lambda \in \mathbb{UX}_\infty^2$.

Because the unit $\tilde{\mathcal{K}}$ is a multiplication of a unit $U$ with $\Lambda = \text{col}(\mathcal{F}, \mathcal{W})$, also $\Lambda$ has to be a unit. Therefore:

$$\mathcal{K} = \ker \left[ \overline{\mathcal{W}}^T \right] = \ker \left( \mathcal{W}^T \right) \ker \left( \overline{\mathcal{W}}^T \right),$$
so a possible controller is described as the kernel of the rational function

\[ C = WK \]

that belongs to \( \mathcal{R}H_{\infty}^- \).

Then, the controller behavior is given by:

\[ \mathcal{C} = \{ w \in \mathcal{L}_2 \mid C(s)w(s) = 0 \} = \ker C. \]

**Step 5:** This controller is not the only one that realizes the desired controlled behavior \( \mathcal{K} \), because another multiplication with a unit \( Q \in \mathcal{U}H_{\infty} \) is possible:

\[ \mathcal{K} = \ker \left\{ \left[ \begin{array}{c} Q_1 \frac{1}{s} Q_2 \end{array} \right] \right\} = \ker \left( \left[ Q_1 P + Q_2 WK \right] \right), \]

where \( Q_1, Q_2 \in \mathcal{R}H_{\infty}^- \) and \( Q_2 \) is full rank.

**Result:** With this pre-multiplication, all possible controllers are given by the parametrization:

\[ \mathcal{C}_{\text{par}} = \{ Q_1 P + Q_2 WK \mid Q_1, Q_2 \in \mathcal{R}H_{\infty}^-, Q_2 \text{ full rank} \}, \]

such that the controller behavior \( \mathcal{C} = \ker C \) with \( C \in \mathcal{C}_{\text{par}} \).

Now this algorithm is known, the proof of Theorem 4.1 can be given:

**Proof:** Proof of Theorem 4.1:

(\( \Rightarrow \)): This is trivial.

(\( \Leftarrow \)): If \( \mathcal{K} \subseteq \mathcal{P} \), then there exists a \( F \) as in Theorem 4.2 or 4.3, which can be used in Algorithm 1 that estimates a controller.

\[ \Box \]

**C. Partial interconnection - Problem**

Now the full interconnection situation is studied, an extension to the partial interconnection problem can be made. This makes it possible to take aspects like disturbances as noise into account. This expands the scope of applications to e.g. robust control problems.

In this partial interconnection case, depicted in Fig. 4(a), the behavior \( \mathcal{P}_{\text{full}} \) consists of trajectories \( (w, c) \in \mathcal{L}_2 \) in which \( w \) is interpreted as the set of manifest variables and \( c \) as the set of “controller” variables. Refer to \( \mathcal{P}_{\text{full}} \subseteq (\mathcal{W} \times \mathcal{C})^\mathcal{F} \) as the full plant behavior (here \( \mathcal{C} \) is a variable signal space).

Any such system defines a manifest plant behavior:

\[ \mathcal{P}_{\text{manifest}} = \{ w \in \mathcal{L}_2 \mid \exists c \in \mathcal{L}_2 : (w, c) \in \mathcal{P}_{\text{full}} \}. \]

If \( \mathcal{P}_{\text{full}} \) is in \( \mathcal{L}_{\pm}^\mathcal{F} \) (or in \( \mathcal{M}_{\pm}^\mathcal{F} \)) then it is immediate that \( \mathcal{P}_{\text{manifest}} \) also belongs to \( \mathcal{L}_{\pm}^\mathcal{F} \) (or to \( \mathcal{M}_{\pm}^\mathcal{F} \)).

**Problem 4.2:** Given two linear left-shift invariant systems \( \mathcal{P}_{\text{full}} \) and \( \mathcal{K} \) in the class \( \mathcal{L} \) (or \( \mathcal{L}_+ \) or \( \mathcal{L}_- \)), where \( \mathcal{P}_{\text{full}} \subseteq (\mathcal{W} \times \mathcal{C})^\mathcal{F} \) and \( \mathcal{K} \subseteq \mathcal{W}^\mathcal{F} \). Find, if it exists, a linear left-shift invariant system \( \mathcal{C} \) in the class \( \mathcal{L} \) (or \( \mathcal{L}_+ \) or \( \mathcal{L}_- \)) such that:

\[ \mathcal{K} = \{ w \in \mathcal{L}_2 \mid \exists c \in \mathcal{L}_2 : (w, c) \in \mathcal{P}_{\text{full}}, \ c \in \mathcal{C} \}. \]

Any such system is said to implement \( \mathcal{K} \) for \( \mathcal{P}_{\text{full}} \) by partial interconnection.

A similar problem formulation applies for the model classes \( \mathcal{M}, \mathcal{M}_+ \) and \( \mathcal{M}_- \) and for \( (w, c) \in \mathcal{H}_2^+ \) or \( (w, c) \in \mathcal{H}_2^- \).

**Note:** As shown in Fig. 4(a), the system \( \mathcal{C} \) restricts \( \mathcal{C} \) only (and not \( w \)), the full plant behavior \( \mathcal{P}_{\text{full}} \) restricts \( w \) as well as \( c \) and the controlled behavior restricts \( w \) only.

Now, the full plant behavior \( \mathcal{P}_{\text{full}} \) has to be introduced using stable- and anti-stable rational representations, as done for behaviors in the previous sections:

\[ \mathcal{P}_{\text{full}}(P) : = \{ (w, c) \in \mathcal{L}_2 \mid P(s) \left[ \begin{array}{c} w(s) \\ c(s) \end{array} \right] = 0 \} \]

\[ = \ker P \subset \mathcal{L}_2, \]

\[ \mathcal{P}_{\text{full}, +}(P) : = \{ (w, c) \in \mathcal{H}_2^+ \mid P(s) \left[ \begin{array}{c} w(s) \\ c(s) \end{array} \right] = 0 \} \]

\[ = \ker P \subset \mathcal{H}_2^+, \]

\[ \mathcal{P}_{\text{full}, -}(P) : = \{ (w, c) \in \mathcal{H}_2^- \mid P(s) \left[ \begin{array}{c} w(s) \\ c(s) \end{array} \right] = 0 \} \]

\[ = \ker P \subset \mathcal{H}_2^- \]

where \( P = [P_1 \ P_2] \in \mathcal{R}H_{\infty}^+ \).

The separation of the matrix \( P \) into \( P_1 \) and \( P_2 \) is such that:

\[ P(s) \left[ \begin{array}{c} w(s) \\ c(s) \end{array} \right] = P_1(s)w(s) + P_2(s)c(s). \]

A similar definition can be made using anti-stable representations \( \hat{P} = [P_1 \ P_2] \in \mathcal{R}H_{\infty}^- \):

\[ \mathcal{P}_{\text{full}}(\hat{P}) : = \{ (w, c) \in \mathcal{L}_2 \mid \hat{P}(s) \left[ \begin{array}{c} w(s) \\ c(s) \end{array} \right] = 0 \} \]

\[ = \ker \hat{P} \subset \mathcal{L}_2, \]

\[ \mathcal{P}_{\text{full}, +}(\hat{P}) : = \{ (w, c) \in \mathcal{H}_2^+ \mid \hat{P}(s) \left[ \begin{array}{c} w(s) \\ c(s) \end{array} \right] \in \mathcal{H}_2^+ \} \]

\[ = \ker \hat{P} \subset \mathcal{H}_2^+, \]

\[ \mathcal{P}_{\text{full}, -}(\hat{P}) : = \{ (w, c) \in \mathcal{H}_2^- \mid \hat{P}(s) \left[ \begin{array}{c} w(s) \\ c(s) \end{array} \right] = 0 \} \]

\[ = \ker \hat{P} \subset \mathcal{H}_2^- \]

Later on, the so called “hidden behavior” \( \mathcal{N} \) is required in the algorithm. This behavior is named “hidden”, because it is not possible to estimate \( \mathcal{N} \) from the controller’s view. There is only control on the variables \( c \), which are distinguished as well as \( w \).
controller variables that will in general be independent of \( w \). Using stable- and anti-stable rational representations, this behavior is given by:

\[
N(P) := \{ w \in L_2 \mid (w, 0) \in P_{\text{full}}(P) \} = \{ w \in L_2 \mid P_1(w(s) = 0) = \ker P_1 \subset L_2, \]

\[
N_e(P) := \{ w \in H^+ \mid (w, 0) \in P_{\text{full},+}(P) \} = \{ w \in H^+ \mid P_1(w(s) = 0) = \ker P_1 \subset H^+_\phi \}
\]

\[
N_i(P) := \{ w \in H^- \mid (w, 0) \in P_{\text{full},-}(P) \} = \{ w \in H^- \mid P_1(w(s) = 0) = \ker P_1 \subset H^-\phi \}
\]

where \( P = [P_1 \ P_2] \in \mathcal{R}H^+_\phi \) and

\[
N(\hat{P}) := \{ w \in L_2 \mid (w, 0) \in P_{\text{full}}(\hat{P}) \} = \{ w \in L_2 \mid \hat{P}_1(w(s) = 0) = \ker \hat{P}_1 \subset L_2, \]

\[
N_e(\hat{P}) := \{ w \in H^+ \mid (w, 0) \in P_{\text{full},+}(\hat{P}) \} = \{ w \in H^+ \mid \hat{P}_1(w(s) = 0) = \ker \hat{P}_1 \subset H^+_\phi \}
\]

\[
N_i(\hat{P}) := \{ w \in H^- \mid (w, 0) \in P_{\text{full},-}(\hat{P}) \} = \{ w \in H^- \mid \hat{P}_1(w(s) = 0) = \ker \hat{P}_1 \subset H^-\phi \}
\]

where \( \hat{P} = \begin{bmatrix} \hat{P}_1 & \hat{P}_2 \end{bmatrix} \in \mathcal{R}H^-\phi \).

Now the controller implementation theorem that is introduced by Willems [14] can be used:

**Theorem 4.4**: Given the systems \( K, N, P_{\text{full}} \in L_\phi \) (or \( M_\phi \)), there exists a controller \( C \in L_\phi \) (or \( M_\phi \)), which implements \( K \) for \( P_{\text{full}} \) by partial interconnection if and only if \( N \subset K \subset P_{\text{manifest}} \).

To prove this theorem, the theorems used in the previous subsection, the elimination-theorem for obtaining the manifest behavior and the algorithm, that is to be introduced in the next section, are used.

First, an elimination theorem will be given (and proved), which makes it possible to obtain a manifest behavior, which only contains trajectories \( w \), that fulfills the full behavior:

**Theorem 4.5**: Suppose that \( P_{\text{full}} = \ker P \) with \( P = [P_1 \ P_2] \in \mathcal{R}H^+_\phi \). If there exists a unit \( U(s) \in UH^-\phi \) such that:

\[
U(s)P_2(s) = \begin{bmatrix} 0 & P_2(s) \end{bmatrix},
\]

where \( P_2(s) \in \mathcal{R}H^-\phi \) is outer and the zero block has \( p \) rows, then the manifest behavior is given by:

\[
P_{\text{manifest}} = \{ w \in L_2 \mid P_1(w(s) = 0) = \ker P_1 \},
\]

\[
P_{\text{manifest}} = \{ w \in H^+ \mid P_1(w(s) = 0) = \ker P_1 \},
\]

\[
P_{\text{manifest}} = \{ w \in H^- \mid P_1(w(s) = 0) = \ker P_1 \},
\]

where \( P_{\text{manifest}} \in \mathcal{R}H^+_\phi \).

This also holds for \( w \in H^+_\phi \) and \( w \in H^-\phi \), which results in the the manifest plant behaviors described by:

\[
P_{\text{manifest}} = \{ w \in L_2 \mid P_1(w(s) = 0) = \ker P_1 \},
\]

\[
P_{\text{manifest}} = \{ w \in H^+ \mid P_1(w(s) = 0) = \ker P_1 \},
\]

\[
P_{\text{manifest}} = \{ w \in H^- \mid P_1(w(s) = 0) = \ker P_1 \},
\]

where \( P_{\text{manifest}} \in \mathcal{R}H^-\phi \).

The proof of this theorem can also be found in the appendix. A similar elimination theorem involves behaviors that are described using anti-stable rational mappings:

**Theorem 4.6**: Suppose that \( P_{\text{full}} = \ker P \) with \( P = [P_1 \ P_2] \in \mathcal{R}H^-\phi \). If there exists a unit \( U(s) \in UH^-\phi \) such that:

\[
U(s)P_2(s) = \begin{bmatrix} 0 & P_2(s) \end{bmatrix},
\]

where \( P_2(s) \in \mathcal{R}H^-\phi \) is outer and the zero block has \( p \) rows, then the manifest behavior is given by:

\[
P_{\text{manifest}} = \{ w \in L_2 \mid P_1(w(s) = 0) = \ker P_1 \},
\]

where \( P_1(s) \in \mathcal{R}H^-\phi \) consists of the first \( p \) rows in:

\[
U(s)P_1(s) = \begin{bmatrix} P_1(s) \ P_2(s) \end{bmatrix}.
\]

This also holds for \( w \in H^+\phi \) and \( w \in H^-\phi \), which results in the the manifest plant behaviors described by:

\[
P_{\text{manifest}} = \{ w \in L_2 \mid P_1(w(s) = 0) = \ker P_1 \},
\]

\[
P_{\text{manifest}} = \{ w \in H^+ \mid P_1(w(s) = 0) = \ker P_1 \},
\]

\[
P_{\text{manifest}} = \{ w \in H^- \mid P_1(w(s) = 0) = \ker P_1 \},
\]

where \( P_{\text{manifest}} \in \mathcal{R}H^-\phi \).

The proof of this theorem is similar to the one of Theorem 4.5, so it’s not useful to include it here in this paper.

**D. Partial interconnection - Algorithm**

Now the algorithm for deriving controller representations can be given. This is, as in the full interconnection case, done using trajectories \( (w, c) \in L_2 \) and rational anti-stable mappings in \( \mathcal{R}H^-\phi \):

**Algorithm 2**: Given \( P \in \mathcal{R}H^+\phi \) that defines the systems \( P_{\text{full}} \) and \( K \):

\[
P_{\text{full}} = \{ (w, c) \in L_2 \mid P(s) \mid \begin{bmatrix} w(s) \\ c(s) \end{bmatrix} = 0 \} = \ker P,
\]

\[
K = \{ w \in L_2 \mid K(s)w(s) = 0 \} = \ker K,
\]

where \( P = [P_1 \ P_2] \).

**Aim**: Find \( C \in \mathcal{R}H^-\phi \) that defines the system:

\[
C := C(C) := \{ c \in L_2 \mid C(s)(c(s) = 0) = \ker C,
\]

such that \( C \) implements \( K \) for \( P_{\text{full}} \) by partial interconnection through \( c \).
Step 1:
Represent the manifest plant behavior \( P_{\text{manifest}} \) by applying the elimination theorems (Theorems 4.5 and 4.6) on the full plant behavior \( P_{\text{full}} \). So, find \( P'_1 \in RH\), such that:
\[
P_{\text{manifest}} = \{ w \in L_2 \mid P'_1(s)w(s) = 0 \} = \ker P'_1.
\]

Step 2:
Verify whether there holds that the “hidden behavior” \( N \) is a subset of the controlled behavior \( K \) (so, \( N \subset K \)). If so, there should exist a mapping \( X \in RH\) such that \( K = XP_1 \) (see Theorems 4.2 and 4.3). If not, the algorithm stops and no controller can be found.

Step 3:
Now one has to verify if \( K \subset P_{\text{manifest}} \), so there should exist a \( Y \in RH\) such that \( P'_2 = YK \). If not, the algorithm stops here.

Step 4:
Compress the number of columns of the found matrix \( Y \) by post-multiplying with a unit \( U \in RH\) (as done in step 2 of Algorithm 1):
\[
\mathbf{Y} = YU = [Y_1, 0],
\]
where \( Y_1 \in RH\) and full rank.

Step 5:
Extend this matrix \( \mathbf{Y} \) using \( \mathbf{W} = [0, 1] \) such that:
\[
\mathbf{X} = \begin{bmatrix} \mathbf{Y} \\ \mathbf{W} \end{bmatrix} = \begin{bmatrix} Y_1 \\ 0 \\ 1 \end{bmatrix},
\]
where \( \mathbf{W} = \mathbf{W}U^{-1} \) and \( \mathbf{X} \in RH\) is a unit.

Step 6:
Compute a possible controller, also using a similar step as in Algorithm 1, such that:
\[
\bar{C}(s)w(s) = 0 \quad \Rightarrow \quad \bar{C} = WXP_1,
\]
and because an interconnection is made between the controller and the plant using the interconnect variables \( c \), one has to use:
\[
P_1w = -P_2c,
\]
which results in the mapping \( C(s)c(s) = 0 \), where:
\[
C = WXP_2 \in RH\).
\]

Now this algorithm is known, the proof of Theorem 4.4 can be given:

**Proof:**
\( (\Rightarrow) \): This is trivial.
\( (\Leftarrow) \): If \( \mathcal{N} \subset K \subset P_{\text{manifest}} \), then it is possible to obtain the desired controlled behavior by partial interconnection with a controller using the introduced algorithm.

V. Example: Quadratic Cost
In many control situations, the desired behavior of the system has to minimise a certain optimisation problem. One example is the well known linear quadratic regulator (LQR), which is used in this section. As the name says, the optimisation problem for this type of regulator is a quadratic one. In the first subsection, this optimisation problem will be introduced and converted into a Hamiltonian system. Afterwards, the algorithm for the full interconnection situation will be applied to a simple numerical MIMO example. As an intermezzo, a method of model reduction on this Hamiltonian system is described in Section V-C.

A. Optimisation Problem
In this example, the plant behavior \( P \) of an unstable plant \( \Sigma_P \) is described by the state-space realisation:
\[
\Sigma_P : \begin{bmatrix} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = x(t), \end{bmatrix} \quad x(t_0) = x_0, \tag{5}
\]
where \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \).

**Notational Remark:** To make the equations more clear, the functions related with time, like the \( x(t), u(t) \) and \( \lambda(t) \) which is introduced later on, are shortened in notation, so like \( x, u, \lambda \).

The desired controlled behavior \( K \) consists of all pairs \((u, y)\) that minimise the cost function:
\[
J(x, u) = \frac{1}{2} \left( \int_{t_0}^{t_f} f^T(t)Qx(t)+u^T(t)Ru(t) dt + z(t_f)Ex(t_f) \right), \tag{6}
\]
subject to the system equations of the plant model \( \Sigma_P \) in (5). In the cost function, \( 0 \leq Q \in \mathbb{R}^{n \times n}, 0 < R \in \mathbb{R}^{m \times m} \) are weighting matrices and the matrix \( E \in \mathbb{R}^{n \times n} \) will be estimated later on.

As discussed in [4, 11], this optimisation problem can be converted into a Hamiltonian system. The first step in this conversion is the introduction of the dual Lagrangian optimisation problem:
\[
l(\lambda) = \min_{x,u} L(x, u, \lambda) \]
\[
= \min_{x,u,\lambda} \left[ \int_{t_0}^{t_f} F(x, u) + \lambda^T (Ax + Bu - \dot{x}) dt + \Phi(x) \right]_{t=t_f}
\]
\[
= \min_{x,u,\lambda} \left[ \frac{1}{2} \int_{t_0}^{t_f} f^T(t)Qx(t)+u^T(t)Ru(t) dt + z(t_f)Ex(t_f) \right]_{t=t_f}, \tag{7}
\]
where \( \langle \cdot, \cdot \rangle \) is the \( L_2 \) type inner product \( \langle (x, y) = \int_{t_0}^{t_f} x^2 y dt \rangle \), \( l(\lambda) \) is named the dual cost, \( L(x, u, \lambda) \) is called the Lagrangian, \( \lambda : \mathbb{R} \to \mathbb{R}^n \) are the Lagrange multipliers (or also named co-states) and \( H(x, u, \lambda) \) the Hamiltonian (energy) function.

The optimal solution of this problem is defined as \( p^* = (x^*, u^*, \lambda^*) \) and can be obtained when all partial derivatives of the Lagrangian are equal to zero:
\[
\nabla L(p^*) = 0.
\]
Using this aspect, it is possible to rewrite the complete problem into the following dynamical system $\Sigma_H$:

$$\Sigma_H : \begin{bmatrix} \dot{x}^* \\ \dot{\lambda}^* \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix}$$ \tag{6}

with $u^* = -R^{-1}B^T \lambda^*$, $x(0) = x_0$ and $\lambda^*(t_f) = E x^*(t_f)$.

In the autonomous part (6), $H \in \mathbb{R}^{n \times n}$ is called the Hamiltonian matrix, which contains a stable- and anti-stable part (where $n = 2x$). To obtain a $L_2^\infty$ solution for this autonomous system, only the stable part is interesting. This Hamiltonian system can be reduced to a stable one, because the representation is not minimal. A minimal realisation can be obtained by applying the following state transformation [11]:

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} I & 0 \\ S & I \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} \iff \begin{bmatrix} x \\ \sigma \end{bmatrix} = \begin{bmatrix} I & 0 \\ -S & I \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix},$$

where $S$ is the stable solution of the algebraic Ricatti equation (ARE):

$$A^T S + S A - S B^T S R^{-1} + Q = 0.$$ 

This makes the state variable $x$ independent of the "co-state" variable $\sigma$, so this results in a stable dynamical system that describes the controlled behavior $\Sigma_K$, given by:

$$\Sigma_K : \begin{bmatrix} \dot{x} \\ u \end{bmatrix} = \begin{bmatrix} A - BR^{-1}B^T S \\ -R^{-1}B^T S \end{bmatrix} x,$$ \tag{7}

B. Full interconnection algorithm

$\Sigma_P$ (5) and $\Sigma_K$ (7) are using the time domain, however $L_2^\infty$ behaviors using rational kernel representations are required. Because the controlled system is autonomous (and stable), the left-shift invariance property is required, which restricts us to use anti-stable mappings for $P.K$ and also $C$. For the given system descriptions, this results in:

$$P(s) = \begin{bmatrix} -1 & (sI - A)^{-1}B \\ 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty,$$

$$K(s) = \begin{bmatrix} (sI - (A - BR^{-1}B^T S))^{-1} & 0 \\ R^{-1}B^T S (sI - (A - BR^{-1}B^T S))^{-1} \end{bmatrix} \in \mathcal{RH}_\infty,$$

where $u(s) = [y(s) \ w(s)]^T$, $\alpha, \beta > 0$ and $\alpha \neq \beta$. Because the controlled behavior is stable and an anti-stable mapping is required, the anti-stable "poles" $\alpha$ and $\beta$ are introduced. Of course, no "pole-zero" cancellation should occur when $\alpha$ and $\beta$ are chosen. Using those representations, the full interconnection algorithm can be applied to the problem:

**Step 1:** The first step in the full interconnection algorithm is to verify whether $K \subset P$, which should be the case. Indeed there does exist a $F(s) \in \mathcal{RH}_\infty$, such that $P(s) = F(s)K(s)$:

$$F(s) = \Gamma(s) \Lambda(s) \in \mathcal{RH}_\infty,$$

where

$$\Gamma(s) = \begin{bmatrix} -I & (sI - A)^{-1}B \end{bmatrix}(sI - (A - BR^{-1}B^T S)), \text{ and}$$

$$\Lambda(s) = (sI - A)^{-1}B(sI - \beta).$$

**Step 2:** The next step is to column reduce this $F(s)$. This can be done using algorithms such as in [3]. There is not a general expression for this column reduced form, as it is for $F(s)$, so some values for the matrices have to be substituted, which are in this example:

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad S = \begin{bmatrix} -4 & 5 \\ 15 & -7 \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}^+ \quad \alpha \neq \beta.$$

This results in $F(s) = \begin{bmatrix} 0 & \frac{1}{4} \\ 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty$, which can be column reduced to $F(s) = F(s)U(s) = [F_0(s) \ 0]$, where

$$F(s) = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty$$

and

$$U(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{UH}_\infty.$$

**Step 3-4:** Then, as discussed in Step 4 of Algorithm 1, a possible controller behavior $C = \ker C_1$ is expressed as:

$$C_1(s) = W(s)K(s) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty.$$

As mentioned before, the behavioral framework does not require a separation of the variable $w$ into inputs and outputs. This can be seen in the result above, because the controller tries to "restrict" the outputs of the plant in the first row when a separation in the variable space is made. So, mathematically the interconnection with this controller results in the desired controlled behavior, but this representation is not directly implementable for a real system, because outputs of a plant can’t be used as inputs. Therefore, another representation for this controller behavior has to be found in the next step.

**Step 5:** Another controller has to be found using the matrices $Q_1(s)$ and $Q_2(s)$ as defined in the algorithm. When those matrices are chosen to be:

$$Q_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \in \mathcal{RH}_\infty \quad \text{and} \quad Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \in \mathcal{RH}_\infty,$$

the resulting controller is equal to:

$$C(s) = Q_1(s)P(s) + Q_2(s)C_1(s) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty,$$

which can be implemented. As expected, this controller is equal to the general LQR-solution, namely:

$$u(t) = -\begin{bmatrix} 4 & -\frac{1}{2} \\ 15 & 8 \end{bmatrix} x(t).$$

**Note:** Denote that the values in the Ricatti solution $S$ and the values in the estimated rational expressions are rounded to integers for simplification.
C. Intermezzo: Model reduction

It is possible to reduce the controlled behavior description (containing stable- and anti-stable parts) using modal truncation. The first step for this reduction is to apply an eigenvalue decomposition on the Hamiltonian matrix (6):

\[ HV = VA \quad \iff \quad H = V A V^{-1}, \]

where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) and \( V = [V_{\text{anti-stable}}, V_{\text{stable}}] \) are the eigenvalues and eigenvectors of \( H \). To make this section more clear, the autonomous system is described as:

\[ \dot{\xi} = H \xi \quad (\text{so } \xi = [\xi_c] \text{ compared with } (6)). \]

The idea of model reduction is to reduce the states which have less influence on the complete system dynamics. This means that the eigenvectors that are less dominant will be truncated, which implies that the eigenvectors coupled with the smallest eigenvalues have to be removed. The resulting set of eigenvalues in the reduced order model is given by:

\[ \Lambda_r = \text{diag}(\lambda_1, \ldots, \lambda_r, -\lambda_{r+1}, \ldots, -\lambda_n), \]

which implies the set of eigenvectors:

\[ V_r = [V_{r, \text{anti-stable}}, V_{r, \text{stable}}], \]

where \( r < n \).

To reduce the number of states, a projection has to be applied to the original state vector. This projection, named \( \Pi_{V_r} \), is visualised in Fig. 5. Here, the state vector \( \xi \in \mathcal{V} \) is a direct sum of the states that are used in the reduced order model \( \{\xi_r \in \mathcal{V}_r\} \) and the truncated states, which are orthogonal to the ones in the reduced order system \( \{\xi_c \in \mathcal{V}_c\} \):

\[ \xi = \xi_r \oplus \xi_c, \]

where \( \xi_r := \Pi_{V_r} \xi, \quad \xi_c := \Pi_{V_c} \xi \quad \text{and} \quad (\xi_r, \xi_c) = 0. \)

Because of some properties that are required for a projection, we define \( \Pi_{V_r} \) and \( \Pi_{V_c} \) as:

\[ \Pi_{V_r} := V_r (V_r^T V_r)^{-1} V_r^T, \]
\[ \Pi_{V_c} := I - \Pi_{V_r} = V_c (V_c^T V_c)^{-1} V_c^T, \]

where \( V_r \) is a matrix containing the eigenvectors that are truncated. Applying those projections on the Hamiltonian system in (6), this results in the following reduced order system that needs to be solved:

\[
\begin{align*}
\dot{\alpha}_r &= (V_r^T V_r)^{-1} V_r^T H V_r \alpha_r, \\
u &= R^{-1} B^T V_r \alpha_r.
\end{align*}
\]

Unfortunately, it is not so easy to obtain the stable part of this reduced order system (yet). This is required in the algorithms for obtaining controllers, because it is not possible to obtain \( \mathcal{L}_2 \) behaviors for autonomous systems that contain unstable- as well as stable poles.

VI. Conclusions and recommendations

We considered the problem of controller synthesis for specific classes of \( \mathcal{L}_2 \) functions. Operators in the classes \( \mathcal{R} \mathcal{H}_\infty^2 \) of stable rational functions and \( \mathcal{R} \mathcal{H}_\infty^{\infty} \) of anti-stable rational functions define linear right-shift invariant \( \mathcal{L}_2 \) behaviors and linear left-shift invariant \( \mathcal{L}_2 \) behaviors by considering their kernel spaces. Given two \( \mathcal{L}_2 \) behaviors \( \mathcal{P} \) and \( \mathcal{K} \) we solve the question to synthesize a third \( \mathcal{L}_2 \) system \( \mathcal{C} \) that realises \( \mathcal{K} \) in the sense that the full- or partial interconnection of \( \mathcal{P} \) and \( \mathcal{C} \) satisfies \( \mathcal{K} = \mathcal{P} \cup \mathcal{C} \). Necessary and sufficient condition for the existence of a “controller” \( \mathcal{C} \) are the inclusions \( \mathcal{K} \subset \mathcal{P} \) or \( \mathcal{N} \subset \mathcal{K} \subset \mathcal{P}_{\text{manifest}} \) respectively. The general quadratic cost problem is given as example to demonstrate the full-interconnection algorithm for the construction of a rational representation of \( \mathcal{C} \). As intermezzo, a type of model reduction that can be applied to a Hamiltonian matrix is introduced. As shown in the example, it is possible to obtain a controller, which is valid in a mathematical manner, but not implementable in a real situation. Therefore, an extension on the algorithm is required such that it is possible to find implementable controllers directly. Also some more research is required for the partial-interconnection case. It is possible that there are some limitations due to some requirements on the matrices in the elimination theorem. Studies already started for infinite smooth continuous behaviors in [8]. A nice aspect of this partial interconnection case is that disturbances like noise can be taken into account, which may yield in robust control problems. Because, as in the given example, Hamiltonian functions can be used to describe controlled behaviors \( \mathcal{K} \), it can be possible to take dissipativity properties into account for the controller synthesis problems, because the Hamiltonian has some specific properties about energies. Also some more research has to be done on the model reduction and the separation of the stable- and unstable parts for this type of autonomous systems.

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APPENDIX

PROOF OF THEOREM 4.2

Proof: (iv $\Rightarrow$ (i, ii, iii)):

- iv $\Rightarrow$ i:
  \[ \mathcal{K}, \mathcal{P} \subset \mathcal{L}_2. \]
  Suppose $P = FK$. Take $w \in \mathcal{K} \subset \mathcal{L}_2$. Then, $v = Kw = 0$, so also $Pw = FKw = Fv = 0$. This implies that $P(s)v(s) = 0$, so $w \in \mathcal{P}$. Since $w \in \mathcal{K}$ is arbitrary, it follows that $\mathcal{K} \subset \mathcal{P}$.

- iv $\Rightarrow$ ii:
  \[ \mathcal{K}, \mathcal{P} \subset \mathcal{H}_2^+, \text{ so } w \in \mathcal{H}_2^+. \]
  This proof is identical to the case when $\mathcal{K}, \mathcal{P} \subset \mathcal{L}_2$.

- iv $\Rightarrow$ iii:
  \[ \mathcal{K}, \mathcal{P} \subset \mathcal{H}_2^+, \text{ so } w \in \mathcal{H}_2^+. \]
  Again, suppose $P = FK$ and take $w \in \mathcal{K}$. Define $v = Kw$. Then $v \in \mathcal{H}_2^+$ and hence $Pw = FKw = Fv$. Now, $F \in \mathcal{R}\mathcal{H}_2^\perp$ and $v \in \mathcal{H}_2^+$. Consequently, $Fv \in \mathcal{H}_2^+$ and it follows that $Pw \in \mathcal{H}_2^+$. Using (3) this gives that $w \in \mathcal{P}$. Since $w \in \mathcal{K}$ is arbitrary, $w \in \mathcal{P}$ which gives the result.

(iv $\Leftarrow$ (i, ii, iii)):

- iv $\Leftarrow$ i:
  \[ \mathcal{K}, \mathcal{P} \subset \mathcal{L}_2, \text{ so } w \in \mathcal{L}_2: \]
  Using the definition of $\mathcal{K}$, one can write:
  \[ \mathcal{K} = \{ w \in \mathcal{L}_2 \mid (Kw, v)_{\mathcal{L}_2} = 0 \forall v \in \mathcal{L}_2 \} \]
  \[ = \{ w \in \mathcal{L}_2 \mid (w, K^*v)_{\mathcal{L}_2} = 0 \forall v \in \mathcal{L}_2 \} = (\text{im } K^*)^\perp, \]
  where $K^*$ is the dual or adjoint operator defined by $K^*(v) = K^T(s^{-1})v$. Something similar can be applied to the plant behavior. So, $\mathcal{K} \subset \mathcal{P}$ implies that $\mathcal{P}^\perp \subset \mathcal{K}^\perp$ and using the previous definition of $\mathcal{K}$, this results in
  \[ (\text{im } P^*) \subseteq (\text{im } K^*), \]
  where the bar denotes the closure in $\mathcal{L}_2$.
  For rational operators the latter implies that:
  \[ (\text{im } P^*) \subset (\text{im } K^*), \]
  because in that case the images are closed. Then we can say that for some $\varepsilon_2^2$, $P^*e_i \in \text{im } K^*$, so there exists a $v_i$ such that:
  \[ P^*e_i = K^*v_i. \]
  This can be extended to a set of $v_i$’s, such that:
  \[ P^* = K^*X \text{ with } X = (v_1, \ldots, v_p) \in \mathcal{R}\mathcal{H}_2^\perp \subset \mathcal{R}\mathcal{H}_\infty. \]
  Then, we can rewrite this to $P = X^*K$, where $F$ is equal to the dual operator $X^* \in \mathcal{R}\mathcal{H}_\infty^\perp$.

- iv $\Leftarrow$ ii:
  \[ \mathcal{K}, \mathcal{P} \subset \mathcal{H}_2^+, \text{ so } w \in \mathcal{H}_2^+: \]
  This proof is similar to the one in the previous item, except that now the $\mathcal{H}_2^+$ inner product is used. However, $\mathcal{H}_2^+$ inherited this inner product from $\mathcal{L}_2$.

Equality condition:

Using the previous items, one can say that $\mathcal{P} = \mathcal{K}$ if and only if $P = U_1K$ and $K = U_2P$ with both $U_1$ and $U_2$ in $\mathcal{R}\mathcal{H}_\infty^\perp$. Moreover, if $U_1$ and $U_2$ satisfy these conditions, then $P = U_1U_2K$ and $K = U_2U_1P$. If $P$ and $K$ are full rank, we find that $U_1 = U_2^{-1}$, which completes the proof.

PROOF OF THEOREM 4.5

Proof: Partitioning using the unit gives us the following equivalent description for the full behavior $\mathcal{P}_{\text{hall}}$:

\[ P_1'(s)v(s) = 0, \quad (8a) \]
\[ P_1''(s)v(s) = -P_2''(s)c(s). \quad (8b) \]

Equation (8a) introduces a constraint on the manifest variable $w(s)$.

So, if $w(s)$ is such that $(w(s), c(s)) \in \mathcal{P}_{\text{hall}}$ for some $c(s)$, then $w(s)$ has to satisfy (8a). The proof is complete if one shows that (8b) is redundant in the sense that $w(s) \in \mathcal{L}_2$ is not constrained by (8b).

That is,

\[ \forall w(s) \exists c(s) \text{ such that } P_1''(s)v(s) = -P_2''(s)c(s). \quad (9) \]

The sufficient condition for this is that $P_2''(s)$ is an outer matrix. This matrix can be square, but then one knows that $P_2''(s)$ is a unit which is invertible, resulting in:

\[ c(s) = P_2''(s)c(s) \]

This completes the proof. However, if $P_2''(s)$ is not square (but “wide”), it is possible to rewrite $P_2''(s)c(s)$ as:

\[ P_2''(s)c(s) = [P_2''(s), P_2''(s)]^{c(s)}_{c(s)} \]

where $P_2''(s)$ is a square outer matrix (and so a unit). Then it is possible to find a $c(s)$ that holds (9) by setting $c'(s) = 0$.

REFERENCES


Appendix C

Presentation

The following pages contain the handouts of my final MSc presentation given on the 19th of June 2008
Controller Synthesis for $L_2$ behaviors using rational kernel representations

MSc presentation

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June 19, 2008

Outline

1. Introduction
   - Motivation
   - Behavioral approach
2. Rational representations of behaviors
3. Controller synthesis
   - Full interconnection
   - Partial interconnection
4. Example: LQR problem
   - Full interconnection controller synthesis
   - Model reduction
5. Conclusions

Motivation: Reduction problem

“Classical” control problems:
Two different strategies:

No guarantees about stability, performance and error in $\Sigma_K$

Our control strategy:

My work $\Rightarrow$ Synthesis problem (and a bit on model reduction)
Introduction
Rational representations of behaviors
Controller synthesis
Example: LQR problem
Conclusions

Motivation: synthesis problem

Classic control problem:

\[
\begin{align*}
\text{Plant} & \quad \text{Controller} \\
\rightarrow & \quad \rightarrow \\
\text{ref} & \quad \text{yu} \\
\downarrow & \quad \downarrow \\
\text{y} & \quad \text{y} \\
\end{align*}
\]

- Described as a controlled system:
  - Dynamics of plant are known
  - Control objectives are known
- Find, if possible, the controller that achieves objectives: using a behavioral approach!

Behaviors - General

General description of a behavior:

Assume a phenomenon that produces certain events

A possible event (voltage & current): \((V, I)\)

The totality of feasible events is named the universum \(V\)

A mathematical model of the phenomenon restricts the outcomes to \(B \subseteq V\), which is the behavior

**Ohm’s law:** \(V = IR\)

\[
\begin{align*}
(V, I) & \quad \rightarrow \\
\end{align*}
\]

**Ohm’s law has to hold \(\Rightarrow\) restriction:** \(B = \{(V, I) \in V \mid V = IR\}\)

Behaviors - Dynamical systems

Dynamical system:

\[
\Sigma = (T, W, B)
\]

which is e.g. a system with electrical components \((R, L, C, \ldots)\),

\[
\begin{align*}
T & \subseteq \mathbb{C} \\
W & = \mathbb{R}^{w_1 \times w_2} \\
B & \subseteq W^T
\end{align*}
\]

frequency space
signal space
behavior

Define the event:

\[
\begin{align*}
w_1 & \quad w_2 \\
\end{align*}
\]

\[
\begin{align*}
\text{Behavior} & \text{ The behavior } B \text{ is the set of trajectories } w(s) \text{ that fulfill the constraints compatible with the dynamical laws} \\
\Rightarrow \text{ How to describe the restrictions applied on } w(s)? \\
\text{Using rational elements!}
\end{align*}
\]
**Introduction**

Rational representations of behaviors

Controller synthesis

Example: LQR problem

Conclusions

---

**L2 behaviors for dynamical systems**

Apply restrictions using a dynamical system:

\[ \Sigma = (T, W, P) \]

L2 behavior:

\[ P = \{ w(s) \in L_2 \mid P(s)w(s) = 0 \} = \ker P(s), \]

where \( P \) is a rational function and \( w \) is in the frequency domain

Example: Transfer function

\[ H(s) = -\frac{s+1}{s+2} \rightarrow P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} = 0 \rightarrow \begin{bmatrix} s+1 \\ s+2 \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} = 0 \]

---

**Problem description**

Problem

Given \( P, K \subset L_2 \) described by anti-stable rational functions \( P(s), K(s) \in RH_{\infty} \):

\[ \begin{align*}
  & P = \ker P(s) \\
  & K = \ker K(s)
\end{align*} \]

Find \( C(s) \in RH_{\infty} \) such that the controller \( C \subset L_2 \):

\[ C = \ker C(s) \]

after full interconnection with \( P \) results in \( K \).

---

**Algorithm**

One has to verify the following steps:

- **Exists?**
  Does there exist a \( C \) such that \( K = C \wedge w P \)

- **Find a controller \( C_0(s) \)?**
  Find a controller behavior \( C = \ker C_0(s) \)

- **Find all controllers \( C(s) \)?**
  Find the set of all controller behaviors \( C = \ker C(s) \)
Rational representations of behaviors

Controller synthesis

Example: LQR problem

Conclusions

Full interconnection - Algorithm

Partial interconnection - Algorithm

Full interconnection - Problem description

Partial interconnection - Problem description

Interconnection of behaviors visualized:

\[ \Sigma_P \xrightarrow{w} \Sigma_C \]

Interconnection of two dynamical systems results in intersection of their behaviors!

Full impl. theorem

Given the plant- and “desired” behaviors \( P, K \), then there exists a controller, with behavior \( C \), that implements \( K \) for \( P \) by full interconnection \( \iff K \subset P \)

Algorithm - Exists?

Verify if \( K \subset P \), restricted by rational elements \( K \) and \( P \)

YES: if there exists a \( F \), such that \( P = FK \) --- Step 2

NO: if this \( F \) does not exist --- Stop algorithm

Column reduce the found rational matrix \( F(s) \)

Apply some matrix multiplications to find a certain \( W(s) \)

Then, the restriction by the controller is given by

\[ C(s) = W(s)K(s) \]

Find all controllers \( C(s) \):

Then it is possible to describe a set of controller behaviors:

\[ C(s) = Q_1(s)P(s) + Q_2(s)C_0(s) \]

Known are:

- the full plant behavior \( \mathcal{P}_{\text{full}} = \{(w, c) \mid P(s) \begin{bmatrix} w(s) \\ c(s) \end{bmatrix} = 0 \} \),
- the desired controlled behavior \( \mathcal{K} = \{w \mid K(s)w(s) = 0\} \),

Find, if there exists:

- a controller behavior \( \mathcal{C} = \{c \mid C(s)c(s) = 0\} \) such that \( \mathcal{K} = \{w \mid \exists c \in \mathcal{C} \text{ such that } (w, c) \in \mathcal{P}_{\text{full}}\} \) by partial interconnection through \( c \)

Example: Robust control problems

Problem description

Given two systems \( \mathcal{P}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{C})^T \subseteq \mathbb{L}_2 \) and \( \mathcal{K} \subseteq W^T \subseteq \mathbb{L}_2 \) in the class of left-shift invariant systems. Find, if possible, a controller behavior \( \mathcal{C} \subseteq \mathbb{L}_2 \) such that:

\[ \mathcal{K} = \{w \mid \exists c \in \mathcal{C} \text{ such that } (w, c) \in \mathcal{P}_{\text{full}}\} \]

Any such system is said to implement \( \mathcal{K} \) for \( \mathcal{P}_{\text{full}} \) by partial interconnection

\[ \Rightarrow \text{Due to time limitations, algorithm not in presentation} \]
Introduction

Rational representations of behaviors
Example: LQR problem

Full interconnection controller synthesis
Model reduction

Simple MIMO LQR example

Plant dynamics:

\[
\begin{array}{ccc}
R_l & & R_u \\
V_1 & C & V_2 \\
\end{array}
\]

Inputs : \((V_1, V_2)\)

Outputs/states : \((V_C, I_L)\)

\[
\begin{bmatrix}
V_C \\
I_L \\
\end{bmatrix}
= \begin{bmatrix}
-\frac{1}{V_2} & -\frac{1}{V_1} \\
\frac{1}{V_2} & 0 \\
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_L \\
\end{bmatrix}
+ \begin{bmatrix}
\frac{1}{V_2} & \frac{1}{V_1} \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
V_1 \\
\end{bmatrix}
\]

Desired objectives:

\[
\min_u J(x, u) = \min_u \int_{t_0}^{t_f} x^T Q x + u^T R u + x(t_f) E x(t_f) \ dt
\]

subject to : \(x(0) = x_0\) and the system dynamic laws hold

Controller found by algorithm

Possible controllers:

Controller \(C = C_0(s) w(s)\) found by algorithm:

\[
C_0(s) = \begin{bmatrix}
-\frac{1}{V_2} & -\frac{1}{V_1} & 0 & 0 \\
\frac{1}{V_2} & 0 & 0 & 0 \\
\end{bmatrix}
\in \mathbb{R}^{2 \times 2}
\]

knowing that \(w(s) = [V_C(s), I_L(s), V_1(s), V_2(s)]^T\)

Using weighting matrices \(Q_1\) and \(Q_2\) also the “desired” LQR gain \(C_1(s)\) can be found:

\[
C_1(s) = \begin{bmatrix}
0.8118 & 1.08 & 0 \\
1.17 & 1.28 & 0 \\
\end{bmatrix}
\in \mathbb{R}^{2 \times 2}
\]

Using the behavioral approach it is also possible to e.g. find anti-causal controllers

Full interconnection LQR case

Interconnection variable: \(w(s) = [V_C(s), I_L(s), V_1(s), V_2(s)]^T\)

Plant behavior:

Anti-stable rational representation: \(P(s) w(s) = 0\),

where \(P(s) = \begin{bmatrix}
-4.690 & 0 & 0 & 0 \\
-4.690 & 1 & 0 & 0 \\
-2.00 & 2 & 1 & 0 \\
-2.00 & 2 & 1 & 0 \\
\end{bmatrix}\) \(\in \mathbb{R}^{4 \times 4}\)

Desired controlled behavior:

Anti-stable rational representation \(K(s) w(s) = 0\),

\[
K(s) = \begin{bmatrix}
0.8118 & 1.08 & 0 \\
1.17 & 1.28 & 0 \\
\end{bmatrix}
\in \mathbb{R}^{2 \times 2}
\]

\(\Rightarrow\) If exists, find the set of controllers that after full interconnection with the plant result in the controlled behavior!

Model reduction on desired objectives

Large scale systems \((\#ODE's > 10^8)\) result in a large set of control objectives \(\Rightarrow\) Approximation of these objectives!

Model reduction:

Reduction of control objectives: modal truncation:

\(\checkmark\) Apply eigenvalue decomposition on \(\Sigma_K\)

\(\checkmark\) Truncate of states with less influence

\(\checkmark\) Apply projection to the controlled system \(\Sigma_K\)

\(\Rightarrow\) Approximated controlled system \(\hat{\Sigma}_K\)

Reduction method with full interconnection problem:

This model reduction accomplishes \(K_{\text{reduced}} \subseteq K\).

So, if \(K \subset P\), there also exists a \(C\) such that \(P \wedge w C = K_{\text{reduced}}\).
Conclusions and results of MSc work

- Studied the “classical” behavioral approach
- Introduced the $L_1$ behavior
- Theoretical proof of controller synthesis using $L_2$ behaviors (full- and partial interconnection problems)
- Paper submission for IEEE CDC 2008 conference
- Synthesis algorithms implemented in MATLAB
- Example used: LQ optimal control
- Model reduction technique on Hamiltonian system applied
- Looked for separation technique using EVD for Hamiltonian systems

Further work

- Use theory for constrained LQ optimal control problems
- More analysis on separation of stable- and anti-stable part of Hamiltonian systems
- Synthesis of controllers after applying model reduction on plant or controlled system
- Make use of dissipativeness properties of Hamiltonian system and plant
- Extend behavioral framework for PDEs
- ...

Enough to do the next four years!

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