INTERPOLATIONAL AND EXTREMAL PROPERTIES OF $L$-SPLINE FUNCTIONS

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL EINDHOVEN, OP GEZAG VAN DE RECTOR MAGNIFICUS, PROF. IR. J. ERKELENS, VOOR EEN COMMISSIE AANGEWIEZEN DOOR HET COLLEGE VAN DEKANEN IN HET OPENBAAR TE VERDEIGEN OP VRIJDAG 16 APRIL 1982 TE 16.00 UUR

DOOR

HENRICUS GERHARDUS TER MORSCHIE

GEBOREN TE ENTER
Dit proefschrift is goedgekeurd
door de promotoren
Prof.Dr.Iz. P. Schurer
en
Prof.Dr. P.W. Steutel
Aan Marijke,
Judith en Robert-Jan
Met dank aan mijn ouders
CONTENTS

GENERAL INTRODUCTION .......................................................... I-IV

1. BASIC CONCEPTS AND PRELIMINARY RESULTS ......................... 1
   1.1. Introduction and summary ........................................... 1
   1.2. Notations and conventions ........................................ 5
   1.3. L-spline functions ................................................ 8
   1.4. Some basic concepts in the theory of L-spline functions .... 11

2. THE B-SPLINE FUNCTIONS .................................................. 27
   2.1. Introduction and summary ......................................... 27
   2.2. Fundamentals about B-splines .................................. 33
   2.3. The total positivity property of B-splines ................. 38
   2.4. On recurrence relations for B-splines ....................... 44
   2.5. The minimum supremum norm of a polynomial B-spline .... 49

3. ON RELATIONS BETWEEN FINITE DIFFERENCES AND DERIVATIVES OF
   CARDINAL L-SPLINE FUNCTIONS .................................... 55
   3.1. Introduction and summary ....................................... 55
   3.2. The exponential L-splines .................................... 56
   3.3. Relations between finite differences and derivatives .... 62
   3.4. Some applications of Theorems 3.3.2 and 3.3.3 .......... 67

4. ON EXISTENCE AND CONVERGENCE PROPERTIES OF INTERPOLATING CARDINAL
   L-SPLINES AND INTERPOLATING PERIODIC CARDINAL L-SPLINES .... 75
   4.1. Introduction and summary ....................................... 75
   4.2. On the existence and unicity problem of cardinal
        L-spline interpolation ........................................ 76
   4.3. Periodic cardinal L-spline interpolation .................... 81
   4.4. An error estimate for cardinal L-spline interpolation .... 87
   4.5. An error estimate for periodic cardinal L-spline
        interpolation ................................................ 93
5. ON THE LANDAU PROBLEM FOR SECOND AND THIRD ORDER DIFFERENTIAL OPERATORS

5.1. Introduction and summary 102
5.2. Optimal differentiation algorithms 105
5.3. Some general properties of the sets $F_m(p_n, d)$ and $F_m(p_n, J, l)$ 106
5.4. On the relation between the Landau problem and the sets $\gamma_m^*(p_n)$ and $\gamma_m^*(p_n)$ 112
5.5. The Landau problem for second order differential operators 113
5.6. The Landau problem for third order differential operators 126

6. ON THE LANDAU PROBLEM FOR PERIODIC FUNCTIONS

6.1. Introduction and summary 132
6.2. A few preliminary lemmas and the sets $\tilde{F}(p_n, T)$ and $\tilde{F}(p_n, T)$ 134
6.3. Some properties of the sets $\tilde{F}(p_n, T)$ and $\tilde{F}(p_n, T)$ 137
6.4. Perfect Euler $\varepsilon$-splines as extremal functions 144
6.5. A parametrization of the set $\gamma^D(T', l)$ 148

7. PERFECT $\varepsilon$-SPLINES AND THE LANDAU PROBLEM ON THE HALF LINE

7.1. Introduction and summary 151
7.2. Perfect $\varepsilon$-splines and $\varepsilon$-approximate perfect $\varepsilon$-splines 153
7.3. A representation theorem for the set $F_{\varepsilon}(p_n)$ 161
7.4. An extremal property of perfect $\varepsilon$-splines 167
7.5. A characterization of $\gamma_n^*(p_n)$ 170

REFERENCES 184

LIST OF SYMBOLS 192
SUBJECT INDEX 194
AUTHOR INDEX 198
SAMENVATTING 200
CURRICULUM VITAE 204
GENERAL INTRODUCTION

No doubt, the theory and application of spline functions (splines for short) has been a flourishing branch of approximation theory during the last few decades. Although Bernoulli and Euler already used very simple splines, i.e., polynomials, for the approximate solution of differential equations, it is in general agreed upon that a systematic investigation of splines began with the work done by I.J. Schoenberg during the second World War. It took some time till it was widely recognized that splines have interesting extremal properties, and are a good tool for the numerical approximation of functions as well. Since 1960 a large number of papers have been published. The first book on the subject by Ahlberg, Nilson and Walsh [1] dates from 1967 and in recent years a few more have appeared (cf., for instance, Schumaker [56], which also contains an extensive bibliography).

In their original form splines are piecewise polynomials, in general of rather low degree, tied together with a certain degree of smoothness at the so-called knots. A natural generalization is obtained if the polynomials are replaced by functions in the kernel of a linear differential operator $p_n(D)$ of the form

$$p_n(D) = D^n + a_{n-1}D^{n-1} + \ldots + a_1D + a_0,$$

where $D$ is the ordinary differentiation operator and the coefficients $a_i$ $(i = 0,1,\ldots,n-1)$ are real. The associated splines are then called $C$-splines; the polynomial splines are obtained if $p_n(D) = D^n$.

There is an extensive literature on the problem of interpolation by means of polynomial splines. If the knots of the interpolating $C$-spline and the points of interpolation, the so-called nodes, are equally spaced on $\mathbb{R}$, then, following Schoenberg's terminology, one speaks of cardinal $C$-spline interpolation. The interpolation problem may then be formulated as follows. Let $f$ be defined on $\mathbb{R}$, let $h$ be the distance between consecutive knots (the mesh distance) and let $a \in (0,h)$ be prescribed; then one is asked to determine an $C$-spline, corresponding to $p_n(D)$ and with knots at $0,ah,2ah,\ldots$, 

that interpolates $f$ at the nodes $0, 2h, 4h, \ldots$. Questions concerning existence and uniqueness of such an $L$-spline interpolant, and the problem of determining (best possible) error estimates naturally present itself.

Cardinal polynomial spline interpolation has been investigated in detail by Schoenberg [31] in the case $u = h/2$ and $u = h$. Results for arbitrary values of $u \in (0, h]$ in the case of periodic cardinal polynomial spline interpolation are contained in Ter Morsche [40]. Assuming that the characteristic polynomial $p_n$ of $p_n(D)$ has only real zeros, Nichelli [37] has generalized Schoenberg's results for cardinal polynomial spline interpolation to cardinal $L$-spline interpolation.

In the first part of this thesis we study (periodic) cardinal $L$-spline interpolation without the assumption that $p_n$ has only real zeros. When extending the theory to arbitrary operators $p_n(D)$ one encounters the concept of disconjugacy. A differential operator $p_n(D)$ is said to be disconjugate on an interval $(a, b)$ if $p_n(D)$ can be factorized in a sequence of first order differential operators $D_1, D_2, \ldots, D_n$ of the form $D_i = w_i D_{i-1}$, where $w_i$ is a positive function defined on $(a, b)$; apparently, this is true on any interval $(a, b)$ if $p_n$ has only real zeros. In general, it will be assumed that $p_n(D)$ is disconjugate on an appropriate interval, as in the absence of this property some problems are substantially more difficult.

Chapters 3 and 4 deal with existence and uniqueness of an $L$-spline interpolant, and error estimates are derived that are best possible. The estimates obtained in general are of the form

$$|f(x) - s_n(x)| \leq c \|f_0(x) - s_0(x)\| \|p_n(D) f\| \quad (x \in \mathbb{R}),$$

where $c > 0$ is an appropriate constant, $\| \cdot \|$ is the supremum norm on $\mathbb{R}$, $s_n$ is the $L$-spline interpolant to $f$ and $f_0$ is a so-called perfect Euler $L$-spline. Perfect $L$-splines are characterized by the fact that their "$p_n(D)$-derivative" has constant absolute value $c$ and jumps from $\pm c$ to $\mp c$ at the knots. Perfect Euler $L$-splines $f_0$ additionally satisfy the functional relation $f_0(x+h) = f_0(x) \quad (x \in \mathbb{R})$ and are such that their "$p_n(D)$-derivative" is $\pm 1$. We emphasize that our analysis of the (periodic) cardinal $L$-spline interpolation problem makes essential use of specific relations between derivatives and finite differences of cardinal $L$-splines. As a simple illustration we mention the relation

$$s''(x_{i+1}) + 4s''(x_{i+1}) + s''(x_i) = \frac{\Delta^2 s(x_{i+2}) - 2s(x_{i+1}) + s(x_i)}{h^2} \quad (k \in \mathbb{Z}),$$
which holds for polynomial cubic splines, \( \mathbf{N} \) being the equally spaced knots and \( h \) being the mesh distance. The relations we derive in Chapter 3 are far-reaching generalizations of (1).

The second part of this thesis deals with extremal properties of perfect \( L \)-splines in connection with so-called Landau problems. This name has its origin in a few interesting inequalities that Landau [30] derived in 1913 for twice differentiable functions. If \( f \) and \( f' \) are bounded on \( \mathbb{R} \) (on \( \mathbb{R}^+_0 = [0, +\infty) \)) and if we write \( \| f \|_s = \| f \|_{L^2_s} \), then

\[
\| f' \|_s \leq \sqrt{2/s} \| f \|_{L^2_s}, \quad \| f' \|_s \leq 2\sqrt{\| f \|_{L^2_s}^2 + \| f'' \|_{L^2_s}^2},
\]

where the constants \( \sqrt{2/s} \) and \( 2 \) are best possible. A generalization of the first inequality to higher derivatives is due to Kolmogorov [27]; the corresponding problem on \( \mathbb{R}^+_0 \) has been solved by Schoenberg and Cavarretta [54].

A still further generalization reads as follows. Let \( p_n \) be a monic polynomial of degree \( n \), let \( J = \mathbb{R} \) be a closed interval, and let \( m \) be a positive number. Further, let \( \mathbb{F}_m(p_n,J) \) denote the set of functions \( f \) with \( f^{(n-1)} \) absolutely continuous on every compact subinterval of \( J \), \( \| f \|_s \leq m \), and \( \| f^{(n)} \|_{L^1_J} \leq 1 \). The generalized Landau problem then amounts to determining the best possible upper bound for \( \| f^{(k)}(0) \|_{L^1_J} \) on \( \mathbb{F}_m(p_n,J) \), where \( p_k(0) \) is a given linear differential operator of order \( k \leq n-1 \). The cases \( J = \mathbb{R} \) (full-line case) and \( J = \mathbb{R}^+_0 \) (half-line case) are of particular interest. In what follows we assume that \( J = \mathbb{R} \) or \( J = \mathbb{R}^+_0 \).

Our analysis of the generalized Landau problem is based on an investigation of the set

\[
\Gamma_m(p_n,J,0) := \left\{ (f(0), f'(0), \ldots, f^{(n-1)}(0)) \mid f \in \mathbb{F}_m(p_n,J) \right\}.
\]

It is shown that \( \Gamma_m(p_n,J,0) \) is a compact, convex subset of \( \mathbb{R}^n \) having \( 0 \) as an interior point. Solving a Landau problem is then equivalent to maximizing a linear function on \( \Gamma_m(p_n,J,0) \). By the support hyperplane theorem for convex sets, every boundary point of \( \Gamma_m(p_n,J,0) \) has the property that an appropriate linear function attains its maximum on \( \Gamma_m(p_n,J,0) \) at that point. Consequently, every \( f \in \mathbb{F}_m(p_n,J) \) with the property that \( (f(0), f'(0), \ldots, f^{(n-1)}(0)) \) is an extremal function for a Landau problem, i.e., for some \( p_k \), the function \( f \) maximizes \( \| p_k(D)f \|_{L^1_J} \) on \( \mathbb{F}_m(p_n,J) \). It is therefore of importance to describe (parametrize) the set \( \Gamma_m(p_n,J,0) \). This we do for
general second order and for some specific third order differential operators in the cases \( J = \mathbb{R} \) and \( J = \mathbb{R}_0^+ \). There is a separate chapter on Landau problems for periodic functions, where, among other things, a generalization of a result of Nemat-Nasser [46] is given. Landau problems on the half line are discussed in the last chapter, under the additional hypotheses that \( p_0 \) has only real zeros and that \( p_n(0) = 0 \). The problem we pose is to minimize \( \|p_n(D)\|_\infty \) with respect to all functions in \( L_p[\mathbb{R}_0^+, R_+^n] \) satisfying the initial conditions \( f^{(i)}(0) = a_i \) (\( i = 0, 1, \ldots, n-1 \)) for prescribed \( a_i \). It is shown, by means of a so-called representation theorem, that a perfect \( L \)-spline with a specific oscillation property furnishes the solution.

Landau problems are of importance with respect to the optimal recovery of the derivatives of smooth functions (cf. Micchelli and Rivlin [39, p. 27]).

The foregoing is intended to introduce the main themes of the thesis and to give a rough sketch of the problems that are dealt with. We refrain from giving here a detailed summary of the contents of the seven chapters. Instead of this we refer to the first sections of the various chapters which are introductory and summarize the main results obtained. A few remarks are in order with respect to Chapters 1 and 2. We have strived to make this thesis reasonably self-contained. To achieve this, preliminary material of a general kind is brought together in Chapter 1; most of it is standard in approximation theory, apart from a generalization of ours of the Budan-Fourier theorem to piecewise continuous functions. This useful theorem is, among other things, applied to give a rather simple proof of the so-called total positivity property of a sequence of consecutive B-splines. Chapter 2 introduces the nonpolynomial \( B \)-splines and contains various properties, notably recurrence relations, of these very useful functions. One of these recurrence relations is used to obtain the knot distributions for which the supremum norm of a polynomial \( B \)-spline is minimal.

Examples are given in each chapter to illustrate and clarify our assertions. In order to facilitate the readability of this thesis, at the end a list of symbols, an author index and a subject index are added.
1. BASIC CONCEPTS AND PRELIMINARY RESULTS

1.1. Introduction and summary

The purpose of this chapter is to collect concepts and preliminary results that will be needed in this thesis. In Section 1.2 some general notations are listed and a number of classes of functions are introduced. Section 1.3 gives the definition of an $L$-spline function, together with the definitions of some particular $L$-splines such as cardinal $L$-splines and perfect $L$-splines. Section 1.4 contains a variety of concepts in the theory of $L$-spline functions, e.g., the concept of disconjugacy of a differential operator, Taylor's formula and Peano's remainder formula, the classical Budan–Fourier theorem and a generalized version of it, various forms of a Chebyshev system, the notions of solvent and unsolvent families, and divided differences. Most of them are standard in approximation theory, apart from the generalized Budan–Fourier theorem. In order to obtain this generalization a specific device for counting zeros of piecewise continuous functions is used. The generalized Budan–Fourier theorem is applied to $L$-splines in Subsection 1.4.6 in order to count its zeros.

1.2. Notations and conventions

1.2.1. Formula indication

Formulas are numbered independently from theorems, lemmas, corollaries and definitions. When referring, for example, to the third formula of Section 2 in Chapter I, we shall write (1.2.3).
1.2.2. Notations

Here only those notations are explained that are used throughout this thesis. A more extensive list of symbols is given on pp. 192, 193.

\[ \mathbb{N} \] : the set of positive integers, \( \mathbb{N} = \{1, 2, 3, \ldots\} \).

\[ \mathbb{N}_0 \] : the set of nonnegative integers, \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \).

\[ \mathbb{Z} \] : the set of integers.

\[ \mathbb{R} \] : the set of real numbers.

\[ \mathbb{C} \] : the set of complex numbers.

\( [a, b] \) : the closed interval \( \{ x \in \mathbb{R} \mid a \leq x \leq b \} \).

\( (a, b) \) : the open interval \( \{ x \in \mathbb{R} \mid a < x < b \} \); similarly \( (a, b], [a, b) \).

\[ \mathbb{R}_0^+ \] : the set \( (0, \infty) \).

\[ \mathbb{R}^n \] : the Euclidean space of dimension \( n \) consisting of column vectors.

\( \underline{x} \) : the transpose of a vector \( x \).

\( V^\circ \) : the interior of a set \( V \).

\( \partial V \) : the boundary of a set \( V \).

\( \emptyset \) : the empty set.

\( \delta_{i,j} \) : the Kronecker symbol, i.e., \( \delta_{i,j} = 1 \) and \( \delta_{i,j} = 0 \) if \( i \neq j \).

\( f(x) \) : \( \lim_{h \to 0} f(x+h) \).

\( f(x) \) : \( \lim_{h \to 0} f(x-h) \).

\( [x] \) : the largest integer not exceeding \( x \).

\( \text{sgn} \) : the sign function, i.e.,

\[
\text{sgn}(x) = \begin{cases} 
1 & (x > 0), \\
0 & (x = 0), \\
-1 & (x < 0). 
\end{cases}
\]

\( := \) : is used in a definition if a new symbol occurs on the left-hand side.

\( \Box \) : marks the end of a proof.
1.2.3. Functions

As we mainly deal with real-valued functions, the range of a function is assumed to be \( \mathbb{R} \), unless otherwise stated.
If \( f(x) \) is the value of a function \( f \) at \( x \), then the function is denoted by \( f \) or by \( x \rightarrow f(x) \) or by \( f(x) \).
If \( f \) is a function of several variables, say two, then the function of one variable obtained from \( f \) by fixing one of the two variables, say \( x \), is denoted by \( y \rightarrow f(x,y) \) or by \( f(x,\cdot) \).

We proceed by defining some classes of functions. In these definitions \( J \) stands for an interval and \( n \in \mathbb{N} \), unless otherwise stated.

\( \mathcal{C}(J) \) : the set of continuous functions defined on \( J \).

\( \mathcal{C}^{(n)}(J) \) : the set of functions \( f \) defined on \( J \) having continuous \( n \)-th derivatives, i.e., \( f^{(n)} \in \mathcal{C}(J) \) where \( n \in \mathbb{N}_0 \).

\( \mathcal{PC}(J) \) : the set of functions \( f \) defined on \( J \) that are continuous on \( J \) with the possible exception of finitely many points in any bounded subinterval of \( J \), and such that at every point \( x \) of discontinuity \( f(x^+) \) and \( f(x^-) \) both exist with \( f(x) = f(x^+) \).

\( \mathcal{AC}^{(n)}(J) \) : the set of functions \( f \in \mathcal{C}^{(n-1)}(J) \) for which the \( (n-1) \)-st derivatives \( f^{(n-1)} \) are absolutely continuous on every compact subinterval of \( J \).

\( \mathcal{PC}^{(n)}(J) \) : the set of functions \( f \in \mathcal{AC}^{(n)}(J) \) for which the \( (n-1) \)-st derivatives \( f^{(n-1)} \) are integrals of functions in \( \mathcal{PC}(J) \), i.e., for every \( f \in \mathcal{PC}^{(n)}(J) \) there exists a function \( g \in \mathcal{PC}(J) \) such that

\[
\int_{t_0}^{t} g(t) \, dt \quad (t \in J, \ t_0 \in J).
\]

The \( n \)-th derivative is defined on \( J \) with the possible exception of finitely many points in any bounded subinterval of \( J \).
At any point \( t \in J \) we define \( f^{(n)}(t) := g(t) \).

\( L_{\infty}(J) \) : the set of measurable functions \( f \) which are essentially bounded on \( J \), i.e., for every function \( f \in L_{\infty}(J) \) there exists a number \( M > 0 \) such that \( |f(t)| \leq M \) (a.e.).

\( \mathcal{M}^{(n)}(J) \) : the set of functions \( f \in \mathcal{AC}^{(n)}(J) \) for which \( f^{(n)} \in L_{\infty}(J) \).
1.2.4. The supremum norm of a function

If a function $f$ is essentially bounded on $J$, then the essential supremum of $|f|$ on $J$ is denoted by $\|f\|_\infty$.

If $J = \mathbb{R}$ or $J = \mathbb{R}_0^+$, then the notation for the supremum norm is shortened by writing $\|f\| := \|f\|_\infty$.

1.2.5. Interpolation of Hermite data

Let $\mathbf{t} = (t_0, t_1, \ldots, t_{n-1})^T \in \mathbb{R}^n$ with $t_0 \leq t_1 \leq \ldots \leq t_{n-1}$. If $f$ is a function that is sufficiently often differentiable, then $f(t) = f(t_0, t_1, \ldots, t_{n-1})^T \in \mathbb{R}^n$ defined as follows: if $t_{j-1} < t_j = t_{j+1} = \ldots = t_{j+k-1} < t_{j+k}$, then $a_{j+i} := f^{(i)}(t_j)$ ($i = 0, 1, \ldots, k-1$); in particular, if the $t_j$ are distinct then $a_j := f(t_j)$ ($j = 0, 1, \ldots, n-1$). The sequence $t_j, t_{j+1}, \ldots, t_{j+k-1}$ with $t_{j-1} < t_j = t_{j+1} = \ldots = t_{j+k-1} < t_{j+k}$ will be called a coincident block of length $k$.

We say that a function $f$ interpolates the data $(y_0, y_1, \ldots, y_{n-1})$ at the points $t_0, t_1, \ldots, t_{n-1}$ when $f(t_0, t_1, \ldots, t_{n-1}) = (y_0, y_1, \ldots, y_{n-1})^T$. This kind of interpolation is called Hermite interpolation.

1.2.6. Determinants

Let $\mathbf{t} = (t_0, t_1, \ldots, t_{m-1})^T \in \mathbb{R}^m$ with $t_0 \leq t_1 \leq \ldots \leq t_{m-1}$ and let $q_0, q_1, \ldots, q_{m-1}$ be $m$ functions that are sufficiently often differentiable, then

$$
\begin{vmatrix}
q_0 & q_1 & \cdots & q_{m-1} \\
t_0 & t_1 & \cdots & t_{m-1}
\end{vmatrix} = \det(q_0, q_1, \ldots, q_{m-1}),
$$

where $\det(q_0, q_1, \ldots, q_{m-1})$ is the determinant of the $m \times m$ matrix, the $j$-th column of which is given by $q_j(t_0, t_1, \ldots, t_{m-1})$. 

1.3. $L$-spline functions

Usually, spline functions (throughout the thesis the term "spline" will be used as a synonym for "spline function") are defined as functions with a certain degree of smoothness consisting of piecewise polynomials tied together at the so-called knots. More generally, the polynomials are replaced by functions satisfying a given linear homogeneous differential equation $Lf = 0$ (cf. Michelli [37]). This leads to the definition of an $L$-spline.

The space of all polynomials with real coefficients and of degree at most $n \leq N$ is denoted by $\pi_n$. A polynomial $p_n$ in $\pi_n$ is called monic if $p_n$ has degree $n$ and if its leading coefficient is equal to one.

Let $p_n \in \pi_n$ be monic, i.e., $p_n(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$, then the linear differential operator $f \mapsto f^{(n)} + a_{n-1}f^{(n-1)} + \ldots + a_0 f$ is denoted by $D^n = a_{n-1}D^{n-1} + \ldots + a_0 I$ or by $p_n(D)$ for short, with $Df := f'$ and $D^k f := D^{k-1} f$ $(k \geq 2)$.

The kernel of the operator $p_n(D)$, denoted by $\text{Ker}(p_n)$ or $\text{Ker}(p_n(D))$, is defined as follows.

**DEFINITION 1.3.1.**

$$\text{Ker}(p_n) := \{ f \in C_c^{(n)} [\mathbb{R}] \mid p_n(D)f(t) = 0 \ (a.e.) \}.$$ 

**DEFINITION 1.3.2.** Let $n$ be defined on $[a,b]$. We say that $s$ is an $L$-spline function of order $n$, if there exists a monic polynomial $p_n \in \pi_n$ and a finite sequence of points $x_1, x_2, \ldots, x_k$ with $a = x_0 < x_1 < x_2 < \ldots < x_k < x_{k+1} = b$ and $x_i < x_{i+1}$ $(i = 1, \ldots, n-2)$, such that

1) on every nonempty interval $(x_i, x_{i+1})$ ($i = 0, 1, \ldots, k$) the function $s$ coincides with an element of $\text{Ker}(p_n)$,

2) if $x_{i-1} < x_i = x_{i+1} = \ldots = x_{i+k-1} < x_{i+k} \equiv 0$, i.e., if $x_i$ has multiplicity $k$, then $s$ has a continuous $(n-1)$-th derivative in a neighbourhood of $x_i$. If $i = n$ then $s(x_i) := s(x_{i+1})$.

If $p_n(z) = z^n$ then the $L$-spline $s$ consists of piecewise polynomials and it will be called a polynomial spline of order $n$ (or of degree $n-1$).

The points $x_1, x_2, \ldots, x_k$ are called the knots of the $L$-spline.
If a knot has multiplicity one, then it is called a simple knot. Occasionally, the distinct points among the knots \( x_1, x_2, \ldots, x_k \) are denoted by \( y_1, y_2, \ldots, y_k \) and their multiplicities by \( v_1, v_2, \ldots, v_k \), respectively.

We also deal with \( L \)-splines defined on \( \mathbb{R} \) or on \( \mathbb{R}^* \). In these cases the number of knots is allowed to be infinite; however, a finite accumulation of knots is not permitted.

The following definition concerns an \( L \)-spline function that is fundamental in the theory of splines.

**Definition 1.3.3.** Let \( q \in \text{ker}(p_n) \) be the function satisfying

\[
p(0) = q'(0) = \ldots = q^{(n-2)}(0) = 0, \quad q^{(n-1)}(0) = 1.
\]

Then the function \( \varphi_+ \in C^{(n-1)}(\mathbb{R}) \) is defined by

\[
\varphi_+(t) := \begin{cases} 
q(t) & (t \geq 0), \\
0 & (t < 0).
\end{cases}
\]

We call \( q \) the fundamental function corresponding to the operator \( p_n(D) \) and \( \varphi_+ \) the Green's function corresponding to \( p_n(D) \). The dependence of the functions \( q \) and \( \varphi_+ \) on \( p_n \) is not expressed in the notation.

The function \( \varphi_+ \) can be used to represent \( L \)-splines. For instance, if \( s \) is an \( L \)-spline with knots located at \( y_1, y_2, \ldots, y_k \) with \( y_1 < y_2 < \ldots < y_k \) and with multiplicities \( v_1, v_2, \ldots, v_k \), respectively, then \( s \) can be written as

\[
(1.3.1) \quad s(t) = q(t) + \sum_{i=1}^{k} \sum_{j=0}^{v_i-1} w_{i,j} q^{(j)}(t-y_i),
\]

where \( q \in \text{ker}(p_n) \) and the real numbers \( w_{i,j} \) are uniquely determined (cf. Karlin [23, p. 317]).

For numerical purposes, the representation by means of \( q \) can be bad (cf. De Boor [5, p. 104]). The so-called \( B \)-spline functions constitute a basis for \( L \)-splines that is appropriate for both practical and theoretical purposes. Chapter 2 is devoted to these functions.

We shall give special attention to \( L \)-splines with equally spaced simple knots. With respect to this class the following definition is in order.
DEFINITION 1.3.4. An $L$-spline function $s$ corresponding to the operator $p_n(D)$ is called a cardinal $L$-spline function of order $n \in \mathbb{N}$ if there exists a number $h > 0$ such that

i) $p_n(D)s(t) = 0 \ (i h < t < (i+1)h ; \ i \in \mathbb{Z})$,

ii) $s \in C^{(n-2)}(\mathbb{R})$ if $n \geq 2$ and $s \in PC(\mathbb{R})$ if $n = 1$.

The number $h$ in the definition of a cardinal $L$-spline is called the mesh distance of the knot distribution.

The set of all cardinal $L$-splines corresponding to the operator $p_n(D)$ and with mesh distance $h$ is, in general, denoted by

$$\{ p_n(D), h \}$$

occasionally, the notation $S(p_n(D), h)$ will also be used.

Usually, the term cardinal is reserved for $L$-splines with knots at the integers, but by a simple transformation of scale $t = ht$ the knots are transformed to the integers, while the differential operator $p_n(D)$ is replaced by $p_n(hD)$.

The cardinal polynomial splines are studied in detail by Schoenberg [51] in his monograph. Parts of the theory are extended by the author of this thesis in [40] and, independently, by Micelli [37].

Finally, the so-called perfect $L$-splines will be defined. These splines often occur as "extremal functions" with respect to certain extremal problems in the space $W^m(J)$ in case the supremum norm is used (cf. Chapters 5, 6, 7).

DEFINITION 1.3.5. A function $s$, defined on the interval $J$, is called a perfect $L$-spline function corresponding to the operator $p_n(D)$ if $s$ is an $L$-spline of order $n+1$ corresponding to the operator $Dp_n(D)$ with the property that a number $c$ exists such that for all $t \in J$, knots being excluded,

$$|p_n(D)s(t)| = c.$$ 

It follows from this definition that the "$p_n(D)$-derivative" of $s$ has constant absolute value $c$ and jumps from $-c$ to $+c$ at the knots. Note that a perfect $L$-spline corresponding to the operator $p_n(D)$ is an $L$-spline of order $n+1$ corresponding to the operator $Dp_n(D)$. 
1.4. Some basic concepts in the theory of $L$-spline functions

In order to make this thesis reasonably self-contained, in this section we will collect some results that will be frequently used. In general, proofs are omitted in case a reference is available or if it concerns proofs that are standard in the theory of $L$-spline functions.

1.4.1. Disconjugacy of a differential operator

DEFINITION 1.4.1. The differential operator $p_n(D)$ is said to be disconjugate on an interval $J$ if there exist $n$ strictly positive functions $w_i \in C^{n+1-1}(J)$ ($i = 1, 2, \ldots, n$) such that $p_n(D)$ can be factorized as

\[
   p_n(D) = D_n D_{n-1} \ldots D_1,
\]

where the differential operator $D_i$ is defined by

\[
   D_i := \frac{w_i}{w_{i+1}} D \left( \frac{1}{w_i} \right) \quad (i = 1, 2, \ldots, n).
\]

The interval $J$ is called an interval of disconjugacy.

The following characterization was proved by Pólya in 1922.

LEMMA 1.4.2 (Pólya [46]). The differential operator $p_n(D)$ is disconjugate on an open interval $(a, b)$ if and only if no nontrivial function $f \in \ker(p_n)$ has more than $n-1$ zeros in $(a, b)$, counting multiplicities.

If $p_n$ has only real zeros, i.e., if $p_n(D) = (D - \lambda_1 I) \ldots (D - \lambda_n I)$ with $\lambda_i \in \mathbb{R}$, then one has

\[
   p_n(D)f(t) = e^{t} \frac{d}{dt} e^{-t} \frac{d}{dt} e^{\lambda_2 t} \frac{d}{dt} e^{-\lambda_2 t} \ldots e^{\lambda_n t} \frac{d}{dt} e^{-\lambda_n t} f(t),
\]

so $p_n(D)$ is disconjugate on the whole real line. If $p_n$ has at least one non-real zero, then an interval of disconjugacy is finite. As an example, let us consider the operator $p_2(D) = D^2 + 1$. One easily verifies that for all $a \in \mathbb{R}$ and $f \in AC^2(\mathbb{R})$

\[
   p_2(D)f(t) = \frac{1}{\sin(t-a)} D \sin^2(t-a) D \frac{f(t)}{\sin(t-a)}.
\]
Hence intervals of the form \((a, a+\ell)\) are intervals of disconjugacy for \(p_3(D)\).

Since \(p_3(D)\) has constant real coefficients, the maximal length \(\ell\) of an interval of disconjugacy does not depend on the location of that interval on the real line. It turns out (cf. Troch [62, Theorem 1]) that the maximum intervals on which each nontrivial function in \(\text{Ker}(p_3)\) has at most \(n-1\) zeros counting multiplicities, are half open intervals \([a, a+\ell)\) or \((a, a+\ell]\).

For second order differential operators the maximal length \(\ell\) is simply the distance between two consecutive zeros of the corresponding fundamental function \(\varphi\). For third order differential operators the function \(\varphi\) also determines the number \(\ell\). This is a consequence of Lemma 2 in Troch [62], which for third order differential operators can be stated as follows.

**Lemma 1.4.3** (Troch [62]). Let \(p_3 \in \mathbb{R}[x]\) be a monic polynomial having a nonreal zero, and let \(x_0 \in \mathbb{R}\) be a nearest point to zero for which \(\varphi(x_0) = 0\) and \(x_0 \neq 0\). Then \(\ell = |x_0|\).

If the order of the differential operator exceeds three, the zeros of its fundamental function \(\varphi\) do not in general determine the number \(\ell\) as in the case of second and third order differential operators. This is shown by the following example.

**Example.** If \(p_4(D) = D^4 - 1\), then \(\varphi(t) = \frac{1}{2}(\sinh t - \sin t)\) and thus \(\varphi(t) \neq 0\) when \(t \neq 0\). As \(t \mapsto \sin t \in \text{Ker}(p_4)\), by Lemma 1.4.2 the number \(\ell\) for \(p_4(D)\) is finite.

It is well known that a trigonometric polynomial of degree \(n\), i.e., a function belonging to \(\text{Ker}(p_{2n+1})\) with

\[
(1.4.3) \quad p_{2n+1}(D) = D(D^2 + 1)(D^2 + 42) \cdots (D^2 + n^2),
\]

cannot have more than \(2n\) zeros in an interval \([a, b]\) with \(b-a \leq 2\pi\), unless it is identically zero. As shown by the function \(t \mapsto \sin(nt)\), the number \(\ell\) for the operator \((1.4.3)\) is equal to \(2\pi\).

For more detailed information concerning the property of disconjugacy the reader is referred to the book of Coppel [14].
1.4.2. Taylor's formula and Peano's remainder formula for \( p_n(D) \)

For the differential operator \( p_n(D) \) Taylor's formula can be stated as follows.

**Lemma 1.4.4.** Let \( f \in \mathcal{C}^{(n)}([a,b]) \) and let \( q \in \text{Ker}(p_n) \) be the unique function satisfying \( q^{(k)}(a) = f^{(k)}(a) \) \( (k = 0, 1, \ldots, n-1) \). Then

\[
(f(t) = q(t) + \int_a^b e^{-i(t-\tau)}p_n(D)q(\tau)d\tau \quad (a \leq t \leq b),
\]

where the function \( e^{-i} \) is given by Definition 1.3.3.

We proceed by giving Peano's remainder formula for \( p_n(D) \).

**Lemma 1.4.5.** Let \( \lambda \) be a linear functional defined on \( \mathcal{C}^{(n)}([a,b]) \) of the form

\[
\lambda(f) = \sum_{j=1}^m \sum_{k=0}^{n-1} \alpha_{k,j} e^{(k)}(x_j) \quad (f \in \mathcal{C}^{(n)}([a,b])),
\]

where \( m \in \mathbb{N}, \ n \in \mathbb{N}, \ \alpha_{k,j} \in \mathbb{R} \) and \( a \leq x_1 < x_2 < \ldots < x_m \leq b \).

If \( \lambda(q) = 0 \) for all \( q \in \text{Ker}(p_n) \), then one has

\[
(1.4.5) \quad \lambda(f) = \int_a^b K(\tau)p_n(D)f(\tau)d\tau,
\]

where the kernel \( K \) is defined by \( K(\tau) := \lambda(t \rightarrow e^{-i(t-\tau)}) \).

A proof of Lemma 1.4.5 if \( p_n(D) = D^2 \) is contained in Davis [16, p. 70].

In the case of a general differential operator \( p_n(D) \) a proof of Peano's remainder formula can be given along similar lines using Taylor's formula of Lemma 1.4.4. We note that Lemma 1.4.5 does not state Peano's remainder formula in its full generality.
1.4.3. The theorem of Budan-Fourier

The classical form of the Budan-Fourier theorem gives an upper bound for the number of zeros in an open interval \((a, b)\) of a polynomial of degree \(n\) by means of the number of sign changes in the sequences \(p^{(k)}(a)\) \(k=0\) and \(p^{(k)}(b)\) \(k=0\) (cf. Karlin [21, p. 317]). It reads as follows:

**THEOREM 1.4.6 (Budan-Fourier).** Let \(p \in \mathbb{P}_n\) be a polynomial of exact degree \(n \in \mathbb{N}\). The total number of zeros of \(p\) in \((a, b)\), counting multiplicities, is bounded by

\[
S^-(p(a), \ldots, p^{(n)}(a)) - S^+(p(b), \ldots, p^{(n)}(b)),
\]

where \(S^-\) and \(S^+\) are defined by Definition 1.4.7.

An extension to polynomial splines with simple knots is given by De Boor and Schoenberg [6, p. 6] and to polynomial splines with knots of arbitrary multiplicity by Melkman [36].

The version of the Budan-Fourier theorem we present here is a generalization of the classical theorem to the set of functions \(\mathbb{P}_n^{(a,b)}\). Before we can formulate our result we have to agree on the way of counting zeros. This gives rise to the following definitions.

**DEFINITION 1.4.7.** Let \(a_1, a_2, \ldots, a_n\) be a sequence of real numbers. Then

\[
S^-(a_1, a_2, \ldots, a_n)
\]

denotes the number of sign changes in \(a_1, a_2, \ldots, a_n\) by deleting zero entries. The maximum number of sign changes that can be obtained by replacing the zero entries in \(a_1, a_2, \ldots, a_n\) by \(+1\) or \(-1\) is denoted by

\[
S^+(a_1, a_2, \ldots, a_n).
\]

**DEFINITION 1.4.8.** Let \(f\) be a function defined on \((a, b)\). Then

\[
S^-(f, (a,b)) := \sup \{S^-(f(t_1), \ldots, f(t_n)) \mid a < t_1 < t_2 < \ldots < t_n < b, n \in \mathbb{N}\}.
\]

We shall call \(S^-(a_1, a_2, \ldots, a_n)\) the number of strong sign changes and \(S^+(a_1, a_2, \ldots, a_n)\) the number of weak sign changes of \(a_1, a_2, \ldots, a_n\).
One has the following equality (cf. De Boor and Schonberg [6, p. 3]):

\[ S^n(a_1, a_2, \ldots, a_{n-1}) + S^{-1}(a_1, a_2, \ldots, a_n) = n - 1. \]

The next step is to define the way of counting zeros in an open interval (a,b) of functions in \( PC_{(n)}([a,b]) \) \((n = 0, 1, \ldots)\). This will be done as follows. First the procedure is described for the class \( PC([a,b]) \); one then extends it to the class \( PC_{(n)}([a,b]) \).

As zeros in (a,b) of a function \( f \in PC([a,b]) \) are considered: points \( x_0 \in (a,b) \) where \( f(x_0^-)f(x_0) < 0 \) and intervals \( (c,d) \subset (a,b) \) such that \( f \) is identically zero in \( (c,d) \).

The function \( f \) is said to "vanish" at a point \( x_0 \) if \( x_0 \) is a zero of \( f \) as defined above; so \( f \) may vanish at \( x_0 \) even when \( f(x_0) \neq 0 \). The function \( f \) is said to "vanish identically" in an interval \( J \) if \( f \) vanishes at every point of \( J \).

We shall only take into account those zeros \( a \) (a point \( x_0 \) or an interval \( [c,d] \subset (a,b) \)) having the property that \( f \) does not vanish identically in any interval containing \( \{x_0\} \) \(([c,d])\) as a proper subset. These zeros will be called strong zeros.

**Definition 1.4.9.** Let \( f \in PC([a,b]) \) and let \( a \) be a strong zero of \( f \) in \((a,b)\). Then the multiplicity \( N_0(f,a) \) of \( a \) is determined as follows.

- If \( S^n(f,U) = \infty \) in each neighbourhood \( U \subset (a,b) \) of \( a \), then \( N_0(f,a) := \infty \).
- If \( S^n(f,U) = 1 \) for all sufficiently small neighbourhoods \( U \subset (a,b) \) of \( a \), then \( N_0(f,a) := 1 \).
- If \( S^n(f,U) = 0 \) for all sufficiently small neighbourhoods \( U \subset (a,b) \) of \( a \), then \( N_0(f,a) := 0 \).

Note that in this definition \( \pm, 1 \) and 0 are the only possible values of \( S^n(f,U) \). Clearly, for a sufficiently small neighbourhood \( U \) of a strong zero \( a \) we have either \( S^n(f,U) = 0 \) or \( S^n(f,U) = 1 \) or \( S^n(f,U) = \infty \). According to the counting rule as given above and taking into account Definition 1.4.8, the total number of strong zeros in \((a,b)\), counting multiplicities, of a function \( f \in PC([a,b]) \) is equal to \( S^n(f,(a,b)) \).

The following example illustrates the way of counting zeros for functions \( f \in PC([a,b]) \).
The strong zeros are 2, 3, [4,5], 6, [7,8] with multiplicities 0, 1, 1, 0, respectively. Hence $S^*(f,(0,9)) = 3$.

Our next purpose is to define the multiplicity of a strong zero of a function $f \in \mathcal{C}(n)^{(n)}([a,b])$ $(n \geq 1)$ with respect to a sequence of differential operators $D_1, D_2, \ldots, D_n$ given by

\[(1.4.7) \quad D_k^i = \nu_k \frac{D^i}{\nu_k} \quad (i = 1, \ldots, n),\]

where $\nu_k \in C^{n+1}((a,b))$ is a strictly positive function.

We define the operators $D^{[i]}$ as follows:

\[(1.4.8) \quad D^{[0]} = I, \quad D^{[i]} = D_k^1 D_{k-1}^1 \cdots D_k^1 \quad (i = 1, \ldots, n).\]

**DEFINITION 1.4.10.** Let $f \in \mathcal{C}(n)^{(n)}([a,b])$ $(n \geq 1)$ and let $a$ be a strong zero of $f$ in $(a,b)$. The multiplicity of $a$ with respect to the sequence of differential operators $D_1, D_2, \ldots, D_n$, denoted by $M(D_1^1, f, a)$, is defined as follows.

- If $D^{[i]} f(a) = 0$ for $i = 0, 1, \ldots, k-1$ and $D^{[k]} f(a) \neq 0$ with $k \leq n-1$, then $M(D_1^1, f, a) = k$.
- If $D^{[i]} f(a) = 0$ for $i = 0, 1, \ldots, n-1$ and $D^{[n]} f$ does not vanish at $a$, then $M(D_1^1, f, a) = n$.
- If $D^{[i]} f$ vanishes at $a$ for $i = 0, 1, \ldots, n$, then $M(D_1^1, f, a) = n + \nu_k [D^{[n]} f, a]$.

**REMARK 1.** If $a$ is a strong zero of $f \in \mathcal{C}(n)^{(n)}([a,b])$ with an odd multiplicity $M(D_1^1, f, a)$ then $f$ changes sign at $a$, and if $M(D_1^1, f, a)$ is even then $f$ does not change sign at $a$.

**REMARK 2.** If $a$ is a strong zero of $f \in \mathcal{C}(n)^{(n)}([a,b])$ with $M(D_1^1, f, a) = k \leq n-1$, then, of course, $a$ reduces to a single point $x_0$ and $f(x_0) = f^{(k)}(x_0) = \cdots = f^{(k-1)}(x_0) = 0$, $f^{(k)}(x_0) \neq 0$. Moreover, $x_0$ then is an isolated zero, i.e., there exists a neighbourhood of $x_0$ such that $f$
vanishes in \( O \) only at the point \( x_0 \).

On the other hand, if \( x_0 \in (a,b) \) satisfies \( f(x_0) = f'(x_0) = \ldots = f^{(n-1)}(x_0) = 0 \) and \( f^{(n)}(x_0) \neq 0 \) with \( k \leq n-1 \), then it is easy to verify that \( M((D_1)^0, f, 0) = k \) for any sequence of differential operators \( D_1, D_2, \ldots, D_n \) given by \( (1.4.7) \).

**Remark 3.** We show that there may exist two different sequences of differential operators \( (D_1)^0 \) and \( (D_2)^0 \) such that for some function \( f \in C^{(n)}(a,b) \) and some \( x_0 \in (a,b) \), \( M(D_i)^0, f, x_0) \neq \{n,n+1\} \), whereas \( M((D_2)^0, f, x_0) = \infty \). To this end we take \( n = 1 \), \( D_1 = D \) and \( D_2 = \frac{1}{w_1} \frac{1}{w_1} \) with \( w_1 \) given by

\[
    w_1(x) = \begin{cases} 
    1 + x^3 \sin \left( \frac{\pi}{x} \right) & \text{if } |x| \leq 1, \ x \neq 0, \\
    1 & \text{if } x = 0.
    \end{cases}
\]

If \( f \in C^{(1)}([-1,1]) \) satisfies \( f(0) = 0 \), then

\[
f(x) = \left. \frac{1}{w_1(x)} \right|_0^x \int_0^t \frac{1}{w_1(t)} D_1 f(t) \, dt
\]

and thus

\[
f'(x) = \left. \frac{1}{w_1(x)} \right|_0^x \int_0^t \frac{1}{w_1(t)} D_1 f(t) \, dt.
\]

Substituting a function \( f \) for which \( D_1 f(x) = \cos^2(1/x) \), we may easily verify that \( M(D_1)^0, f, 0) = 2 \) and \( M(D_2)^0, f, 0) = \infty \).

**Remark 4.** If \( a \) is a strong zero of \( f \in C^{(n)}(a,b) \) \( (n \leq 1) \) with \( M(D_1)^0, f, a) \neq \{n,n+1\} \), then \( S^-(D^n)^0, f, a) = 0 \) or \( S^-(D^n)^0, f, 0) = 1 \) in a neighbourhood \( U \) of \( a \). From this it follows that \( a \) is an isolated zero.

The total number of strong zeros in \( (a,b) \) of \( f \in C^{(n)}(a,b) \) \( (n \in \mathbb{N}) \), counting multiplicities according to Definitions 1.4.9 and 1.4.10, is denoted by

\[
(1.4.9) \quad Z((D_1)^0, f, (a,b)) = .
\]

Note that if \( n = 0 \) then (1.4.9) may be replaced by \( S^-(f, (a,b)) \).
Let \( f \in \mathop{PC}^{(n)}([a,b]) \). Taking into account (1.4.8) one has
\[
\text{D}^j f \in \mathop{PC}^{(n-j)}([a,b]),
\]
and the total number of strong zeros in \((a,b)\) of
\[
\text{D}^j f,
\]
counting multiplicities in accordance with Definitions 1.4.9 and
1.4.10, is denoted by
\[
(1.4.10) \quad Z((D^{n-j}_a)^j f, (a,b)) \quad (j = 0, 1, \ldots, n).
\]
Note that if \( j = n \) then (1.4.10) may be replaced by
\[
S^{-}(D^n f, (a,b)).
\]
The following lemma may be considered as a generalization of Rolle's theorem.

**Lemma 1.4.11.** Let \( f \in \mathop{PC}^{(n)}([a,b]) \) \((n \geq 1)\). Then
\[
(1.4.11) \quad Z((D^{n-j}_a)^j f, (a,b)) \leq Z((D^{n-j}_a)^{j+1} f, (a,b)) + 1 \quad (j < n).
\]

**Proof.** Let \( a_1 \) and \( a_2 \) be two distinct strong zeros of \( f \). Assume that \( a_1 \) and
\( a_2 \) are points \( x_1 \) and \( x_2 \) with \( x_1 < x_2 \). Since
\[
0 = f(x_2) = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f'(t) \, dt = 0,
\]
and \( D^{[1]} f \) does not vanish identically in \((x_1, x_2)\), it follows that
\[
S^{-}(D^{[1]} f, (x_1, x_2)) \geq 1
\]
and therefore
\[
Z((D^{n-j}_a)^j f, (x_1, x_2)) \geq 1.
\]
If \( a_1 \) is a strong zero of multiplicity \( k \geq 2 \) of \( f \), then it follows immediately from Definition 1.4.10 that \( a_1 \) is a zero of multiplicity \( k - 1 \) of
\( D^{[1]} f \). This proves our lemma, since the case that at least one of the
strong zeros is an interval can be treated in a similar way.

**Corollary 1.4.12.** Let \( f \in \mathop{PC}^{(n)}([a,b]) \) \((n \geq 1)\), then
\[
(1.4.12) \quad Z((D^{n-j}_a)^j f, (a,b)) \leq S^{-}(D^n f, (a,b)) + n.
\]

**Proof.** Obviously, the inequality
\[
(1.4.13) \quad Z((D^{n-j}_a)^{j+1} f, (a,b)) \leq Z((D^{n-j}_a)^j f, (a,b)) + 1
\]
is a direct consequence of Lemma 1.4.11. Repeated application of (1.4.13) yields (1.4.12).

In the proof of our version of the Sudan-Fourier theorem the following lemma is needed.

**Lemma 1.4.13.** Let \( f \in \mathcal{C}^\infty([a,b]) \). If \( D^{[n]} f(a) \neq 0 \), then

\[
(1.4.14) \quad S^+(f(a), D^{[1]} f(a), \ldots, D^{[n]} f(a)) = \lim_{x \to a} S^+(f(x), D^{[1]} f(x), \ldots, D^{[n]} f(x)) = \lim_{x \to a} S^+(f(b), D^{[1]} f(b), \ldots, D^{[n]} f(b)).
\]

If \( D^{[n]} f(b-) \neq 0 \), then

\[
(1.4.15) \quad S^+(f(b), D^{[1]} f(b), \ldots, D^{[n]} f(b)) = \lim_{x \to b} S^+(f(x), D^{[1]} f(x), \ldots, D^{[n]} f(x)) = \lim_{x \to b} S^+(f(b), D^{[1]} f(b), \ldots, D^{[n]} f(b)).
\]

**Proof.** Since \( D^{[n]} f(a) \neq 0 \) there exists a number \( \epsilon_1 > 0 \) such that \( D^{[i]} f(x) \neq 0 \) for \( 0 < x < a + \epsilon_1 \). Hence

\[
S^+(f(x), D^{[1]} f(x), \ldots, D^{[n]} f(x)) = S^+(f(a), D^{[1]} f(a), \ldots, D^{[n]} f(a)) = S^+(f(b), D^{[1]} f(b), \ldots, D^{[n]} f(b)) = \lim_{x \to a} S^+(f(x), D^{[1]} f(x), \ldots, D^{[n]} f(x)).
\]

For \( x \in (a, a + \epsilon_1) \),

\[
(1.4.16) \quad S^+(D^{[1]} f(a), \ldots, D^{[n]} f(a)) = S^+(D^{[1]} f(b), \ldots, D^{[n]} f(b)).
\]

Application of (1.4.16) to consecutive segments of \( (f(a), \ldots, D^{[n]} f(a)) \) yields (1.4.14).

If \( D^{[n]} f(b-) \neq 0 \) then there exists a number \( \epsilon_2 > 0 \) such that \( D^{[i]} f(x) \neq 0 \) for \( 0 < x < b - \epsilon_2 \). Furthermore, if \( D^{[j+1]} f(b) = \ldots = D^{[j+k]} f(b) = 0 \) for \( 0 < x < b - \epsilon_2 \) and \( D^{[j+k]} f(b-) \neq 0 \), then

\[
S^+(D^{[j]} f(b), \ldots, D^{[j+k]} f(b)) = sgn(D^{[j+k]} f(b)),
\]

\( k = 1, 2, \ldots, x, b - \epsilon_2 < x < b \). Hence

\[
S^+(D^{[j]} f(b), \ldots, D^{[j+k]} f(b)) = S^+(D^{[j]} f(b), \ldots, D^{[j+k]} f(b)) = \lim_{x \to a} S^+(f(x), D^{[1]} f(x), \ldots, D^{[n]} f(x)).
\]

which implies (1.4.15).
We remark that a particular case of the previous lemma, i.e., when \( D_i = D_i^* \) for \( i = 1, 2, \ldots, n \), can be found in de Boor and Schoenberg [5, p. 6], in Karlin and Michelli [24], and in Schwahnweck [56, p. 163].

**Theorem 1.4.14 (Generalized Budan-Fourier Theorem).** Let \( f \in P^n([a,b]) \) and let \( D_1, \ldots, D_n^* \) be given by (1.4.7) and (1.4.8), respectively. If \( f(a)D_1^*f(b) \neq 0 \), then

\[
(1.4.17) \quad Z(D_j^*, f(a), f(b)) \leq S^-(D_n^* f, (a, b)) + S^+(f(a), D_1^* f, (a, b)) + \ldots + S^+(f(b), D_n^* f, (b, b)) + \ldots + S^+(f(b), D_n^* f, (b, -)) + \ldots + S^+(f(b), D_n^* f, (b, b)) .
\]

**Proof.** First we prove (1.4.17) under the additional condition that

\[
(1.4.18) \quad D_i^* f(a)D_i^* f(b) \neq 0 \quad (i = 0, 1, \ldots, n-1) .
\]

We assert that for \( j = 0, 1, \ldots, n \)

\[
(1.4.19) \quad Z(D_j^*, f(a), f(b)) \leq S^+(D_j^* f, (a, b)) + S^-(D_j^* f, (b, b)) .
\]

Varying \( n \) in (1.4.19) one can see that it suffices to prove (1.4.19) for \( j = 0 \). If \( Z(D_0^*, f(a), f(b)) \geq 1 \) and \( Z(D_1^*, f(a), f(b)) = 0 \) then (1.4.19) holds trivially. Now let us assume that

\[
Z(D_0^*, f(a), f(b)) = Z(D_1^*, f(a), f(b)) = 0 .
\]

Then \( sgn(f(a)) = sgn(f(b)) \) and \( sgn(D_0^* f(a)) = sgn(D_0^* f(b)) \) and again (1.4.19) holds trivially. So, the case \( Z(D_0^*, f(a), f(b)) \geq 1 \) remains.

Since \( f \) is continuous we can find points \( c \) and \( d \) in \((a, b)\) such that \( f(c)/f(d) > 1 \) and \( f(d)/f(c) > 1 \). If \( Z(D_0^*, f(a), f(c)) = 0 \), then \( f(c) > f(a) \) in case \( D_0^* f(a) > 0 \) and \( f(c) < f(a) \) in case \( D_0^* f(a) < 0 \). Hence \( S^+(f(a), D_0^* f(a)) = 1 \). A similar reasoning applied to the interval \((d, b)\) leads to the following assertion: if \( Z(D_0^*, f(a), f(b)) = 0 \), then \( S^-(f(b), D_0^* f(b)) = 0 \). So we have...
\[ S^-(f(a), D^{-1} f(b)) = S^+(f(a), D^{[1]} f(b)) \geq 1 - 2 \left( \frac{1}{2} D^{[1]} f(a), f(a, c) \right) \]

and

\[ S^+(f(b), D^{[1]} f(b)) \leq 2 \left( D^{[1]} f(b), f(b, d) \right) . \]

These inequalities combined with (1.4.11) give

\[ 2 \left( \frac{1}{2} D^{[1]} f(a), f(a, b) \right) = 2 \left( D^{[1]} f(a), f(a, c) \right) \leq 1 + 2 \left( \frac{1}{2} D^{[1]} f(a), f(c, d) \right) \leq 1 + 2 \left( \frac{1}{2} D^{[1]} f(a), f(a, b) \right) - 2 \left( \frac{1}{2} D^{[1]} f(a), f(a, c) \right) - 2 \left( \frac{1}{2} D^{[1]} f(a), f(d, b) \right) \leq 2 \left( \frac{1}{2} D^{[1]} f(a), f(a, b) \right) + S^-(f(a), D^{[1]} f(a)) = S^+(f(b), D^{[1]} f(b)) . \]

We thus obtain (1.4.19). Repeated application of (1.4.19) yields

\[ 2 \left( D^{[1]} f(a), f(a, b) \right) \leq S^-(f(a), D^{[1]} f(a)) + \sum_{j=0}^{n-1} S^-(f(a), D^{[j+1]} f(a)) + \frac{1}{2} \sum_{j=0}^{n-1} S^+(f(b), D^{[j+1]} f(b)) . \]

In view of this and taking into account condition (1.4.18), inequality (1.4.17) follows from the identity

\[ S^-(a_0, a_1, \ldots, a_n) = \frac{1}{2} \sum_{j=0}^{n-1} S^+\left(a_j, a_{j+1}\right) \quad (a_j \in \mathbb{R}, a_j \neq 0) . \]

If condition (1.4.18) is not satisfied, then because of the assumption \( D^{[n]} f(a) D^{[n]} f(b) \neq 0 \) it follows that for all sufficiently small \( \varepsilon > 0 \) one has \( D^{[1]} f(a+c) D^{[1]} f(b-c) \neq 0 \) \((i = 0, 1, \ldots, n)\). Therefore, (1.4.17) holds on the interval \((a+c, b-c)\). Since \( 2 \left( D^{[1]} f(a), f(a, b) \right) = 2 \left( D^{[1]} f(a), f(a+c, b-c) \right) \), for sufficiently small \( \varepsilon > 0 \), Theorem 1.4.14 is obtained by letting \( \varepsilon \) tend to zero and using Lemma 1.4.13. \( \square \)

Remark. Theorem 1.4.14 implies the classical Bock-Fourier theorem 1.4.6, this can be seen as follows. Since \( p^{[n]}(a) \neq 0 \) one has that \( S^-(p^{[n]}(a), (a, b)) = 0 \). Furthermore, the multiplicity of a zero of \( p \) with respect to the operators \( D^{[1]} \) is as usual determined by consecutive derivatives. Hence (1.4.17) yields Theorem 1.4.6.
1.4.4. Chebyshev systems

**Definition 1.4.15.** Let \( J \) be an interval and let \( \varphi_0, \varphi_1, ..., \varphi_{n-1} \) be \( n \) functions in \( C(J) \). The set \( \{ \varphi_0, \varphi_1, ..., \varphi_{n-1} \} \) is called a Chebyshev system on \( J \) if (cf. Section 1.2.8 for notation)

\[
\begin{vmatrix}
\varphi_0 & \varphi_1 & \cdots & \varphi_{n-1} \\
t_0 & t_1 & \cdots & t_{n-1}
\end{vmatrix} > 0
\]

for each choice of the \( n \) points \( t_i \in J \) with \( t_0 < t_1 < ... < t_{n-1} \).

If the determinant in the foregoing definition is nonnegative for all \( t_0 < t_1 < ... < t_{n-1} \) and if the functions \( \varphi_0, \varphi_1, ..., \varphi_{n-1} \) are linearly independent, then \( \{ \varphi_0, \varphi_1, ..., \varphi_{n-1} \} \) is called a weak Chebyshev system on \( J \).

**Definition 1.4.16.** The Chebyshev system \( \varphi_0, \varphi_1, ..., \varphi_{n-1} \) is called an extended Chebyshev system of order \( p \) on \( J \) if \( \varphi_1 \in C^{(p-1)}(J) \) \( (i = 0, 1, ..., n-1) \) and if

\[
\begin{vmatrix}
\varphi_0 & \varphi_1 & \cdots & \varphi_{n-1} \\
t_0 & t_1 & \cdots & t_{n-1}
\end{vmatrix} > 0
\]

for all \( t_0 \leq t_1 \leq ... \leq t_{n-1} \) \((t_i \in J)\), with coincident blocks of length not exceeding \( p \) (cf. Section 1.3.5).

If the determinant in Definition 1.4.16 is nonnegative for every choice of \( t_0 \leq t_1 \leq ... \leq t_{n-1} \) with coincident blocks of length not exceeding \( p \) and if the functions \( \varphi_0, \varphi_1, ..., \varphi_{n-1} \) are linearly independent, then these functions are called an extended weak Chebyshev system of order \( p \) on \( J \).

In what follows we give examples of Chebyshev systems together with a number of lemmas that will be used in the sequel of this thesis.

**Example 1.** If \( \varphi \) is the fundamental function corresponding to the operator \( p_n(D) \), then the functions \( \varphi, \varphi', ..., \varphi^{(n-1)} \) are linearly independent on every interval \( J \), but they only form an extended Chebyshev system of order \( n \) on intervals where \( p_n(D) \) is disconjugate. This assertion follows from Theorem 4.3 in Karlin and Studden [25, p. 24]. An important consequence of the fact that \( \varphi, \varphi', ..., \varphi^{(n-1)} \) form an extended Chebyshev system on intervals of disconjugacy is given in the following lemma.
LEMMA 1.4.17. Let \((a,b)\) be an interval of disconjugacy for the operator 
\(p_n(D)\), let \(a < x_1 < x_2 < \ldots < x_n < b\) and let \(a_1, a_2, \ldots, a_n\) be real numbers. Then 
there exists a unique function \(f \in \text{Ker}(p_n)\) such that

\[
f(x_1, x_2, \ldots, x_n) = (a_1, a_2, \ldots, a_n) \quad \text{T},
\]

i.e., \(f\) is the Hermite interpolate of the data \(a_1, a_2, \ldots, a_n\) (cf. Section
1.2.6).

PROOF. Since

\[
\begin{vmatrix}
  q & q' & \ldots & q^{(n-1)} \\
  x_1 & x_2 & \ldots & x_n
\end{vmatrix} > 0,
\]

the null function is the unique function interpolating zero data. This guarantees the existence and the uniqueness of a function \(f \in \text{Ker}(p_n)\) with
the required properties. \(\square\)

LEMMA 1.4.18. Let \((a,b)\) be an interval of disconjugacy for the operator 
\(p_n(D)\) and let \(n\) distinct points \(x_1 < x_2 < \ldots < x_n\) be given in \((a,b)\). Let 
\(\varphi\) be the fundamental function of \(p_n(D)\) and define the functions \(\varphi_k\)
\((k = 1, \ldots, n)\) by

\[\varphi_k(t) := \varphi(t-x_k) \quad (t \in \mathbb{R}).\]

Then the functions \(\varphi_1, \ldots, \varphi_n\) form an extended Chebyshev system of order \(n\)
on every interval of disconjugacy for the operator \(p_n(D)\).

PROOF. Only a sketch of the proof will be given. First we show that the functions \(\varphi_1, \ldots, \varphi_n\) form a basis for the space \(\text{Ker}(p_n)\). If not, then there 
would exist constants \(c_1, \ldots, c_n\) with \((c_1, \ldots, c_n) \neq (0,0,\ldots,0)\) such that

\[
\sum_{k=1}^{n} c_k \varphi_k^{(j)}(x_n-x_1) = 0 \quad (j = 0,1,\ldots,n-1),
\]

and we would have

\[
\begin{vmatrix}
  q & q' & \ldots & q^{(n-1)} \\
  x_n-x_{n-1} & x_n-x_1
\end{vmatrix} = 0;
\]
which contradicts the fact that \( \varphi, \varphi'_{1}, \ldots, \varphi_{(n-1)} \) is an extended Chebyshev system on the interval \((0, b-\alpha)\) (cf. Example 1, p. 19).

By a nonsingular linear transformation the basis \( \varphi, \varphi'_{1}, \ldots, \varphi_{(n-1)} \) can be transformed into the basis \( \varphi_{1}, \ldots, \varphi_{n} \). It then suffices to note that nonsingular linear transformations transform Chebyshev systems into Chebyshev systems.

A variant of the above lemma is obtained by dropping the assumption that \( x_{1}, \ldots, x_{n} \) are distinct. If there is a coincident block \( x_{k} = \ldots = x_{k+j-1} < x_{k+j} \) then the functions \( \varphi_{i}, \varphi'_{i+1}, \ldots, \varphi_{i+j-1} \) in (1.4.20) only have to be replaced by

\[
(1.4.21) \quad \varphi_{i+j}^{(k)}(t) = \varphi_{i}^{(k)}(t - x_{i}) \quad (k = 0, 1, \ldots, j-1; \ t \in \mathbb{R}).
\]

Then \( \varphi_{1}, \ldots, \varphi_{n} \) again form an extended chebyshev system of order \( n \) on every interval of disconjugacy.

EXAMPLE 2. Let \( m \) distinct points \( x_{1} < x_{2} < \ldots < x_{m} \) be given and let \( [x_{i}, x_{i+1}] \) be an interval of disconjugacy for the operator \( p_{n}(D) \). The functions \( \varphi_{i} \) \( (i = 1, \ldots, m) \) are defined by \( \varphi_{1}(t) = \varphi_{n}(t - x_{1}) \) \( (t \in \mathbb{R}) \). Then on every interval of disconjugacy the functions \( \varphi_{1}, \ldots, \varphi_{n} \) form an extended weak Chebyshev system of order \( n-1 \).

This assertion is a consequence of a result of Karlin, which for this particular case takes on the following form.

**Lemma 1.4.19 (Karlin [21, p. 503]).** Let \( (a, b) \) be an interval of disconjugacy for the operator \( p_{n}(D) \). Let \( x_{0} < x_{1} < \ldots < x_{m} \) be points located in \( (a, b) \) with \( x_{i} < x_{i+1} \) \( (i = 0, 1, \ldots, m) \). Furthermore, let \( t_{0} < t_{1} < \ldots < t_{m} \) satisfy the conditions

1) \( a \leq t_{0} \leq t_{1} \leq \ldots \leq t_{m} \leq b \),
2) \( t_{i} < t_{i+1} \) \( (i = 0, 1, \ldots, m-1) \),
3) \( t_{i-j} < t_{i} < t_{i+1} < \ldots < t_{i+j-1} < x_{i+j} \) \( \text{for } i, j, k \in \mathbb{N} \) and if \( x_{k} = t_{1} \), then \( i+j \leq n+1 \).

Finally, let the functions \( \varphi_{0}, \ldots, \varphi_{n} \) be defined by \( \varphi_{0}^{(k)} = \varphi_{n}^{(k)}(t - x_{1}) \), \( \varphi_{i}^{(k)} = x_{i+1} - x_{i} \) \( (i = 0, 1, \ldots, n) \).

Then ...
Moreover, for \( n \geq 2 \) strict inequality in (1.4.22) holds if and only if
\[
(1.4.23) \quad x_i < t_i < x_{i+1} \quad (i = 0, 1, \ldots, n),
\]
where \( x_j := \infty \) if \( j > n \).
For \( n = 1 \) strict inequality holds if and only if \( x_i \leq t_i < x_{i+1} \)
\( (i = 0, 1, \ldots, n) \).

If the computation of (1.4.22) requires the \((n-1)\)st or \(n\)th derivative of
\( \varphi_+ \) at zero we define \( \varphi_+^{(n-1)}(0) := \varphi^{(n-1)}(0) \) and \( \varphi_+^{(n)}(0) := \varphi^{(n)}(0) \).

1.4.5. Solvent and unisolvent families of functions

DEFINITION 1.4.20. Let \( \Omega \subset \mathbb{R}^n \), let \( J \) be an interval and let \( R_\Omega(J) \) be a set
of functions \( f(\omega, \cdot) \) indexed by \( \omega \in \Omega \). The set \( R_\Omega(J) \) is called a solvent
family of order \( p \) if
1) \( f(\omega, \cdot) \in C^{p-1}(J) \quad (\omega \in \Omega) \),
2) for any sequence of \( n \) data \( a_1, a_2, \ldots, a_n \) and any sequence of \( n \) points
\( x_0 \neq x_1 \neq \cdots \neq x_{n-1} \) with coincident blocks of length not exceeding \( p \)
there exists a point \( \omega \in \Omega \) such that
\[
R(\omega; x_1, x_2, \ldots, x_n) = (a_1, a_2, \ldots, a_n).
\]

If \( \omega \in \Omega \) in 2) is uniquely determined, then \( R_\Omega(J) \) is said to be a uni-
solvent family of order \( p \).

The above definition is contained in Pinkus [47, p. 69]. An example of a
unisolvent family of order \( p \) is the linear span of an extended Chebyshev
system of order \( p \); an example of a solvent family consisting of a partic-
ular set of \( L \)-splines shall be given in Chapter 7, p. 154.
1.4.6. The number of zeros of an \( L \)-spline function

Rolle's theorem is one of the basic tools in the theory of polynomial interpolation. When counting zeros of \( L \)-splines we have to deal with situations where the given \( L \)-spline may be identically zero on a subinterval without being the null function. Just for that reason zeros are counted according to the rules as given in Definitions 1.4.9 and 1.4.10. We recall that the multiplicity of a zero of a function depends on the choice of the smoothness class in which the function is considered. Given an arbitrary \( L \)-spline \( s \) of order \( n \) defined on a finite interval \([a,b]\), one always has \( s \in \mathcal{C}^0([a,b]) \). Between consecutive knots the function \( s \) coincides with a function in \( \text{Ker}(p_n) \) and since the number of knots in \((a,b)\) is finite, it follows that the total number of strong zeros of \( s \) in \((a,b)\) is also finite.

In what follows we give in particular attention to \( L \)-spline functions with simple knots. To begin with, Corollary 1.4.12 yields the following

**Lemma 1.4.21.** Let \( s \) be an \( L \)-spline function on \([a,b]\) of order \( n \) with simple knots. If the corresponding operator \( p_n(D) \) is disconjugate on \([a,b]\), then (cf. pp. 11-14 for notation)
\[
Z(D^{[n-1]}_1, s, (a,b)) \leq Z(D^{[n-1]}_1, s, (a,b)) + n - 1,
\]
where \( p_n(D) = D_n \cdots D_1 \) and \( D^{[n-1]}_1 = D_{n-1} \cdots D_1 \) (cf. (1.4.9)).

**Proof.** We note that \( D^{[n-1]}_1 \in \mathcal{C}^{(n-1)}([a,b]) \) and thus Corollary 1.4.12 may be applied with \( n \) replaced by \( n-1 \).

The function \( D^{[n-1]}_1 s \) satisfies the equation \( p_n(D) s = D^{[n-1]}_1 s = 0 \) between each two consecutive knots in \((a,b)\), hence (cf. (1.4.2)) \( D^{[n-1]}_1 s(t) = \Phi_n (t) \). As a consequence we have \( Z(D^{[n-1]}_1, s, (a,b)) \leq k \), where \( k \) is the number of knots in \((a,b)\). This observation combined with Lemma 1.4.21 gives the following

**Corollary 1.4.22.** Let \( s \) be an \( L \)-spline on \([a,b]\) of order \( n \) with simple knots. If the corresponding operator \( p_n(D) \) is disconjugate on \([a,b]\), then
\[
Z(D^{[n-1]}_1, s, (a,b)) \leq k + n - 1,
\]
where \( k \) is the number of knots of \( s \) in \((a,b)\).
The Buban-Fourier theorem 1.4.14 applied to $\ell$-splines yields

**Theorem 1.4.23.** Let $s$ be an $\ell$-spline function on $[a,b]$ of order $n$ with simple knots. If the corresponding operator $p_n(D)$ is disconjugate on $[a,b]$ and if $D^{[n-1]}s(a)D^{[n-1]}s(b) \neq 0$, then

$$Z(D_1^{[n-1]}s, (a,b)) < \infty,$$

$$s \leq s^-(D^{[n-1]}s, (a,b)) + s^+(s(a), D^{[1]}s(a), \ldots, D^{[n-1]}s(a)) +$$

$$- s^+(s(b), D^{[1]}s(b), \ldots, D^{[n-1]}s(b)),$$

where $D^{[1]}$ ($i = 1, 2, \ldots, n-1$) is given by (1.4.8).

**Corollary 1.4.24.** Let $s$ be an $\ell$-spline on $[a,b]$ of order $n$ with simple knots. If the corresponding operator $p_n(D)$ is disconjugate on $[a,b]$ and if $D^{[n-1]}s(a)D^{[n-1]}s(b) \neq 0$, then

$$Z((D_1)^n, s, (a,b)) < \infty,$$

$$s \leq s^-(s(a), D^{[1]}s(a), \ldots, D^{[n-1]}s(a)) +$$

$$- s^+(s(b), D^{[1]}s(b), \ldots, D^{[n-1]}s(b)),$$

where $k$ is the number of knots of $s$ in $(a,b)$.

1.4.7. The operator $p^*_n(D)$

The operator $p^*_n(D)$ is defined by

(1.4.24) $p^*_n(D) = (-1)^n p_n(-D)$.

If $\psi$ is the fundamental function corresponding to the operator $p_n(D)$, then evidently $t\rightarrow (-1)^{n-1} \psi(-t)$ is the fundamental function corresponding to the operator $p^*_n(D)$. Furthermore, we note that intervals of disconjugacy for $p_n(D)$ are also intervals of disconjugacy for $p^*_n(D)$, as $(t\rightarrow \mathcal{I}(-t)) \in \ker(p^*_n)$ whenever $f \in \ker(p_n)$.
1.4.8. Divided differences

Let $n+1$ points $x_0 \leq x_1 \leq \ldots \leq x_n$ with $x_0 < x_n$ be given and let $[x_0, x_n]$ be an interval of disconjugacy for the operator $p_n(D)$. Due to Lemma 1.4.17 the set $V \subset \mathbb{R}^{n+1}$ defined by

$$V := \{ f(x_0, x_1, \ldots, x_n) \mid f \in \text{Ker}(p_n) \}$$

has dimension exactly $n$, and therefore there exists a vector $\mathbf{a} \in \mathbb{R}^{n+1}$, uniquely determined up to a multiplicative constant, such that

$$\begin{align*}
\sum_{t=0}^{n} f(x_0, x_1, \ldots, x_n) @^t \mathbf{a} = 0 \quad (f \in \text{Ker}(p_n)).
\end{align*}$$

To ensure the uniqueness of $\mathbf{a}$ a further condition is needed. Hence, we require that

$$\begin{align*}
\sum_{t=0}^{n} g(x_0, x_1, \ldots, x_n) @^t \mathbf{a} = 1,
\end{align*}$$

where $g$ is a function satisfying $p_n(D)g(t) = n! (t \in \mathbb{R})$.

**DEFINITION 1.4.25.** If $\mathbf{a} \in \mathbb{R}^{n+1}$ is the unique vector satisfying (1.4.25) and (1.4.26), then the expression $\sum_{t=0}^{n} f(x_0, x_1, \ldots, x_n) @^t \mathbf{a}$ is called the divided difference of $f$ with respect to the points $x_0, x_1, \ldots, x_n$ and the operator $p_n(D)$; it is denoted by $K[f; x_0, x_1, \ldots, x_n]$.

In the particular case that $p_n(D) = D^n$ we obtain the ordinary divided difference common in numerical analysis and for this the notation $f(x_0, x_1, \ldots, x_n)$ will be used.

It follows from (1.4.25) and (1.4.26) that

$$f[x_0, x_1, \ldots, x_n] = \begin{vmatrix}
\phi & \phi' & \ldots & \phi^{(n-1)} & f \\
x_0 & x_1 & \ldots & x_{n-1} & x_n \\
\phi & \phi' & \ldots & \phi^{(n-1)} & g \\
\phi_0 & x_1 & \ldots & x_{n-1} & x_n
\end{vmatrix},$$

If $p_n(D) = D^n$ and $x_0 < x_1 < \ldots < x_n$, then formula (1.4.27) reduces to the well-known formula (cf. Davis [16, p. 40])

$$f[x_0, x_1, \ldots, x_n] = \sum_{j=0}^{n} \frac{D}{D^j} f(x_j),$$

where $D$ is the shifted $n$th divided difference operator.
where \( W(t) := (t-x_0) \cdots (t-x_n) \) \((t \in \mathbb{R})\).

Writing \( \mathbf{a} = (a_0, a_1, \ldots, a_n)^T \), we may compute the numbers \( a_j \) from formula (1.4.27) by expanding the determinant in the numerator of (1.4.27) with respect to its last column. One then obtains

\[
\begin{vmatrix}
\varphi & \varphi' & \cdots & \varphi^{(n-1)} \\
 x_0 & x_1 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_n \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 x_0 & x_1 & \cdots & x_{i-1} & x_{n-1} & x_n \\
 \end{vmatrix}
\]

\[a_i = (-1)^{n+1} \begin{vmatrix}
\varphi & \varphi' & \cdots & \varphi^{(n-1)} \\
 x_0 & x_1 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_n \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 x_0 & x_1 & \cdots & x_{i-1} & x_{n-1} & x_n \\
 \end{vmatrix} \frac{v(n)}{g}.
\]

According to Example 1 on p. 19, the functions \( \varphi, \varphi', \ldots, \varphi^{(n-1)} \) form an extended Chebyshev system of order \( n \) on \([x_0, x_n]\). This implies that the determinant in the numerator of (1.4.29) has constant sign, hence

\[S(a_0, a_1, \ldots, a_n) = n.\]
2. THE B-SPLINE FUNCTIONS

2.1. Introduction and summary

The subject of this chapter is the class of the so-called basic $\mathcal{L}$-spline functions, B-spline functions or B-splines for short. This name finds its origin in the fact that for some spaces of $\mathcal{L}$-splines these functions form a basis; e.g. in the case of polynomial splines the well-known polynomial B-splines form a basis. For various properties of polynomial B-splines the reader is referred to Lectures 1 and 2 of Schoenberg's monograph [51]. Fundamental properties of B-splines corresponding to a general operator $p_n(D)$ are studied in a paper by Schmidt, Lancaster and Watkins [50].

For the sake of completeness, in Section 2.2 we prove some fundamental properties of B-splines. Section 2.3 is concerned with the so-called total positivity property of a consecutive sequence of B-splines. Applying our version of the Budan-Fourier theorem to $\mathcal{L}$-splines (Theorem 1.4.25) we give a rather simple proof of this property. Using the Fourier transform in Section 2.4, we derive various recurrence relations for B-splines.

In the final Section 2.5 one of these recurrence relations is used to investigate the behaviour of the supremum norm of the polynomial B-spline of order $n$ as a function of its knot distribution. The optimal distribution of the knots, i.e., the distribution for which the supremum norm is minimal, is determined; this answers a question of Meinardus [34, p. 174].

2.2. Fundamentals about B-splines

2.2.1. Definition of a B-spline function

The B-spline will be defined by using the concept of divided differences as given in Definition 1.4.35. In order to obtain B-splines corresponding to the operator $p_n(D)$ we need divided differences with respect to the operator $p_n(D)$ (cf. Subsection 1.4.7).
Let \( x_0 \leq x_1 \leq \ldots \leq x_n \) be a sequence of points with \( x_0 < x_n \) and let \([x_0,x_n]\) be an interval of disconjugacy for the operator \( p^*_{n}(D) \). Since 
\[
\mathfrak{I}[p^*_{n}(D; x_0,x_1,\ldots,x_n)] = 0 \text{ for all } f \in \text{Ker}(p^*_{n}),
\]
it follows from Lemma 1.4.5 (Peano's remainder formula) that a function \( K \) exists such that
\[
(2.2.1) \quad K(t)p^*_{n}(D)\xi(t) = \int_{x_0}^{x_n} K(r)p^*_{n}(D)\xi(r) \, dr \quad (f \in \mathcal{C}^{(n)}([x_0,x_n])).
\]
The kernel \( K \) is given by \( K(t) := f_0[p^*_{n}; x_0,x_1,\ldots,x_n] \), where \( f_0 \) is defined by \( f_0(t) := \varphi^*_n(t-t) \) and \( \varphi^*_n \) is the Green's function corresponding to the operator \( p^*_{n}(D) \) (cf. Definition 1.3.3).
The function \( \varphi^*_n \) can be expressed in terms of \( \varphi_n \) and \( \varphi^* \) by means of
\[
(2.2.2) \quad \varphi^*_n(t) + (-1)^{n+1} \varphi_n(t) = \varphi^*(t) \quad (t \in \mathbb{R}).
\]
Since \( \mathfrak{I}[p^*_{n}; x_0,x_1,\ldots,x_n] = 0 \) it follows that \( K(t) = f_1[p^*_{n}; x_0,x_1,\ldots,x_n] \) with \( f_1(t) := (-1)^n \varphi_n(t-t) \).

Using these preliminaries we arrive at the following definition.

**Definition 2.2.1.** Let \( x_0 \leq x_1 \leq \ldots \leq x_n \) be a sequence of points with \( x_0 < x_n \) and let \([x_0,x_n]\) be an interval of disconjugacy for the operator \( p^*_{n}(D) \). The \( m \)-spline function \( M[p^*_{n}; x_0,x_1,\ldots,x_n] \) of order \( n \) corresponding to the operator \( p^*_{n}(D) \) and having knots \( x_0, x_1, \ldots, x_n \) is for \( t \in \mathbb{R} \) defined by
\[
(2.2.3) \quad M[p^*_{n}; x_0,x_1,\ldots,x_n; t] := n!(-1)^n \varphi^*_n(t-t)[p^*_{n}; x_0,x_1,\ldots,x_n].
\]

On account of this definition and Formula (1.4.27) it is easily verified (cf. (1.3.1)) that \( M[p^*_{n}; x_0,x_1,\ldots,x_n] \) is indeed an \( n \)-spline function of order \( n \) corresponding to the operator \( p^*_{n}(D) \) and having the knots \( x_0, x_1, \ldots, x_n \).

In accordance with Definition 2.2.1, the \( m \)-spline corresponding to the operator \( p^*_{n}(D) \) and having knots \( x_j, x_{j+1}, \ldots, x_{j+n} \) will be denoted by \( M[p^*_{n}; x_j, x_{j+1}, \ldots, x_{j+n}] \). As \( m \)-splines will be heavily used in this chapter, it is convenient to have other notations available that are more compact. We define
\[
M[p^*_{n}; x_j, x_{j+1}, \ldots, x_{j+n}] := M[p^*_{n}; x_0, x_1, \ldots, x_n].
\]

And in case of polynomial \( m \)-splines of order \( n \) we introduce the notation
\[ M_n,j := M(\mathbb{B}^n; x_j, x_{j+1}, \ldots, x_{j+n}) . \]

The B-spline \( M_n(p_n) \) with equally spaced knots \( x_i = nh \) \((i = 0, 1, \ldots, n)\)
where \( h \) is a positive number is denoted by \( M_n(p_n) \), and in the case of polynomial B-splines of order \( n \) we introduce the notation
\[ M_n := M(\mathbb{B}^n; 0, h, \ldots, nh) . \]
Obviously one has
\[ M(p_n) = m_{n,h} \phi_{n,h}(t) \]
if
\[ p_n(\mathbb{B}) = \mathbb{B}(\mathbb{B}^2 + w \mathbb{I}) \cdots (\mathbb{B}^2 + n w \mathbb{I}) \quad (n = 2m + 1 ; \ w \neq 0) \]
or
\[ p_n(\mathbb{B}) = \mathbb{B}(\mathbb{B}^2 + \frac{1}{4} w \mathbb{I}) \cdots (\mathbb{B}^2 + \frac{1}{4} w \mathbb{I}) \cdots (\mathbb{B}^2 + \frac{2m-1}{2} w \mathbb{I}) \quad (n = 2m ; \ w \neq 0) , \]
the corresponding B-splines are called **trigonometric B-splines**.

If
\[ p_n(\mathbb{B}) = \mathbb{B}(\mathbb{B}^2 - w \mathbb{I}) \cdots (\mathbb{B}^2 - n w \mathbb{I}) \quad (n = 2m + 1 ; \ w \neq 0) \]
or
\[ p_n(\mathbb{B}) = \mathbb{B}(\mathbb{B}^2 - \frac{1}{4} w \mathbb{I}) \cdots (\mathbb{B}^2 - \frac{1}{4} w \mathbb{I}) \cdots (\mathbb{B}^2 - \frac{2m-1}{2} w \mathbb{I}) \quad (n = 2m ; \ w \neq 0) , \]
the corresponding B-splines are called **hyperbolic B-splines**.

In the case of polynomial B-splines of order \( n \) we have \( p_n(\mathbb{B}) = \mathbb{B}^n \) and (cf. definition 1.3.3) \( q(t) = \frac{1}{(n-1)!} t^{n-1} \), hence

\[ (2.2.4) \quad x_n(t) = (-1)^n n(t-\alpha)^{n-1} \left[ x_0, x_1, \ldots, x_n \right] . \]

Furthermore, if \( x_0 < x_1 < \ldots < x_n \) then according to (1.4.28) Formula (2.2.4) may be written as

\[ (2.2.5) \quad x_n(t) = (-1)^n n \sum_{j=0}^{n-1} \frac{1}{W(x_j)} (t-x_j)^{n-1} , \]

where \( W(t) := (t-x_0) \cdots (t-x_i) \). This representation precisely agrees with Formula 2.7 in Schoenberg [51, p. 2].
2.2.2. Some general properties of B-splines

We begin this subsection with a representation formula for \( N_0(p_n,t) \). For that purpose the distinct points among the knots \( \kappa_0 < \kappa_1 < \ldots < \kappa_n \) are denoted by \( \gamma_0 < \gamma_1 < \ldots < \gamma_{n'} \), their multiplicities by \( \nu_0, \nu_1, \ldots, \nu_{n'} \), respectively. It then follows from (2.2.3), (1.4.29), and (1.4.30) that

\[
(2.2.6) \quad N_0(p_n,t) = \sum_{k=0}^{n'} \sum_{j=0}^{n-k} w_k \nu_j \delta^{(j)}(t - \gamma_k) \quad (t \in \mathbb{R}),
\]

where the coefficients \( w_k \) when ordered as \( w_0, w_1, \ldots, \nu_0, \nu_1, \ldots \) alternate in sign and are uniquely determined.

In Schoenberg\[51, pp. 2.3\] it is shown that the polynomial B-spline \( N_{n,0} \) has the properties \( N_{n,0}(t) > 0 \) if \( t < \kappa_0 \) or \( t > \kappa_n \), \( N_{n,0}(t) = 0 \) if \( t < \kappa_0 \) or \( t > \kappa_n \), and \( \int_{\kappa_0}^{\kappa_n} N_{n,0}(t) dt = 1 \). In what follows we prove that the B-spline \( N_0(p_n,t) \) has the same properties.

As an immediate consequence of (2.2.1), (2.2.2), and (2.2.3) B-splines satisfy the relation

\[
(2.2.7) \quad \delta(p_n^{*}p_n, x_0, \ldots, x_n) = \frac{1}{n!} \int_{x_0}^{x_n} N_0(p_n,t)p_n^{*}(t) \delta(t) dt.
\]

Applying (2.2.7) to a function \( g \) for which \( p_n^{*}(t)g(t) = n! \quad (t \in \mathbb{R}) \) and using (1.4.26) we obtain

\[
(2.2.8) \quad \int_{x_0}^{x_n} N_0(p_n,t) dt = 1.
\]

As \( \delta^*(\tau - t) = 0 \quad (\tau \geq t), \quad \delta^*(\tau - t) = \delta(t - \tau) = (-1)^{n-1} \delta^*(t - \tau) \) and \( \delta^*(\tau - t) p_n^{*}(x_0, x_1, \ldots, x_n) = 0 \), Formula (2.2.3) implies that

\[
(2.2.9) \quad N_0(p_n, t) = 0 \quad (t / \in (x_0, x_n))
\]

Thus a B-spline has compact support, and in view of (2.2.8)

\[
\int_{-\infty}^{\infty} N_0(p_n,t) dt = 1.
\]
We proceed by showing that B-splines are positive on their supports, i.e.,
that
\[ (2.2.10) \quad M_0(p; t) > 0 \quad (t \in (x_0, x_n)) . \]

In the case of simple knots this can be established by using the generalized
Duhamel-Fourier theorem 1.4.23 for f-spline functions (cf. Lemma 2.3.1, p. 34); however in the case of multiple knots we need Lemma 1.4.19 to derive
(2.2.10). A proof of (2.2.10) for the general case will now be given.

PROOF. Since \( p_n(D) \) is disconjugate on \([x_0, x_n]\) and since the maximal intervals of disconjugacy are open, a number \( b > x_n \) exists such that \( p_n(D) \) is disconjugate on \([x_0, b]\). Now assume that \( M_0(p; t_0) = 0 \) for some \( t_0 \in (x_0, x_n) \).

Choose \( n \) arbitrary points \( t_1 < t_2 < \ldots < t_n \) in \((x_0, b)\) and define \( x_{i+1} := b \]
\( (i = 1, 2, \ldots, n) \). Then \( M_0(p; t_i) = 0 \) \( (i = 0, 1, \ldots, n) \). In view of (2.2.6) we have
\[ (2.2.11) \quad \begin{vmatrix} t_0 & t_1 & \ldots & t_n \\ \phi_0 & \phi_1 & \ldots & \phi_n \end{vmatrix} = 0 , \]

where the functions \( \phi_i \) are defined as in Lemma 1.4.19. However, since
\( x_i < t_1 < x_{i+1} \) \( (i = 0, 1, \ldots, n) \) it follows from Lemma 1.4.19 that the
determinant in (2.2.11) is positive; this contradiction proves that \( M_0(p; \cdot) \)
has no zeros in \((x_0, x_n)\). It then follows from (2.2.8) and the continuity of
\( M_0(p; \cdot) \) on \((x_0, x_n)\) that \( M_0(p; t) > 0 \) on \((x_0, x_n)\).

\[ \square \]

REMARK. From (2.2.8), (2.2.9), and (2.2.10) it follows that a B-spline is a probability density on \((x_0, x_n)\). For probabilistic interpretations in the
polynomial case we refer to Feller [17, p. 26].

2.2.3. The basic property of B-splines

Whereas formula (1.3.1) represents an f-spline in terms of the Green's
function \( \phi_\alpha \), one may also show that B-splines can be used to represent an
arbitrary f-spline. This property of B-splines is the content of the following

THEOREM 2.2.2. Let the sequence of knots \((x_\nu)_{\nu=0}^{\infty}\) satisfy \( x_0 < x_{\nu+1} \),
\( x_\nu < x_{\nu+1} \) \( (\nu \in \mathbb{N}) \), \( \lim_{\nu \to \infty} x_\nu = -\infty \) and \( \lim_{\nu \to -\infty} x_\nu = +\infty \). Furthermore, let the
operator $p_n(0)$ be discontinuous on each interval $[x_\nu, x_{\nu+1}]$ $(\nu \in \mathbb{Z})$. Then for every $L$-spline $s$ corresponding to the operator $p_n(0)$ and having knots $(x_\nu)_{\nu=-\infty}^{\infty}$ there exists a unique sequence $(c_\nu)_{\nu=-\infty}^{\infty}$ of real numbers such that

$$(2.2.12) \quad s(t) = \sum_{\nu=-\infty}^{\infty} c_\nu M_\nu(p_n; t) \quad (t \in \mathbb{R}).$$

**Proof.** Without loss of generality we may assume that $x_0 < x_1$. We first show that the $L$-spline $M_\nu(p_n; t)$ $(\nu = 1, \ldots, r)$, when restricted to the interval $[x_0, x_1)$ form a basis for $\ker(p_n)$. For this it is sufficient to prove that these $n$ functions are linearly independent on $(x_0, x_1)$. For this purpose let us assume that a linear combination

$$(2.2.13) \quad f(t) = \sum_{\nu=1}^{r} \lambda_\nu M_\nu(p_n; t)$$

vanishes identically on $(x_0, x_1)$. The function $f$ is an $L$-spline with knots $x_{1-n}, x_{2-n}, \ldots, x_0, x_1, \ldots, x_r$. According to the representation (1.3.1) for $f$ in terms of $q_s$, there exists a vector $d \in \mathbb{R}^n$ such that

$$f(t) = d^T \alpha_\nu(t-x_{1-n}, t-x_{2-n}, \ldots, t-x_0) \quad (x_{1-n} \leq t < x_1).$$

Since $f(t) = d^T \alpha_\nu(t-x_{1-n}, t-x_{2-n}, \ldots, t-x_0) = 0$ $(x_0 < t < x_1)$ it follows from Lemma 1.4.17 that $d = 0$. Hence $f(t) = 0$ for all $t \in [x_{1-n}, x_1]$. Formula (2.2.13) together with (2.2.9) implies that one successively has $\lambda_1 = \lambda_2 = \ldots = \lambda_r = \lambda_0 = 0$.

Consequently, there exists a unique sequence of numbers $c_1, c_2, \ldots, c_r$ such that

$$s(t) = s_0(t) := \sum_{\nu=1}^{r} c_\nu M_\nu(p_n; t) \quad (x_0 \leq t < x_1)$$

Hence $s = s_0$ is an $L$-spline with knots $(x_\nu)_{\nu=-\infty}^{\infty}$, which is identically zero on $(x_0, x_1)$, and thus $s - s_0$ can be written uniquely as $s - s_0 = s_1 + s_2$, where $s_1$ and $s_2$ vanish on $(-\infty, x_0)$ and $(x_0, x_1)$, respectively. If $x_1$ has multiplicity $1$, then by (1.3.1) $s_1$ is a linear combination of the functions $t \mapsto \alpha_\nu(t-x_1)$ $(\nu = 0, 1, \ldots, r-1)$ on the interval $(x_0, x_1)$ and thus in view of (2.2.6) $s_1$ can be written uniquely as a linear combination of the functions $M_\nu(p_n; t)$ $(\nu = 1, 2, \ldots, r)$. Repeated application of this argument leads to the unique representation.
\[ z_1(t) = \sum_{\nu=1}^{w} c_{\nu} N_{\nu}(p_{n}^{*}t) \quad (t \in \mathbb{R}) . \]

In a similar way it can be shown that \( z_2 \) can be represented uniquely as

\[ z_2(t) = \sum_{\nu=1}^{w} c_{\nu} M_{\nu}(p_{n}^{*}t) \quad (t \in \mathbb{R}) . \]

As \( s = s_0 + s_1 + s_2 \) this establishes the theorem.

\[ \square \]

2.3. The total positivity property of \( B \)-splines

Given an operator \( p_n(D) \) that is disconjugate on \( \mathbb{R} \) and given a sequence of simple knots \( (\kappa_{\nu})_{\nu=0}^{w} \), a result of Karlin [21, Theorem 4.1, p. 527] implies that the corresponding sequence of \( B \)-splines \( (M_{\nu}(p_{n}^{*}))_{\nu=0}^{w} \) (cf. p. 28 for notation) has the property

\[
\begin{vmatrix}
  \zeta_0 & \zeta_1 & \ldots & \zeta_w \\
  0 & M_{0}(p_{n}^{*}) & \ldots & M_{w}(p_{n}^{*}) \\
  \cdots & \cdots & \cdots & \cdots 
\end{vmatrix} \geq 0
\]

(2.3.1)

for all \( \kappa_0 < \kappa_1 < \ldots < \kappa_w \), \( \forall \nu \), \( \forall \nu_0 < \nu_1 < \ldots < \nu_w \). We recall that the determinant in (2.3.1) is defined in Subsection 1.2.6.

Property (2.3.1) is known as the total positivity property of the sequence of \( B \)-splines \( (M_{\nu}(p_{n}^{*}))_{\nu=0}^{w} \).

An important consequence of the total positivity property of the sequence \( (M_{\nu}(p_{n}^{*}))_{\nu=0}^{w} \) is contained in Theorem 1.3 (cf. Karlin [21, p. 531]), which can be stated as follows: if an \( L \)-spline \( s \) can be represented in the form \( s(t) = \sum_{\nu=0}^{w} c_{\nu} M_{\nu}(p_{n}^{*}t) \), then \( S^-(s, \mathbb{R}) \) is \( S^-((c_{\nu})_{\nu=0}^{w}) \); this is known as the variation diminishing property of \( B \)-splines.

With respect to the required disconjugacy condition for the operator \( p_n(D) \) we recall (cf. Section 1.4) that \( p_n(D) \) is disconjugate on \( \mathbb{R} \) if and only if all zeros of \( p_n \) are real. In view of this it follows that in particular the sequence of polynomial \( B \)-splines \( (M_{\nu}, \mathbb{V}_{\nu=0}^{w}) \) with simple knots is totally positive.

Using Theorem 1.4.23 (the Budan-Fourier theorem for \( L \)-splines) we shall give a rather simple proof of (2.3.1) under the stronger condition that
Let \( x_0 < x_{v+1} < \ldots < x_{v+n} \) be such that \([x_0, x_{v+n}]\) is an interval of discontinuity for the operator \( p^n \). Then for every sequence of real numbers \( x_1, x_2, \ldots, x_r \) the number of strong zeros of \( \sum_{j=0}^{r} a_j M_{v+j}(p^n) \), counting multiplicities according to Definition 1.4.10, is at most \( r \), i.e. (cf. pp. 14 for notation)

\[
\sum_{j=0}^{r} a_j M_{v+j}(p^n) \leq r,
\]

where \( p^n(1) = P_n \ldots D_1 \) (cf. (1.4.1), (1.4.2)).

PROOF. The proof will be given by induction with respect to the variable \( r \).

Let \( f_0 = x_0(p^n) \) and for \( r \geq 1 \) let \( f_r \) be defined by \( f_r = \sum_{j=0}^{r} a_j M_{v+j}(p^n) \).

Inequality (2.3.2) in the case \( r = 0 \) asserts that \( M_{v}(p^n) \) has no strong zeros in \([x_0, x_{v+n}]\). Using Lemma 1.4.19 we have shown this to be true (cf. (2.2.10) for knots with arbitrary multiplicity. However, in the case of simple knots the Budan-Fourier theorem for \( \epsilon^- \)-splines can be used to prove that \( f_0 > 0 \) on \([x_0, x_{v+n}]\), since then \( f_0 \in P_{v+n-1}([x_0, x_{v+n}]) \). It follows from (2.2.6) that \( f_0 \) can be written in the form

\[
f_0(t) = \sum_{k=0}^{n} \omega_k a_k (t-x_{v+k}) \quad (t \in \mathbb{R}),
\]

where

\[
S_0(\omega_0, \omega_1, \ldots, \omega_n) = n.
\]

In view of Definitions 1.3.4 and 2.2.1 one has

\[
D^{[1]} f_0 (x_0) = D^{[1]} f_0 (x_{v+n}) = 0 \quad (i = 0, 1, \ldots, n-2).
\]

Taking into account Definition 1.3.3 and (2.3.3) together with (2.3.4), we conclude that \( D^{[n-1]} f_0 (x_0) = \omega_0 \neq 0 \). As \( \mathbb{R}^{n-1} \omega_0 \mathbb{Q}(t-x_{v+k}) = 0 \) \((t \in \mathbb{R})\) it follows that

\[
D^{[n-1]} f_0 (x_{v+n}) = \sum_{k=0}^{n-1} \omega_k g^{(n-1)} (x_{v+n} - x_{v+k}) = \omega_n \neq 0,
\]

the last inequality again being a consequence of (2.3.4). Applying Theorem
1.4.23 we get
\[ \mathcal{Z}(D_r^{n-1}, r_f_0, \{x_{r_0}, x_{r_0+n}\}) \leq G_{n-1}^{n-1} E_0, (x_{r_0}, x_{r_0+n}) + 0 - (n-1) \leq \]
\[ \leq n - 1 - (n-1) = 0 , \]
as an upper bound for \( G_{n-1}^{n-1} E_0, (x_{r_0}, x_{r_0+n}) \) is given by the number of knots in \( (x_{r_0}, x_{r_0+n}) \) (cf. Corollary 1.4.24). This proves (2.3.2) for \( r = 0 \). We proceed by an induction argument with respect to \( r \). In view of (2.2.9), \( F_r \) is identically zero outside the interval \([x_{r0}, x_{r0+n}]\). If \( x_0 = 0 \) or \( x_r = 0 \), then by the induction hypothesis we may conclude that
\[ \mathcal{Z}(D_r^{n-1}, r_f_0, \{x_{r0}, x_{r0+n}\}) \leq n + r - 1 + 0 - (n-1) = r . \]
This completely proves the lemma.

Next we shall prove (2.3.1) under the assumption that \((v_0, v_1, \ldots, v_r)\) is a sequence of consecutive integers \((v_0, v_1, \ldots, v_r)\).

**Theorem 2.3.2.** Let \( p_n \in \mathbb{Z}_n \) \( (n \geq 2) \) be a monic polynomial having only real zeros and let \((x_v)_{v=1}^r\) be a sequence of simple knots. Then for every \( r \in \mathbb{Z}_n \) and for every sequence of numbers \( \xi_0, \ldots, \xi_r \) with \( \xi_0 < \xi_1 < \cdots < \xi_r \) and every \( v \in \mathbb{Z}_n \)
\begin{equation}
M_v(p_{n-1}) M_{v+1}(p_{n-1}) \cdots M_{v+r}(p_{n+1})
\begin{bmatrix}
\xi_0 & \xi_1 & \cdots & \xi_r
\end{bmatrix}
\geq 0 .
\end{equation}

Moreover, strict inequality in (2.3.5) prevails if and only if
\begin{equation}
x_v \xi_0 < \xi_1 < x_v \xi_0 + n \quad (1 = 0, 1, \ldots, r) .
\end{equation}

**Proof.** As a preliminary remark we want to emphasize that throughout the proof Formula (2.2.9), i.e., \( M(v_r, t) = 0 \) for \( t \neq (x_v, x_{v+n}) \), plays a crucial role.

If for some \( i_0 \in \{0, 1, \ldots, r\} \) (2.3.6) does not hold, then either \( x_{i_0} \leq x_{i_0+n} \) or \( x_{i_0} > x_{i_0+n} \) and thus in view of (2.2.9)
\[ M_{j+1} (p_n; t_i) = M_j (p_n; t_i) = \cdots = M_1 (p_n; t_i) = 0 \quad (j = 0, 1, \ldots, l_0) \]

or

\[ M_{j+1} (p_n; t_i) = M_j (p_n; t_i) = \cdots = M_1 (p_n; t_i) = 0 \quad (j = l_0, l_0 + 1, \ldots, r) . \]

Therefore, if \( l_0 \leq r \), it is easily seen that the first \( l_0 + 1 \) columns in the determinant of (2.3.5) are dependent. Hence the determinant is zero; in the other case a similar argument applies. Consequently, (2.3.5) is established if (2.3.6) does not hold.

There remains to be proved that the determinant is positive if (2.3.6) holds. This will be done by induction with respect to the variable \( r \).

If \( r = 0 \) then (2.3.5) and (2.3.6) assert that the \( B \)-spline \( M_j (p_n; \cdot) \) is positive on \( (x_{u}, x_{u+1}) \), which is obviously true in view of (2.2.10).

We proceed by defining the function \( f \) as

\[
(2.3.7) \quad f(t) := \begin{vmatrix}
M_0 (p_n; t) & \cdots & M_{l_0} (p_n; t) \\
M_0 (p_n; t_i) & \cdots & M_{l_0} (p_n; t_i) \\
\vdots & \ddots & \vdots \\
M_0 (p_n; t_{l_0}) & \cdots & M_{l_0} (p_n; t_{l_0}) 
\end{vmatrix} \quad (t \in \mathbb{R}) ,
\]

where (cf. (2.3.6)) \( \xi_{l_0+1} < \xi_i < \xi_{l_0+n} \quad (i = 1, 2, \ldots, r) \).

In view of this definition and (2.3.5) one has to prove that \( f(t_0) > 0 \) for all \( t_0 < \xi_i \) and \( t_0 > (x_{u}, x_{u+n}) \). It follows from (2.3.7) that \( f(t) = \Sigma_{j=0}^{l_0} a_j M_j (p_n; t) \) and \( f(t_i) = 0 \quad (i = 1, 2, \ldots, r) \). Now suppose that also \( f(t_0) = 0 \). As \( p_n \) has only real zeros, it follows that \( p_n (D) \) is disconjugate on \( N \). Hence Lemma 2.3.1 is applicable and one has

\[ \Sigma_{j=1}^{l_0} f_j (x_{u}, x_{u+n+2}) \leq 0 . \]

Consequently, at least one of the zeros \( \xi_{l_0}, \xi_{l_0+2} \) is not strong. Let \( \xi_{l_1} \) be the smallest zero among \( \xi_{l_0}, \xi_{l_0+2} \), which is not strong. Then by the definition of a strong zero (cf. p. 12) \( f \) vanishes identically between two consecutive knots \( x_{k-1} \) and \( x_{k} \) with \( k \leq l_{l_1} + (x_{k-1}, x_{k}) \), since \( f \) is an \( E \)-spline function. It follows from the proof of Lemma 2.1.2 that for \( t \in (-\infty, x_{l_1}) \)

\[
(2.3.8) \quad f(t) = \begin{cases}
\frac{k-n-1}{l_0} a_j M_j (p_n; t) & (k \geq n+1) \\
0 & (k < n+1) 
\end{cases}
\]
If \(k \geq n+1\), then in view of (2.3.2) one has \(z(\mathcal{D}_n)_{k-n-1} f_n(x, x) \leq k-n-1\). Since, moreover, \(x_1 < x_{n+1-n} < x_1 < x_{n+1-n+1}\) and \(x \in [x_{n+1-n+1}, x_{n+1}]\), one has \(k-n-1 \geq 1-n-1 = k-1\), and again one has to conclude that at least one of the zeros \(\zeta, \zeta_1, \ldots, \zeta_{k-1}\) is not strong. This, however, contradicts the assumption that \(\zeta_1\) is the smallest zero among \(\zeta, \zeta_1, \ldots, \zeta_{k-1}\) which is not strong, and thus \(k \geq n+1\) does not hold. Consequently, \(k < n+1\) and therefore by (2.3.8) \(f\) vanishes identically on \((-\infty, x]\). Since \(\ell_{1} \in (x_{n+1-n}, x_{n+1-n+1})\) and \(\ell_{k} \in [x_{n+1-n+1}, x_{n+1}]\) one has \(k-1 \geq 1-n+1 = n\). Hence \(k \geq n\) and \(f\) vanishes identically on \((x, x_{n+1})\). However, taking into account (2.3.8) and (2.2.9), on \((x, x_{n+1})\) we may write

\[
f(t) = \begin{vmatrix}
M_{v+1}(p_{v+1}) & \cdots & M_{v+1}(p_{n+1}) \\
\xi_1 & \cdots & \xi_k
\end{vmatrix},
\]

which by the induction hypothesis and (2.2.10) is strictly positive on \((x, x_{n+1})\). This contradicts the fact that \(f\) is identically zero on \((x, x_{n+1})\). So \(f(t) \neq 0\) and it remains to prove that \(f(t) > 0\). This can be done as follows. Since \(n \geq 2\) and all knots are simple the B-splines \(M_{v}(p_{n-i})\) \((v \in \mathbb{N})\) are continuous on \(\mathbb{R}\). Therefore, the determinant \(\Delta = (\ell_{0}, \ell_{1}, \ldots, \ell_{k})^T \in \mathbb{R}^{n+1}\), and so it does not change sign on the open and connected set \((x_{i}, x_{i+1})\) \((i = 0, 1, \ldots, \ell_{k})\). By taking the specific configuration \(\ell_{0} < \ell_{1} < \cdots < \ell_{k}\), we see that (2.3.5) is equal to \(\Delta_{0} \in \mathbb{R}_{++}^{n+1} \prod_{v=0}^{n} M_{v}(p_{n-i}) > 0\).

REMARK 1. If \(n = 1\), then a straightforward calculation shows that for all \(v \in \mathbb{N}\)

\[
\begin{vmatrix}
M_{v}(p_{1}) & M_{v+1}(p_{1}) & \cdots & M_{v+1}(p_{n}) \\
\ell_{0} & \ell_{1} & \cdots & \ell_{k}
\end{vmatrix} > 0
\]

for all \(\ell_{0} < \ell_{1} < \cdots < \ell_{k}\) with strict inequality if and only if \(x_{i+1} \geq \xi_{i} < x_{i+1}\) \((i = 0, 1, \ldots, k)\).

REMARK 2. If not all zeros of \(p_{n}\) are real then an additional condition for the knot sequence \(x_{i}\) is needed in order to obtain (2.3.5). We note that for \(r = 0\) Theorem 2.3.2 reduces to property (2.2.10). There we made the assumption that \(p_{n}(x)\) is disconjugate on \([x_{n}, x_{n+1}]\). It is an open
problem whether the assumption that \( p_{n}(D) \) is disconjugate on \([x_{0},x_{v+n}]\)
\((v \in \mathbb{Z})\) guarantees (2.3.5) for all \( r \geq 1 \).

2.4. On recurrence relations for B-splines

2.4.1. The Fourier transform of a B-spline

given the B-spline \( M_{v}(p_{n}^{*}) \) with knots \( x_{0}, x_{v+1}, \ldots, x_{v+n} \), its Fourier transform is defined as

\[
\phi_{n,v}(z) := \int_{-\infty}^{\infty} M_{v}(p_{n}^{*})e^{2\pi i z t} dt \quad (z \in \mathbb{C}) .
\]

In view of (2.2.8) one obviously has

\[
\phi_{n,v}(0) = 1 .
\]

Lemma 2.4.1. Let \( \phi_{n,v} \) be the Fourier transform of \( M_{v}(p_{n}^{*}) \), then the function \( \mathfrak{h}_{n,v} \) defined by

\[
\mathfrak{h}_{n,v}(z) := p_{n}(z) \phi_{n,v}(z) \quad (z \in \mathbb{C})
\]

has the property

\[
(\mathbb{D} + x_{0}I)(\mathbb{D} + x_{v+1}I) \cdots (\mathbb{D} + x_{v+n}I)\mathfrak{h}_{n,v}(z) = 0 \quad (z \in \mathbb{C}) .
\]

Proof. As a consequence of (2.4.1) one has

\[
\mathfrak{h}_{n,v}(z) = (-1)^{n} \int_{-\infty}^{\infty} M_{v}(p_{n}^{*})p_{n}(z) e^{2\pi i z t} dt
\]

and thus in view of (2.2.7) and (1.4.27) we conclude that (2.4.4) holds. \( \square \)

Remark. For polynomial B-splines (2.4.3) and (2.4.2) imply that \( \mathfrak{h}_{n,v}^{(k)}(0) = 0 \)
\((k = 0, 1, \ldots, n-1)\), \( \mathfrak{h}_{n,v}^{(n)}(0) = n! \). Hence, as a consequence of (2.4.4),
\( \mathfrak{h}_{n,v}^{(n)} \) is the fundamental function corresponding to the operator

\( (\mathbb{D} + x_{0}I)(\mathbb{D} + x_{v+1}I) \cdots (\mathbb{D} + x_{v+n}I) \) (cf. Cox [15]; Ter Morshede [42, p. 5]).
2.4.2. Some recurrence relations for B-splines

The first of a series of recurrence relations for B-splines is contained in the following

**Lemma 2.4.2.** Let \( x_0 < x_1 < \ldots < x_{n+1} \) be a sequence of knots with \( x_0 < x_n \) and \( x_1 < x_{n+1} \). Let \( \lambda \in \mathbb{R} \) be such that \( p_{n+1}(D) \) and \( p_n(D) \) satisfy the conditions

i) \( p_{n+1}(D) = (D - \lambda I)p_n(D) \),

ii) \( p_n(D) \) is disconjugate on both \([x_0, x_n]\) and \([x_1, x_{n+1}]\), and \( p_{n+1}(D) \) is disconjugate on \([x_0, x_{n+1}]\).

Then one has the recurrence relation

\[
(2.4.5) \quad (D - \lambda I)M_0(p_{n+1}; t) = \alpha M_1(p_n; t) - (\alpha + \lambda)M_0(p_n; t) \quad (t \in \mathbb{R}),
\]

where

\[
(2.4.6) \quad \alpha = \begin{cases} 
\lambda \Phi_{n,0}^{(\lambda)}(0) / \Phi_{n,1}^{(\lambda)}(0) - \Phi_{n,0}^{(\lambda)}(0)^{-1} & (\lambda \neq 0), \\
(\Phi_{n,1}^{(0)}(0) - \Phi_{n,0}^{(0)}(0))^{-1} & (\lambda = 0).
\end{cases}
\]

**Proof.** The function \((D - \lambda I)M_0(p_{n+1}; \cdot)\) is an \( \ell \)-spline corresponding to the operator \( p_n(D) \) and having the knots \( x_0, \ldots, x_{n+1} \). Because of (2.2.9) it is identically zero outside \([x_0, x_{n+1}]\), and in view of Lemma 2.2.2 it can be written as a linear combination of the two \( \ell \)-splines \( M_0(p_n; \cdot) \) and \( M_1(p_n; \cdot) \).

In order to obtain the coefficients in this linear combination we take the Fourier transform of both sides in (2.4.5) yielding (as \( \Phi_{n,\nu}^{(0)}(0) = 1 \) for \( n \in \mathbb{N} \) and \( \nu \in \mathbb{Z} \) the coefficients must sum to \( -\lambda \))

\[
(2.4.7) \quad (z - \lambda)\Phi_{n+1,0}(z) = \alpha \Phi_{n,1}(z) - (\alpha + \lambda)\Phi_{n,0}(z).
\]

Letting \( z + \lambda \) gives (2.4.6) and inversion yields (2.4.5). \( \square \)

For polynomial B-splines, i.e., when \( p_{n+1}(D) = Dp_n(D) = D^{n+1} \), formulas (2.4.5) and (2.4.6) take the form

\[
(2.4.8) \quad \Phi_{n+1,0}(t) = \frac{n+1}{x_{n+1} - x_0} \Phi_{n,0}(t) - \frac{n}{x_{n+1} - x_0} \Phi_{n,1}(t) \quad (t \in \mathbb{R}).
\]

In order to verify (2.4.8) we compute \( \alpha \) as given by the second formula of (2.4.6). This involves the evaluation of \( \Phi_{n,\nu}^{(0)}(0) \), which can be performed
as follows. In view of (2.4.3) \( \Phi^{(n)}_{n,v}(z) = z^n \Phi^{(n)}_{n,v}(z) \), one has (cf. remark on p. 39) that

\[
\begin{align*}
\Phi^{(1)}_{n,v}(0) &= 0 \quad (i = 0, 1, \ldots, n-1), \\
\Phi^{(n)}_{n,v}(0) &= n!,
\end{align*}
\]

and

\[
\Phi^{(n+1)}_{n,v}(0) = (n+1)! \Phi^{(n)}_{n,v}(0).
\]

On the other hand, (2.2.4) implies that

\[
\Phi^{(n+1)}_{n,0}(0) = n! \{ x_0 + x_1 + \ldots + x_n \}.
\]

Consequently,

\[
\Phi^{(n+1)}_{n,0}(0) = \frac{1}{n+1} \{ x_0 + x_1 + \ldots + x_n \}
\]

and thus

\[
a = -\frac{(n+1)}{n+1} x_0.
\]

For numerical purposes the recurrence relation

\[
(2.4.9) \quad \frac{1}{n+1} \Phi^{(n+1)}_{n+1,0}(t) = \frac{1}{n+1} \left( \frac{t-x_0}{n} \Phi^{(n)}_{n,0}(t) + \frac{x_{n+1}-t}{n} \Phi^{(n+1)}_{n,1}(t) \right).
\]

where \( t \in \mathbb{R} \), is of importance, as it is useful to compute B-splines in a stable manner (cf. Cox [15]).

It turns out that (2.4.9) may be derived from Theorem 2.4.3 (cf. p. 42). The Fourier transform can be a useful tool to construct formulas similar to (2.4.9) for B-splines corresponding to specific operators \( \Phi(D) \). For instance, in the case of trigonometric B-splines (cf. p. 29) a formula similar to (2.4.9) is due to Xoch and Lyche [26, p. 188].

As an illustration, we here derive a recurrence relation for hyperbolic B-splines (cf. p. 29). So, let \( n = 2m \) \( (m \in \mathbb{N}) \) and let \( \omega \in \mathbb{R} \) with \( \omega \neq 0 \). The operators \( \Phi_n(D) \) and \( \Phi_{n+1}(D) \) are defined by

\[
\begin{align*}
\Phi_n(D) &:= \prod_{j=1}^{m} \left( D^2 - (j-1)^2 \omega^2 \right), \\
\Phi_{n+1}(D) &:= D \prod_{j=1}^{m} \left( D^2 - (j\omega)^2 \right).
\end{align*}
\]
For simplicity we assume the knots $x_0, x_1, \ldots, x_{n+1}$ to be simple. We recall (cf. (2.4.1)) that the Fourier transforms of $K_N(p_{n+1}^*), K_N(p_n^*),$ and $N_n(p_n^*)$ are denoted by $\theta_{n+1,0}, \theta_n, \theta_{n,1}$, respectively. Furthermore, let $H_{n+1,0}, H_n, H_{n,1}$, according to (2.4.3), be the functions related to $\theta_{n+1,0}, \theta_n, \theta_{n,1}$. Now constants $a_1$ and $a_2$ exist such that

\[(2.4.10)\quad a_1 H_{n,0}(z - \frac{u}{2}) + a_2 H_{n,1}(z - \frac{u}{2}) = H_{n+1,0}(z) ; \]

the existence of these constants is guaranteed, because $H_{n,0}(z - \frac{u}{2})$ and $H_{n,1}(z - \frac{u}{2})$ satisfy (2.4.4) with $\nu = 0$, and $n$ replaced by $n + 1$. Hence by (2.4.3) $E_n(z - \frac{u}{2})$ and $H_{n,1}(z - \frac{u}{2})$ vanish at the zeros of $P_n(z - \frac{u}{2})$, which polynomial is a divisor of $F_{n+1}$. Consequently, an appropriate linear combination of $E_n(z - \frac{u}{2})$ and $H_{n,1}(z - \frac{u}{2})$ vanishes at the zeros of $P_{n+1}$.

We note that the coefficients $b_{1,i}, b_{1,0}$ and $b_{1,1,i}$ in the expansions

\[(2.4.11)\quad \begin{cases}
E_{n+1,0}(z) = b_{0,0} e^{-x_0 z} + b_{1,0} e^{-x_1 z} + \cdots + b_{n,0} e^{-x_{n+1} z}, \\
H_{n,0}(z) = b_{1,0} e^{-x_0 z} + b_{1,0} e^{-x_1 z} + \cdots + b_{n,0} e^{-x_{n} z}, \\
H_{n,1}(z) = b_{1,1} e^{-x_1 z} + b_{2,1} e^{-x_2 z} + \cdots + b_{n+1,1} e^{-x_{n+1} z},
\end{cases}
\]

are even functions of $u$. Substituting (2.4.11) into (2.4.10) and equating the coefficients of $e^{-x_i z}$, we obtain

\[a_1 = c_1(u) e^{-x_0 u/2}, \]

\[a_2 = c_2(u) e^{-x_n u/2}, \]

where

\[c_1(-u) = c_1(u), \quad c_2(-u) = c_2(u).\]

Multiplying (2.4.10) with $p_n(z - \frac{u}{2})$ and using the relations $p_{n+1}(z) = (z + nu)p_n(z - \frac{u}{2})$ and (2.4.3), we get

\[(2.4.12)\quad \begin{cases}
c_1(u) e^{-x_0 u/2} \phi_{n,0}(z - \frac{u}{2}) + c_2(u) e^{-x_n u/2} \phi_{n,1}(z - \frac{u}{2}) = \\
\quad = (z + nu) \phi_{n+1,0}(z).
\end{cases}\]
Replacing $\omega$ by $-\omega$ yields the identity

\begin{equation}
(2.4.13) \quad c_1(\omega) e^{x_n^{\omega/2}} n_n z_{i, n}(z + \frac{\omega}{2})^2 = c_2(\omega) e^{x_n^\omega} n_n z_{i, 1}(z + \frac{\omega}{2}) = (z - \omega) \Phi_{n+1, 0}(z).
\end{equation}

The required recurrence relation is now obtained by subtracting (2.4.13) from (2.4.12) and then taking Fourier inverses. For the hyperbolic B-splines under consideration this gives rise to the following:

**Theorem 2.4.3.**

\begin{equation}
(2.4.14) \quad M_0(p_{n+1}; t) = \frac{c_1(\omega)}{\nu_n} \sinh \left( \frac{\omega}{2} (t - x_n) \right) M_0(p_n; t) + \frac{c_2(\omega)}{\nu_n} \sinh \left( \frac{\omega}{2} (x_n - t) \right) M_1(p_n; t) \quad (t \in \mathbb{R}).
\end{equation}

In order to apply (2.4.14) the coefficients $c_1(\omega)$ and $c_2(\omega)$ must be evaluated explicitly. Furthermore, we note that (2.4.9) may be obtained from (2.4.14) by letting $\omega$ tend to zero.

Our next purpose is to derive yet another recurrence relation for polynomial B-splines, which will be needed in the following section.

**Lemma 2.4.4.** Let $M_{n, 0}^r$ be the polynomial B-spline of order $n$ with knots $x_0, \ldots, x_n$. Let $x$ be an arbitrary point in $[x_0, x_n]$ and let $r \in \{1, 2, \ldots, n\}$ be such that $x \in (x_{r-1}, x_r)$. Furthermore, let $M_{n+1}^r(x) := M(D^n x_0^1, \ldots, x_{r-1}, x_r, x_{r+1}, \ldots, x_n; x^n)$ be the polynomial B-spline of order $n+1$ with knots $x_0, \ldots, x_{r-1}, x_r, x_{r+1}, \ldots, x_n$. Then

\begin{equation}
(2.4.15) \quad M_{n+1}^r(x) = \frac{n}{n} M_{n+1}^r(x^n) + \frac{n + 1}{n} M_{n, 0}^r(x^n). \quad (t \in \mathbb{R}).
\end{equation}

**Proof.** The fundamental functions corresponding to the operators

\begin{equation}
(D + x_0^1) \ldots (D + x_{r-1}^1) (D + x_r^1) (D + x_{r+1}^1) \ldots (D + x_n^1)
\end{equation}

and

\begin{equation}
(D + x_0^1) (D + x_{r+1}^1) \ldots (D + x_n^1)
\end{equation}

are denoted by $\Phi_{n+1, 0}$ and $\Phi_n$, respectively. The Fourier transforms $\Phi_{n+1, 0}(x)$
and $\Phi_{n,0}$ of $\Phi_{n+1}(z)$ and $\Phi_{n,0}$ are given by the relations (cf. (2.4.3) and the remark on p. 38)

$$
\Phi_{n+1}(z) = \frac{1}{(n+1)!} \frac{d^{n+1}}{dx^{n+1}} \Phi_{n+1}(x)(z) , \quad \Phi_n(z) = \frac{z^n}{n!} \Phi_{n,0}(z).
$$

The functions $\Phi_{n+1}$ and $\Phi_n$ are related by $(D + xI)\Phi_{n+1} = \Phi_n$. Hence

$$
(2.4.16) \quad \Phi_{n+1}(z) = \frac{z}{n+1} \Phi_n(z) + \frac{z^n}{n!} \Phi_{n,0}(z).
$$

Inversion of (2.4.16) yields (2.4.15).

\[\square\]

### 2.4.3. B-splines with equidistant knots

For equidistant knots $0, h, \ldots, nh$ ($h > 0$) the Fourier transform of the B-spline function $B_h(\{p_n\}^*)$ may be easily obtained as follows. If $p_n(z) = \sum_{j=1}^{n+1} (2 - s_j) \delta_j$ then, according to (2.4.3) and (2.4.4),

$$
B_{n,0}(z) = \text{a scalar multiple of the function}
$$

$$
\sum_{j=1}^{n} \left( e^{-h s_j} - e^{-h s_j^*} \right).
$$

The function $\Phi_{n,0}$ is then given by

(2.4.17) \quad $\Phi_{n,0}(z) = \frac{1}{n+1} \prod_{j=1}^{n} \left( e^{-h s_j} - e^{-h s_j^*} \right),$

where $d_n$ is a constant such that $\Phi_{n,0}(0) = 1$ (cf. (2.4.2)).

It follows from (2.4.17) that $B_h(\{p_n\}^*)$ equals the convolution product $f_1 \ast f_2 \ast \cdots \ast f_n$, where

$$
(2.4.18) \quad f_k(t) = \begin{cases} 
-1 & (0 < t < h; s_k = 0), \\
- \frac{1}{h} & (0 < t < h; s_k 
eq 0), \\
0 & (t > h).
\end{cases}
$$

In case $\delta_k \in \mathbb{R}$ one has $f_k := B_h(D - s_kI)^*$.

We observe that (2.4.5) and (2.4.6) may also be simplified in the case of equidistant knots. For this purpose we introduce the shift operator $E.$
2.4.5. Given a positive number \( h \), the shift operator \( E \) is defined as follows. For each real-valued function \( f \) defined on \( \mathbb{R} \) the function \( Ef \) is given by

\[ Ef(t) := f(t + h) \quad (t \in \mathbb{R}). \]

As usual, \( E^0 := I \) and \( E^n := E(E^{n-1}) \quad (n \in \mathbb{N}). \)

Now let \( p_{n+1}(z) = (z - \lambda)p_n(z) \) with \( \lambda \in \mathbb{R} \). Taking into account (2.4.17) one has

\[ (z - \lambda)q_{n+1,0}(z) = \frac{z}{\lambda} - 1 - \frac{1}{e^{-z}} - \frac{1}{1 - e^{-z} - \lambda - 1} \cdot (e^{-z} - e^{-z} - \lambda). \]

Inversion yields the identity

\[ \frac{(z - \lambda)B_{n+1,0}(t) = \frac{1}{1 - e^{-z}} - \frac{1}{1 - e^{-z} - \lambda} \cdot (e^{-z} - e^{-z} - \lambda). \]

If \( \lambda = 0 \), then \( \lambda(1 - e^{-z})^{-1} \) has to be replaced by \( h^{-1} \). For polynomial \( B \)-splines with \( h = 1 \) recurrence relation (2.4.19) takes the simple form

\[ B_{n+1,1}(t) = (E - I)B_{n,1}(t - 1), \]

which relation is contained in Ter Horst [41, p. 211].

2.5. The minimum sup-norm of a polynomial \( B \)-spline

2.5.1. The normalized polynomial \( B \)-spline basis

Let \( s \) be a polynomial spline of order \( n \) defined on \([a,b]\) and having the knots \( \gamma_1 < \gamma_2 < \gamma_3 < \ldots < \gamma_k \) with multiplicities \( \nu_1, \nu_2, \ldots, \nu_k \), respectively. We recall that \( s \) can be represented by the so-called truncated power basis (cf. (1.3.1)), where in the case of polynomial splines \( \varphi_i(t) = \varphi_i^{(n)}(t) \) is \( t^{n-1}/(n-1)! \). In order to express the function \( s \) as a linear combination of polynomial \( B \)-splines the points \( a \) and \( b \) are considered to be knots of multiplicity \( n \), so we introduce additional knots \( x_0 = x_1 = \ldots = x_{n+1} = a \) and \( x_{n+k+1} = \ldots = x_{k+1} = b \), where \( k = n + \nu_1 + \ldots + \nu_k - 1 \).

If we put \( x_{n+i} = x_{n+i+1} = \ldots = x_{n+i+1} = Y_i \), \( x_{n+i+1} = x_{n+i+1} = \ldots = Y_i \), \( x_{n+i+1} = x_{n+i+1} = \ldots = Y_i \), \( x_{n+i+1} = x_{n+i+1} = \ldots = Y_i \), the function \( s \) can (cf. (2.2.12)) uniquely be represented as
\[ s(t) = \sum_{i=0}^{b-a} c_i N_{n,i}(t) \quad (a < t < b) . \]

Hence the polynomial B-splines \( N_{n,i} \) \((i = 0, 1, \ldots, k)\) form a basis for the set of all polynomial splines of order \( n \) with fixed knots \( y_1, \ldots, y_k \) with multiplicities \( v_1, v_2, \ldots, v_k \), respectively.

As a slightly different basis for the polynomial spline functions one often uses the so-called normalized polynomial B-splines \( N_{n,i} \) \((i = 0, 1, \ldots, k)\)
(cf. De Boor [3]), which are related to the polynomial B-splines \( N_{n,i} \) by

\[ N_{n,i}(t) = \left( \frac{x_i - x_{i+n}}{a} \right) N_{n,i}(t) \quad (t \in \mathbb{R}) . \]

The normalized polynomial B-splines satisfy the important identity

\[ \sum_{i=0}^{b-a} N_{n,i}(t) = 1 \quad (a < t < b) , \]

which is due to Marsden [33].

2.5.2. The condition number of a normalized B-spline basis

As a measure for the quality of a basis serves the condition number: broadly speaking, it measures the possible relative change in \( c \) as a result of a relative change in the spline \( s \).

The condition number \( K(x_0, x_1, \ldots, x_{n+1}) \) of the normalized polynomial B-spline basis \( N_{n,i} \) \((i = 0, 1, \ldots, k)\), relative to the knot sequence \( x_0 \leq x_1 \leq \cdots \leq x_{n+1} \) with \( a \leq x_0 \), \( x_{n+1} \leq b \) and \( x_i < x_{i+n} \) \((i = 0, 1, \ldots, k)\), is defined by
(cf. De Boor [3])

\[ K(x_0, x_1, \ldots, x_{n+1}) := \left( \sup_{\|c\| = 1} \|s\|_{[a,b]} \bigg/ \inf_{\|\delta\| = 1} \|s + \delta\|_{[a,b]} \right)^{-1} , \]

where \( c = (c_0, c_1, \ldots, c_k)^T \in \mathbb{R}^{k+1} \), \( \|c\| = \max_{\text{Odis}\mathbb{R}} |c_\ell| \) and \( s = \sum_{i=0}^{k} c_i N_{n,i} \).

Since

\[ |s(t)| \leq \|c\| \sum_{i=0}^{b-a} N_{n,i}(t) = \|c\| \]

because of Marsden's identity, we conclude that the numerator in (2.5.2) equals one. Hence

\[ K(x_0, x_1, \ldots, x_{n+1}) = \left( \inf_{\|\delta\| = 1} \|s + \delta\|_{[a,b]} \right)^{-1} . \]
The condition number $K_n$ of the normalized polynomial B-spline basis of order $n$ is then defined as the supremum of $K(x_0, x_1, \ldots, x_{n+1})$ over all knot sequences in $[a, b]$, i.e.,

$$ (2.5.4) \quad K_n : = \sup \left\{ K(x_0, x_1, \ldots, x_{n+1}) \mid a \leq x_0 \leq \ldots \leq x_{n+1} \leq b, \quad x_i < x_{i+n} \ (i = 0, 1, \ldots, t) \right\}. $$

Lower and upper bounds for $K_n$ have been given by De Boor [3] and Lyche [32].

It follows from (2.5.3) that the problem of computing $K_n$ is equivalent to the problem of minimizing the sup norm of a specific linear combination of normalized polynomial B-splines over all knot sequences.

With respect to the general problem, which seems difficult to solve, our objective is modest: instead of trying to solve it, we shall investigate in this subsection the simpler problem, formulated by Meinardus [34, p. 174], of minimizing the sup norm of a single polynomial B-spline of order $n$.

Using recurrence relation (2.4.15) we are able to give a complete solution. We note that the exposition given closely follows a preliminary report of ours (cf. Tor Möller [42]).

### 2.5.3. The computation of the minimum supremum norm

Throughout this subsection let $n \in \mathbb{N}$ be fixed and let $x_0, x_1, \ldots, x_n$ be a knot sequence with $0 = x_0 \leq x_1 \leq \ldots \leq x_n = 1$. Our aim is to determine

$$ (2.5.5) \quad \inf_{0 \leq x_1 \leq \ldots \leq x_{n-1} \leq 1} M_{n, 0}, $$

where

$$ \| M_{n, 0} \| = \sup_{t \in [0, 1]} M_{n, 0}(t). $$

Before going into details, we give four simple examples of polynomial B-splines corresponding to specific knot distributions and their sup norms. These examples serve as an illustration and will be needed later on.

**Example 1.** $x_1 = x_2 = \ldots = x_{n-1} = x \in (0, 1)$,

$$ M_{n, 0}(t) = \begin{cases} n \left( \frac{t}{x} \right)^{n-1} & (0 \leq t \leq x), \\ n \left( \frac{1-t}{1-x} \right)^{n-1} & (x < t \leq 1). \end{cases} $$
\[ \| M_{n,0} \| = n \, . \]

**EXAMPLE 2.** \( x_1 = x \in (0,1), \ x_2 = x_3 = \ldots = x_{n-1} = 1, \)

\[ M_{n,0}(t) = \begin{cases} \frac{n}{x} e^{n-1} (0 \leq t < x) \\ \frac{n}{x} \left( \frac{t}{1-x} \right)^{n-1} \left( \frac{t-x}{1-x} \right)^{n-1} (x \leq t < 1) \end{cases} \]

(2.5.6) \[ \| M_{n,0} \| = n \left[ \frac{x}{1-(1-x)^{n-2}} \right]^{n-2} \, . \]

**EXAMPLE 3.** \( 0 = x_1 = \ldots = x_{k-1}, \ x_k = x_{k+1} = \ldots = x_{n-1} = 1 (1 \leq k \leq n), \)

\[ M_{n,0}(t) = k \binom{n}{k} t^{n-k} (1-t)^{k-1} (0 \leq t \leq 1) \, , \]

(2.5.7) \[ \| M_{n,0} \| = k \binom{n}{k} \frac{(n-k)(n-k-1)\ldots 3\ldots 1}{(n-1)\ldots 3\ldots 1} =: L(n,k) \, . \]

**EXAMPLE 4.** \( x_j = j/n \ (j = 1, \ldots, n-1), \)

\[ M_{n,0}(t) = \frac{n}{(n-1)!} \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} \frac{1}{j} \left( \frac{t}{n} \right)^j \prod_{i=1}^{j-1} \frac{j-i}{i} (t < \infty) \, . \]

(2.5.8) \[ \| M_{n,0} \| = M_{n,0}(1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{n,0}(1u) e^{-\frac{u^2}{2}} du = \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{n}{(1-e^{-u})^{(n-1)/2}} \frac{1}{(4\pi)^{n/2}} du = \]

\[ = \frac{n}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\sin \frac{\omega}{2}}{\omega/2} \right)^n du \sim \left( \frac{6n}{\pi} \right)^{1/2} (n \to \infty) \, . \]

Computational details concerning the Examples 1,2,3,4 are omitted here. We only remark that the formulas in (2.5.8) are obtained by using (2.4.17) with \( \beta_1 = \beta_2 = \ldots = \beta_n = 0, \) followed by Fourier inversion; the asymptotic
behaviour of the integral in the right-hand side of (2.5.8) follows by an application of formula (4.2.4) in De Bruijn [6, p. 655].

In order to compare the asymptotic behaviour of \( \|M_{n,0}\| \) in Example 4 with that of \( \|M_{n,0}\| \) in Example 3, we apply Stirling's formula in (2.5.7). For that purpose the following two cases are considered:

1) if \( k \) is fixed, then

\[
L(n,k) \propto \frac{(-1)^{k-1}(k-1)!}{(k-1)^{(k-1)}} n \quad (n \to +\infty);
\]

2) if \( \lim \frac{k}{n} = a \) with \( 0 < a < 1 \), then

\[
L(n,k) \propto \left( \frac{n}{2\pi n(1-a)} \right)^{\frac{1}{2}} \quad (n \to +\infty).
\]

Comparing (2.5.8) and (2.5.10) we note that for sufficiently large \( n \) the sup norm corresponding to equally spaced knots is larger than the sup norm corresponding to a distribution of knots as given in Example 3 with \( a = 1 \) in (2.5.10).

Hence, for large values of \( n \), an equally spaced knot distribution does not furnish the minimum sup norm.

In what follows we shall establish that the knot distribution as given in Example 3 with \( k = \left\{ \frac{n+1}{n} \right\} \) provides the minimum sup norm over all knot sequences \( 0 = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n = 1 \). To this end recurrence relation (2.4.15) is needed, viz.

\[
M_{n+1}(x)(t) = \frac{t-x}{n} M_{n+1}(x)(t) + \frac{B+1}{n} M_{n,0}(t) \quad (t \in \mathbb{R}).
\]

This relation may be considered as a differential equation for the B-spline \( M_{n+1}(x)(t) \), the solution of which is given in the following

**Lemma 2.5.1.**

\[
M_{n+1}(x)(t) = \begin{cases} 
(n+1)(x-t)^n & \int_0^t \frac{M_{n,0}(t)}{(x-t)^{n+1}} \, dt \\ (0 < t < x) \\
\frac{n+1}{x} M_{n,0}(x) & (t = x) \\
(n+1)(t-x)^n & \int_t^1 \frac{M_{n,0}(t)}{(t-x)^{n+1}} \, dt \\ (x < t < 1) 
\end{cases}
\]
If $M_{n,0}$ is continuous on $\mathbb{R}$, then differentiation in (2.5.11) followed by integration by parts shows that $M_{n+1,0}(x)$ can be expressed in terms of $M_{n,0}$ as follows

$$
M_{n+1,0}(x) = \begin{cases} 
(n+1)(x-t)^{n-1} \int_0^t (x-t)^{-n} dM_{n,0}(t) & (0 \leq t < x), \\
\frac{n+1}{n-1} M_{n,0}(x) & (t = x) \text{ if } M_{n+1,0}(x) \text{ is differentiable at } x, \\
(n+1)(x-t)^{n-1} \int_t^1 (t-x)^{-n} dM_{n,0}(t) & (x < t \leq 1).
\end{cases}
$$

(2.5.12)

As a first step in obtaining an optimal knot distribution, we fix the knots $x_1, \ldots, x_{n-1}$ and study how the value of $M_{n+1,0}(x)$ depends on the position of the knot $x$.

As $x_0 = 0$ and $x_n = 1$, the multiplicity of a knot in the interior of $[0,1]$ obviously does not exceed $n-1$, hence $M_{n,0}$ is continuous in $(0,1)$. If $M_{n,0}$ is not continuous on $\mathbb{R}$, then either $x_0$ or $x_n$ is a knot of multiplicity $n$.

In this case the dependence of $M_{n+1,0}(x)$ on $x$ is given by formula (2.5.6), with $n$ replaced by $n+1$. Having observed this, we assume from now on that $M_{n,0}$ is continuous on $\mathbb{R}$.

Let $x_0^* = (0,1)$ be the point where $M_{n,0}$ attains its maximum value and let $x_{n+1}^*(x) \in (0,1)$ denote the point where $M_{n+1,0}(x)$ has its maximum. The following result shows how $x_{n+1}^*(x)$ depends on $x$.

**Lemma 2.5.2.**

If $x > x_n^*$ then $x_n^* < x_{n+1}^*(x) < x$.

If $x < x_n^*$ then $x < x_n^*(x) < x_n^*$.

If $x = x_n^*$ then $x_n^*(x) = x_n^*$.

**Proof.** If $x > x_n^*$, then in view of the first formula of (2.5.12) we have

$$
M_{n+1,0}(x_n^*) = (n+1) \int_{x_n^*}^x \frac{(x-t)^{n-1}}{(x-t)^n} \frac{dM_{n,0}(t)}{dM_{n,0}(t)} > 0.
$$
It can easily be shown by Rolle's theorem that $\lim_{n \to \infty} M_{n+1}(x) = 0$ and thus the preceding inequality implies that $x_n < x_n^{n+1}(x)$. 
Now let $x < x_n^*$. Using again (2.5.12) we conclude that $M_{n+1}(x) < 0$. Hence $x_n^* > x_n^*(x)$. Taking into account the sign of $M_{n+1}(x)$ (cf. (2.5.12)) we likewise deduce that $x > x_n^*$ implies $x > x_n^{n+1}(x)$, $x < x_n^*$ implies $x < x_n^{n+1}(x)$ and, finally, $x = x_n^*$ implies $x = x_n^{n+1}(x) = x_n^*$. A combination of these results yields the lemma. \[\Box\]

We proceed by assuming that $x < x_n^*$, since the two cases $x < x_n^*$ and $x < x_n^*$ are very similar. Recalling that $M_{n+1}(x)$ attains its maximum at $x_n^{n+1}(x)$, we deduce from (2.4.15) that
\[(2.5.13) \quad M_{n+1}(x) = \frac{n+1}{n} M_{n+1}(x_n^{n+1}(x)).\]

Since $M_{n+1}$ is decreasing on $(x_n^*, 1)$ one has
\[\inf_{x_n^* < x < 1} M_{n+1}(x) = \frac{n+1}{n} M_{n+1}(x_n^{n+1}(1)),\]
where
\[x_n^* = \sup_{x_n^* < x < 1} x_n^{n+1}(x) .\]

We shall prove that (cf. (2.5.10)) $x_n^* = x_n^{n+1}(1)$. For this purpose the following lemma is needed.

**Lemma 2.5.3.** If $x < y < z < 1$, then $x_n^{n+1}(y) < x_n^{n+1}(z)$.

**Proof.** From (2.5.12) it follows that for $x > x_n^*$
\[(2.5.14) \quad \int_0^1 (t-x)^{-\gamma} dM_{n+1}(t) = 0 .\]

Now let $x_n^* < y < z < 1$. An application of the mean value theorem yields
\[\int_0^1 (t-z)^{-\gamma} dM_{n+1}(t) = \left(\int_0^z + \int_{x_n^*}^y + \int_y^z\right) (t-y)^{-\gamma} dM_{n+1}(t) = \]
\[
\begin{align*}
&= \left( \frac{\tau - y}{\tau_2 - z} \right) \int_0^{x_n^*} (t - y)^{-n} dM_{n,0}(\tau) + \left( \frac{\tau - y}{\tau_2 - z} \right) \int_{x_n^*}^{y} (t - y)^{-n} dM_{n,0}(\tau) = \\
&= \left\{ \left( \frac{\tau - y}{\tau_2 - z} \right) - \left( \frac{1 - y}{\tau_1 - z} \right) \right\} \int_{x_n^*}^{y} (t - y)^{-n} dM_{n,0}(\tau), \\
\end{align*}
\]

for some values \( \tau_1 \in (0, x_n^*) \) and \( \tau_2 \in (x_n^*, x_{n+1}^*) \).

As \( \tau = \left( \frac{\tau - y}{\tau_2 - z} \right) \) is decreasing on \((0, y)\) we have

\[
x_{n+1}^*(y) = \text{sgn} \left( \int_0^y (t - z)^{-n} dM_{n,0}(\tau) \right) = (-1)^n.
\]

The function \( h \) defined by

\[
h(t) := \int_0^t (\tau - z)^{-n} dM_{n,0}(\tau)
\]

changes sign on \([0, z]\) only at \( t = x_{n+1}^*(z) \); in fact, we have

\[
\text{sgn}(h(t)) = \begin{cases} 
(-1)^n & (0 < t < x_{n+1}^*(z)) \\
(-1)^{n+1} & (x_{n+1}^*(z) < t < z)
\end{cases}.
\]

So from (2.5.15) it follows that \( x_{n+1}^*(y) < x_{n+1}^*(z) \).

\[\square\]

**COROLLARY 2.5.4.**

(2.5.16) \[
\min_{x_{n+1}^* \leq 1} \|M_{n+1,0}(x)\| = \frac{n+1}{n} M_{n,0}(x_{n+1}^*(1))
\]

(2.5.17) \[
\min_{0 < x_{n+1}^* \leq 1} \|M_{n+1,0}(x)\| = \frac{n+1}{n} M_{n,0}(x_{n+1}^*(0)).
\]

**PROOF.** Equality (2.5.16) follows immediately from Lemma 2.5.3 and (2.5.13). Equality (2.5.17) follows from the observation that the cases \( x \notin x_n^* \) and \( x \notin x_n^* \) are essentially the same.

\[\square\]
Corollary 2.5.4 implies that the function \( x \mapsto \| M_{n+1} \|_{n+1} \) attains its minimum at \( x = 0 \) or \( x = 1 \).

However, the preceding analysis is carried out under the assumption that \( M_{n,0} \) is continuous on \( \mathbb{R} \) (cf. p. 49). If \( M_{n,0} \) is not continuous on \( \mathbb{R} \), then either \( x_0 \) or \( x_n \) is a knot of multiplicity \( n \). Assume that \( x_n \) is a knot of multiplicity \( n \), then in view of (2.5.6), with \( n \) replaced by \( n+1 \), one has

\[
\| M_{n+1} \|_{n+1} = (n+1) \left( \frac{n+1}{n-1} \right)^{\frac{n+1}{n}} .
\]

From this expression we conclude that in case \( n \geq 2 \) the function \( x \mapsto \| M_{n+1} \|_{n+1} \) attains its minimum value at \( x = 1 \). Evidently, the assumption that \( x_0 \) is a knot of multiplicity \( n \) similarly implies that \( \| M_{n+1} \|_{n+1} \) attains its minimum value at \( x = 0 \) in case \( n \geq 2 \).

The main result in this section is contained in the theorem below.

**Theorem 2.5.5.** Let \( M_{n,0} \) be the polynomial B-spline of order \( n \) with knots \( 0 = x_0 < x_1 < \ldots < x_n = 1 \). Then

\[
\min_{0 < x_1 < \ldots < x_{n-1} < 1} \| M_{n,0} \| = \begin{cases} \left( \frac{k}{k+1} \right)^{k+1} & (n = 2k), \\
\left( \frac{k+1}{k+1} \right)^{k+1} & (n = 2k+1) \end{cases}
\]

Moreover, the knot distributions for which the minimum sup norm is attained are given by

\[x_0 = x_1 = \ldots = x_{k-1} = 0, \quad x_k = x_{k+1} = \ldots = x_n = 1,\]

where

\[k = \begin{cases} \frac{n+1}{2} & (n \text{ odd}), \\
\frac{n}{2} \text{ or } \frac{n}{2} + 1 & (n \text{ even}) \end{cases}\]

**Proof.** The case \( n = 1 \) is trivial. Furthermore, \( \| M_{2,0} \| = 2 \), independent of the position of \( x_1 \). Let now \( M_{n,0} \) be a polynomial B-spline of order \( n \) (\( n \geq 2 \)) and assume there is a knot \( x \) in the interior of \([0,1]\). By deleting this knot we get the B-spline \( M_{n}^{(n-1)}(x_0, x_1, \ldots, x_{l-1}, x_l, x_{l+1}, \ldots, x_{n-1}, x_n) \). By Corollary 2.5.4 and Lemma 2.5.3 one may conclude that
Thus a knot distribution with knots in the interior of \([0,1]\) cannot be optimal and therefore we may restrict ourselves to knot distributions with all knots located at the end points 0,1. Example 3 on p. 47 furnishes such a knot distribution and so it can be used to obtain the minimum sup norm. Consequently, it remains to compute \(\min_{L(n,k)} \left< \frac{n}{2} \right>\) (cf. (2.5.7)).

Theorem will be completely proved if we show that

\[
\min_{L(n,k)} \left< \frac{n}{2} \right> = \min_{L(n,k)} \left< \frac{n}{2} \right> = \min_{L(n,k)} \left< \frac{n}{2} \right> \cdot
\]

This can be done as follows. In view of (2.5.7) and writing \(\lambda = n - k\), we have

\[
\frac{L(n,k+1)}{L(n,k)} = \left(\frac{1 + \frac{1}{k}}{1 - \frac{1}{k}}\right)^k = \left(\frac{1 + \frac{1}{k}}{1 - \frac{1}{k}}\right)^k.
\]

Since the sequence \((1 + \frac{1}{k})^k\) is increasing it follows that

\[
L(n,k+1)/L(n,k) < 1 \text{ if } k < \lambda, \text{ hence } k < n/2. \text{ This implies (2.5.10).} \]

Remark 1. With respect to the supremum of the sup norm of \(M_{n,0}\) our analysis shows that

\[
\sup_{0 \leq x_1, \ldots, x_{n-1} \leq 1} \|M_{n,0}\| = n.
\]

Indeed, the knot distributions as given by Example 1 have this extremal property.

Remark 2. An elementary computation shows that the sequences \(\rho_{2k+1}\) and \(\rho_{2k}\) defined by

\[
\rho_{2k+1} := (2k+1)^{-1} L(2k+1,k+1), \quad \rho_{2k} := (2k)^{-1} L(2k,k)
\]

are both decreasing with (cf. (2.5.10)) \(\lim_{k \to \infty} \rho_{2k} = \lim_{k \to \infty} \rho_{2k+1} = \left(\frac{2}{n}\right)^{1/2}\). Hence

\[
\min_{0 \leq x_1, \ldots, x_{n-1} \leq 1} \|M_{n,0}\| \sim \left(\frac{2n}{1}\right)^{1/2} = (n \to \infty).\]
REMARK 3. It can also be shown that the sequence \( \{a_k\}_{k=1}^\infty \) satisfies
\[
\rho 2^{k-1} \leq \rho_k \leq \rho 2^{k-1} \quad (k = 2, 3, \ldots).
\]

Hence, for \( n = 3, 4, \ldots \),
\[
\min_{0 \leq x_1 \leq \cdots \leq x_{n-1} \leq 1} \|M_{n,0}\| \leq \frac{\rho}{\rho} n^{\frac{1}{2}} \;
\]
where the constant \( \rho = 0.8888 \ldots \) cannot be replaced by a smaller one. On the other hand, for \( n = 1, 2, \ldots \),
\[
(2.5.19) \quad \min_{0 \leq x_1 \leq \cdots \leq x_{n-1} \leq 1} \|M_{n,0}\| \geq \left(\frac{2\rho}{\pi}\right)^\frac{1}{n} \;
\]
where the constant \( (2/\pi)^\frac{1}{2} = 0.7978 \ldots \) cannot be replaced by a larger one.

REMARK 4. For an arbitrary knot distribution \( x_0 \leq x_1 \leq \cdots \leq x_n \) with \( x_0 < x_n \), the function
\[
t \mapsto (x_n - x_0)^{n-1}(x_0 + (x_n - x_0)t)
\]
is a polynomial B-spline of order \( n \) having its first knot at 0 and its last knot at 1. In view of (2.5.19) one has
\[
(2.5.20) \quad \|M_{n,0}\| \geq \left(\frac{2\rho}{\pi}\right)^\frac{1}{n} (x_n - x_0)^{n-1} \quad (n = 1, 2, \ldots ) \;
\]

REMARK 5. As a consequence of (2.5.20) and (2.5.1) one has
\[
\|N_{n,0}\| \geq \left(\frac{2\rho}{\pi}\right)^\frac{1}{n} \quad (n = 1, 2, \ldots ) \;
\]
where \( N_{n,0} \) is the normalized polynomial B-spline of order \( n \) as defined by (2.5.1).

Hence, (2.5.3) and (2.5.4) imply that \( \|K_n\| \geq (\rho n^2)^\frac{1}{n} \). However, this lower bound of \( K_n \) is poor since it has been shown by Yumura [32] that
\[
(2.5.21) \quad K_n \geq \frac{n-1}{n} \frac{\rho}{\pi} n^\frac{3}{2} \;
\]
We observe that the lower bound in (2.5.21) is obtained by taking the knots in \([0,1]\) distributed as equally as possible over the endpoints 0, 1.
3. ON RELATIONS BETWEEN FINITE DIFFERENCES AND DERIVATIVES OF CARDINAL Z-SPLINE FUNCTIONS

3.1. Introduction and summary

It is well known [cf. Ahlberg, Nilson and Walsh [1, pp. 10, 12]] that for a cubic spline, i.e., a polynomial spline $s$ of order four (or degree three) with equally spaced knots $x_i = ih$ ($h > 0$, $i \in \mathbb{N}$) the following relations hold:

\begin{align}
(3.1.1) & \quad \left\{ \begin{array}{l}
\left( s''(x_{i+2}) + 4a''(x_{i+1}) + b''(x_i) \right) = \frac{6}{h^2} \left( s(x_{i+2}) - 2s(x_{i+1}) + s(x_i) \right), \\
( s'(x_{i+2}) + 4a'(x_{i+1}) + b'(x_i) \right) = \frac{3}{h} \left( s(x_{i+2}) - s(x_i) \right).
\end{array} \right.
\end{align}

For quintic splines, i.e., polynomial splines of degree five, relations similar to (3.1.1) have been derived by Schwer [57] and for polynomial splines of arbitrary order by Fyfe [18]. Note that in these relations the spline function $s$ and its derivatives are evaluated at points that coincide with the knots. These relations are often used in problems of interpolating a given function by means of splines in case the points of interpolation and the knots coincide. From now on the points of interpolation will be called nodes.

In order to obtain results with respect to the problem of existence and convergence of (periodic) spline interpolation in the case each node is located between two consecutive knots, relations similar to (3.1.1) are required. This is done by Ter Horst and Hors [40]. Relations of the kind (3.1.1) can be written in a compact form using the shift operator $\delta$ as introduced in Definition 2.4.5; in the case of polynomial splines we refer to Ter Horst [41].

The objective of this chapter is to derive relations similar to (3.1.1) for cardinal $Z$-splines with respect to an arbitrary differential operator $p_n(D)$. Fundamental for the derivation of these relations, which will be called relations between finite differences and derivatives of cardinal $Z$-splines,
are the so-called exponential $f$-spline functions. In this respect it is appropriate to cite Schoenberg's remark about these functions for polynomial splines (cf. Schoenberg [51, p. V]): "the role played by the exponential function in calculus is taken over in equidistant spline theory by the exponential spline". A similar observation applies to the exponential $f$-splines in the case of the more general, $f$-spline theory.

The contents of this chapter may be briefly summarized as follows.
Section 3.2 introduces exponential $f$-splines and contains various properties of them. Part of it is preliminary material for Section 3.3, in which a few fundamental relations between finite differences and derivatives are derived.
In the final Section 3.4 a lemma will be given that is of importance for the problem of cardinal $f$-spline interpolation, to be dealt with in Chapter 4.
Moreover, a number of simple differentiation formulas are deduced.

3.2. The exponential $f$-splines

3.2.1. Definition of exponential $f$-splines

Throughout this chapter $p_n \in \mathbb{R}$ will be a monic polynomial and $h$ a positive number. Associated with $p_n$ and $h$, the mesh distance of the knot distribution, we introduce the polynomial $\tilde{p}_n$ as follows: if $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n$ are the, not necessarily real, zeros of $p_n$, then $\tilde{p}_n$ is given by

\begin{equation}
\tilde{p}_n(z) := \prod_{j=1}^{n} \left( z - \tilde{a}_j \right) \quad (z \in \mathbb{C}).
\end{equation}

Furthermore, we recall (cf. 1.3.2)) that $S(p_n, h)$ denotes the set of cardinal $f$-splines corresponding to the operator $p_n (\cdot)$ and with mesh distance $h$.

In order to define the exponential $f$-splines we need the following elementary result, a proof of which may be given by induction with respect to the variable $n$.

**Lemma 3.2.1.** If $\lambda \in \mathbb{R}$ is such that $\tilde{p}_n(\lambda) \neq 0$, then the null function is the only function $f \in \text{Ker}(p_n)$ satisfying

\[ f(t+h) = \lambda f(t) \quad (t \in \mathbb{R}). \]
Given an arbitrary \( c \neq 0 \) it follows that under the condition of Lemma 3.2.1 there exists a unique \( f \in \text{ker}(p_n) \) such that

\[
(3.2.2) \quad s^{(1)}(h) - \lambda s^{(1)}(0) = c d_{i,n-1} \quad (i = 0, 1, \ldots, n-1).
\]

**Definition 3.2.2.** Let \( \lambda \in \mathbb{R} \) be such that \( \widetilde{p}_n(\lambda) \neq 0 \). Then a nontrivial \( \ell \)-spline \( s \in S(p_n, h) \) is called an exponential \( \ell \)-spline function if it satisfies the functional equation

\[
(3.2.3) \quad s(t + h) = \lambda s(t) \quad (t \in \mathbb{R}).
\]

Since an exponential \( \ell \)-spline \( s \) belongs to \( C^{(n-2)}(\mathbb{R}) \) it follows that

\[
s^{(1)}(h) - \lambda s^{(1)}(0) = 0 \quad (i = 0, 1, \ldots, n-2),
\]

and thus by (3.2.2) an exponential \( \ell \)-spline is uniquely determined by (3.2.3) up to a multiplicative constant.

If \( p_n(0) \) is disconjugate on \( [0, nh] \), then an exponential \( \ell \)-spline corresponding to \( p_n(0) \) can easily be expressed in terms of \( \mathbf{B} \)-splines. This is the content of the following

**Theorem 3.2.3.** Let the operator \( p_n(0) \) be disconjugate on \( [0, n h] \) and let \( \lambda \in \mathbb{R} \) be such that \( \widetilde{p}_n(\lambda) \neq 0 \). If \( s \in S(p_n, h) \) is an exponential \( \ell \)-spline, then there exists a constant \( c \neq 0 \) such that

\[
(3.2.4) \quad s(t) = c \sum_{v=0}^{n} \lambda^v B_n(p_n; t-vh) \quad (t \in \mathbb{R}),
\]

where (cf. p. 29 for notation) \( B_n(p_n, \cdot) \) denotes the \( \mathbf{B} \)-spline of order \( n \) corresponding to the operator \( p_n(0) \) and having equidistant knots \( 0, h, \ldots, nh \).

**Proof.** Since \( p_n(0) \) is disconjugate on \( [ih, nh] \) for every \( i \in \mathbb{N} \), we may apply Lemma 2.2.2 in order to obtain

\[
s(t) = \sum_{v=0}^{\infty} c_v B_n(p_n; t-vh) \quad (t \in \mathbb{R}),
\]

where the coefficients \( c_v \) are uniquely determined.

Since (3.2.3) holds these coefficients satisfy the relation

\[
c_{v+1} = \lambda c_v \quad (v \in \mathbb{N}).
\]

Hence \( c_v = c \lambda^v \) for some constant \( c \neq 0 \). \[\square\]
If \( I \in \text{Ker}(p_n) \) has the property that it coincides with an exponential \( \mathcal{E} \)-spline on \( [0,h] \), then \( I \) is called an exponential \( \mathcal{E} \)-polynomial. Furthermore, exponential \( \mathcal{E} \)-polynomials corresponding to \( p_n(D) = D^n \) are simply called exponential polynomials.

It follows from (3.2.2) that an exponential \( \mathcal{E} \)-polynomial satisfies the relation

\[
(3.2.5) \quad f(t+h) - \lambda f(t) = c_0(t),
\]

where \( (\text{cf. p. 6}) \) \( c \) is the fundamental function corresponding to \( p_n(D) \) and \( c \) is a constant different from zero.

On the other hand, if \( p_n^{-1}(\lambda) \neq 0 \) and if \( c \) is prescribed, then there exists exactly one function \( f \in \text{Ker}(p_n) \) satisfying (3.2.5).

We proceed by giving an example of a particular \( \mathcal{E} \)-spline corresponding to the operator \( p_n^{-1}(D) = D^{n+1} \). Given \( c \in \mathbb{R} \) with \( c \neq 0 \), let \( p \in \mathcal{T}_n \) be the polynomial of degree exactly equal to \( n \) satisfying

\[
(3.2.6) \quad p(t+h) + p(t) = \frac{c}{n!} t^n \quad (t \in \mathbb{R}).
\]

The polynomial \( p \) is well known and can be written as

\[
(3.2.7) \quad p(t) = \frac{1}{n!} h^n p_n(t),
\]

where \( p_n \) is the classical Euler polynomial of degree \( n \) (cf. Hörmander [45, p. 23]) given by the generating function

\[
\frac{e^{zt}}{e^z + 1} = \sum_{n=0}^{\infty} p_n(t) \frac{z^n}{n!} \quad (|z| < 2\pi).
\]

The exponential \( \mathcal{E} \)-spline corresponding to \( D^{n+1} \) and having the property that it coincides with \( p \) on \( [0,h] \) satisfies (3.2.3) with \( \lambda = -1 \). This is a specific so-called Euler \( \mathcal{E} \)-spline, the general definition of which reads as follows.

**Definition 3.2.4.** Let \( p_n^{-1}(-1) \neq 0 \). If \( s \in S(p_n, h) \) satisfies the functional equation

\[
s(t+h) = s(t) + \lambda s(t) \quad (t \in \mathbb{R}),
\]

then \( s \) is called an Euler \( \mathcal{E} \)-spline function. In particular, if \( p_n(D) = D^n \) then \( s \) is called an Euler spline function of order \( n \) (or degree \( n+1 \)).
3.2.2. The normalized exponential \( \ell \)-spline

We recall that an exponential \( \ell \)-polynomial is uniquely determined up to a multiplicative constant. Prescribing the constant \( c \) in (3.2.5) in a particular way gives rise to the normalized exponential \( \ell \)-polynomial.

**Definition 3.2.5.** Let \( \lambda \in \mathbb{R} \) be such that (cf. (3.2.1)) \( \tilde{p}_n(\lambda) \neq 0 \). The normalized exponential \( \ell \)-polynomial \( \tilde{\phi}_\lambda(p_n; h, \cdot) \) is the unique exponential \( \ell \)-polynomial satisfying

\[
\tilde{\phi}_\lambda(p_n; h, t; h) = \lambda \tilde{\phi}_\lambda(p_n; h, t) = \bar{p}_n(\lambda) \tilde{\phi}_1(t) \quad (t \in \mathbb{R}),
\]

where, as usual, \( \tilde{\phi}_\lambda(p_n; h, t) \) is the fundamental function corresponding to the operator \( p_n(n) \).

The normalized exponential \( \ell \)-polynomial is now used to define the normalized exponential \( \ell \)-spline function in the following way.

**Definition 3.2.6.** Let \( \lambda \in \mathbb{R} \) be such that \( \tilde{p}_n(\lambda) \neq 0 \). The normalized exponential \( \ell \)-spline function \( \tilde{\psi}_\lambda(p_n; h, \cdot) \) is the unique exponential \( \ell \)-spline satisfying the two conditions

i) \( \tilde{\psi}_\lambda(p_n; h, t; h) = \lambda \tilde{\psi}_\lambda(p_n; h, t) \quad (t \in \mathbb{R}), \)

ii) \( \tilde{\psi}_\lambda(p_n; h, t) = \tilde{\phi}_\lambda(p_n; h, t) \quad (0 \leq t < h). \)

If confusion is unlikely we shall simply use the notations \( \tilde{\psi}_\lambda \) and \( \tilde{\phi}_\lambda \) instead of \( \tilde{\psi}_\lambda(p_n; h, \cdot) \) and \( \tilde{\phi}_\lambda(p_n; h, \cdot) \).

3.2.3. Some properties of the function \( \tilde{\psi}_\lambda \)

In the following lemma a number of properties of the normalized exponential \( \ell \)-polynomial \( \tilde{\phi}_\lambda \) is collected. In accordance with Definition 3.2.5 it is assumed throughout this subsection that \( \tilde{p}_n(\lambda) \neq 0 \).

**Lemma 3.2.7.** The function \( \tilde{\psi}_\lambda \) as defined in Definition 3.2.5 enjoys the following properties:

i) \( \tilde{\psi}_\lambda \) has the representation
\[ (3.2.9) \quad \hat{\psi}_n(t) = \frac{\hat{p}_n(\lambda)}{2\pi i} \oint_C \frac{e^{tz}}{(\lambda - e^{h}z)^n} \, dz \quad (t \in \mathbb{R}) , \]

where \( C \) is any contour in the complex plane surrounding the zeroes of \( p_n \) but excluding the poles of \( z \mapsto e^{tz}(\lambda - e^{h}z)^{-1} \).

1) For any \( t \in \mathbb{R} \) the function \( \lambda \mapsto \hat{\psi}_n(t) \) is a polynomial of degree at most \( n-1 \) with leading coefficient \( \hat{p}_n(t) \).

2) If \( p_n(0) \) is disconjugate on \([0, nh]\), then \( \hat{\psi}_n \) admits the representation

\[ (3.2.10) \quad \hat{\psi}_n(t) = c \sum_{\nu=0}^{n-1} \lambda^\nu \hat{p}_n(t+(n-\nu)h) \quad (0 \leq t < h) , \]

where the constant \( c \) does not depend on \( \lambda \).

4) Let the polynomial \( p_n^* \) be defined by \( p_n^*(t) := (-1)^n p_n(-t) \) (cf. (1.4.24)), then

\[ (3.2.11) \quad \hat{\psi}_n(p_n^*(h), h-t) = (-1)^n \hat{p}_n^*(0) \lambda^{n-1} \hat{\psi}_n(p_n^*(h), t) \quad (t \in \mathbb{R}) . \]

5) For sufficiently small \( |\lambda| \) the function \( \hat{\psi}_n \) can be written in the form

\[ (3.2.12) \quad \hat{\psi}_n(t) = -\frac{\hat{p}_n(\lambda)}{2\pi i} \sum_{k=0}^n \hat{p}_n(t-(k+1)h) \lambda^k \quad (t \in \mathbb{R}) . \]

**Proof.**

1) Let \( f \), considered as a function of \( t \), be defined by the right-hand side of (3.2.9). Then \( f \in \text{Ker}(p_n) \) since the contour \( C \) excludes the poles of \( z \mapsto e^{tz}(\lambda - e^{h}z)^{-1} \). Moreover, one readily verifies that

\[ f(t+h) - \lambda f(t) = -\frac{\hat{p}_n(\lambda)}{2\pi i} \oint_C \frac{e^{tz}}{p_n(z)} \, dz . \]

It is well known that

\[ (3.2.13) \quad \hat{p}_n(t) = \frac{1}{2\pi i} \oint_C \frac{e^{tz}}{p_n(z)} \, dz , \]

so we conclude from (3.2.9) and the uniqueness of \( \hat{\psi}_n \) that \( \hat{\psi}_n = f \), i.e., that (3.2.9) holds.

2) Assume for simplicity that \( p_n \) has distinct zeroes \( \beta_1, \beta_2, \ldots, \beta_n \), then according to the residue theorem property (3.2.9) implies
\[ (3.2.14) \quad \psi_\lambda(t) = \frac{\rho_\lambda(\lambda)}{p_n(\lambda)} \sum_{i=1}^{n} \frac{1}{p_n'(\lambda)} \frac{e^{-t \lambda_i^{2}}}{(\lambda - e^{-\lambda_i^{2}})} \]

In view of (3.2.1), we observe that \( \lambda \mapsto \psi_\lambda(t) \) is a polynomial in \( \lambda \) of degree at most \( n-1 \) with leading coefficient

\[ \frac{\rho_\lambda(\lambda)}{p_n(\lambda)} \]

which, by the residue theorem applied to (3.2.13), equals \( \varphi(t) \). Thus property ii) is established in case \( p_n \) has distinct zeros.

If \( p_n \) has multiple zeros, then iii) may be obtained by considering the distribution of the zeros at hand as the limit of a sequence of distributions of distinct zeros.

iii) This property follows from (3.2.5) and the observation that

\[ M(p_n, \ldots, \lambda h + h; t) = B_n(p_n, t - \lambda h) \]  

iv) In view of (3.2.6), the function \( \psi_\lambda(p_n^* h, \cdot) \) satisfies the functional relation

\[ \theta_\lambda(p_n^* h, h - t) = \lambda \theta_\lambda(p_n^* h, h - t) \]  

Moreover, we note that

\[ \frac{\rho_\lambda(\lambda)}{p_n(\lambda)} \left[ \frac{1}{p_n'(\lambda)} \right] \, \varphi(t) = (-1)^{n-1} \psi(t) \]  

Consequently,

\[ \psi_\lambda(p_n^* h, h - t) = \frac{1}{\lambda} \psi_\lambda(p_n^* h, h - t) = (-1)^{n-1} \frac{\rho_\lambda(\lambda)}{p_n(\lambda)} \lambda^{n-1} \psi(\lambda - \frac{1}{\lambda}) \psi(t) \]

and hence (cf. (3.2.6))

\[ \psi_\lambda(p_n^* h, h - t) = (-1)^{n-1} \frac{\rho_\lambda(\lambda)}{p_n(\lambda)} \lambda^{n-1} \psi(\lambda - \frac{1}{\lambda}) \psi(t) \]

v) This property follows from (3.2.9) if we replace \( e^{x\lambda} (\lambda - e^{x\lambda})^{-1} \) by the series

\[ \sum_{k=0}^{\infty} e^{-(\lambda - 1)(h + 1)k} \lambda^k \]

which converges on the contour \( C \) if \( |\lambda| \) is sufficiently small.
Assuming \( p_n(D) = D^n \) (polynomial case), together with \( h = 1 \), we observe that (3.2.12) takes the form

\[
(3.2.15) \quad \psi_\lambda(D^n;1,t) = (1-t)^n \sum_{k=0}^{n} \frac{k+1-\lambda}{(n-1)!} k^t \lambda^k \quad \text{for} \ |\lambda| < 1.
\]

Comparing this result with formula (3.10) for \( p_{n-1} \) in Ter-Morsche [40, p. 212], we observe that \( \psi_\lambda(D^n;1,\cdot) \) agrees with \( p_{n-1} \) up to the multiplicative constant \( (n-1)! \). Consequently, properties of normalized exponential polynomials (we recall that in the polynomial case the \( \delta \) is omitted from the notation) can be obtained from results that have been proved for \( p_{n-1} \) in Ter-Morsche [40, p. 212]. In the following subsection we list some of these properties.

3.2.4. Some properties of the normalized exponential polynomials

In this subsection \( \psi_\lambda \) denotes the normalized exponential polynomial, i.e.,

\[ \psi_\lambda := \psi_\lambda(D^n;h,\cdot). \]

According to Ter-Morsche [40, p. 212] and taking into account the results of the previous subsection, we observe that the function \( \psi_\lambda \) has the following properties:

\[
\begin{align*}
(3.2.16) \quad \psi_\lambda(t+h) - \lambda \psi_\lambda(t) &= \frac{(\lambda-1)^n h^{n-1} \psi_\lambda(t)}{(n-1)!}, \\
(3.2.17) \quad \psi_\lambda(0) &= \frac{n^{n-1}}{(n-1)!} \psi_\lambda(1),
\end{align*}
\]

in which \( \psi_{n-1} \) is the so-called Euler-Frobenius polynomial of degree \( n-2 \) (cf. Schoenberg [51, p. 22]), where an interesting survey of the main properties of these polynomials is given. The first few Euler-Frobenius polynomials are

\[
\begin{align*}
(3.2.18) \quad \psi_0(\lambda) &= 1, & \psi_3(\lambda) &= \lambda^2 + 4\lambda + 1, \\
\psi_1(\lambda) &= 1, & \psi_4(\lambda) &= \lambda^3 + 11\lambda^2 + 11\lambda + 1, \\
\psi_2(\lambda) &= \lambda + 1, & \psi_5(\lambda) &= \lambda^4 + 26\lambda^3 + 66\lambda^2 + 26\lambda + 1.
\end{align*}
\]

\[
(3.2.19) \quad \psi_{n-1}(t) = \frac{(-2h)^{n-1}}{(n-1)!} E_{n-1}\left(\frac{t}{h}\right),
\]

where \( E_{n-1} \) is the Euler polynomial of degree \( n-1 \) (cf. p. 58),
(3.2.20) \[ \frac{\partial}{\partial t} \tilde{\psi}_\lambda(t^n, h, t) = (\lambda - 1) \psi_{\lambda, h}^{(n-1)}(t^n, h, t) \, . \]

(3.2.21) \[ \psi_{\lambda, h}^{(n)}(t) = \frac{1}{(\lambda - 1)^n} \sum_{k=0}^{n-1} \binom{n-1}{k} (\lambda - 1)^k \sum_{\lambda_{1-1-k}} h h^{n-1-k} \, . \]

(3.2.22) \[ \psi_{\lambda, h}^{(n)}(1 - \epsilon) = \epsilon^{n-1} \psi_{\lambda, h}^{(n)}(1 - \epsilon) \, . \]

3.2.5. Extension of the definition of the normalized exponential L-polynomial

We recall (cf. p. 59) that the function \( \tilde{\psi}_\lambda(p_n, h, \cdot) \) is defined for those values of \( \lambda \) for which \( \tilde{\psi}_\lambda(p_n, \cdot) \neq 0 \) and that (cf. property ii) of Lemma 3.2.7) \( \lambda \rightarrow \tilde{\psi}_\lambda(p_n, h, t) \) is a polynomial. This gives rise to the following

DEFINITION 3.2.8. Let \( \alpha \in \mathbb{C} \) be such that \( \tilde{\psi}_\lambda(p_n, h, \cdot) = 0 \). Then the normalized exponential L-polynomial \( \psi_{\alpha}(p_n, h, \cdot) \) is defined by

(3.2.23) \[ \psi_{\alpha}(p_n, h, t) := \lim_{\lambda \rightarrow \alpha} \tilde{\psi}_\lambda(p_n, h, t) \quad (t \in \mathbb{R}) \, . \]

In view of (3.2.6) the function \( \psi_{\alpha} \) satisfies the functional relation

(3.2.24) \[ \psi_{\alpha}(p_n, h, te) = \psi_{\alpha}(p_n, h, t) \quad (t \in \mathbb{R}) \, . \]

If there is precisely one zero \( \beta \) of \( p_n \) satisfying \( \beta h = \epsilon \), then, since \( p_n(\beta h) \psi_{\alpha}(t) = 0 \ (t \in \mathbb{R}) \), there exists a constant \( c_\alpha \) such that

(3.2.25) \[ \psi_{\alpha}(p_n, h, t) = c_\alpha e^{\beta t} \quad (t \in \mathbb{R}) \, . \]

The constant \( c_\alpha \) in (3.2.25) may be obtained by applying the residue theorem in (3.2.6); we observe that we only have to compute the residue at \( z = \beta \).

Hence

(3.2.26) \[ c_\alpha = \lim_{\lambda \rightarrow \beta} \tilde{\psi}_\lambda(p_n, \cdot) \text{ Res}_{z = \beta} \frac{1}{(1 - e^{-z} h)p_n(z)} = q_{n-k}(\beta) \neq 0 \, , \]

where the polynomial \( q_{n-k} \) is given by \( q_{n-k}(z) = (z - \beta)^{k} p_n(z) \), \( k \) being the multiplicity of \( \beta \).

If there exist at least two different zeroes \( \beta_1 \) and \( \beta_2 \) such that \( e^{\beta_1} = e^{\beta_2} = \epsilon \), then a short calculation shows that
\begin{equation}
\lim_{\lambda \to 0} p_n^*(\lambda) \text{ Res}_{z=0} \frac{e^{tz}}{(\lambda - e^{it})p_n(z)} = 0 \quad (t \in \mathbb{R}).
\end{equation}

Consequently, \( \psi_{\lambda}(p_n^*, h, t) = 0 \quad (t \in \mathbb{R}). \)

REMARK. If \( p_n^*(a) = 0 \) and \( a = e^{ih} < 0 \), then \( h \) is a nonreal zero of \( p_n^* \). Since \( p_n \) is a polynomial having real coefficients, the complex conjugate \( \bar{h} \) of \( h \) is a zero of \( p_n \) which also satisfies \( e^{ih} = a \). It follows from the foregoing that \( \psi_{\lambda}(p_n^*, h, t) = 0 \quad (t \in \mathbb{R}). \)

3.2.6. On the zeros of the normalized exponential \( L \)-polynomials

The first objective of this subsection is to prove a result similar to the following

**Lemma 3.2.9** (Michelli [37, p. 210]). Let \( p_n \in \text{nst} \) be a monic polynomial having only real zeros. Then for each \( h > 0 \) and each \( \lambda < 0 \) the function \( \psi_{\lambda}(p_n^*, h, \cdot) \) has exactly one zero in \((0, h)\).

If \( p_n \) has zeros not all of which are real, an additional condition has to be formulated to ensure that \( \psi_{\lambda} \) has precisely one zero in \((0, h)\). This is specified in the following

**Lemma 3.2.10.** If the operator \( p_n(D) (n \geq 2) \) is disconjugate on \([0, (n-1)h]\), then for every \( \lambda < 0 \) the function \( \psi_{\lambda}(p_n^*, h, \cdot) \) has precisely one zero in \((0, h)\) or \( \psi_{\lambda}(p_n^*, h, \cdot) \) is identically zero.

**Proof.** We may assume that \( \psi_{\lambda} \) is a nontrivial function. Since \( \psi_{\lambda} \in \text{ker}(p_n^*) \) is an analytic function, each of its zeros is a strong zero in \( \mathbb{H} \) having multiplicity not exceeding \( n - 1 \) with respect to any sequence of differential operators \( (D_{\lambda})_{n-1} \). As the operator \( p_n(D) \) is disconjugate on \([0, (n-1)h]\) we may write (cf. (1.4.8)): \( p_n(D) = D^{n-1} + \cdots + D_{n-1}D_1 \). Taking into account Corollary 1.4.22 one has

\begin{equation}
\|(D_{\lambda})_{n-1}^v, (0, (n-1)h)\| \leq n - 2 + n - 1 = 2n - 3.
\end{equation}

If \( \psi_{\lambda}(0) = \psi_{\lambda}(h) \neq 0 \) then \( \psi_{\lambda} \) has an odd number of zeros in \((0, h)\). Consequently, the assumption that \( \psi_{\lambda} \) has at least three zeros in \((0, h)\), combined
with the functional relation (cf. 1), p. 59) \( \psi(t+h) = \lambda^h(t) \), implies
\[
\mathcal{Z}(D_{\lambda}^{-1} I_{Y_{\lambda}}(0,(n-1)h)) \leq 3n - 3
\]
which contradicts (3.2.27). Hence \( \psi_\lambda \) has precisely one zero in \([0,h)\).
If \( \psi_\lambda(0) = -\psi_\lambda(h) = 0 \) and if \( \psi_\lambda \) has at least one zero in \((0,h)\), then \( D_{\lambda} \psi_\lambda \) has at least two zeros in \((0,h)\). Hence,
\[
\mathcal{Z}(D_{\lambda}^{-1} I_{Y_{\lambda}}(0,(n-1)h)) \geq 2n - 2
\]
On the other hand, it follows from Corollary 1.4.22 that
\[
\mathcal{Z}(D_{\lambda}^{-1} I_{Y_{\lambda}}(0,(n-1)h)) \leq n - 2 + n - 2 = 2n - 4
\]
Again we conclude that \( \psi_\lambda \) has precisely one zero in \([0,h)\). This proves the lemma.

Next \( \psi_\lambda \) is considered as a function of \( \lambda \). In view of Lemma 3.2.7 (property iii)) the function \( \lambda \mapsto \psi_\lambda(t) \) is a polynomial of degree \( n - 1 \) in case \( \psi(t) \not= 0 \) and of degree at most \( n - 2 \) if \( \psi(t) = 0 \).

With respect to the polynomial case \( p_\lambda(0) = D^n \), it is shown in Ter Horst in [41] that the polynomial \( \lambda \mapsto \psi_\lambda(t) \) has \( n - 1 \) distinct negative zeros if \( 0 < t < h \) and \( n - 2 \) distinct negative zeros if \( t = 0 \). Michelli established that the same conclusion holds if \( p_\lambda \) has only real zeros. In fact, one has

**Lemma 3.2.11** (Michelli [37, p. 211]). Let \( p_\lambda \in \Pi_n \) be a monic polynomial having only real zeros. Then for all \( t \in [0,h) \) the polynomial \( \lambda \mapsto \psi_\lambda(p_\lambda(t), t) \) has only distinct negative zeros.

If not all zeros of \( p_\lambda \) are real, there are, to the best of our knowledge, no general results available with respect to this problem. The following simple example shows what kind of situations may occur.

**Example.** Let the operator \( p_\lambda(0) \) be given by \( p_\lambda(0) = D(D^2 + 1) \). Using (3.2.9) we obtain
\[
(3.2.28) \quad \psi_\lambda(t) = (1 - \cos t) \lambda^2 + (\cos(t+h) + \cos(t-h) - 2 \cos h) \lambda + 1 - \cos(t-h)
\]
This formula yields the following conclusions:
1) If \( h = \pi \) and \( t = \pi / 2 \), then \( \psi_3(t) = \lambda^2 + 2\lambda + 1 \) and thus \( \lambda = -1 \) is a negative zero of multiplicity two;

2) If \( h = 2\pi \) and \( t = 3\pi / 2 \), then \( \psi_3(t) = \lambda^2 - 2\lambda + 1 \) and thus \( \lambda = 1 \) is a positive zero of multiplicity two;

iii) If \( 0 < h < \pi \) and \( t = h / 2 \), then \( \lambda = \psi_3(t) \) has two distinct negative zeros.

In fact, the assumption \( 0 < h < \pi \) ensures that \( \lambda \mapsto \psi_3(t) \) has two distinct negative zeros for all \( t \in (0, h) \).

With respect to the general case we conjecture that the disconjugacy of \( p_n(t) \) on \([0, nh]\) will be sufficient to prove the assertion in Lemma 3.2.11.

3.2.7. The perfect Euler I-splines

We recall that perfect Euler I-splines are introduced in Definition 1.3.5.

A particular kind of these splines, the so-called perfect Euler I-splines, play a significant role in Chapters 5 and 6. Their definition reads as follows.

**DEFINITION 3.2.12.** Let the mesh distance \( h \) be such that \( \bar{p}_n(-1) \neq 0 \). A function \( s \), defined on \( \mathbb{R} \), is said to be a perfect Euler I-spline of order \( n + 1 \) corresponding to the operator \( p_n(t) \) if it satisfies the conditions

\[
\begin{align*}
\text{(i)} & \quad s \in C^{(n-1)}(\mathbb{R}), \\
\text{(ii)} & \quad s(t+h) = s(t) \quad (t \in \mathbb{R}), \\
\text{(iii)} & \quad p_n(0)s(0) = -1 \quad (0 < t < h),
\end{align*}
\]

It follows from Lemma 3.2.1 that the condition \( \bar{p}_n(-1) \neq 0 \) ensures that \( s \) is uniquely determined by (3.2.29). A perfect Euler I-spline will be denoted by \( E(p_n|h, s) \).

In view of (3.2.9) there exists a nonzero constant \( c \) such that for \( 0 < t < h \)

\[
E(p_n|h, t) = \frac{c}{2\pi i} \oint_C \frac{e^{xz}}{(1 + e^{hz})p_n(z)} \, dz,
\]

where \( C \) is any contour in the complex plane surrounding the zeros of \( p_n(z) \) but excluding the poles of \( z \mapsto e^{xz} (1 + e^{hz})^{-1} \).
Since \( p_n(D) E(p_n; h, t) = -1 \) \((0 < t < h)\) the constant \( c \) is given by

\[
c^{-1} = -\frac{1}{2\pi i} \oint_{C} \frac{dz}{(1 + e^{zh})z} = -\frac{1}{2}.
\]

Hence \( c = -2 \) and so

\[
(3.2.30) \quad E(p_n; h, t) = -\frac{1}{\pi i} \oint_{C} \frac{e^{zh}}{(1 + e^{zh})2p_n(z)} \, dz \quad (0 < t < h).
\]

3.3. Relations between finite differences and derivatives

We recall that the introductory section of this chapter contains a number of relations between derivatives and finite differences of a polynomial cubic spline. The purpose of this section is to generalize this kind of relations to cardinal \( L \)-splines corresponding to an arbitrary operator \( p_n(D) \). The main results are Theorems 3.3.2 and 3.3.3, the contents of which for the polynomial case \( p_n(D) = D^n \) may be found in Ter Horsho [41].

**DEFINITION 3.3.1.** Let the normalized exponential \( L \)-polynomial be given by (cf. property ii), p. 66)

\[
(3.3.1) \quad \phi_{\lambda}(p_n; h, t) = \sum_{\nu=0}^{n-1} a_{\nu}(t) x^{\nu}.
\]

Then the operator \( \Phi_{L}(p_n; h, t) \) is defined by

\[
(3.3.2) \quad \Phi_{L}(p_n; h, t) = \sum_{\nu=0}^{n-1} a_{\nu}(t) x^{\nu},
\]

where (cf. Definition 2.4.5 and p. 44 for notational \( h \) \((x) := f(x + h) \) \((x \in \mathbb{R})\).

We shall now prove the following fundamental result from which the required relations between derivatives and finite differences may easily be derived.

**THEOREM 3.3.2.** Let \( s \in S(p_n; h) \), the class of cardinal \( L \)-splines corresponding to the operator \( p_n(D) \) and with mesh distance \( h \). Then
\[ (3.3.3) \quad \Psi_n(p_n; h, y) = \sum_{k=0}^{m} (h + k) = \psi_E(p_n; h, x) = \psi_E(p_n; h, x^2 (h + y)) , \]

where \( n \in \mathbb{Z} \) and \( 0 \leq k \leq h, 0 \leq y \leq h. \)

**Proof.** The proof will be divided into several parts. We first show that (3.3.3) is valid for functions \( s \in \text{Ker}(p_n) \). Then (3.3.3) is shown to hold for the Green's function \( \varphi_n \) corresponding to \( p_n(h) \), under the additional condition that \( p_n \) has distinct zeros. Having established this, we point out that this additional condition can be dropped without difficulty. The last step then is to prove (3.3.3) for a general cardinal \( \mathcal{C} \)-spline; this is done by making use of its representation (1.3.1).

As for the first part of the proof, let \( s \in \text{Ker}(p_n) \). It is then sufficient to prove (3.3.3) for the functions \( t \rightarrow t^le_t \) (\( j = 0, 1, \ldots, k-1 \)), where \( \delta \in \mathcal{C} \) is a zero of multiplicity \( k \) of \( p_n \). We can even restrict ourselves to a proof of (3.3.3) for the function \( t \rightarrow t^le_t \), since (3.3.3) then holds for \( t \rightarrow (t - u)^le_t \) (\( u \in \mathbb{Z} \)) and it suffices to observe that \( t^le_t \) is a linear combination of these functions and that \( \psi_E \) is a linear operator. Now let \( \epsilon \) be an arbitrary positive number. We define the polynomial \( p_{n, \epsilon} \) by

\[ p_{n, \epsilon}(z) = (z - \delta)^{n-k} p_n(z) , \]

where (cf. p. 63) \( q_{n-k}(z) = (z - \delta)^{-k} p_n(z) \).

Moreover, let the functions \( s_{j, \epsilon} \) (\( j = 1, \ldots, k \)) be defined by

\[ s_{j, \epsilon}(t) = \epsilon (t - (j-1)\epsilon) \quad (t \in \mathbb{R}) . \]

According to (3.3.2) and (3.2.25)

\[ (3.3.4) \quad \psi_E(p_{n, \epsilon}; h, y) s_{j, \epsilon}(h + x) = \psi_E(p_{n, \epsilon}; h, y) s_{j, \epsilon}(h + x) = \psi_E(p_{n, \epsilon}; h, x^2) s_{j, \epsilon}(h + y) . \]

We now introduce the function \( \delta \) as the divided difference (cf. Subsection 1.4.3) of the function \( t \rightarrow e_t \) with respect to the \( k \) points \( \beta - (k-1)\epsilon, \beta - (k-2)\epsilon, \ldots, \beta \). In view of (1.2.7) one has (cf. p. 29 for notation)

\[ s_{j, \epsilon}(t) = \frac{k-1}{(k-1)!} \int_{\beta}^{t} e_{\beta - (k-1)\epsilon} \, dt . \]
and so

\[(3.3.5) \quad \lim_{\epsilon \to 0} s_{\epsilon}(t) = \frac{k-1}{(k-1)!} e^{st}.
\]

Since $s_{\epsilon}$ is a linear combination of $s_{1,\epsilon}, \ldots, s_{k,\epsilon}$, it follows from (3.3.4) that

\[(3.3.6) \quad \varphi_{\epsilon}(p_{n,\epsilon}^1, y)s_{\epsilon}(u + x) = \varphi_{\epsilon}(p_{n,\epsilon}^1, x)s_{\epsilon}(u + y).
\]

Letting $\epsilon \to 0$ in (3.3.6) and using (3.3.5) one obtains (3.3.3), thus showing that (3.3.3) holds for all $s \in \text{Ker}(p_{n,\epsilon}^1)$.

The next step is to prove (3.3.3) for the Green's function $\varphi_{\epsilon}$ corresponding to the operator $p_{n}^1(0)$. In view of (3.3.1) this implies that we have to establish

\[(3.3.7) \quad \sum_{\nu=0}^{\nu-1} a_{\nu}(y)\varphi_{\epsilon}(u + x + y) = \sum_{\nu=0}^{\nu-1} a_{\nu}(y)\varphi_{\epsilon}(u + x + y),
\]

where $u \in \mathbb{Z}$ and $0 \leq x \leq h$, $0 \leq y \leq h$.

We observe that the two cases can be dealt with easily: if $u < -n$ then (3.3.7) holds trivially; if $u > 0$ then $\varphi_{\epsilon}$ may be replaced by $\varphi_{\epsilon} \in \text{Ker}(p_{n,\epsilon}^1)$, for which (3.3.7) has already been shown to be true.

Now let $u = -k$ in (3.3.7), with $k = 1, 2, \ldots, n-1$. It then follows from the above considerations that we have to show that

\[(3.3.8) \quad \varphi_{\epsilon}(p_{n,\epsilon}^1, y)\varphi_{\epsilon}(x - kh)
\]

is symmetric in $x$ and $y$ for $k = 1, 2, \ldots, n-1$.

Assume first that $p_{n}$ has distinct zeros $\delta_{1}, \delta_{2}, \ldots, \delta_{n}$ ($\delta_{1} < \delta_{2} < \cdots$). We define the polynomials $q_{j} \in \mathbb{P}_{n-1}$ ($j = 1, \ldots, n$) by

\[(3.3.9) \quad q_{j}(z) := (z - \delta_{j})^{-1} p_{n}(z)
\]

and thus (cf. (3.2.1))

\[\varphi_{\epsilon}(z) = \prod_{1<j} (z - \delta_{j}^{-1}).
\]

It follows from (3.2.9) and the residue theorem that $\varphi_{\epsilon}(p_{n,\epsilon}^1, y)$ can be written as

\[(3.3.10) \quad \varphi_{\epsilon}(p_{n,\epsilon}^1, y) = \sum_{j=1}^{n} \frac{\delta_{j}^{-1}}{p_{n}^1(\delta_{j})} q_{j}(y).
\]
Hence

\[ (3.3.11) \quad \psi_E(p_n;h,y)q_n(x-kh) = \sum_{j=1}^{n-1} \sum_{k=1}^{h-1} \frac{\phi_j(y) \psi_k(y)}{p_n(z) \psi_k(z)} \xi_{j,k} q_n(x+(j-k)h) , \]

\[ \xi_{j,k} \text{ being the coefficient of } z^j \text{ in } q_n^*(z). \]

Another application of the residue theorem yields that \( \xi_{j,k} \) can be represented in the form

\[ (3.3.12) \quad \xi_{j,k} = \frac{1}{2\pi i} \oint_C \frac{\tilde{p}_n(z)}{(z-e^{j\theta})^{k+1}} \, dz \]

where \( C \) is a closed contour surrounding \( z = 0 \).

Substituting (3.3.12) into (3.3.11) we obtain (cf. (3.2.13))

\[ (3.3.13) \quad \psi_E(p_n;h,y)q_n(x-kh) = \frac{1}{2\pi i} \oint_C \frac{\psi_z(p_n;h,y)}{z^{k+1}} \sum_{j=1}^{n-1} \frac{\phi_j(y) \psi_k(y)}{\phi_j(z) \psi_k(z)} \xi_{j,k} q_n(x+(j-k)h) \, dz . \]

Now let the function \( H \) be defined by

\[ (3.3.14) \quad H(x) := \sum_{k=1}^{n-1} \frac{\phi_k(y)}{z^{k+1}} \quad (x \in \mathbb{R}) . \]

Then \( p_n(z)H(x) = 0 \) for \( x \in \mathbb{R} \) and

\[ U(x+h) = xH(x) = \psi(x+(n-k)h) - \frac{\phi_k(y)}{z^{k+1}} . \]

On account of this and (3.2.8) one has

\[ (3.3.15) \quad U(x) = \frac{1}{p_n(z)} \left\{ \frac{\psi_z(p_n;h,x)}{z^k} \frac{\phi_k(y)}{z^{k+1}} \right\} . \]

Denoting by \( \gamma \) a closed contour surrounding all zeros of \( \tilde{p}_n(z) \) and \( z = 0 \), we observe that

\[ (3.3.16) \quad \oint_{\gamma} \frac{\psi_z(p_n;h,y)}{z^{n}} \frac{\phi_k(y)}{p_n(z)} \, dz = 0 , \]

since the numerator of the integrand is, with respect to \( z \), a polynomial of degree at most \( 2n - 2 \) (cf. Lemma 3.2.7, property ii), whereas the denominator has degree \( 2n \).

Finally, substituting (3.3.15) into (3.3.13) and using (3.3.16) we have
\[ (3.3.17) \quad \psi_B(p_n, h, y) \psi_B(x - kh) = \frac{1}{2\pi i} \int \frac{\psi_B(p_n, h, y) \psi_B(z)}{z^k \sigma_B(z)} \, dz, \]

which clearly is symmetric with respect to \( x \) and \( y \). Consequently, (3.3.3) holds for the Green's function \( \psi_B \) under the additional assumption that the zeros of \( p_n \) are distinct.

If \( p_n \) has multiple zeros then formula (3.3.17) may be obtained by a limiting procedure; we omit the details.

In order to prove (3.3.3) for a general cardinal \( \ell \)-spline \( s \), we note (cf. (1.3.11)) that in the interval \([sh, (u+1)h]\) \( s \) can be represented as a linear combination of a function \( g \in \text{Ker}(p_n) \) and the functions \( t \mapsto \psi_B(t-jh) \) \((j = u+1, \ldots, u+n-1)\). Since all functions in that representation satisfy (3.3.3) we conclude that (3.3.3) holds for any cardinal \( \ell \)-spline \( s \in S(p_n, h) \).

This completely proves the theorem.

\( \Box \)

REMARK. The proof of Theorem 3.3.2 can be greatly simplified if we assume that \( p_n(D) \) is disconjugate on \([0, nh]\). In that case the \( \tau \)-spline representations \( (2.2.10) \) and \( (2.1.13) \) are available. The validity of (3.3.3) can then be shown as follows:

\[ \psi_B(p_n, h, y)s(uh+x) = c \sum_{\nu=0}^n \psi_B(p_n, y+(n-1-\nu)h)s(uh+\nu h+x) = \]

\[ = c \sum_{\nu=0}^n \psi_B(p_n, y+(n-1-\nu)h)\prod_{i=0}^{\nu-1} \sigma_B(p_n, x+(\nu-i)h) = \]

\[ = c \sum_{\nu=0}^n \psi_B(p_n, y+(n-1-\nu)h)\psi_B(p_n, x+(n-1-\nu)h) = \]

\[ = c \sum_{\nu=0}^n \psi_B(p_n, y+(n-1-\nu)h)\psi_B(p_n, x+(n-1-\nu)h) = \]

\[ = c \sum_{\nu=0}^n \psi_B(p_n, y+(n-1-\nu)h)s(uh+\nu h+y) = \psi_B(p_n, h, x)s(uh+y) \]

where in the course of writing down these identities we have replaced \( \nu \) by \( \sigma := n-1-\nu+1 \). Changing the order of summation is permitted, since all but a finite number of elements are zero.
The following theorem contains the required relations between derivatives and finite differences of cardinal $L$-splines.

**Theorem 3.3.3.** Let $s \in \mathcal{S}(p_n, h)$ and let $p_n$ be of the form $p_n = p_{n-k} \psi_n^k$, where $p_{n-k} \in \mathcal{P}_{n-k}$ and $\psi_n^k \in \mathcal{P}_{n-k}$ are both monic polynomials. Then

$$
\psi_n^k(p_n; h, y) \phi_n^k(D)s(\nu h + x) = \psi_n^k(p_{n-k}; h, x) \phi_n^k(D)s(\nu h + y),
$$

where $\nu \in \mathbb{Z}$ and $0 \leq x \leq h$, $0 \leq y \leq h$.

**Proof.** We note that $p_n^k(D) \phi_n^k(p_n; h, x)$ is an exponential $L$-polynomial corresponding to the operator $p_{n-k}^k(D)$ and satisfying the functional relation (cf. (3.2.8))

$$
p_n^k(D) \psi_n^k(p_n; h, x) = p_n^k(D) \phi_n^k(p_n; h, x) = -p_n^k(D) \phi_n^k(D) \psi_n^k(x) .
$$

Since $p_n^k(D) \psi_n^k$ is the fundamental function corresponding to the operator $p_{n-k}^k(D)$ it follows from (3.2.8) that

$$
p_n^k(D) \psi_n^k(p_n; h, x) = \psi_n^k(p_{n-k}; h, x),
$$

$x \in \mathbb{R}$.

By taking the "$p_{n-k}^k(D)$-derivative" of both sides of (3.3.3) with respect to $x$ and using (3.3.19), relation (3.3.18) immediately follows.

As an application of (3.3.18) we give a relation between second derivatives and second order differences of quintic splines: this relation also occurs in Schurer [57], where it is derived by different means.

For that purpose the Euler-Frobenius polynomials $\Pi_5$ and $\Pi_3$ as given by (3.2.18) are needed. In view of (3.2.17) one has

$$
\psi_3^j(D^2h, y) = \frac{3}{31} \Pi_5^j(\nu), \quad \psi_3^j(D^2h, 0) = \frac{5}{31} \Pi_5^j(\nu) .
$$

Substituting (3.3.20) into (3.3.18) with $x = y = 0$ we obtain, for any $\nu \in \mathbb{Z}$,

$$
s^\nu((\nu+4)h) + 26s^\nu((\nu+3)h) + 66s^\nu((\nu+2)h) + 26s^\nu((\nu+1)h) + s^\nu(\nu h) =
$$

$$
= \frac{20}{h^2} \left\{ 6p_2(E)s((\nu+4)h) + 4p_2(E)s((\nu+3)h) + p_2(E)s((\nu+2)h) \right\} =
$$

$$
= \frac{20}{h^2} \left\{ s((\nu+4)h) + 2s((\nu+3)h) - 6s((\nu+2)h) + 2s((\nu+1)h) + s(\nu h) \right\} .
$$

For $\nu = 0$ this relation does indeed agree with formula (12) in Schurer [57].
3.4. Some applications of Theorems 3.3.2 and 3.3.3

Chapter 4 deals with the problem of cardinal $f$-spline interpolation, i.e., the problem of determining an $s \in \mathcal{S}(p_n, h)$ in prescribed function values at the nodes $u \odot +u$ ($u \in \mathbb{Z}$), where $a \in (0, h]$ is fixed. Replacing \( y \) by $a$ in (3.3.3), we may interpret relation (3.3.3) as a difference equation for the unknown $s$, the characteristic polynomial of which is given by $z \rightarrow \varphi_z(p_n; h, a)$. Being interested in functions $s \in \mathcal{S}(p_n, h)$ that are of power growth, i.e.,

$|c(t)| = O(|t|^{\gamma}) \quad (|t| \rightarrow \infty)$

for some $\gamma \geq 0$, the following lemma will be needed in Chapter 4.

**Lemma 3.4.1.** Let $q_n \leq \mathcal{V}_n$ be a monic polynomial that has no zeros on the unit circle in the complex plane. Furthermore, let the difference equation

(3.4.1) $q_n^{(a)} x_y = b_y \quad (u \in \mathbb{Z})$

be given, where $Ex_y = \mathcal{V}_y$.

If the sequence $(b_y)_{-\infty}^\infty$ has the property that for some nonnegative $\gamma$,

(3.4.2) $|b_y| = O(|y|^{\gamma}) \quad (|y| \rightarrow \infty)$,

then (3.4.1) has a unique solution $(x_y)_{-\infty}^\infty$ with the property

(3.4.3) $|x_y| = O(|y|^{\gamma}) \quad (|y| \rightarrow \infty)$.

Moreover, this solution can be expressed in the form

(3.4.4) $x_y = \sum_{j=-\infty}^{\infty} \omega_j b_{y+j} \quad (u \in \mathbb{Z})$,

where the complex numbers $\omega_j$ are the coefficients in the Laurent expansion

(3.4.5) $q_n^{-1}(z) = \sum_{j=-\infty}^{\infty} \omega_j z^j$

that converges on a ring containing the unit circle in its interior.

**Proof.** By (3.4.5) and the conditions of the lemma the series $\sum_{j=-\infty}^{\infty} \omega_j |y|^{\gamma+j}$ converges. It is now easy to verify that (3.4.4) is a solution of (3.4.1) with the asserted property (3.4.3). It remains to prove the uniqueness of the solution. Taking into account the general solution of the homogeneous equation $q_n^{(a)} x_y = 0 \quad (u \in \mathbb{Z})$, we conclude that in this case only the trivial
solution \( x_0 = 0 \) (\( \mu \in \mathbb{R} \)) satisfies (3.4.3). This proves the unicity of the solution. \( \square \)

Lemma 3.4.1 may also be applied to relation (3.3.10) to the effect that \( p_k(0)s(\nu) \) is expressed in terms of function values of the form \( s(\nu + y) \) (\( \nu \in \mathbb{Z} \)). Such an expression is called a differentiation formula. By way of illustration we give two simple examples concerning cubic splines and quadratic splines.

EXAMPLE 1. The first relation in (3.1.1) may be written in the form

\[
(3.4.5) \quad \Pi_3(\nu) = \frac{3}{h} \left( \frac{2}{h} - 1 \right) s(\nu) \quad (\nu \in \mathbb{Z}) .
\]

As (cf. (3.2.10))

\[
\Pi_3(\omega) = \omega^2 + 4\omega + 1 = (\omega + 2 + \sqrt{3})(\omega + 2 - \sqrt{3}) ,
\]

\( \Pi_3 \) has no zeros on the unit circle and thus by putting \( x_0 := s'(0)h \),

\( b_0 := \frac{3}{h} \left( \frac{2}{h} - 1 \right) s(0) \) and applying (3.4.1) one obtains

\[
(3.4.7) \quad s'(0) = \frac{3}{h} \sum_{j=-\infty}^{\infty} \omega_j \left( s(jh + 2h) - s(jh) \right) .
\]

In this formula \( s \) is the cardinal cubic spline with knots \( \nu \in \mathbb{Z} \) and satisfying \( s(\nu) = 0 \) (\( \nu = 0 \)) for some nonnegative constant \( \gamma \). It is easy to verify that the coefficients \( \omega_j \) in (3.4.7) are, according to (3.4.5), explicitly given by

\[
\omega_j = \begin{cases} 
\frac{(-1)^{j+1}(2\gamma)^{-1}(2 + \gamma)^{j-1}}{h} & (j \geq 0), \\
\frac{(-1)^{j+1}(2\gamma)^{-1}(2 - \gamma)^{j-1}}{h} & (j < 0).
\end{cases}
\]

EXAMPLE 2. Let \( p_3(\nu) = \delta^3, \quad n = 3, \quad k = 1, \quad \gamma = \frac{1}{3} \) and \( y = 0 \). Relation (3.3.10) then takes the form

\[
(3.4.8) \quad (E + 1)s'(jh + yh) = \frac{1}{h} \left( E + 1 \right) \left( E - 1 \right) s(\nu) .
\]

As (cf. (3.2.10)) \( \Pi_3(\nu) = 0 \), we cannot apply Lemma 3.4.1. However, in this particular case it is easy to see that \( s'(jh) = \frac{1}{h} \left( s(\nu) - s(0) \right) \) holds for every cardinal quadratic spline with knots \( \nu \in \mathbb{Z} \).
4. ON EXISTENCE AND CONVERGENCE PROPERTIES OF INTERPOLATING CARDINAL $L$-SPLINES AND INTERPOLATING PERIODIC CARDINAL $L$-SPLINES

4.1. Introduction and summary

Let an arbitrary sequence of real numbers $(y_u)_{-m}^{m}$ be given. We consider the following interpolation problem:

Determine a function $s$ in $S(p_m, h)$, the set of cardinal $L$-splines corresponding to $p_m(0)$ and with mesh distance $h$, that satisfies

$$(4.1.1) \quad s(u + \alpha) = y_u \quad (u \in \mathbb{Z}),$$

where $\alpha \in (0, h)$ is a prescribed number.

If the sequence $(y_u)_{-m}^{m}$ has period $N$, i.e., if $y_{u+N} = y_u$ ($u \in \mathbb{Z}$) and if $s \in S(p_m, h)$ is uniquely determined by (4.1.1), then evidently $s$ is a periodic function with period $Nh$. For reasons of brevity, a periodic function with period $T$ will occasionally be called a $T$-periodic function and, similarly, a periodic sequence with period $N$ will be called an $N$-periodic sequence. Obviously, the problem of periodic $L$-spline interpolation is related in a natural way to the more general problem of cardinal $L$-spline interpolation (4.1.1). The subject of cardinal polynomial spline interpolation in the cases $h = 1, \alpha = 1$ and $h = 1, \alpha = \frac{1}{2}$ is studied in detail by Schoenberg [51] in his monograph (see in particular lecture 4).

Meir and Sharma [35] investigated the problem of periodic cubic spline interpolation with variable $\alpha \in (0, h)$ and proved the existence and uniqueness of a periodic interpolating cubic spline for $0 < \alpha \leq 1/3h$ and for $2/3h \leq \alpha < h$.

A few years ago we also dealt (cf. Ter Morsche [40, p. 188]) with the periodic version of (4.1.1) in the polynomial case and gave a complete solution of the existence problem. The main result of Ter Morsche [40] is stated in the following theorem.
THEOREM 4.1.1. Let an N-periodic sequence of real numbers \((y_u)_{u=-\infty}^{\infty}\) be given and let \(a \in [0,1]\) be an arbitrary prescribed number. Then there exists a unique \(N\)-periodic spline \(s \in S(\mathbb{R}^N, 1)\) satisfying the interpolation property 
\[ s(wh + a) = y_u \quad (u \in \mathbb{Z}) \]
in each of the following cases:

i) \(N\) is odd, \(n \in \mathbb{N}\) arbitrary, \(a\) arbitrary.
ii) \(N\) is even, \(n\) is even, \(a \neq 1/2\).
iii) \(N\) is even, \(n\) is odd, \(a \neq 1/2\).

The contents of this chapter may be briefly summarized as follows. In Section 4.2 results are derived concerning existence and uniqueness for the cardinal \(L\)-spline interpolation problem (4.1.1). We note that specific cases of this general problem have been dealt with earlier; for polynomial spline interpolation this was done in Terzopoulos [41], whereas Micchelli [37] investigated (4.1.1) polynomials \(p_n\) having only real zeros. Section 4.3 is devoted to the problem of existence and uniqueness of interpolating periodic cardinal \(L\)-splines. There an analogue of Theorem 4.1.1 is given for periodic cardinal \(L\)-splines corresponding to operators \(p_n(D)\) with the property 
\[ p_n(D) = p_n^s(D) \]
The last two sections, Sections 4.4 and 4.5, deal with error estimates for cardinal \(L\)-spline interpolation and periodic cardinal \(L\)-spline interpolation, respectively.

4.2. On the existence and uniqueness problem of cardinal \(L\)-spline interpolation

Given an arbitrary sequence of real numbers \((y_u)_{u=-\infty}^{\infty}\) and an arbitrary positive number \(h\), one may ask under what conditions a function \(s \in \mathcal{S}(p_n, h)\) exists satisfying (4.1.1). As the following lemma shows, the existence of such an interpolant is always guaranteed if the operator \(p_n(D)\) is disconjugate on \([0,h]\).

**Lemma 4.2.1.** Let \(p_n(D)\) \(n \geq 2\) be disconjugate on \([0,h]\). Then for every sequence \((y_u)_{u=-\infty}^{\infty}\) and for every \(a \in (0,h]\) the set of functions \(s \in \mathcal{S}(p_n, h)\) satisfying (4.1.1) is a linear manifold of dimension \(n-1\) if \(a \in (0,h)\), and of dimension \(n-2\) if \(a = h\).

**Proof:** We first assume that \(a \in (0,h)\). It is well known that the set of functions \(f \in \text{ker}(p_n)\) satisfying \(f(a) = y_0\) is a linear manifold of dimension \(n-1\). Let \(f\) be such a function. In order to construct a function...
s \in S(p_n,h) \) that coincides with \( f \) on \([0,h]\) and for which \( s(uh+\alpha) = y_u \) \((u \in \mathbb{Z})\) one may proceed as follows. On \([h,2h]\) we put \( s(t) = f(t) + c_n s(t-h) \)
(cf. 1.3.1), where the coefficient \( c_n \) is uniquely determined by the requirement \( s(h+\alpha) = y_1 \), which can be met since \( s(\alpha) \neq 0 \). On \([-h,0]\) one has \( s(t) = f(t) + c_{-1} s(t) \). As \( s(a-h) \neq 0 \), the number \( c_{-1} \) can be chosen such that \( s(-h+\alpha) = y_{-1} \). This process can be continued to yield a function \( s \in S(p_n,h) \) satisfying \( s(uh+\alpha) = y_u \) \((u \in \mathbb{Z})\), which is uniquely determined by the function \( f \). Hence, for \( 0 < a < h \), the set of functions \( s \in S(p_n,h) \) satisfying (4.1.1) is a linear manifold of dimension \( n - 1 \). If \( a = h \), the set of functions \( f \in \ker(p_n) \) for which \( f(0) = y_{-1} \) and \( f(h) = y_0 \) is a linear manifold of dimension \( n - 2 \). This, in fact, is a consequence of Lemma 1.4.17. Therefore the set of functions \( s \in S(p_n,h) \) satisfying \( s(uh) = y_{u-1} \) \((u \in \mathbb{Z})\) also is a linear manifold of dimension \( n - 2 \).

In case \( n = 1 \) it is easy to see that the interpolating \( L \)-spline \( s \in S(p_n,h) \) is uniquely determined for all \( a \in (0,h] \), since in this case \( s \) is by definition continuous to the right.

As a consequence of the previous lemma unicity cannot be expected for the interpolation problem (4.1.1), unless further restrictions on the interpolating \( L \)-spline are imposed. Following Schoenberg [51] and Michelli [37] we restrict ourselves to sequences \( (y_u)_{u=0}^{\infty} \) of power growth, i.e.,
\[
|y_u| = O(|u|^{-\gamma}) \quad (|u| \rightarrow \infty)
\]
for some nonnegative constant \( \gamma \). The interpolant \( s \) is then required to satisfy \( |s(t)| = O(|t|^{-\gamma}) \quad (|t| \rightarrow \infty) \). In connection with this we have the following result.

**Theorem 4.2.2.** Let \( p_n \) \((n \geq 2)\) be a nonzero polynomial and let \( \gamma \geq 0 \). Furthermore, let \( (y_u)_{u=0}^{\infty} \) be a sequence satisfying \( |y_u| = O(|u|^{-\gamma}) \quad (|u| \rightarrow \infty) \) and let \( a \in (0,h] \).

If (cf. Definition 3.2.5) \( \varphi_n(p_n,h,a) \) has no zeros on the unit circle, then there exists a unique \( s \in S(p_n,h) \) satisfying the conditions
\[
(4.2.1) \begin{cases} 
1) \quad s(uh+\alpha) = y_u \quad (u \in \mathbb{Z}), \\
2) \quad s(t) = O(|t|^{-\gamma}) \quad (|t| \rightarrow \infty).
\end{cases}
\]

If \( \varphi_n(p_n,h,a) \) has zeros on the unit circle then a function \( s \in S(p_n,h) \) satisfying 1) and 2) is either nonexistent or nonunique.
PROOF. Assume that \( \psi_z(p_n, h; a) \neq 0 \) on \( \{ z \in \mathbb{C} \mid |z| = 1 \} \). First we construct a so-called fundamental solution for the interpolation problem, i.e., a function \( L \) satisfying

\[
\begin{align*}
(4.2.3) \quad L(\nu h + a) &= \delta_{\nu, 0} \quad (\nu \in \mathbb{Z}), \\
(4.2.4) \quad |L(t)| &\text{ decays exponentially if } |t| \to \infty.
\end{align*}
\]

Since \( \psi_z(p_n, h; a) \) has no zeros on the unit circle, we may define numbers \( c_{\nu} \in \mathbb{C} \) (\( \nu \in \mathbb{Z} \)) by means of the Laurent expansion

\[
(4.2.5) \quad \psi_z^{-1}(p_n, h; a) = \sum_{\nu=-\infty}^{\infty} c_{\nu} z^\nu
\]

that converges on a ring in the complex plane containing the unit circle in its interior. Consequently, a number \( r \in (0, 1) \) exists such that

\[
(4.2.6) \quad c_{\nu} = O(r^{|\nu|}) \quad (|\nu| \to \infty).
\]

We recall (cf. Lemma 3.2.7, property ii)) that

\[
\psi_z(p_n, h; x) = \frac{1}{2 \pi i} \sum_{k=0}^{n-1} a_k(x) z^k \quad (z \in \mathbb{C}, \quad z \neq 0).
\]

Moreover, as a consequence of Formula (3.2.9) and taking into account the second part of the proof of Lemma 3.2.7, \( a_k \in \mathrm{Ker}(p_n) \) (\( k = 0, 1, \ldots, n-1 \)). Hence we may write for \( z \) in a ring containing the unit circle in its interior

\[
(4.2.5) \quad \frac{\psi_z(p_n, h; x)}{\psi_z(p_n, h; a)} = \sum_{\nu=-\infty}^{\infty} L_{\nu}(x) z^\nu,
\]

with

\[
(4.2.6) \quad L_{\nu}(x) = \sum_{k=0}^{n-1} c_{\nu-k} a_k(x) \quad (x \in \mathbb{R}).
\]

Since \( \psi_z(p_n, h) = z \psi_z((p_n; h, 0) \quad (j = 0, 1, \ldots, n-2) \) it follows from (4.2.5) that

\[
L_{\nu}^{(j)}(h) = L_{\nu-1}^{(j)}(0) \quad (j = 0, 1, \ldots, n-2).
\]

In view of this and (4.2.6) the function \( L \) defined by

\[
(4.2.7) \quad L(t) := L_{\nu}(t - \nu h) \quad (0 < t \leq \nu h; \quad \nu \in \mathbb{Z}),
\]

satisfies the properties (4.2.3) and (4.2.4). Hence, by the uniqueness theorem for the Fredholm integral equation of the second kind, we obtain

\[
\psi_z(p_n, h; a) = \int_{[0, 1]} L(t) d\nu h + a.
\]

Therefore, \( \psi_z(p_n, h; a) \neq 0 \) on \( \{ z \in \mathbb{C} \mid |z| = 1 \} \).
is a cardinal $\mathcal{L}$-spline: $L \in \mathcal{S}(\mathcal{P}_n,h)$. Furthermore, (4.2.5) implies that
\[ L(u+h) = L(u) = \delta_{u,0}, \quad \text{i.e., } L \text{ satisfies } i) \text{ of (4.2.2). Because of (4.2.4) and (4.2.6) one has} \]
\[ L(u,h) = 0(t \in \mathcal{U}/(|u| = \infty)). \]
Hence $L$ also satisfies 'ii' of (4.2.2) and therefore $L$ is the fundamental solution. Now, if $(y_u)_{u \in \mathbb{Z}}$ satisfies $y_u = O(|u|)$ ($|u| \to \infty$), then the function $s$ defined by
\[ s(t) = \sum_{\gamma \in \mathbb{Z}} y_\gamma L(t-\gamma h) \quad (t \in \mathbb{R}) \]
belongs to $\mathcal{S}(\mathcal{P}_n,h)$ and satisfies (4.2.1).

In order to prove the unicity of $s$, we note that a function $s \in \mathcal{S}(\mathcal{P}_n,h)$ having the required interpolation property satisfies the relation (cf. (3.3.3) and (3.3.1))
\[ \sum_{\gamma = 0}^{n-1} a_\gamma(s)(u) = \sum_{\gamma = 0}^{n-1} a_\gamma(s)y_{u+\gamma} \quad (u \in \mathbb{Z}, \ 0 \leq x < h). \]
Since $(\sum_{\gamma = 0}^{n-1} a_\gamma(s)y_{u+\gamma})_{u \in \mathbb{Z}}$ is a sequence of power growth, Lemma 3.4.1 applies. This yields the unicity of $s$.

Now assume that $\psi_{u}([u]_h, \omega) = 0$ for $\omega \in \mathcal{I}$ with $|\omega| = 1$. We observe that the second part of the theorem will be proved if we show the existence of a nontrivial bounded function $s$ in $\mathcal{S}(\mathcal{P}_n,h)$ for which $s(u) = 0$ ($u \in \mathbb{Z}$). Consider the function $\psi_{u}([u]_h, \omega)$ and assume that $\psi_{u}([u]_h, \omega)$ is a nontrivial function. Then, the complex-valued function $\hat{\psi}_{u}([u]_h, \omega)$ which coincides with $\hat{\psi}_{u}([u]_h, \omega)$ on $[0,h]$ and for which relation (1) of Definition 3.2.6 holds (with $\omega$ replaced by $\omega_u$) is bounded and such that
\[ \hat{\psi}_{u}([u]_h, \omega) = 0 \quad (u \in \mathbb{Z}). \]
Consequently, the real part or the imaginary part of $\hat{\psi}_{u}([u]_h, \omega)$ is a nontrivial bounded function in $\mathcal{S}(\mathcal{P}_n,h)$ vanishing at the points $\omega_u + u$ ($u \in \mathbb{Z}$). If $\psi_{u}([u]_h, \omega)$ is identically zero then (cf. subsection 3.2.5) $\psi_{u}([u]_h, \omega) = 0$ and there exist at least two different zeros $z_1$ and $z_2$ of $\psi$ such that $e^{z_1} = e^{z_2} = z_0$. Since $|z_0| = 1$ it is then easy to find a nontrivial $g \in \ker(\mathcal{P}_n)$ having the property $g(u) = 0$ ($u \in \mathbb{Z}$). This completely proves the theorem.

As a simple illustration we compute the fundamental solution in the case of trigonometric spline interpolation of order three (cf. p. 65). The
corresponding differential operator is given by $p_2(D) = D(D^2 + 1)$ and the normalized exponential $L$-polynomial by (3.2.28). We consider interpolation at the midpoints of the mesh intervals, i.e., we take $\alpha = \frac{1}{2}h$. A simple computation yields

$$\Psi_2(p_2; h, \frac{1}{2}h) = (1 - \cos(\frac{1}{2}h))(z^2 + \frac{1}{2}(2 \cos(\frac{1}{2}h) + 1)z + 1).$$

For $h \in (0, \pi)$ it is easily verified (cf. the example on p. 65) that $z \mapsto \Psi_2(p_2; h, \frac{1}{2}h)$ has two negative zeros $a_1$ and $a_2$ satisfying $a_1 < 0 < a_2$. Consequently, Theorem 4.2.2 applies. Partial fraction expansion yields

$$(4.2.10) \quad \Psi_2^{-1}(p_2; h, \frac{1}{2}h) = \frac{(a_2 - a_1)^{-1}}{(1 - \cos(\frac{1}{2}h))}\left( \sum_{k=0}^{\infty} a_1^{k-1} z^{k} + \sum_{k=0}^{\infty} a_2^{k-1} z^{k} \right),$$

i.e., the Laurent series that converges on the unit circle.

In accordance with its definition (cf. p. 79) the fundamental solution of the interpolation problem at hand is the bounded function $L \in S(p_n; h)$ satisfying $L(v + x) = \delta_{x,0} (v \in \mathbb{C})$. By means of (4.2.5), (4.2.6) and (4.2.7) $L$ may be computed on $[v h, v h + h]$ and we obtain

$$(4.2.11) \quad L(v h + x) = \frac{(a_1 - a_2)^{-1}}{(1 - \cos(\frac{1}{2}h))(1 - \cos(\frac{1}{2}h))}\left( \cos x + \cos(x + h) - 2 \cos h) a_1 + \cos x + \cos(x + h) - 2 \cos h) a_2 + 1 \right),$$

where $v \in \mathbb{C}$ and $0 \leq x \leq h$.

We recall (cf. Lemma 3.2.11) that the polynomial $z \mapsto \Psi_2(p_2; h, 0)$ has distinct, negative zeros if $p_2$ has only real zeros and $\alpha \in (0, h)$. It follows from (3.2.11) that for $\alpha = h$ the polynomial $z \mapsto \Psi_2(p_2; h, h)$ has zero at 0 while all its other zeros are negative. As a consequence, using Theorem 4.2.2, we obtain the following result, which has also been proved by Michelli [37] by a different method.

**Corollary 4.2.3.** Let $p_n \in \mathbb{P}_n$ be a monic polynomial having only real zeros, and let $\gamma$ be a nonnegative constant. Corresponding to every sequence of real numbers $(\gamma_n^n)_{n=1}^\infty$ satisfying $\gamma_n = 0(1|\gamma^n|)(1|\gamma^n| = 1)$ there exists a unique $s \in S(p_n; h)$ with the properties

1. $s(v h + a) = \gamma_n \quad (v \in \mathbb{C})$.}
11) \( \phi(z) = \emptyset(\{z\}^\gamma) \quad (|z| < \infty) \),

if and only if \( \varphi_{-1}(p_0; h, a) \neq 0 \).

In fact, Corollary 4.2.3 states that the interpolation problem (4.1.1) has a unique solution \( s \in S(p_0, h) \) satisfying the growth property 11) of (4.2.1) if and only if the Euler L-spline \( \psi_{-1}(p_0; h, a) \) does not vanish at \( a \).

If \( p_0 \) has zeros not all of which are real, general information about the location of the zeros of \( \varphi_{-1}(p_0; h, a) \) is lacking, and, consequently, corollaries to Theorem 4.2.2 similar to Corollary 4.2.3 are difficult to obtain.

We finally remark that Jakimovski and Russell [20] have derived some general results on the existence and unicity of polynomial splines that interpolate a sequence \( (p_j)_{m} \) at non-equidistant points; for details the reader is referred to their paper.

4.3. Periodic cardinal L-spline interpolation

The object of this section is to investigate existence and uniqueness properties of interpolating periodic cardinal L-splines.

Definition 4.3.1. A function \( s \) defined on the interval \([a, b]\) is said to be a periodic cardinal L-spline function of order \( n \) if \( s \) is an L-spline with equally spaced knots \( x_j = a + j(b - a)/N \) \((j = 0, 1, \ldots, N; N \in \mathbb{N})\), and if \( s \) satisfies the periodicity conditions

\[
(4.3.1) \quad s^{(j)}(a) = s^{(j)}(b) \quad (j = 0, 1, \ldots, n-2)
\]

It is well known [cf. Ahlberg, Nilson and Walsh [1, p. 165]] that for a given periodic function there exists a unique periodic polynomial spline of even order (odd degree) that interpolates \( f \) at the points \( x_j \) \((j = 0, 1, \ldots, N)\); in this case the knots and the nodes coincide. This, in general, is not true if the periodic splines are of odd order (even degree), i.e., interpolating periodic splines of odd order with the knots coinciding with the nodes may not exist. However, Subbotin [61] proved that an arbitrary periodic function can be interpolated by a unique periodic cardinal polynomial spline of odd order, if the nodes are at the midpoints of the mesh.
intervals \([x_{i-1}, x_i]\) \((i = 1, 2, \ldots, N)\).

If we take \(a = 0\) and \(h = (b - a)/N\), the periodic continuation of \(s\) to \(\mathbb{R}\) is simply an \(N\)-periodic cardinal \(L\)-spline. We shall denote the periodic cardinal \(L\)-splines in \(S(p_n, h)\) with period \(Nh\) by

\[
(4.3.2) \quad S(p_n, h, N).
\]

Since \(S(p_n, h, N)\) is a set of bounded functions in \(S(p_n, h)\), the following result is a consequence of Theorem 4.2.2.

**Lemma 4.3.2.** Let \(p_n\) \((n \geq 2)\) be a monic polynomial and let \(a \in (0, h]\) be such that the polynomial \(z \mapsto \varphi(z, p_n, h, a)\) has no zeros on the unit circle. Then for every \(N\)-periodic sequence \(\{y_k\}_{k=-\infty}^{\infty}\) there exists a unique \(s \in S(p_n, h, N)\) satisfying \(s(uh + a) = y_k\) \((u \in \mathbb{Z})\).

**Proof.** According to Theorem 4.2.2 there exists a unique bounded function \(s \in S(p_n, h)\) satisfying \(s(uh + a) = y_k\). Since \(t \mapsto s(t + Nh)\) also satisfies this interpolation property, it follows that \(s(t + Nh) = s(t)\) \((t \in \mathbb{R})\) and thus \(s \in S(p_n, h, N)\). 

According to Theorem 4.2.2 the condition that \(z \mapsto \varphi(z, p_n, h, a)\) has no zeros on the unit circle is necessary and sufficient to ensure existence and unicity of a bounded \(L\)-spline interpolating a bounded sequence of numbers. For periodic cardinal \(L\)-spline interpolation this condition is sufficient; however, it is not necessary. In fact, the next theorem states that the condition that \(z \mapsto \varphi(z, p_n, h, a)\) has no zeros on the unit circle can then be replaced by the weaker condition that \(z \mapsto \varphi(z, p_n, h, a)\) has no zeros at the \(N\)-th roots of unity, i.e., \(\varphi(z, p_n, h, a) \neq 0\) if \(z = e^{i\pi/N}\).

Two proofs will be given. The first one is rather long and is constructive in character. The second one is short and simple, but nonconstructive. Elements of the longer proof will be needed later; this is one of the main reasons why it is included here.

**Theorem 4.3.3.** Let \(p_n\) \((n \geq 2)\) be a monic polynomial and let \(N \in \mathbb{N}\). Furthermore, let \(\{y_k\}_{k=-\infty}^{\infty}\) be an \(N\)-periodic sequence and let \(a \in (0, h]\).

If \(z \mapsto \varphi(z, p_n, h, a)\) has no zeros at the \(N\)-th roots of unity, then there exists a unique \(s \in S(p_n, h, N)\) satisfying the interpolation condition \(s(uh + a) = y_k\) \((u \in \mathbb{Z})\).
If $z \rightarrow \Phi_z(p_n,h,a)$ has a zero at an $N$-th root of unity, then a function $s \in S(p_n,h,N)$ satisfying the interpolation condition is either nonexistent or nonunique.

**Proof.**

1. Assume first that $\Phi_z(p_n,h,a) \neq 0$ if $z^N = 1$. Our aim is to construct a function $s \in S(p_n,h,N)$ satisfying the given interpolation condition and, then, to prove that $s$ is uniquely determined. Let (cf. (3.3.1))

$$
\begin{align*}
\psi_u(x) &= \psi_u(p_n,h,x)(x + a), \\
\psi_u(x) &= s(x + bh) \quad (u = 0, 1, \ldots, N-1).
\end{align*}
$$

An application of (3.3.3) for $u = 0, 1, \ldots, N-1$, together with the periodicity conditions for the function $s$, leads to a system of linear equations $Aw = v$ of the form

$$
\begin{pmatrix}
\begin{array}{cccc}
a_0(a) & \cdots & a_{N-1}(a) \\
a_{N-1}(a) & a_0(a) & \cdots & a_{N-2}(a) \\
\vdots & \vdots & \ddots & \vdots \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
s(a) \\
s(a + h) \\
\vdots \\
\end{array}
\end{pmatrix}
=
\begin{pmatrix}
\begin{array}{c}
v_0(x) \\
v_1(x) \\
\vdots \\
v_{N-1}(x)
\end{array}
\end{pmatrix},
$$

where the $a_k(a)$ ($k = 0, 1, \ldots, N-1$) are given by (3.3.1) and $a_k(a) = 0$ if $k > n$.

In order to solve (4.3.4) we note that the matrix $A$ in (4.3.4) is a circulant matrix, which we denote by $C(a_0(a), a_1(a), \ldots, a_{N-1}(a))$. Hence (cf. Aitken [2, p. 124]) $A$ can be written in the form

$$
(4.3.5) \quad A = a_0(a)I + a_1(a)Q + \ldots + a_{N-1}(a)Q^{N-1},
$$

where $Q$ is the $N \times N$ circulant matrix $C(0, 1, 0, \ldots, 0)$.

We further need the following well-known fact about circulant matrices (cf. Aitken [2, p. 124]): $C(0, 1, 0, \ldots, 0)$ has eigenvalues $\lambda_k := e^{2\pi i k/N}$ ($k = 0, 1, \ldots, N-1$) with associated independent eigenvectors $(1, e^{i\pi k/N}, \ldots, e^{i(N-1)\pi k/N})^T$ ($k = 0, 1, \ldots, N-1$). Consequently, in view of (4.3.5) the eigenvalues $\rho_k$ of $A$ are $\rho_k = \sum_{j=0}^{N-1} a_k(a)e^{i\pi kj/N}$. Hence (cf. (3.3.1))

$$
(4.3.6) \quad \rho_k = \psi_{u_k}(p_n,h,a) \quad (k = 0, 1, \ldots, N-1; \ N \geq n).
$$
First let $N > n$. As $a_K^N = 1$ it follows from the assumption on $\phi_k(p_n,h,N)$ that $c_k \neq 0$ $(k = 0, 1, \ldots, N-1)$, and thus $A$ is nonsingular. Hence (cf. (4.3.4))

$$
\zeta = A^{-1} \gamma .
$$

As the inverse of a circulant matrix is again a circulant matrix (cf. Allken [2, p. 124]), $A^{-1}$ has the form $C(b_0, b_1, \ldots, b_{N-1})$. Consequently, for $u = 0, 1, \ldots, N-1$ and $0 < h < h$ one has

$$
(4.3.7) \quad s(uh + x) = \sum_{j=0}^{N-1} b_{u,j} \phi_k(p_n,h,N)s(jh + a),
$$

where $(b_0, b_1, \ldots, b_{N-1})$ denotes the $(j+1)$-th row of $A^{-1}$. Substituting $s(jh + a) = y_j$ into the right-hand side of (4.3.7) we obtain a function $s$ defined on $[0, Nh)$. Now $s$ is extended to $\mathbb{R}$ by the relation $s(t + Nh) = s(t)$ $(t \in \mathbb{R})$. The next step is to prove that $s \in S(p_n,h,N)$ and that $s$ has the interpolation property $s(uh + x) = y_{u}$ $(u = 0, 1, \ldots, N-1)$. First we note that on each interval $(uh, (u+1)h)$ $(u \in \mathbb{Z})$ $s$ coincides with a function in $Ko(p_n)$. Furthermore,

$$
s^{(k)}(uh) = s^{(k)}(0)\quad (k = 0, 1, \ldots, n-2), \quad u = 1, \ldots, N-1.
$$

Hence $s^{(k)}(nh) = s^{(k)}(nh-h)$ $(k = 0, 1, \ldots, n-2)$ and so $h, 2h, \ldots, (N-1)h$ are simple knots of $s$. In a similar way we can show that $s^{(k)}(0+) = s^{(k)}(nh)$ $(k = 0, 1, \ldots, n-2)$. Consequently, $s \in S(p_n,h,N)$.

We proceed with the proof of $s(uh + x) = y_{u}$ $(u \in \mathbb{Z})$. According to (4.3.7) we must show that

$$
\sum_{j=0}^{N-1} b_{u,j} \phi_k(p_n,h,N)s(jh + a) = y_{u} .
$$
This assertion, however, follows immediately from the fact that
\( C(b_0, b_1, \ldots, b_{n-1}) \) is the inverse of \( A \). This completes the proof of the
existence part of Theorem 4.3.3 in case \( N \geq n \). Taking into account the
construction of \( s \) we conclude that \( s \) is the unique function in \( S(p_n; h, N) \n\)
having the desired interpolation properties.

Now let \( n > N \) and let \( k \) be an integer such that \( kN \geq n \). By the proof given
above there is a unique function \( s \in S(p_n; h, kN) \) such that \( s(\nu h + a) = y_\nu \)
(\( \nu \in \mathbb{Z} \)). Since \( t \mapsto kN(t + \nu h) \) also has these properties it follows that
\( s(t) = s(t + \nu h) \) (\( t \in \mathbb{R} \)). Hence \( s \in S(p_n; h, N) \). This completes the proof of
the first part of the theorem.

Now let us assume that \( \psi_{z_0}(p_n; h, a) = 0 \) for \( z_0 \in \mathbb{C} \) with \( z_0^{N} = 1 \). We note that
the second part of the theorem will be proved if one shows the existence of
a nontrivial periodic function in \( S(p_n; h, N) \) having the property \( s(\nu h + a) = 0 \).
This can be done in a similar way as in the proof of Theorem 4.2.2.

II. The restriction of an \( \ell \)-spline \( s \in S(p_n; h) \) to the interval \([0, Nh] \) can be
represented as (cf. (3.3.1))

\[
\sum_{j=1}^{N-1} w_j g_j(t - jh)
\]

for an appropriate \( g \in \text{ker}(p_n) \) and appropriate coefficients \( w_j \).
Since \( \dim(\text{ker}(p_n)) = n \), \( s \) is determined on \([0, Nh] \) by \( n + N - 1 \) parameters.
The interpolation condition \( s(\nu h + a) = y_\nu \) (\( \nu = 0, 1, \ldots, N-1 \)) yields \( N \) linear
equations for the \( n + N - 1 \) parameters. The periodicity condition \( s^{(1)}(0) = \)
(\( s^{(1)}(Nh) \) (\( \nu = 0, 1, \ldots, n-2 \)) yields \( n - 1 \) additional linear equations. Con-
sequently, the solvability of an \((n + N - 1) \times (n + N - 1) \) linear system of
equations must be examined. It is well known that a unique solution exists
if and only if the corresponding homogeneous system only has the trivial
solution. Therefore one has to prove that the null function is the only
function in \( S(p_n; h, N) \) satisfying \( s(\nu h + a) = 0 \) (\( \nu \in \mathbb{Z} \)). In view of (3.3.1)
this amounts to showing that the difference equation

\[
\sum_{\nu=0}^{n-1} a_{\nu}(0) s(\nu h + \nu h + x) = 0 \quad (\nu \in \mathbb{Z}; \ 0 \leq x \leq h)
\]
does not have a nontrivial \( Nh \)-periodic solution. However, this follows (cf.
the last few lines of the first proof) from the fact that \( \sum_{\nu=0}^{n-1} a_{\nu}(0) z^\nu =
\nu_z(p_n; h, a) \) has, by assumption, no zeros at the \( N \)-th roots of unity.  \( \square \)
Theorem 4.3.3 has an interesting consequence for differential operators $p_n(D)$ satisfying $p_n(n) = p_n(0)$ and such that $p_n$ only has real zeros. It then follows from (3.2.11) that

$$
\psi_n(p_n;h,0) = (-1)^n p_n(0) z^{-1} \psi_z(p_n;h,a).
$$

Since $p_n^* = p_n$ implies $p_n(0) = (-1)^n$, one has

$$
\psi_n(p_n;h,0) = z^{-1} \psi_z(p_n;h,a).
$$

Hence, if $n$ is even then $\psi_z(p_n;h,0) = 0$ and in view of Lemma 3.2.9 the function $\psi_z(p_n;h,\cdot)$ has no other zeros in $(0,h)$. If $n$ is odd, then

$$
\psi_z(p_n;h,0) = \psi_z(p_n;h,h) = \psi_z(p_n;h,0).
$$

Hence $\psi_z(p_n;h,0) = 0$ and $h$ is the only zero in $(0,h)$ of $\psi_z(p_n;\cdot,\cdot)$. These observations combined with Theorem 4.3.3 lead to the following generalization of Theorem 4.1.1.

**THEOREM 4.3.4.** Let $p_n \in \pi_n (n \geq 2)$ be a monic polynomial having only real zeros, and let $p_n^* = p_n$. Then for $a \in (0,h)$ and an $N$-periodic sequence $(\gamma_u)^\infty$ there exists a unique periodic cardinal $L$-spline $s \in S(p_n,h,N)$ satisfying the interpolation property $s(u+a) = \gamma_u (u \in \mathbb{Z})$ in each of the following cases:

i) $N$ is odd, $n$ is arbitrary, and arbitrary.

ii) $N$ is even, $n$ is even, $a \neq h$.

iii) $N$ is even, $n$ is odd, $a \neq h$.

We emphasize the fact that case i) of this theorem holds without the condition $p_n^* = p_n$. This is an immediate consequence of Theorem 4.3.3; we state it here as

**COROLLARY 4.3.5.** Let $p_n \in \pi_n (n \geq 2)$ be a monic polynomial having only real zeros and let $N$ be an arbitrary odd integer. Then for any $a \in (0,h)$ and any $N$-periodic sequence $(\gamma_u)^\infty$ there exists a unique $s \in S(p_n,h,N)$ satisfying $s(u+a) = \gamma_u (u \in \mathbb{Z})$. 

4.4. An error estimate for cardinal $L$-spline interpolation

Relation (3.3.3) holds for all $x \in S(p_n, h)$ and by fortiori also for all functions in $Ker(p_n)$. In view of Peano's remainder formula (cf. Lemma 1.4.5) one has

\begin{equation}
\psi_{\nu}(p_n; h, y) f^*(x) = \psi_{\nu}(p_n; h, x) f^*(h + x) = \psi_{\nu}(p_n; h, x) f^*(h + y) + \int_0^h K_{x,y}(t) \psi_{\nu}(p_n; 0) f^*(h + y + t) dt,
\end{equation}

(4.4.1)

for all $f \in MC^{(r)}(R)$ and an appropriate kernel which, because of its dependence on $x$ and $y$, is denoted by $K_{x,y}$.

In order to determine $K_{x,y}$ we make use of (3.3.2) and take $\nu = 0$ in (4.4.1). It then follows from Lemma 1.4.5 that

$$K_{x,y}(x) = \frac{1}{n!} \sum_{k=0}^{n-1} \{ a_x(y) \psi_{\nu}(k h + x - t) - a_x(x) \psi_{\nu}(k h + y - t) \},$$

where, as usual, $\psi_{\nu}$ is the Green's function corresponding to $p_n(D)$.

Hence, for $t \in \mathbb{R}$,

\begin{equation}
K_{x,y}(x) = \frac{1}{n!} \sum_{k=0}^{n-1} \{ a_x(y) \psi_{\nu}(t - (n-k)h - x) - a_x(x) \psi_{\nu}(t - (n-k)h - y) \}.
\end{equation}

(4.4.2)

Formula (4.4.2) shows that $K_{x,y}$ is an $L$-spline defined on $R$ having the knots $h-x, 2h-x, \ldots, nh-x, nh-y, 2h-y, \ldots, nh-y$. Using (3.3.3) and (4.4.2) one can easily verify that the function $K_{x,y}$ enjoys the following properties:

\begin{enumerate}
  \item $K_{x,y}(x) = 0$ \quad ($x \notin (0, nh)$),
  \item $K_{x,y}(x h) = 0$ \quad ($y = 0, 1, \ldots, n$),
  \item $K_{x,y}(x) = 0$ \quad ($x \in \mathbb{R}$), in case $x = y$.
\end{enumerate}

(4.4.3)

Next we investigate the sign structure of $K_{x,y}$; information about this and properties (4.4.3) are needed in the sequel to derive an error estimate for cardinal $L$-spline interpolation, which is the ultimate aim of this section. Details about the zeros of $K_{x,y}$ are given in the following.

**Lemma 4.4.1.** Let the differential operator $p_n(D)$ ($n \geq 2$) be disconjugate on $[0, nh]$. If $0 < x < y < h$ then the function $K_{x,y}$ as given by (4.4.2) has
simple strong zeros in \( h, 2h, \ldots, (n-1)h \). Moreover, these are the only zeros of \( \zeta_{X,Y} \) in \( (n-h, nh-x) \).

**Proof.** We first observe that (4.4.1) implies that \( \zeta_{X,Y}(t) = 0 \) for \( t \notin (h, nh-h) \). Furthermore, the number of knots of \( \zeta_{X,Y} \) in \( (nh-x, nh-x) \) is equal to \( 2n-2 \). An interval \( [0, nh] \) is an interval of disconjugacy for \( p_n(t) \) we may write \( p_n(t) = D_n D_{n-1} \cdots D_1 = D_n^{[n-1]} \), where the operators \( D_i \) are given by

\[
D_n^{[m]} = \left( \frac{n-1}{k_{n-1}(h-x+y)} \right) \frac{1}{n-1} K_{n-1}(n-x-y) \neq 0,
\]

since \( \zeta_{X,Y}(x) = \psi(x) \neq 0 \) [cf. (14) of Lemma 3.2.7] and \( \varphi_0(y) = \varphi_1(y) = \frac{1}{n-1} K_{n-1}(n-x-y) \neq 0 \) [cf. (3.2.9) and (3.2.13)]. By Theorem 1.4.23 (cf. pp. 14, 15) one has

\[
(4.4.4) \quad \mathcal{Z}(D_n^{[n-1]} \zeta_{X,Y}, (h, nh-h-y)) \leq 2n-2 - (n-1) = n-1.
\]

We recall (cf. (4.4.3)) that \( \zeta_{X,Y}(y) = 0 \) \( (y = 1, \ldots, n-1) \). So it remains to prove that the points \( h, 2h, \ldots, (n-1)h \) are strong zeros. In order to do so we assume that \( X, \) say, is not a strong zero. Since \( \zeta_{X,Y} \) is an \( \ell \)-spline we then have that \( \zeta_{X,Y} \) vanishes in an interval between two consecutive knots. Accordingly, we may assume that \( \zeta_{X,Y}(t) = 0 \) \( (h-x \leq t \leq (i+1)h-y) \). Consequently, by (4.4.1), for all functions \( f \in \mathcal{AC}^n((0, nh-x)) \) for which \( p_n(t) \) vanishes identically outside the interval \( [h-nh-y, nh-y-x] \) one would have

\[
(4.4.5) \quad \psi(p_n(t)) f(y) = \psi(p_n(t), x+y) f(y).
\]

In view of (3.2.10) formula (4.4.5) may be written as

\[
(4.4.6) \quad \psi(p_n(t)) f(y) = \psi(p_n(t), x+y) f(y) =
\]

\[
= \sum_{u=0}^{n-1} B_n(p_n, x+(n-1-u)h) f(y+vh).
\]

Having established this, our next purpose is to obtain a contradiction by exhibiting a function \( f \in \mathcal{AC}^n((0, nh-x)) \) for which \( p_n(t) \) vanishes outside \( [h-nh-y, nh-y-x] \), but for which (4.4.5) does not hold. To this end we first assume that \( l \geq n/2 \). As a consequence of Lemma 1.4.17 there exists at least one function \( f \in \ker(p_n) \) such that \( f(x+vh) = 0 \).
\((v = 0, 1, \ldots, (n-1)), f_1(y + vh) = 0 (v = 0, 1, \ldots, n-2)\) and
\(f_1(y + (n-1)h) = 1\). Here the assumption \(i \geq n/2\) is used to ensure that the number of interpolation conditions for \(f_1\) is at most \(n\). It is then easy to construct a function \(f \in C^{n-1}([0, nh])\) that coincides with \(f_1\) on 
\([0, (n-1)h], (n-1)h+\), and that vanishes on \([(n-1)h+1, nh]\). Substitution of this function into (4.4.6) yields \(0 = B_n(p_n^1, x+i h)\). However, this violates (2.2.10) since \(0 < x+i h < nh\), and a contradiction is obtained. If \(i < n/2\)
then in a similar way we may prove the existence of a function \(f_2\) with the properties: \(p_n^1, \) vanishes identically outside \([(n-1)h+1, (n-1)h+2\]}, \(f_2\) is identically zero on \([0, (n-1)h+1]\), \(f((n-1)h+1) = 1\), \(f(x+i h) = 0\)
\((v = n-1, \ldots, n-1)\), \(f(y+i h) = 0\) \((v = n-1, \ldots, n-1)\). Then (4.4.6) reduces to \(B_n(p_n^1, (y+i h) = 0,\) which violates (3.2.10) and again a contradiction is obtained. Consequently, the assumption on \(i\) not being a strong zero cannot be maintained. We conclude that \(i\) \((j = 1, \ldots, n-1)\) are strong zeros and that these zeros are simple in view of (4.4.4). The last assertion of the lemma follows from the fact that \(K_{x,y}\) is an \(L\)-spline, and therefore a zero of \(K_{x,y}\) is a strong zero or it belongs to an interval that is a strong zero.

If \(s \in \mathcal{S}(p_n^1)\) interpolates a given function \(f\) at the points \(u + \alpha (u \in \mathbb{R})\), then (4.4.1) combined with (3.3.3) implies that for \(u \in \mathbb{R}\) one has

\[
(4.4.7) \quad \varphi_s(p_n^1, u) = \int_0^{nh} K_{x \alpha}((nh - \tau)) f((nh + \tau) dt.
\]

Formula (4.4.7) may be used to obtain an estimate for \(|f(u) - s(u)| (u \in \mathbb{R})\). For this purpose we assume in addition that \(\|p_n^1(D)\| < \infty\) and, furthermore, that \(\varphi_s(p_n^1, u)\) has no zeros on the unit circle. Then (4.4.7) can be considered as a linear difference equation the solution of which may be obtained in a similar way as in Lemma 3.4.1. Hence it follows from (3.4.4) that

\[
(4.4.8) \quad f((nh + \tau) - s((nh + \tau) = \int_0^{nh} K_{x \alpha}((nh - \tau)) \sum_{j=-\infty}^{\infty} c_j p_n^1(D) f(j+1)h + \tau) dt,
\]

where the coefficients \(c_j\) are given by the Laurent expansion

\[
(4.4.9) \quad \varphi_{s}^{-1}(p_n^1, h, u) = \sum_{j=-\infty}^{\infty} c_j z^j.
\]
that converges on a ring containing the unit circle in its interior.

Changing the order of summation and integration and replacing \( t \) by \( t - jh \), we rewrite formula (4.4.8) for \( 0 \leq x \leq h \) and \( u \in \mathbb{Z} \) in the form

\[
(4.4.10) \quad f(uh + x) - s(uh + x) = \int_{-\infty}^{\infty} K_{x,a}(t) p_n(t) f(uh + t) \, dt,
\]

where

\[
(4.4.11) \quad K_{x,a}(t) := \sum_{j=-\infty}^{\infty} c_j K_{x,a}((n+j)h - t).
\]

Formulas (4.4.10) and (4.4.11), together with i) of (4.4.3), immediately yield

\[
(4.4.12) \quad |E(uh + x) - s(uh + x)| \leq \sum_{j=-\infty}^{\infty} |c_j| \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |K_{x,a}(t)| \, dt \| p_n(t) f \|.
\]

In order to use this estimate one has to evaluate the sum and the integral in the right-hand side of (4.4.12). With respect to the integral, we consider the perfect Euler \( L \)-spline \( E\{p_n, h, \cdot \} \) (cf. Definition 3.2.12). In order to ensure the existence of \( E\{p_n, h, \cdot \} \) it is necessary to show that \( p_n[-1] \neq 0 \). However, if \( p_n[-1] = 0 \) then a nontrivial function \( g \in \text{Ker}(p_n) \) exists such that \( g(t + h) = -g(t) \) \( (t \in \mathbb{R}) \). Consequently, \( g \) would have more than \( n-1 \) zeros in \([0, rh]\) which contradicts the fact that \( p_n(t) \) is disconjugate on \([0, rh]\). Taking into account the sign structure of \( K_{x,a} \) as given by Lemma 4.4.1, writing \( f_0 = E\{p_n, h, \cdot \} \), and using \( f_0(t + h) = -f_0(t) \) \( (t \in \mathbb{R}) \), we conclude from (4.4.1) that

\[
\int_{0}^{\frac{\pi}{h}} |K_{x,a}(\tau)| \, d\tau = \int_{0}^{\frac{\pi}{h}} K_{x,a}(uh + \tau) p_n(\tau) f_0(\tau) \, d\tau =
\]

\[
= |\psi_0(p_n, h, \cdot) f_0(uh + x) - \psi_1(p_n, h, \cdot) f_0(uh + x)| =
\]

\[
= |\psi_{-1}(p_n, h, \cdot) f_0(uh + x) - \psi_{-1}(p_n, h, \cdot) f_0(uh + x)|.
\]

It follows from Definitions 3.2.5 and 3.2.6 that

\[
(4.4.13) \quad \int_{0}^{\frac{\pi}{h}} |K_{x,a}(\tau)| \, d\tau = |\psi_{-1}(p_n, h, \cdot)(f_0(uh + x) - s_0(uh + x))| =
\]
\[ = \left| \psi_{-1}(p_n; h, \phi)(f(0)) - s_0(x) \right| , \]

where

\[ s_0(x) := \frac{f(0)}{\psi_{-1}(p_n; h, \phi)} \psi_{-1}(p_n; h, \phi) , \]

\( s_0 \) being the Euler-L-spline in \( S(p_n, h) \) interpolating \( f \) at the nodes \( \phi + a \) \((a) \in \Omega \).

The next step is to estimate \( \sum_{j=1}^{n} |c_j| \). Let \( z_1, \ldots, z_{n-1} \) be the zeros of the polynomial \( z \to \psi_1(p_n; h, \phi) \). Then in view of property ii) of Lemma 3.2.7 one has

\[ \psi_1(p_n; h, \phi) = q(\phi) \prod_{j=1}^{n-1} (z - z_j) \quad (z \in \Omega) . \]

In case \( |z_1| > 1 \) we write

\[ \frac{1}{z_1 - z} = -\frac{1}{z_1} \sum_{k=0}^{\infty} \left( \frac{z}{z_1} \right)^k \quad (|z| < |z_1|) , \]

whereas \( |z_1| < 1 \) gives rise to

\[ \frac{1}{z_1 - z} = \frac{1}{z_1} \sum_{k=0}^{\infty} \left( \frac{z_1}{z} \right)^k \quad (|z| > |z_1|) . \]

Furthermore,

\[ \sum_{k=0}^{\infty} \left| \frac{1}{z_1} \right|^k = \frac{1}{|z_1| - 1} \quad (|z_1| > 1) , \]

\[ \sum_{k=0}^{\infty} \left| z_1 \right|^k = \frac{1}{1 - |z_1|} \quad (|z_1| < 1) . \]

Using these relations in the Laurent expansion (4.4.9) for \( \psi_1^{-1}(p_n; h, \phi) \) we get

\[ \sum_{j=1}^{n-1} |c_j| \leq q(\phi) \prod_{j=1}^{n-1} (1 - |z_j|) \quad (1) \]

The results obtained above are combined to yield the following

THEOREM 4.4.2. Let \( f \in AC^{m}(\Omega) (n \geq 2) \) and let the operator \( p_n(0) \) be linear. If \( a \in (0, h) \) is such that the polynomial
\[\theta(z) (p_n, h, a) \text{ has no zeros on the unit circle and if } \|p_n(D)f\| < \infty, \text{ then there exists a unique } \theta \in \mathcal{S}(p_n, h) \text{ such that}\]

\[
\begin{align*}
(4.4.16) & \\
&= \begin{cases} \\
(i) & \theta_z(z) = f(z) + \alpha (z \in \mathbb{Z}) , \\
(ii) & |f(x) - \theta_z(x)| \leq |f_0(x) - \theta_z(x)| \|p_n(D)f\| (x \in \mathbb{R}), \\
\end{cases}
\end{align*}
\]

where

\[
x := \left| \frac{n^{-1} \theta_z(p_n, h, a)}{\phi(n) \prod_{j=1}^{n-1} (1 - |z_j|)} \right| ,
\]

\[z_1, z_2, \ldots, z_{n-1}\] are the zeros of \(\theta_z(p_n, h, a)\), \(f_0\) is the perfect Euler \(L\)-envelope corresponding to \(p_n(D)\), \(\theta_0\) is given by \((4.4.14)\) and where \(\phi\) is the fundamental function corresponding to \(p_n(D)\).

PROOF. Formula \((4.4.8)\) can be used to define a function \(\theta\) having the desired properties \((i)\) and \((ii)\). The uniqueness of \(\theta\) follows from \((ii)\), since otherwise there would exist a bounded nontrivial function \(\theta \in \mathcal{S}(p_n, h)\) with \(\theta(z) = 0 (z \in \mathbb{Z})\) which, by Lemma 3.2.1, violates the assumption on the zeros of \(\theta_z(p_n, h, a)\). This proves the theorem. \(\Box\)

If \(p_n\) has only real zeros then, according to Lemma 3.2.11, \(\theta_z(p_n, h, a)\) has only nonpositive zeros and hence

\[
|\phi(n) \prod_{j=1}^{n-1} (1 - |z_j|)| = |x^{-1} \prod_{j=1}^{n-1} (1 - |z_j|)|.
\]

Taking into account \((4.4.15)\) and using \((4.4.9)\) one obtains

\[
\sum_{j=m}^{n} |c_j| = |\phi^{-1}(p_n, h, a)|^{-1} = \left| \sum_{j=m}^{n} (-1)^j c_j \right| = \sum_{j=m}^{n} |c_j| .
\]

Consequently, \(c_j c_{j+1} < 0 (j \in \mathbb{Z})\) and

\[
\sum_{j=m}^{n} |c_j| = |\phi^{-1}(p_n, h, a)|^{-1} .
\]

Inequality \((ii)\) of \((4.4.16)\) may then be written in the form

\[
\begin{align*}
(4.4.17) & \\
|f(x) - s(x)| \leq |f_0(x) - s_0(x)| \|p_n(D)f\| (x \in \mathbb{R}) ,
\end{align*}
\]
where $f_0$ is the perfect Euler $\ell$-spline $E(p_n h, r)$ and $s_{f_0}$ is the Euler $\ell$-spline given by (4.4.14).

Moreover, in this situation (4.4.17) is best possible, since the upper bound is attained for the perfect Euler $\ell$-spline $E(p_n h, r)$. We state the result obtained as

**Corollary 4.4.3.** Let $p_n \in \Pi_n (n \geq 2)$ be a norm polynomial having only real zeros. If $a \in (0, h)$ is such that $\psi_{-1}(p_n h, a) \neq 0$, then for every $f \in AC^{(n)}(\mathbb{R})$ there exists a unique $s_f \in S(p_n h)$ satisfying

\[ s_f(\nu h + a) = f(\nu h + a) \quad (\nu \in \mathbb{Z}), \]

\[
(4.4.18) \\
|f(x) - s_f(x)| \leq |\xi_{f_0}(x) - s_{f_0}(x)| |D f| \quad (x \in \mathbb{R}),
\]

where $f_0$ is the perfect Euler $\ell$-spline $E(p_n h, r)$ and $s_{f_0}$ is the Euler $\ell$-spline interpolating $f_0$ at the nodes $\nu h + a$ $(\nu \in \mathbb{Z})$.

Moreover, the upper bound in (4.4.18) is best possible.

**Remark.** For general operators $p_n (D)$ it is an open problem to determine a best possible upper bound for $|f(x) - s_f(x)|$ $(x \in \mathbb{R})$; to achieve this detailed information about the zeros of $z \mapsto \psi_{-1}(p_n h, a)$ would be needed.

4.5. An error estimate for periodic cardinal $\ell$-spline interpolation

This section is devoted to the problem of periodic cardinal $\ell$-spline interpolation, with emphasis on error estimates. If $f \in AC^{(n)}(\mathbb{R})$ has period $Nh$ and if $z \mapsto \psi_{-1}(p_n h, a)$ has no zeros at the $N$-th roots of unity, then by Theorem 4.3.3 there exists a unique $s_f \in S(p_n h, N)$ interpolating $f$ at the nodes $\nu h + a$ $(\nu \in \mathbb{Z})$. Now the problem arises to compute for all $x \in \mathbb{R}$

\[ \sup_{f \in AC^{(n)}(\mathbb{R}), f \text{ is } Nh\text{-periodic}} \left( |f(x) - s_f(x)| \right). \]

(4.5.1)

If $\psi_{-1}(p_n h, a)$ has no zeros on the unit circle, then (cf. Theorem 4.4.2) the unique bounded cardinal $\ell$-spline $s \in S(p_n h)$ interpolating $f$ at the nodes $\nu h + a$ has, due to its unicity, period $Nh$ and an estimate for (4.5.1) is given by (ii) of (4.4.16). In case $p_n$ has only real zeros, Corollary 4.4.3
applies and the estimate (4.4.18) may be used. However, 11) of (4.4.18) is
best possible for cardinal $L$-spline interpolation and it will be best
possible for periodic cardinal $L$-spline interpolation only if $E(p_n,h,\cdot)$ has
period Nh. Since $E(p_n,h,t+h) = E(p_n,h,t)$ ($t \in \mathbb{R}$) one has $E(p_n,h,t+Nh) =
= (-1)^N E(p_n,h,t)$. Consequently, $E(p_n,h,\cdot)$ has period Nh only if N is an
even number. This gives rise to the following

**THEOREM 4.5.1.** Let $p_n \in \mathcal{C}_n$ ($n \geq 2$) be a monic polynomial having only real
zeros and let $N \in \mathbb{N}$ be even. If $\psi_{-1}(p_n,h,a) \neq 0$, then for every Nh-periodic
function $f \in \mathcal{C}^2_{Nh}(\mathbb{R})$ there exists a unique $s_f \in \mathcal{S}(p_n,h,N)$ such that

\[ s_f((n+\alpha)x) = f((n+\alpha)x). \quad (n \in \mathbb{Z}), \]
\[ |f(x) - s_f(x)| \leq \|f - s_{\psi_{-1}} f\|_{p_n(h,N)} \quad (x \in \mathbb{R}), \]

where $s_{\psi_{-1}}$ is the perfect Euler $L$-spline $E(p_n,h,\cdot)$ and $s_{\psi_{-1}}$ is the Euler $L$-
spline in $\mathcal{S}(p_n,h,N)$ interpolating $s_{\psi_{-1}}$ at the nodes $nh + \alpha$ ($n \in \mathbb{Z}$). Moreover, 11) of (4.4.2) is best possible.

If $p_n$ has only real zeros and if, moreover, $p_n^* = p_n$ then $\psi_{-1}(p_n,h,a) = 0$
for $a = nh$ and $n$ even, and $\psi_{-1}(p_n,h,a) = 0$ for $a = nh$ and $n$ odd (cf. p. 86).
 Consequently, for operators $p_n(D)$ for which $p_n(D) = p_n^*(D)$, Theorem 4.5.1
furnishes the following corollary.

**COROLLARY 4.5.2.** Let $p_n \in \mathcal{C}_n$ ($n \geq 2$) be a monic polynomial having only real
zeros with $p_n^* = p_n$ and let $N \in \mathbb{N}$ be even. Then for every Nh-periodic
function $f \in \mathcal{C}^2_{Nh}(\mathbb{R})$ there exists a unique $s_f \in \mathcal{S}(p_n,h,N)$ such that
the nodes $nh + \alpha$ ($n \in \mathbb{Z}$) in each of the following cases:

1) $n$ is even, $\alpha \neq nh$,
2) $n$ is odd, $\alpha \neq nh$.

Moreover

\[ |f(x) - s_f(x)| \leq \|f(x) - s_{\psi_{-1}} f(x)\|_{p_n(h,N)} \quad (x \in \mathbb{R}), \]

where $s_{\psi_{-1}}$ is the perfect Euler $L$-spline $E(p_n,h,\cdot)$ and $s_{\psi_{-1}}$ is the Euler $L$-
spline in $\mathcal{S}(p_n,h,N)$ interpolating $s_{\psi_{-1}}$ at the nodes $nh + \alpha$ ($n \in \mathbb{Z}$).

Inequality (4.5.3) is best possible.
We now return to the general problem of estimating (4.5.1), only assuming that \( z = \tilde{z}_z(p, h, a) \) has no zeros at the \( N \)-th roots of unity. As in the case of cardinal \( L \)-spline interpolation (cf. (4.4.10)) we shall determine a kernel function \( \tilde{K}_{x, a} \) such that for all \( x \in [0, h) \) and \( \nu \in \mathbb{Z}^d \),

\[
(4.5.4) \quad \tilde{K}(\nu \cdot x - z) = \sum_{\nu \in \mathbb{Z}^d} \tilde{K}_{x,a}(\nu) p_n(\nu) f(\nu + \nu) = \int_0^{Nh} \tilde{K}_{x,a}(\nu) p_n(\nu) f(\nu + \nu) d\nu .
\]

Let \( d_k \) the \( k \)-th component of the vector \( \tilde{d} \in \mathbb{R}^N \) be given by

\[
(4.5.5) \quad d_k := \int_0^{N\nu} K_{x,a}(\nu) p_n(\nu) f(\nu + \nu) d\nu \quad (k = 0, 1, ..., N-1) ,
\]

where \( K_{x,a} \) is given by (4.4.2), let \( e_k = e_k \) and let \( e_k = (e_k(x), e_k(x+h), \ldots, e_k(x+(N-1)h)) \). Using (4.4.7) for \( u = 0, 1, ..., N-1 \) one obtains, similar to (4.3.4), the system of linear equations

\[
(4.5.6) \quad C(s_0, s_1, ..., s_{N-1}; s) \tilde{d} = \tilde{d} .
\]

Since \( \phi_z(p, h, a) \neq 0 \) if \( N = 1 \), the circulant matrix \( C \) has an inverse denoted by \( C \tilde{b}_0, \tilde{b}_1, ..., \tilde{b}_{N-1} \) (cf. p. 84). Hence for \( u = 0, 1, ..., N-1 \) and \( 0 \leq x \leq h \) one has (cf. (4.3.7))

\[
(4.5.7) \quad e_k(\nu, x) = \int_0^{N\nu} \sum_{j=0}^{N-1} b_{\nu,j} K_{x,a}(\nu) p_n(\nu) f(\nu + \nu) d\nu ,
\]

where \( (b_{\nu,j} \ldots b_{\nu,N-1}) \) is the \((j+1)\)-st row of \( C \tilde{b}_0, \tilde{b}_1, ..., \tilde{b}_{N-1} \).

Let the function \( \tilde{K}_{x,a} \) be defined by

\[
(4.5.8) \quad \tilde{K}_{x,a}(\nu) := \sum_{k=-\infty}^{N-1} K_{x,a}(\nu - kN) \quad (x \in \mathbb{R}) .
\]

Evidently, \( \tilde{K}_{x,a} \) is an \( Nh \)-periodic function. Since \( \tilde{K} \) is also an \( Nh \)-periodic function, the integral in the right-hand side of (4.5.7) can be written as

\[
\int_{-\infty}^{\infty} \sum_{j=0}^{N-1} b_{\nu,j} K_{x,a}(\nu) p_n(\nu) f(\nu + \nu) d\nu =
\]
\[ \sum_{k=-\infty}^{n} \frac{(k+1)nh}{\kappa h} \int_{0}^{1} b_{n,k} \kappa_{x,a}(nh - t)p_n(t)f(t + \tau) \, dt = \]
\[ = \sum_{k=-\infty}^{n} \int_{0}^{1} b_{n,k} \kappa_{x,a}(nh - t - \tau) \kappa_{x,a}(nh) \, dt \]
\[ = \int_{0}^{1} b_{n,k} \kappa_{x,a}(\tau)p_n(0)f(t + \tau) \, dt. \]

Replacing \( \tau \) by \( \tau - (j-1)h \) and taking into account (1) of (4.4.3) we obtain
\[ a_{x,h}(x + \tau) = \sum_{j=0}^{n-1} b_{n,j} \kappa_{x,a}(\tau - (j-1)h)p_n(0)f(\tau + \tau) \, dt. \]

The required kernel function \( \kappa_{x,a} \) will now be defined by (cf. (4.5.4))
\[ \kappa_{x,a}(t) := \sum_{j=0}^{n-1} b_{n,j} \kappa_{x,a}(t - (j-1)h) \quad (t \in \mathbb{R}). \]

We observe that \( \kappa_{x,a} \) does not depend on \( u \), since \( (b_{0,0}, ..., b_{u,n-1}) \) is the \( (u+1) \)-st row of the circulant matrix \( C(b_{0,0}, ..., b_{u,n-1}) \) and \( \kappa_{x,a} \) is periodic with period \( \kappa h \). Hence
\[ \kappa_{x,a}(t) = \sum_{j=0}^{n-1} b_{n,j} \kappa_{x,a}(t - jh) \quad (t \in \mathbb{R}), \]
and (4.5.7) may be written as
\[ a_{x,h}(x + \tau) = \int_{0}^{\infty} \kappa_{x,a}(\tau)p_n(0)f(\tau + \tau) \, dt, \]
where \( 0 \leq x \leq h \) and \( u \in \mathbb{Z} \).

The results obtained are collected in the following lemma.

**Lemma 4.5.5.** Let \( a \in (0,h) \), let \( p_n \in \mathbb{P}_n (n \geq 2) \) be a monic polynomial and let \( \mathbb{P}_n \in \mathbb{N} \). If \( v_{\mathbb{P}_n}(a) = 0 \) for \( \mathbb{P}_n = 1 \), then the unique function \( a_{x,h} \in \mathbb{S}(p_n.h,n) \) interpolating an \( n \)-periodic function \( f \in \mathbb{C}(\mathbb{R}) \) at the nodes \( nh + a (u \in \mathbb{Z}) \) satisfies for \( 0 \leq x \leq h \) the relation...
\[(4.5.11)\quad f(uh + u) - s_g(uh + u) = \int_0^{Nh} \tilde{K}_{x,0}(\tau) p_{n}(D) f(uh + \tau) \, d\tau ,\]

where \(\tilde{K}_{x,0}\) is given by \((4.5.10)\).

In order to obtain \((4.5.1)\) one has to determine the supremum of the integral in the right-hand side of \((4.5.11)\) over all \(Nh\)-periodic functions \(f \in AC^{(n)}(\mathbb{R})\) for which \(\|p_{n}(D) f\| \leq 1\). With respect to this problem we need the following lemma.

**Lemma 4.5.4.** Let \(\mathcal{F} \subset C([0,T])\), where \(T\) is a positive number. If there exists a \(T\)-periodic function \(u_0 \in Ker(p_n)\) coinciding with \(\mathcal{F}\) in at most finitely many points and such that for all \(T\)-periodic functions \(g \in Ker(p_n)\)

\[(4.5.12)\quad \int_0^T g(t) \cdot \text{sgn}(F(t) - s_{g_0}(t)) \, dt = 0 ,\]

then

\[(4.5.13)\quad \sup_{\|p_{n}(D) f\| \leq 1} \left\{ \int_0^T F(t)p_{n}(D) f(t) \, dt \mid f \in AC^{(n)}(\mathbb{R}), f \text{ is } T\text{-periodic} \right\} = \int_0^T F(t)p_{n}(D) f_0(t) \, dt ,\]

where \(f_0 \in AC^{(n)}(\mathbb{R})\) is a \(T\)-periodic function satisfying the differential equation

\[(4.5.14)\quad p_{n}(D) f_0(t) = \text{sgn}(F(t) - s_{g_0}(t)) \quad (0 < t < T) .\]

**Proof.** In Lemma 6.2.1 of Chapter 6 it will be shown that the differential equation \(p_{n}(D) f(t) = u(t) \) \((0 < t < T)\), where \(u \in L^1([0,T])\), has a solution satisfying \(f^{(j)}(0) = f^{(j)}(T) \) \((j = 0,1,...,n-1)\) if and only if \(u\) has the property that

\[(4.5.15)\quad \int_0^T g(t) u(t) \, dt = 0\]

for all \(T\)-periodic functions \(g \in Ker(p_n)\).
Let $U$ be the set of functions $u \in L^1([0,T])$ satisfying (4.5.15) and $\|u\|_{L^1([0,T])} \leq 1$. It follows that the supremum in (4.5.15) equals

$$\sup \left\{ \int_0^T F(t)u(t)\,dt \mid u \in U \right\}$$

(4.5.16)

Obviously, for all $T$-periodic functions $g \in \text{Ker}(p_n)$ one has

$$\left| \int_0^T F(t)u(t)\,dt \right| = \left| \int_0^T (F(t) - g(t))u(t)\,dt \right| \leq \int_0^T |F(t) - g(t)|\,dt .$$

Hence the supremum in (4.5.16) is bounded from above by the $L^1$-distance of the continuous function $F$ to the finite dimensional space of functions $g \in \text{Ker}(p_n)$ that are $T$-periodic.

According to a general characterization theorem for $L^1$-approximation (cf. Cheney [12, p. 220]), a function $q_0$ that coincides with $F$ in at most finitely many points and that satisfies (4.5.12), is a best $L^1$-approximation to $F$, i.e.,

$$\int_0^T |F(t) - q_0(t)|\,dt \leq \int_0^T |F(t) - g(t)|\,dt ,$$

for all $T$-periodic functions $g \in \text{Ker}(p_n)$.

Therefore, by taking $u_0 := \text{sgn}(F - q_0)$, it follows that $u_0 \in U$ and thus for all $u \in U$

$$\left| \int_0^T F(t)u(t)\,dt \right| \leq \int_0^T |F(t) - q_0(t)|\,dt =$$

$$= \int_0^T (F(t) - q_0(t))u_0(t)\,dt = \int_0^T F(t)u_0(t)\,dt .$$

By taking $f_0 \in A^{(n)}(\mathbb{R})$ to be the $T$-periodic function that satisfies $F_n(D)f_0(t) = u_0(t)$ $(0 < t < T)$, the lemma is obtained. \qed

REMARK. If $F$ has not more than finitely many points in common with any $T$-periodic function $g \in \text{Ker}(p_n)$, then, according to Schönhage [55, p. 169], a function $g_0$ satisfying (4.5.12) always exists.
Lemmas 4.5.3 and 4.5.4 easily yield the following theorem.

**Theorem 4.5.5.** Let \( a \in (0,h) \), let \( p_n \in \Pi_n \) (\( n \geq 2 \)) be a non-zero polynomial and let \( N \in \mathbb{N} \) be such that \( \Delta_n (p_n, h, a) \neq 0 \) for \( a^N = 1 \). Furthermore, let \( x \in (0, nh) \) and \( x' := x - Nh \in (0, h) \) with \( x' \in \mathbb{N}_0 \).

If there exists an \( Nh \)-periodic function \( \varphi_0 \in \ker(p_n) \) such that \( \varphi_0 \) coincides with \( K_{x', a}(t) \) (cf. (4.5.10)) at least finitely many points in \( (0, nh) \) and, moreover, such that

\[
(4.5.17) \quad \int_0^T \varphi(t) \operatorname{sgn}(K_{x', a}(t) - \varphi_0(t)) \, dt = 0
\]

for all \( Nh \)-periodic functions \( \varphi \in \ker(p_n) \), then

\[
(4.5.18) \quad |f(x) - s_E(x)| \leq \int_0^{Nh} |K_{x', a}(t) - \varphi_0(t)| \, dt \| p_n(D) \| ,
\]

where \( s_E \in S(p_n, h, N) \) is the unique function interpolating the \( Nh \)-periodic function \( f \in \mathcal{C}_c^{1}(\mathbb{R}) \) at the nodes \( nh + a (n \in \mathbb{Z}) \). Moreover, the upper bound in (4.5.18) is best possible.

It seems difficult to obtain detailed information about \( \varphi_0 \) when dealing with a general differential operator \( p_n(D) \). As a consequence, it is not at all easy to evaluate the right-hand side of (4.5.18). Even for periodic piecewise linear interpolation, i.e., \( p_2(D) = \mathbb{P}^1 \), the problem is far from simple. This will be apparent from the following

**Example.** We consider the problem of periodic piecewise linear interpolation, i.e., \( p_2(D) = \mathbb{P}^1 \). Then (cf. (3.2.21) and (3.2.18))

\[
\theta_{a}(p_2, h, t) = t a + h = t .
\]

Taking \( a = h/2 \) and \( N \) odd we may apply Theorem 4.3.4, which ensures the existence of a unique periodic linear spline \( s_E \in S(p_2, h, N) \) interpolating a given \( Nh \)-periodic function \( f \in \mathcal{C}_c^{1}(\mathbb{R}) \) at the midpoints of the mesh intervals. Note that \( \varphi_0 \) is \( p_2(D, h/2) = 0 \).

The function \( K_{x, h/2} \) (cf. (4.4.2)) is now given by

\[
(4.5.19) \quad K_{x, h/2}(t) = \frac{h}{2} (t + h + x)_+ - (h - x) (t - \frac{h}{2})_+ + \frac{h}{2} (t - h - x)_+ - x(t - \frac{h}{2})_+ .
\]
Substituting $x = h$ in (4.5.19) we get a so-called roof function

$$\mathcal{B}_{h,h/2}(t) = \frac{h}{2} \left( (t-h) - 2(t-\frac{h}{2})^+ \right),$$

the graph of which is given in Figure 1.

![Figure 1](image1)

If we take $N = 3$, the circulant matrix $A$ (cf. (4.3.4)) equals $\frac{1}{3} h C(1,1,1)$, and its inverse is $h^{-1} C(-1,-1,1)$.

In view of Formulas (4.5.8) and (4.5.9) the kernel function $\tilde{\mathcal{B}}_{h,h/2}$ is in this particular case given by

$$\tilde{\mathcal{B}}_{h,h/2}(t) = \frac{1}{3} \left( \mathcal{B}_{h,h/2}(t) - \mathcal{B}_{h,h/2}(t-h) + \mathcal{B}_{h,h/2}(t-2h) \right),$$

where

$$\tilde{\mathcal{B}}_{h,h/2}(t) = \sum_{k=0}^{\infty} \mathcal{B}_{h,h/2}((2-3k)h - t) \quad (t \in \mathbb{R}).$$

Taking into account the graph of $\mathcal{B}_{h,h/2}$ as given in Figure 1, we easily verify that $\tilde{\mathcal{B}}_{h,h/2}$ is a periodic function with period $3h$, the graph of which on the interval $[0,3h)$ is shown in Figure 2.

![Figure 2](image2)

In view of (4.5.21) and Figure 2 the graph of the $3h$-periodic function $\tilde{\mathcal{B}}_{h,h/2}$ may now be easily obtained, cf. Figure 3 for a sketch on $[0,3h]$. 
The next step is to determine the function $g_0$ as given by (4.5.17). Taking into account that $g_0 \in \text{Ker}(D^2)$ and that $g_0$ is $3h$-periodic, we conclude that $g_0$ is a constant function, $g_0(t) = c$ say, where by (4.5.12) $c$ is such that

$$\int_0^{3h} \text{sgn}(\tilde{g}_{h,h/2}(t) - c) \, dt = 0$$

One easily finds that $c = h/16$. In view of the graph of $\tilde{g}_{h,h/2}$ as given in Figure 3, one then obviously has on $(0,3h)$

$$\text{sgn}(\tilde{g}_{h,h/2}(t) - \frac{h}{16}) = \begin{cases} -1 & (0 < t < \frac{1}{8}h), \\ 1 & (\frac{1}{8}h < t < \frac{7}{8}h), \\ -1 & (\frac{7}{8}h < t < \frac{9}{8}h), \\ 1 & (\frac{9}{8}h < t < \frac{15}{8}h), \\ -1 & (\frac{15}{8}h < t < 3h). \end{cases}$$

Hence (cf. (4.5.18))

$$|f(h) - s_f(h)| \leq \int_0^{3h} |\tilde{g}_{h,h/2}(t) - \frac{h}{16}| \, dt \|f''\| = \frac{11}{32} h^2 \|f''\|.$$

The function $f_0$ (cf. Lemma 4.5.4) for which the inequality above reduces to an equality is not an Euler $L$-spline corresponding to the operator $D_{P_2}(D) = D^2$, but a perfect $L$-spline (cf. Definition 1.3.5), since $D_{P_2}(D)f_0$ jumps from $\pm 1$ to $\mp 1$ at the non-equipartition points $h/8$, $7h/8$, $9h/8$, $15h/8$. 
5. ON THE LANDAU PROBLEM FOR SECOND AND THIRD ORDER DIFFERENTIAL OPERATORS

5.1. Introduction and summary

As pointed out in the general introduction this thesis consists of two parts. Here we begin the second part, which deals with the Landau problem for linear differential operators. The name "Landau problem" originates from a few interesting inequalities, published by Landau as early as 1913, connecting \( \| f \|, \| f' \|, \) and \( \| f'' \|. \) His main results are contained in the following theorem (cf. pp. 3, 4 for notation).

**THEOREM 5.1.1 (Landau [30]).** Let \( f \in W^{(2)}(\mathbb{R}) \) be such that \( \| f \| = \infty. \) Then

\[
\| f' \| \leq \sqrt{2} \| f'' \|^{1/2}.
\]

Further, let \( f \in W^{(2)}(\mathbb{R}^n_0) \) be such that \( \| f \| \| f'' \| = \infty. \) Then

\[
\| f' \| \leq 2 \sqrt{\| f \| \| f'' \|}.
\]

The constants \( \sqrt{2} \) and \( 2 \) in these inequalities are best possible, i.e., they cannot be replaced by smaller ones.

Later, in 1939, Kolmogorov generalized the first part of Landau's theorem to higher derivatives. His result reads as follows.

**THEOREM 5.1.2 (Kolmogorov [27]).** Let \( f \in W^{(n)}(\mathbb{R}) \) \((n > 2)\) be such that \( \| f \| < \infty. \) Then

\[
\| f^{(k)} \| \leq c_{n,k} \| f^{(1-k/n)} \|^{(1-k/n)} \| f^{(n)} \|^{k/n} \quad (k = 1, 2, \ldots, n-1),
\]

where the constants \( c_{n,k} \) are given by

\[
c_{n,k} := \frac{k}{n-k} \frac{1}{c_{n-k}^{1-k/n}}
\]

with
\[ p_\pm := \begin{cases} 4 \sum_{j=0}^{m} \frac{1}{(2j+1)^{1/2}} & \text{(even)} \\ 4 \sum_{j=0}^{m} \frac{(-1)^j}{(2j+1)^{1/2}} & \text{(odd)} \end{cases} \]

are best possible.

Moreover, Kolmogorov established that the extremal functions of (5.1.1), i.e., those functions for which (5.1.1) reduces to an equality, are the Euler splines. Besides these two theorems it is appropriate to mention an extension of Landau's results to higher derivatives for functions \( f \in W^{(n)}(\mathbb{R}) \), due to Schoenberg and Cavaretta [54]. The emphasis of their work is on the construction of the extremal functions of (5.1.1).

Upper bounds for \( C_{n,k} \) in (5.1.1) with respect to functions defined on \( \mathbb{R}_0 \) are given by Stein [60].

For a lucid exposition of Landau's problem for the operators \( D^2 \) and \( b^2 \) the reader is referred to Schoenberg [52].

In order to put these problems into a more general context we define the set of functions \( F_n(\pi_n,J) \) by

\[(5.1.2) \quad F_n(\pi_n,J) := \{ f \in W^{(n)}(\mathbb{R}) \mid \| f \|_J < \infty, \| D^k f \|_J < 1 \} \]

here \( \pi_n < \pi \) is a monic polynomial, \( J \) is a closed subinterval of \( \mathbb{R} \), \( \pi \) is a positive number, and the set \( W^{(n)}(J) \) is defined on \( \pi J \).

In general, a Landau problem can then be described as follows: determine the best possible upper bound for \( \| D^k f \|_J \) on \( F_n(\pi_n,J) \), where \( D^k f \) is a linear differential operator of order \( k \leq n-1 \). Functions for which this upper bound is attained are again called extremal functions. Recently, Cavaretta [11] established that Euler splines are extremal for the problem of maximizing \( \| (iD^2 + bD^4)^2 \| \) on \( F_n(D^2 \mathbb{R}) \), where \( a \) and \( b \) are arbitrary real numbers and \( 0 \leq i \leq n-2 \). For a result involving a finite interval \( J \) the reader is referred to Chui and Smith [13], who compute the maximum of \( \| f \|_J \) on \( F_n(D^2 \mathbb{R}, J) \) for an interval \( J \) of arbitrary length. Using the approach of Cavaretta [9], occurring in an elementary proof of Kolmogorov's Theorem 5.1.2, Sharma and Trindalario [58] investigated the following special Landau problem: if \( \pi_n < \pi \) is a monic polynomial having only real zeros and if \( p_k \) is a divisor of \( \pi_n \), determine the maximum of
It turns out that a perfect Euler $L$-spline is extremal for this problem. The basic idea of Cavaretto in his elementary proof of Theorem 5.1.2 is first to prove that the Euler splines are extremal with respect to the periodic functions in $F_n(\mathbb{R}^n, \mathbb{R})$, and then to approximate a function $f \in \Gamma_0^\circ_m (\mathbb{R}^n, \mathbb{R})$ by periodic functions.

The contents of this chapter may be roughly described as follows. In Section 5.2 the significance of the Landau problem with respect to optimal differentiation algorithms is briefly explained. Our approach to studying the Landau problem is based on an investigation of the set $\Gamma_0^\circ_m (\mathbb{R}^n, \mathbb{R})$ of $\mathbb{R}^n$ defined by

\[(5.1.3) \quad \Gamma_0^\circ_m (\mathbb{R}^n, \mathbb{R}) \equiv \{ (z(\xi), z'(\xi), \ldots, z^{(r-1)}(\xi))^T \mid f \in F_m(\mathbb{R}^n, \mathbb{R}) \},\]

where $F_m(\mathbb{R}^n, \mathbb{R})$ is given by (5.1.2) and $\xi \in J$.

In Section 5.3 general properties of the sets $F_m(\mathbb{R}^n, \mathbb{R})$ and $\Gamma_0^\circ_m (\mathbb{R}^n, \mathbb{R})$ are derived. For instance, with respect to $F_m(\mathbb{R}^n, \mathbb{R})$ it is shown that any sequence of functions in $F_m(\mathbb{R}^n, \mathbb{R})$ contains a subsequence that converges to an element of $F_m(\mathbb{R}^n, \mathbb{R})$, uniformly on each compact subinterval of $J$. Moreover, simultaneous uniform convergence occurs for the successive derivatives up to the $(n-1)$-st order. The set $\Gamma_0^\circ_m (\mathbb{R}^n, \mathbb{R})$ is shown to be a convex and compact subset of $\mathbb{R}^n$ having $0$ as an interior point. Particular attention is given to the sets $\Gamma_0^\circ_m (\mathbb{R}^n, \mathbb{R})$ and $\Gamma_0^\circ_m (\mathbb{R}^n, \mathbb{R})$. The importance of the sets $\Gamma_0^\circ_m (\mathbb{R}^n, \mathbb{R})$ and $\Gamma_0^\circ_m (\mathbb{R}^n, \mathbb{R})$ for the Landau problem is explained in Section 5.4. In Section 5.5 the Landau problem for second order differential operators is studied in detail. With respect to the full-line case ($J = \mathbb{R}$) our main result is a characterization of the boundary of the set $\Gamma_0^\circ_m (\mathbb{R}^n, \mathbb{R})$ in terms of perfect Euler $L$-splines (Theorems 5.4.3 and 5.4.4). A few examples are given to illustrate the theorems. In the half-line case ($J = \mathbb{R}^+$) it is shown that the role of the perfect Euler $L$-splines is taken over by the so-called one-sided perfect Euler $L$-splines, which are used to parametrize the boundary of the set $\Gamma_0^\circ_m (\mathbb{R}^n, \mathbb{R})$. Various cases with regard to the location of the zeros of $p_2$ are considered. Problems, analogous to those of Section 5.5 for general third order differential operators are much more difficult to deal with. Therefore, Section 5.6 is restricted to third order differential operators $p_3(D)$ with the property $p_3(0) = p_3'(0)$. Operators of this
kind have the form \( p_2(D) = D^3 + cD \). Only the full-line case is studied for
these operators under the additional assumption that \( c > 0 \) (this last
restriction is not essential). The half-line case for operators \( p_2(D) \) of
the indicated form is, up to now, unsolved. In Chapter 7 the half-line case
for the operator \( p_2(D) = D^3 \) is discussed in detail.

5.2. Optimal differentiation algorithms

The purpose of this section is to outline briefly how Landau's problem is
related to the construction of optimal differentiation algorithms. This
problem occurs in the theory of optimal recovery (cf. Micchelli and Rivlin
[39, p. 27]) in connection with the approximation of a differential operator
by means of bounded operators (see also Steklov [59]).

Let \( f \in W^{(n)}(\mathbb{R}) \) (the set of functions \( f \in C^{(n)}(\mathbb{R}) \) for which \( \| f^{(n)} \| < \infty \)).
The objective is to construct an optimal algorithm for the computation of
\( p_k(D)f(0) \) (\( 1 \leq k \leq n-1 \)), based on the fact that \( \| p_k(D)f \| \leq 1 \) and that
required function values of \( f \) can be computed with an error bounded by \( \varepsilon \).
To be more precise: let \( A \) be any mapping from \( L_1(\mathbb{R}) \), the set of essentially
bounded measurable functions into \( \mathbb{R} \), and let \( g \in L_0(\mathbb{R}) \) be such that
\( \| g \| < \varepsilon \) for a given \( \varepsilon > 0 \). Then \( A \) is interpreted as an algorithm and
\( A(g) \) is considered to be an estimate for \( p_k(D)f(0) \). The error in using
algorithm \( A \) is defined by

\[
E_A(\varepsilon) := \sup \{ \| p_k(D)f(0) - A(g) \| : \| f \| \leq 1, \| g \| < \varepsilon, \| p_k(D)f \| \leq 1 \}.
\]

The intrinsic error is defined by

\[
\tilde{E}(\varepsilon) := \inf_{A} E_A(\varepsilon),
\]

and \( \tilde{E} \) is called an optimal algorithm if \( E_A(\varepsilon) = \tilde{E}(\varepsilon) \). The following lemma shows how the intrinsic error \( \tilde{E}(\varepsilon) \) is related to Landau's problem.

**Lemma 5.2.1.** The intrinsic error \( \tilde{E}(\varepsilon) \) as defined in (5.2.1) satisfies the
inequality

\[
\tilde{E}(\varepsilon) \geq \sup \{ \| p_k(D)f \| : f \in \mathcal{F}_\varepsilon(p_k,\mathbb{R}) \},
\]

where \( \mathcal{F}_\varepsilon(p_k,\mathbb{R}) \) is given by (5.1.2).
PROOF. For a function \( f \in F_c(p_n, \mathbb{R}) \) and any \( a \) one has
\[
|p_k(D) f(0) - A(0)| \leq e_A(a),
\]
and, since \( f \in F_c(p_n, \mathbb{R}) \), one also has
\[
|p_k(D) f(0) - A(0)| = |p_k(D) f(0) + A(0)| \leq e_A(a).
\]
Hence
\[
|p_k(D) f(0)| = \|p_k(D) f(0) - A(0) + p_k(D) f(0) + A(0)| \leq e_A(a),
\]
and therefore
\[
(5.2.2) \quad \sup \{ p_k(D) f(0) \mid f \in F_c(p_n, \mathbb{R}) \} \leq e_A(a).
\]
In the following section it will be shown (cf. Lemma 5.3.1) that the functional \( f \mapsto p_k(D) f(0) \) is bounded on \( F_c(p_n, \mathbb{R}) \). Since for every \( f \in F_c(p_n, \mathbb{R}) \) and every \( a \in \mathbb{R} \) the function \( t \mapsto f(t-a) \) also belongs to \( F_c(p_n, \mathbb{R}) \) we conclude that the left-hand side of (5.2.2) equals \( \sup \{ |p_k(D) f(0)| \mid f \in F_c(p_n, \mathbb{R}) \} \). This proves the lemma.

In the specific case that \( p_n(0) = 0 \) and that \( p_n \) has only real zeros, Michelli [33] has obtained optimal differentiation algorithms. It turns out that these differentiation algorithms are exact for perfect \( \varepsilon \)-splines corresponding to the operator \( p_n(D) \) and, consequently, they are of the type considered in Section 3.4.

5.3. Some general properties of the sets \( F_m(p_n, J) \) and \( \Gamma_m(p_n, J, \xi) \)

In this section the sets \( F_m(p_n, J) \) and \( \Gamma_m(p_n, J, \xi) \) as defined by (5.1.2) and (5.1.3) will be considered. With respect to \( F_m(p_n, J) \) a few results will be given, the first of which reads as follows.

LEMMA 5.3.1. For every polynomial \( p_k \in \pi_k, 0 \leq k \leq n \) the functional \( f \mapsto \| p_k(D) f \|_J \) is bounded on \( F_m(p_n, J) \).

PROOF. Let \( 0 < x_1 < x_2 < \ldots < x_n \) be such that \( p_n(D) \) is disconjugate on \( [0, x_n] \). We recall (cf. the proof of Lemma 1.4.18) that the functions \( t \mapsto \varphi(t-x_i) (i = 1, \ldots, n) \), where \( \varphi \) is the fundamental function correspond-
ing to \( p_n(D) \), form a basis for \( \text{Ker}(p_n) \). Since \( p_k(D) \in \text{Ker}(p_n) \) there exist uniquely determined coefficients \( a_1, \ldots, a_n \) such that

\[
p_k(D)q(t) = \sum_{i=1}^{n} a_i \phi(t-x_i) \quad (t \in \mathbb{R}) .
\]

As a consequence of Peano's remainder formula (cf. Lemma 1.4.5) it follows that

\[
p_k(D)f(t) = \sum_{i=1}^{n} a_i f(t-x_i) + \int_{t-x_n}^{t} K(t-t)f(t) \, dt ,
\]

where the kernel function \( K \) is given by

\[
K(t) = p_k(D)\psi(t) = \sum_{i=1}^{n} a_i \phi(t-x_i) \quad (t \in \mathbb{R}) .
\]

Hence

\[
\| p_k(D)f \|_y \leq \sum_{i=1}^{n} |a_i| \| f \|_y + \int_{0}^{1} |K(t)| \, dt \| \phi \|_y .
\]

In view of (5.1.2) this proves the lemma.

We note that a proof of this lemma can also be found in Michelli [38]; however, there it is assumed that \( p_n \) has only real zeros.

Our next result concerns a compactness property of the set \( F_n(p_n,J) \).

**Lemma 5.3.2.** Any sequence of functions \( \{ f_j \}_j \) in \( F_n(p_n,J) \) contains a subsequence \( \{ f_{j_k} \}_k \) that converges to a function \( f \in F_n(p_n,J) \) such that

\[
\lim_{j \to \infty} f_{j_k}^{(i)} = f^{(i)} \quad (i = 0, 1, \ldots, n-1) ,
\]

uniformly on each compact subset of \( J \).

**Proof.** It follows from Lemma 5.3.1 that \( \| f_{j_k}^{(i)} \|_y \leq C (i = 0, 1, \ldots, n; k = 1, 2, \ldots) \) for some constant \( C > 0 \), independent of \( k \) and \( i \). By Ascoli's theorem (cf. Kryszig [38, p. 454]) there exists a subsequence \( \{ f_{j_k} \}_k \) and a function \( f \in AC^\infty(J) \) such that \( f_{j_k}^{(i)} \to f^{(i)} \) \( (j \to \infty) \) for each \( i = 0, 1, \ldots, n-1 \), uniformly on each compact subset of \( J \). Since \( \| f \|_y \leq m \) one has \( \| f \|_y \leq m \).
It remains to prove (cf. (5.1.2)) that \( \|p_n(D)f\|_{J} \leq 1 \). For almost every \( t \in J \) we have
\[
p_n(D)f(t) = e^{(n)}(t) + \sum_{k=0}^{n-1} a_k e^{(k)}(t) = \\
= \lim_{\xi \to t} \frac{1}{t-\xi} \{ e^{(n-1)}(t) - e^{(n-1)}(\xi) + \sum_{k=0}^{n-1} a_k e^{(k)}(\xi) \}.
\]
Hence
\[
p_n(D)f(t) = \lim_{\xi \to t} \frac{1}{t-\xi} \lim_{\delta \to 0} \frac{1}{\delta} \int_{\xi}^{\xi+\delta} \{ p_n(D) f_k(t) + \sum_{k=0}^{n-1} a_k f_k(t) - f_k(t) \} dt = \\
= \lim_{\xi \to t} \frac{1}{t-\xi} \lim_{\delta \to 0} \frac{1}{\delta} \int_{\xi}^{\xi+\delta} \{ p_n(D) f_k(t) + \sum_{k=0}^{n-1} a_k f_k(t) - f_k(t) \} dt.
\]
Since for all \( t \in J \) and almost every \( \tau \in T \)
\[
\left| p_n(D) f_k(t) + \sum_{k=0}^{n-1} a_k f_k(t) - f_k(t) \right| \leq M(t-\tau)
\]
for some constant \( M > 0 \), we conclude that \( \|p_n(D)f(t)\| \leq 1 \) (a.e.), hence \( \|p_n(D)f\|_{J} \leq 1 \).

Having established these properties for the set \( F_n(p_n,J) \), we now turn to \( \Gamma_n(p_n,J,\xi) \) as defined in (5.1.3).

**Theorem 5.1.3.** The set \( \Gamma_n(p_n,J,\xi) \) is a convex and compact subset of \( \mathbb{R}^n \) having \( 0 \) as an interior point.

**Proof.** Since \( F_n(p_n,J) \) is convex the set \( \Gamma_n(p_n,J,\xi) \) (cf. (5.1.3)) is also convex. Furthermore, the compactness property immediately follows from Lemma 5.3.2. In order to prove that \( \Gamma_n(p_n,J,\xi) \) has \( 0 \) as an interior point it suffices to consider the full-line case since \( \Gamma_n(p_n,J,\xi) = \Gamma_n(p_n,J,\xi) \) for all subintervals \( J \subset \mathbb{R} \). Moreover, it is easily seen that \( \Gamma_n(p_n,J,\xi) = \Gamma_n(p_n,J,0) \) for all \( \xi \in \mathbb{R} \). Consequently, we have to prove that \( 0 \) is an interior point of \( \Gamma_n(p_n,J,0) \). To this end, let the functions \( f_k \) (\( k = 0, 1, \ldots, n-1 \)) be defined by
\[
f_k(t) := \frac{t^k}{k!} e^{-t} (t \in \mathbb{R}),
\]
where \( \alpha \) is a positive number, sufficiently small to ensure that 
\( f_i \in \mathcal{F}(p_n, \mathbb{R}) \) \( \{i = 0, 1, \ldots, n-1\} \). Then 
\[
\left(f_1(0), f_1'(0), \ldots, f_1^{(n-1)}(0)\right)^T = \alpha e_i,
\]
where \( e_i \) is the \( i \)-th unit vector.

Hence, \( \alpha e_i \in \mathcal{F}(p_n, \mathbb{R}, 0) \) for \( i = 0, 1, \ldots, n-1 \). Using the convexity of \( \mathcal{F}(p_n, \mathbb{R}, 0) \) one easily concludes that \( \mathcal{F}(p_n, \mathbb{R}, 0) \) is an interior point. This proves the theorem.

Obviously, \( \mathcal{F}_1(p_n, J, \xi) \subset \mathcal{F}_2(p_n, J, \xi) \) if \( m_1 < m_2 \). In the following lemma a condition is given ensuring that \( \mathcal{F}_1(p_n, J, \xi) \) is a proper subset of \( \mathcal{F}_2(p_n, J, \xi) \).

**Lemma 5.3.4.** Let \( m_1 < m_2 \). If there exists a nontrivial function \( f \in \text{Ker}(p_n) \) that is bounded on \( J \), then \( \mathcal{F}_1(p_n, J, \xi) \) is a proper subset of \( \mathcal{F}_2(p_n, J, \xi) \).

**Proof.** We consider a point \( \mathbf{w} = (f(\xi), f'(\xi), \ldots, f^{(n-1)}(\xi))^T \in \mathbb{R}^n \), where \( f \in \text{Ker}(p_n) \) is a nontrivial function bounded on \( J \). Since \( \mathcal{F}(p_n, J, \xi) \) is an interior point of \( \mathcal{F}_2(p_n, J, \xi) \), there exists a positive \( \lambda \) such that \( \lambda \mathbf{w} \in \mathcal{F}_2(p_n, J, \xi) \).

Furthermore, there is a positive \( \mu \) such that \( \|\mathbf{w}\| = \mu > 0 \). It is easy to verify that \( (\lambda + \mu) \mathbf{w} \in \mathcal{F}_2(p_n, J, \xi) \) and \( (\lambda + \mu) \mathbf{w} \notin \mathcal{F}_1(p_n, J, \xi) \), i.e., \( \mathcal{F}_1(p_n, J, \xi) \) is a proper subset of \( \mathcal{F}_2(p_n, J, \xi) \).

**Remark.** If all functions in \( \text{Ker}(p_n) \) are bounded on \( J \), then we can prove the stronger assertion that \( \mathcal{F}_1(p_n, J, \xi) \subset \mathcal{F}_2(p_n, J, \xi) \). This, of course, will always be the case if \( J \) is a bounded interval. Furthermore, \( \mathcal{F}_1(p_n, \mathbb{R}, 0) \) is a proper subset of \( \mathcal{F}_2(p_n, \mathbb{R}, 0) \) if \( p_n \) has at least one zero on the imaginary axis, since then there is a nontrivial bounded function in \( \text{Ker}(p_n) \). If there exists at least one zero of \( p_n \) with a nonpositive real part, then there is a nontrivial function in \( \text{Ker}(p_n) \) bounded on \( \mathbb{R}_0^+ \) and thus Lemma 5.3.4 may be applied to guarantee that \( \mathcal{F}_1(p_n, \mathbb{R}_0^+, \xi) \) is a proper subset of \( \mathcal{F}_2(p_n, \mathbb{R}_0^+, \xi) \).

From now on we restrict ourselves to the full-line case and the half-line case, i.e., to \( J = \mathbb{R} \) and \( J = \mathbb{R}_0^+ \). Since \( \mathcal{F}_m(p_n, \mathbb{R}, 0) = \mathcal{F}_m(p_n, \mathbb{R}, \xi) \) \( (\xi \in \mathbb{R}) \) and
\( \Gamma_{n}(p_{n}, \mathbb{R}^{+}, 0) = \mathcal{T}_{n}(p_{n}, \mathbb{R}^{+}, 0) (i \geq 0) \), the notations will be shortened as follows:

\[(5.3.1) \quad \Gamma_{n}(p_{n}) := \Gamma_{n}(p_{n}, \mathbb{R}, 0), \quad \mathcal{T}_{n}(p_{n}) := \mathcal{T}_{n}(p_{n}, \mathbb{R}^{+}, 0). \]

It is interesting to note that if \( p_{n} \) has no zeroes on the imaginary axis, then any number \( M_{p_{n}} \) exists such that \( \| F_{p_{n}}(\omega) \| \leq 1 \) and \( \| f \| \leq M_{p_{n}} \). This will be proved in Lemma 5.3.3. In order to arrive at this result we give the following definitions.

For \( n \geq 2 \) and \( p_{n} \in \mathbb{R}_{n} \) a sonic polynomial not having purely imaginary zeroes, we define the function \( F_{p_{n}} \) as the inverse Fourier transform of \( p_{n}^{-1}(\omega) \), i.e.,

\[(5.3.2) \quad F_{p_{n}}(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{p_{n}(\omega)} d\omega \quad (t \in \mathbb{R}). \]

Using the well-known formula

\[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega + i\epsilon)^{k+1}} d\omega = \frac{e^{-\epsilon t}}{t^{k+1}} \quad (t \in \mathbb{R}; \ Re \; \epsilon > 0; \ k \in \mathbb{N}),\]

one easily verifies that the function \( F_{p_{n}} \) decreases exponentially as \( |t| \to \infty \), and so the integral in the right-hand side of (5.3.3) is finite. In connection with \( F_{p_{n}} \) the number \( M_{p_{n}} \) is now defined by

\[(5.3.3) \quad M_{p_{n}} := \begin{cases} \int_{-\infty}^{\infty} |F_{p_{n}}(t)| dt & (p_{n} \text{ has no zeroes on the imaginary axis}), \\ = & (\text{otherwise}). \end{cases}\]

**Lemma 5.3.3.** If \( \xi \in F_{n}(\mathbb{R}, \mathbb{R}) \) for some \( n \), then \( \| f \| \leq M_{p_{n}} \), where \( M_{p_{n}} \) is given by (5.3.3).

**Proof.** There is nothing to prove if \( p_{n} \) has a zero on the imaginary axis, since then \( M_{p_{n}} = \infty \). So we may assume that \( p_{n}(\omega) \neq 0 \ (\omega \in \mathbb{R}) \). We note that, apart from the trivial function, every function in \( ker(p_{n}) \) is unbounded on \( \mathbb{R} \). Consequently, a bounded solution of the differential equation \( p_{n}(D)f = u \) is uniquely determined and it can be represented as a convolution integral.
\[ \varepsilon(t) = \int_{-\infty}^{\infty} F_{p_n}(t-\tau)u(\tau)d\tau. \]

Therefore, we may conclude that for \( f \in \mathcal{W}(p_n, \mathbb{R}) \)

\[ (5.3.4) \quad \varepsilon(t) = \int_{-\infty}^{\infty} F_{p_n}(t-\tau)p_n(D)f(\tau)d\tau \quad (t \in \mathbb{R}). \]

Hence, by (5.1.2),

\[ |\varepsilon(t)| \leq \int_{-\infty}^{\infty} |F_{p_n}(t-\tau)|d\tau = \varepsilon_n. \]

\[ \square \]

**Corollary 5.3.6.** \( \Gamma_m(p_n) = \Gamma_{\varepsilon_n}(p_n) \) \( (m \geq \varepsilon_n). \)

**Proof.** Lemma 5.3.5 implies (cf. (5.1.2)) that \( \mathcal{W}(p_n, \mathbb{R}) \subset \mathcal{W}(p_n, \mathbb{R}) \) and thus (cf. (5.1.3) and (5.3.1)) \( \Gamma_m(p_n) \subset \Gamma_{\varepsilon_n}(p_n). \) Moreover, \( \Gamma_{\varepsilon_n}(p_n) \subset \Gamma_m(p_n) \) if \( m \geq \varepsilon_n. \) \( \square \)

In the half-line case there are no nontrivial bounded functions in \( \mathcal{W}(p_n) \) if all zeros of \( p_n \) have positive real parts. If this occurs, the function \( F_{p_n} \) as defined in (5.3.2) vanishes for \( t > 0. \) Consequently, a function \( f \in \mathcal{W}(p_n, \mathbb{R}^+) \) can be represented as

\[ (5.3.5) \quad \varepsilon(t) = \int_{t}^{\infty} F_{p_n}(t-\tau)p_n(D)f(\tau)d\tau \quad (t \geq 0). \]

This leads to the following

**Lemma 5.3.7.** If all zeros of \( p_n \) have positive real parts, then

\[ \Gamma_m^+(p_n) = \Gamma_{\varepsilon_n}^+(p_n) = \Gamma_{\varepsilon_n}(p_n) \] \( (m \geq \varepsilon_n). \)

**Proof.** It follows from (5.3.5) that

\[ |\varepsilon(t)| \leq \int_{t}^{\infty} |F_{p_n}(t-\tau)|d\tau = \int_{0}^{\infty} |F_{p_n}(\tau)|d\tau = \int_{-\infty}^{\infty} |F_{p_n}(\tau)|d\tau = \varepsilon_n. \]
Consequently, \( \bar{\Gamma}_m^{\ast}(p_n) = \bar{\Gamma}_{\mathbb{R}_n}^{\ast}(p_n) \) for all \( m \geq \mathbb{R}_n \). Obviously, \( \bar{\Gamma}_{\mathbb{R}_n}^{\ast}(p_n) \subset \bar{\Gamma}_{\mathbb{R}_n}^{\ast}(p_n) \).

If \( \bar{\rho} \in \bar{\Gamma}_{\mathbb{R}_n}^{\ast}(p_n) \) then a function \( f \in \mathcal{F}_{\mathbb{R}_n}^{\ast}(p_n, \mathbb{R}) \) with \( \bar{u} = (u(0), f'(0), \ldots, f^{(n-1)}(0)) \) can easily be extended to a function \( f \in \mathcal{F}_{\mathbb{R}_n}^{\ast}(p_n, \mathbb{R}) \) by defining \( p_n(t)f(t) = 0 \) for all \( t < 0 \), so \( f \) is then given by (3.3.5) for all \( t \in \mathbb{R} \).

Hence, \( \bar{\rho} \in \bar{\Gamma}_{\mathbb{R}_n}^{\ast}(p_n) \), therefore \( \bar{\Gamma}_{\mathbb{R}_n}^{\ast}(p_n) \supset \bar{\Gamma}_{\mathbb{R}_n}^{\ast}(p_n) \), and thus \( \bar{\Gamma}_{\mathbb{R}_n} = \bar{\Gamma}_{\mathbb{R}_n}^{\ast}(p_n) \). \[ \square \]

5.4. The relation between the Landau problem and the sets \( \bar{\Gamma}_m^{\ast}(p_n) \) and \( \bar{\Gamma}_m(p_n) \)

We recall (cf. Lemma 5.3.1) that for any \( \bar{x} = (\bar{\rho}_0, \bar{\rho}_1, \ldots, \bar{\rho}_{n-1}) \in \mathbb{R}^n \) the functional \( f \mapsto \bar{\rho}_0 f + \bar{\rho}_1 f' + \cdots + \bar{\rho}_{n-1} f^{(n-1)} \) is bounded on \( \mathcal{F}_m(p_n, \mathbb{R}) \).

Furthermore, one easily verifies that

\[
\sup_{f \in \mathcal{F}_m(p_n, \mathbb{R})} \| \bar{\rho}_0 f + \bar{\rho}_1 f' + \cdots + \bar{\rho}_{n-1} f^{(n-1)} \| = \sup_{f \in \mathcal{F}_m(p_n, \mathbb{R})} (\bar{\rho}_0 f(0) + \bar{\rho}_1 f'(0) + \cdots + \bar{\rho}_{n-1} f^{(n-1)}(0)) .
\]

It follows from the definition of the set \( \bar{\Gamma}_m(p_n) \) (cf. (5.3.1) and (5.3.1)) that, in order to obtain the supremum in (5.4.1), one has to compute the supremum of the linear function \( x \mapsto \bar{x}^T x \) on \( \bar{\Gamma}_m(p_n) \). Since \( \bar{\Gamma}_m(p_n) \) is convex and compact this supremum will be attained at a point of \( \bar{\Gamma}_m(p_n) \).

On the other hand, if \( \bar{x} \in \bar{\Gamma}_m(p_n) \) then in view of Theorem 2 in Luenberger [31, p. 133] there is a hyperplane \( H \) containing \( \bar{x} \) such that \( \bar{\Gamma}_m(p_n) \) lies on one side of \( H \). Hence a vector \( \bar{\rho} \in \mathbb{R}^n \) exists such that \( \bar{\rho}^T \bar{x} \leq \bar{\rho}^T \bar{\rho} \) for all \( \bar{\rho} \in \bar{\Gamma}_m(p_n) \). Consequently, if \( f_0 \in \mathcal{F}_m(p_n, \mathbb{R}) \) satisfies \( f_0^{(i)}(\xi) = \bar{\rho}_i \) for some \( \xi \in \mathbb{R} \) then \( f_0 \) is an extremal function (cf. p. 103 for its definition), since in view of (5.4.1) the function \( f_0 \) maximizes \( \| \bar{\rho}_0 f + \bar{\rho}_1 f' + \cdots + \bar{\rho}_{n-1} f^{(n-1)} \| \) on \( \mathcal{F}_m(p_n, \mathbb{R}) \).

Summing up we conclude that a complete description of \( \bar{\Gamma}_m(p_n) \) is intimately connected with solving the Landau problem on \( \mathbb{R} \). A similar conclusion holds for the Landau problem on \( \mathbb{R}_0^+ \).

For \( n = 2 \) we are able to give such complete descriptions, and for \( n = 3 \) under the additional condition that \( p_3 = p_3^* \).

For the case \( n \geq 4 \), treated in Chapter 7, we investigate \( \bar{\Gamma}_m(p_n) \) under the conditions that \( p_n \) has only real zeros and \( p_n(0) = 0 \).
5.5. The Landau problem for second order differential operators

5.5.1. Introductory remarks

Taking into account the contents of Section 5.4 we aim at characterizing the sets $\Sigma_n^p[p_2]$ and $\Sigma_n^{s\star}p_2$. With respect to $\Sigma_n^p[p_2]$, being a closed curve surrounding the origin, it is shown that this set can be parametrized by means of perfect Euler $\ell$-spline $E(p_2;h,\cdot)$ corresponding to the operator $p_2(D)$ (cf. Definition 3.2.12). For this purpose some elementary properties of $E(p_2;h,\cdot)$ are needed, which are contained in Subsection 5.5.2. In Subsection 5.5.3 the main theorems for the full-line case are proved; it also contains a few examples. Subsection 5.5.4 is concerned with a parametrization of $\Sigma_n^{s\star}p_2$ in terms of the so-called one-sided perfect Euler $\ell$-splines $E'(p_2;h,\cdot)$. In general, several cases have to be distinguished corresponding to the various possibilities for the location of the zeros of $p_2$.

As remarked earlier our approach to deal with Landau problems is based on investigating the sets $\Sigma_n^p[p_2]$ and $\Sigma_n^{s\star}p_2$. Consequently, the results of this section and subsequent sections have been obtained within that framework. We observe that for second order differential operators, these results can be derived by different means.

5.5.2. Some elementary properties of $E(p_2;h,\cdot)$

We recall that $E(p_2;h,\cdot)$, the perfect Euler $\ell$-spline corresponding to the operator $p_2(D)$, is uniquely determined by the conditions

\[
\begin{align*}
\text{(i) } & \quad E(p_2;h,\cdot) \in C^{(1)}(\mathbb{R}) , \\
\text{(ii) } & \quad E(p_2;h,t+h) = - E(p_2;h,t) \quad (t \in \mathbb{R}) , \\
\text{(iii) } & \quad p_2(D)E(p_2;h,t) = -1 \quad (0 < t < h) ,
\end{align*}
\]

provided $p_2(-1) \neq 0$.

Remark. If $p_2(-1) = 0$ (cf. (3.2.11)) then a nontrivial function $f \in \text{Ker}(p_2)$ exists such that $f(t+h) = - f(t) \quad (t \in \mathbb{R})$. However, this only occurs if $p_2$ has purely imaginary zeros $\lambda_{1,2} = \pm i\beta \quad (\beta > 0)$ and if $h = (2k+1)\pi/8$ with $k \in \mathbb{N}_0$. 

In order to obtain some elementary properties of $E(p_2; h, \cdot)$, it is of importance to assume that $p_2(0)$ is disconjugate on $(0, h)$. The maximal length of an interval of disconjugacy for the operator $p_2(0)$ is equal to the distance between two consecutive zeros of the corresponding fundamental function $\varphi$ (cf. p. 6). If $p_2$ has zero $\lambda_{1,2} = \alpha + i\beta$ with $\beta > 0$, then $s(t) = e^{-\alpha t} e^{i\beta t}$ and thus the maximal length is equal to $\pi/\beta$. If the zeros of $p_2$ are real, then $p_2(0)$ is disconjugate on $R$. Associated with themonic polynomial $p_2$ we define the number $h^*$ by

$$
(5.5.2) \quad h^* := \begin{cases} 
\frac{\pi}{\beta} & (\lambda_1 = \alpha + i\beta, \beta > 0), \\
-\infty & (\lambda_1 \in R).
\end{cases}
$$

Consequently, every nontrivial function $f \in \ker p_2$ has at most one zero in $(0, h^*)$, counting multiplicities.

It follows from the remark above and the definition of $h^*$ that for all $h < h^*$ the function $E(p_2; h, \cdot)$ is well defined. Moreover, if $h^* < \infty$ then $E(p_2; h^*, \cdot)$ is well defined in case the complex zeros of $p_2$ have nonzero real parts.

For the purpose of parametrizing the set $\mathfrak{M}_0(p_2)$ we introduce the curve $\gamma$ defined by

$$
(5.5.3) \quad \gamma := \{(E(p_2; h, t), E'(p_2; h, t)) | 0 < t < 2h\}.
$$

Because $E(p_2, h, \cdot)$ is periodic with period $2h$ (cf. (5.5.1)) it immediately follows that $\gamma$ is a closed curve. The next lemma is used to deduce another property of $\gamma$.

**Lemma 5.5.1.** Let $(a, b)^T \in \mathbb{R}^2$ with $(a, b) \neq (0, 0)$ and let $h^*$ be defined by (5.5.2). If $0 < h < h^*$ or if $h = h^*$ in case $h^* < \infty$ and $p_2(-1) \neq 0$, then the function $s: (p_2; h, \cdot) + bE'(p_2; h, \cdot)$ has exactly one zero in $(0, h)$, counting multiplicities.

**Proof.** The conditions for $h$ imply that $E(p_2; h, \cdot)$ is well defined and that the operator $p_2(0)$ is disconjugate on $(0, h)$. Let the function $s$ be defined by

$$
s(t) := aE(p_2; h, t) + bh'(p_2; h, t) \quad (t \in \mathbb{R}).
$$

By (5.5.1) one has $p_2(0) s'(t) = 0 \quad (t \in (0, h))$ and $s(t+h) = -s(t) \quad (t \in \mathbb{R})$. If $s(0) \neq 0$ then $s(h)=s(0) = -s^2(0) < 0$ and thus $s$ has an odd number of
zeros in $\mathbb{Q}(0,h)$, counting multiplicities. Therefore, if $s$ has at least three
zeros in $\mathbb{Q}(0,h)$, then $s'$ has at least two zeros in $\mathbb{Q}(0,h)$. However, the op-
erator $p_2(D)$ is disconjugate on $\mathbb{Q}(0,h)$ since $h < h^*$. Hence $s'$ vanishes identi-
cally on $\mathbb{Q}(0,h)$. This contradicts $s(h) = -s(0) \neq 0$. If $s(0) = -s(h) = 0$
then the assumption of at least one zero in $\mathbb{Q}(0,h)$ again leads to $s'(t) = 0$
$\forall t \in \mathbb{Q}(0,h)$. Hence $s(t) = 0 \forall t \in \mathbb{Q}(0,h)$, contradicting $\{s,h\} \neq \{0,0\}$.
Consequently, $s$ has precisely one zero in $\mathbb{Q}(0,h)$, counting multiplicities. □

It follows that, under the condition of the lemma, every half line through the origin intersects $\gamma$ in precisely one point.

Now the aim is to determine the number $h$ such that $\|E(p_2;h,\cdot)\| = w$, where $w$ is a prescribed positive number. For this purpose $M_{P_2}$ as defined by (5.3.3)
is needed. Recalling that $\lambda_1$ and $\lambda_2$ denote the zeros of the monic polynomial
$p_2$, one can easily verify that

$$M_{P_2} = \begin{cases} |p_2(0)|^{-1} = |\lambda_1 \lambda_2|^{-1} \quad (\lambda_1 \lambda_2 \neq 0, \lambda_1 \in \mathbb{R}), \\ (a^2 + b^2)^{-1} \coth(-\frac{|a| b}{2h}) \quad (\lambda_1 = a + ib, a \neq 0, b > 0), \\ 1 \quad (\text{otherwise}). \end{cases}$$

The following lemma shows how $\|E(p_2;h,\cdot)\|$ depends on $h$. As the proof consists
of a straightforward computation, it is omitted here.

**Lemma 5.5.2.** Let $p_2 \in \mathcal{P}_2$ be a monic polynomial and let $h^*$ and $M_{P_2}$ be given
by (5.5.2) and (5.5.4), respectively. Then the function $h \mapsto \|E(p_2;h,\cdot)\|$ is
continuous and strictly increasing on $(0,h^*)$ and it has the properties:

1) If $h = h^*$, then $\lim_{h \to h^*} \|E(p_2;h,\cdot)\| = M_{P_2}$.

2) If $h < h^*$ and $M_{P_2} < 1$, then $\lim_{h \to h^*} \|E(p_2;h,\cdot)\| = \|E(p_2;h^*,\cdot)\| = M_{P_2}$.

3) If $h^* < h < h^*$ and $M_{P_2} = 1$, then $\lim_{h \to h^*} \|E(p_2;h,\cdot)\| = 1$.

Moreover, in all these cases $\lim_{h \to 0} \|E(p_2;h,\cdot)\| = 0$. 
If $h^* = m$ and $M_2 \leq m$, i.e., if the zeros of $p_2$ are real and $p_2(0) \neq 0$, then again a straightforward computation shows that
\[ E(p_2; h, t) = E(p_2; h, t) \quad (h = m), \]
uniformly in $t$ on every bounded subinterval of $\mathbb{R}$, where $E(p_2; h, \cdot)$ is the unique bounded function in $C^1(\mathbb{R})$ determined by
\[ p_2(0)E(p_2; h, t) = -\text{sgn}(t) \quad (t \in \mathbb{R}). \]

It easily follows that $\|E(p_2; h, \cdot)\| = M_2 = |p_2(0)|^{-1}$.

5.5.3. The full-line case ($J = \mathbb{R}$)

In this subsection we shall prove that the curve $\gamma$, defined by (5.5.3), is exactly the boundary $\partial E(p_2)$ we are looking for. The two cases $0 < m < M_2$ and $m = M_2$ are treated separately.

The following result is one of the main theorems of this chapter.

THEOREM 5.5.1. Let $p_2(D)$ be a differential operator of order two and let $m$ be such that $0 < m < M_2$. Then there exists a unique $h$ with $0 < h < h^*$ such that
\[ \partial E(p_2) = \{(E(p_2; h, \cdot), E'(p_2; h, \cdot)) \mid 0 \leq \tau < 2h\}. \]

PROOF. According to Lemma 5.5.2 there exists a unique $h \in (0, h^*)$ such that $\|E(p_2; h, \cdot)\| = m$. Let $(x, y)^T \in \partial E(p_2)$. Then by Lemma 5.5.1 there exists a positive $\sigma$ and a $t_0 \in (0, 2h)$ such that $x = \sigma E(p_2; h, t_0)$ and $y = \sigma E'(p_2; h, t_0)$.

We prove that $\sigma = 1$. As $(x, y)^T \in \partial E(p_2)$ a function $f \in E(p_2; \mathbb{R})$ exists such that $f(t_0) = x$ and $f'(t_0) = y$, without loss of generality we may assume that $t_0 \in (0, h)$. Let $g := f - \sigma E(p_2; h, \cdot)$, then $g(t_0) = g'(t_0) = 0$ and thus according to Taylor's formula (cf. Lemma 1.4.4)
\[ g(t) = \int_{t_0}^{t} g(t - \tau)p_2(D)g(\tau)d\tau, \]
where $g$ is the fundamental function corresponding to $p_2(D)$.

Since $p_2(D)$ is nondegenerate on $(0, h^*)$ and $0 < h < h^*$, one has $g(t) > 0$ $(0 < t < h)$ and $g(t) < 0 (-h < t < 0)$. We shall show that the assumption
\( \sigma > 1 \) leads to a contradiction. As a consequence of (5.5.1) we have
\[
p_2(0)g(t) = p_2(0)\kappa(t) + \sigma > 0 \quad (a.e.) \quad (t < (0,h)) ,
\]
and thus in view of (5.5.6), \( g(t) > 0 \) \( (0 \leq t \leq h, \ t \neq t_0) \). Hence
\[
f(t) > cE(p_2,h,t) \quad (0 < t < h, \ t \neq t_0) .
\]
As by (5.1.2) \( \forall \| \leq m, \) a contradiction is established if we prove that
\( E(p_2, \cdot, \cdot) \) attains its maximum \( m \) at a point in \( (0,h) \). According to Lemma
5.5.1 there is exactly one point \( t_1 \in (0,h) \) such that \( E'(p_2,h,t_1) = 0 \). In view of (3.2.20)
\[
E'(p_2,h,t) = \frac{1}{2i} \int_C \frac{e^z}{(e^z + 1)p_2(z)} \ dz \quad (0 \leq t < h) .
\]
By applying the residue theorem we can compute \( E'(p_2,h,t) \) explicitly. It
then follows from the sign structure of \( E'(p_2,h,\cdot) \) that \( E(p_2, \cdot, \cdot) \) attains
its maximum at a point in \( (0,h) \). Consequently \( \sigma < 1 \) if \( \sigma < 1 \) then because
of the fact that
\[
(x,0,y,0)^T = (E(p_2,h,t_1),E'(p_2,h,t_1))^T \in \Gamma(m,p_2) ,
\]
the point \( (x,0) \) is not a boundary point of \( \Gamma_m(p_2) \), contrary to the assumption
made. Hence \( \sigma = 1 \).

In order to treat the case \( \sigma = \mp_2 \), we have to distinguish between \( h^* < m \)
and \( h^* = m \). If \( h^* < m \) then it can be shown that the set \( \Gamma_m(p_2) \) consists of
the curve \( \gamma \) (cf. (5.5.3)) with \( h = h^* \). The proof is completely similar to
the proof of Theorem 5.5.3. If \( h^* = m \), i.e., if the zeros of \( p_2 \) are real
and \( p_2(0) \neq 0 \), then the function \( E(p_2,m,\cdot) \) may be used to describe \( \Gamma_m(p_2) \).
However, the curve \( \{(E(p_2,m,\cdot),E'(p_2,m,\cdot)) \mid -m < t < m \} \) is not closed: as
can be shown by a simple computation, it only describes half of the set
\( \Gamma_m(p_2) \). By a similar reasoning as given in the proof of Theorem 5.5.3 one
ultimately arrives at the following

**Theorem 5.5.4.** Let \( p_2(D) \) be a differential operator of order two such that
\( \mp_2 = m \). Then \( \Gamma_{mp_2}(p_2) \) is given by
The contents of Theorems 5.5.3 and 5.5.4 are intimately connected with the Landau problem for the operator \( p_2(D) \), as has been pointed out in Section 5.4. Consequently, we have the following

**Corollary 5.5.5.** Let \( p_2(D) \) be a differential operator of order two and let \( 0 \leq m < \text{mp}_2 \). Then there exists an \( h \in (0, h^*) \) such that for arbitrary \((a,b)^T \in \mathbb{R}^2\)

\[
\|a F(p_2,h,\cdot) + b F'(p_2,h,\cdot)\| = \max_{\mathcal{F}_m(p_2,\mathbb{R})} \|a F + b F'\|.
\]

Moreover, if \( m = \text{mp}_2 \) and \( h^* < h \) then (5.5.8) also holds for \( h = h^* \).

We proceed by discussing a few examples intended to illustrate the theorems just given. In each case a complete description of the set \( \Gamma_m(p_2) \) is obtained.

**Example 1.** \( p_2(D) = D^2 \).

As

\[
E(D^2;h,t) = -\frac{1}{2} t^2 + \frac{h}{2} t \quad (0 \leq t \leq h),
\]

the curve \( \gamma \) (cf. (5.5.3)) is given by the equation

\[
|x| = \frac{h^2}{2} - \frac{1}{2} y^2 \quad (-h \leq y \leq h).
\]

Hence, according to Theorem 5.5.3 with \( m = h^2/8 \),

\[
\Gamma_m(D^2) = \{(x,y)^T \mid |x| + |y|^2 = m\}.
\]

As an immediate consequence it follows that

\[
|f'(t)| \leq \sqrt{2(m - \|f(t)\|}) \quad (t \in \mathbb{R} ; f \in H_m(D^2,\mathbb{R})).
\]

which implies Landau's well-known inequality \( \|f''\| \leq \sqrt{2m} \) (cf. Theorem 5.1.1).
EXAMPLE 2. $p_2(D) = D^2 - I$.

In view of (5.5.4) it follows that $Np_2 = 1$. Since

$$E(D^2 - I; h, t) = 1 - \frac{\cosh(t - h/2)}{\cosh(h/2)} \quad (0 \leq t \leq h),$$

the curve $y$ is given by the equation

$$y^2 - x^2 + 2|x| = (\cosh(h/2))^2.$$

Hence, according to Theorem 5.5.3 with $m = 1 = \cosh^{-1}(h/2)$,

(5.5.10) $\mathcal{K}_m(D^2 - I) = \{(x, y) : y^2 - x^2 + 2|x| = 2m - m^2, \ |x| \leq m\}.$

It can be shown, using Theorem 5.5.4 with $h^* = m$, that if $m = 1$ then

(5.5.11) $\mathcal{K}_1(D^2 - I) = \{(x, y) : \ |x| + |y| = 1\}.$

A sketch of the set $\mathcal{K}_m(D^2 - I)$ for $m = 1$ and $m < 1$ is given in Figure 1.

![Figure 1](image)

As a consequence of (5.5.11) the inequality $|f(t)| + |f'(t)| \leq 1$ $(t \in \mathbb{R})$ holds for every bounded $f \in \mathcal{AC}^{(2)}(\mathbb{R})$ satisfying $\|f''\| \leq 1$. Schoenberg [53] obtained the best upper bound for $\|f''\|$ on $F_m(D^2 - I, \mathbb{R})$. Sharma and Trimbalaro [58] observed that the upper bound obtained by Schoenberg is also valid for the functional $\|f'' - af\|$ in case $|a| < (m(2-m))^h$. This
Assertion is also easily inferred from Figure 1, since the restriction on \( a \) agrees with the value of the two tangents at \( (c, m(2-a))^T \).

5.5.4. The half-line case \( (\mathcal{J} = \mathcal{E}_0^+) \)

The role of the perfect Euler \( \mathcal{E} \)-splines for the full-line case is now taken over by the so-called one-sided perfect Euler \( \mathcal{E} \)-splines. For the polynomial case these functions are introduced by Schoenberg and Cavaretta [54] as limits of sequences of special spline functions having knots in \( \mathcal{E}_0^+ \). With respect to the operators \( L^0 \) and \( L^3 \) the one-sided perfect Euler splines are simply related to the perfect Euler splines. In this subsection we only need the one-sided perfect Euler \( \mathcal{E} \)-spline corresponding to an arbitrary differential operator of order two. Therefore this function is introduced in relation to its corresponding perfect Euler \( \mathcal{E} \)-spline.

Let \( F(p_2; h, \cdot) \) be the perfect Euler \( \mathcal{E} \)-spline corresponding to the operator \( p_2(D) \) and let \( 0 < h < h^* \), where \( h^* \) is given by (5.5.2). We consider the \( \mathcal{E} \)-spline \( s \), corresponding to the operator \( L^{p_2}(D) \), with knots \( h, 2h, \ldots \) and coinciding with \( F(p_2; h, \cdot) \) on \( \mathcal{E}_0^+ \). This amounts to removing the knots \( 0, -h, -2h, \ldots \). The function \( F'(p_2; h, \cdot) \) has exactly one zero in \((0, h)\) where \( F(p_2; h, \cdot) \) attains its maximum \( m \) (cf. the proof of Theorem 5.5.3). Now the largest value \( t_0 \) of \( t \) is determined such that \( t_0 < 0 \) and \( s(t_0) = -m \); for the moment we take its existence for granted. Then a one-sided perfect Euler \( \mathcal{E} \)-spline, denoted by \( F_+(p_2; h, \cdot) \), is defined by

\[
F_+(p_2; h, t) := s(t + t_0) \quad (t > 0).
\]

(5.5.12)

We observe that the perfect Euler \( \mathcal{E} \)-spline and its one-sided version coincide for \( t \geq -t_0 \).

With respect to the existence of \( t_0 \) we remark that this is guaranteed in case \( p_2 \) has real zeros, as then \( s' \) has only one zero on \((-\infty, 0)\) and \( \lim_{t \to -\infty} s(t) = -\infty \).

If \( p_2 \) has nonreal zeros an explicit computation is needed to prove the existence of \( t_0 \). By way of example this will be shown for the operator \( p_2(D) = (D + \alpha)^* + 1 \), with \( \alpha > 0 \).

At some value \( t_1 \in (0, h) \) with \( (cf. (5.5.2)) \) \( h < h^* = \tau \), the function \( s \) attains its maximum \( m \). Since \( p_2(D)s(t) = -1 \) \((t < h)\), \( s(t_1) = m \), and \( s'(t_1) = 0 \), it follows that
\[ s(t) = \frac{1}{\sigma^2 + \frac{1}{2}} \left( t + \frac{1}{\sigma^2 + \frac{1}{2}} \right) e^{-\frac{1}{2} (t - t_1)} (\cos(t - t_1) + \sigma \sin(t - t_1)) . \]

Hence
\[ v(t_1 - \tau) = -\frac{1}{\sigma^2 + \frac{1}{2}} - \left( t + \frac{1}{\sigma^2 + \frac{1}{2}} \right) e^{-\sigma \tau} < 0 . \]

This relation ensures that \( t_0 \in [t_1 - \tau, 0] \).

The following theorem emphasizes the importance of the one-sided perfect Euler \( L \)-splines with respect to the Landau problem on the half line.

**Theorem 5.5.6.** Let \( p_2 \) be a monic polynomial and let \( m \) be such that \( 0 < m < m^* \). Then there exists an \( h \in (0, h^*) \) such that for each \( n < \mathbb{R} \)

\[ \left\| \mu E_n(p_2, h, \tau) + E_n^+(p_2, h, \tau) \right\| = \max_{f \in \mathbb{F}_n(p_2, h, \tau)} \left\| \mu f + f' \right\| , \]

where \( \mathbb{F}_n(p_2, h, \tau) \) is given by (5.1.2) and \( E_n(p_2, h, \tau) \) is the one-sided perfect Euler \( L \)-spline defined by (5.5.12).

**Proof.** Because of Lemma 5.5.2 there exists an \( h \in (0, h^*) \) such that \( I E_n(p_2, h, \tau) = m \). Taking into account the construction of \( E_n(p_2, h, \tau) \) on p. 120 we conclude that \( E_n(p_2, h, 0) = -m \) and that \( E_n(p_2, h, \tau) \) increases monotonically from \( -m \) to its first maximum \( m \) at \( t = t_1 \), say. Let \( \varphi \) be the fundamental function corresponding to the operator \( p_2(2) \) and let \( \varphi_1 \in \text{Ker}(p_2) \) be the function satisfying \( \varphi_1(0) = 1 \), \( \varphi_1'(0) = 0 \). Now \( \tau \in [0, t_1] \) is determined in such a way that

\[ (5.5.13) \quad \tau \varphi_1(t_1 - \tau) - \mu \varphi(t_1 - \tau) = 0 . \]

If such a \( \tau \) does not exist, then \( \tau \) is taken to be zero. We assert that

\[ (5.5.14) \quad \left\| \mu E_n(p_2, h, \tau) + E_n^+(p_2, h, \tau) \right\| = \max_{f \in \mathbb{F}_n(p_2, h, \tau)} \left\| \mu f + f' \right\| . \]

In order to prove (5.5.14) we apply Taylor's formula (1.4.4) with \( n = 2 \), \( a = t \), and \( g \in \text{Ker}(p_2) \) given by

\[ g(t) = f(t) \varphi_1(t - \tau) + f'(t) \varphi(t - \tau) \quad (t \in \mathbb{R}) . \]

Consequently,
\[ f(t_1) = f(t_1) \phi(t_1 - \tau) + f'(t_1) \psi(t_1 - \tau) + \int_0^{t_1} \psi(t_1 - \xi)p_2(D)\xi(\xi) d\xi. \]

If \( \tau \in [0,t_1] \) satisfies (5.5.13), then (5.5.15) can be rewritten as

\[ \phi(t_1 - \tau) f'(t_1) + \psi(t_1) = f(t_1) - \int_0^{t_1} \psi(t_1 - \xi)p_2(D)\xi(\xi) d\xi. \]

Since \([0,t_1] = [0,n^*] \) the operator \( p_2(D) \) is disconjugate on \([0,t_1] \) and thus \( \psi(t_1 - \xi) > 0 \) \( (0 \leq \xi < t_1) \). Assuming \( f \in C(p_2, W^1) \), we conclude from (5.5.16) that

\[ \int_0^{t_1} \psi(t_1 - \xi) d\xi = \phi(t_1 - \tau) \leq \frac{1}{\psi(t_1 - \tau)} \left( \psi(t_1) - \int_0^{t_1} \psi(t_1 - \xi) d\xi \right). \]

Substituting the function \( E^0(p_2h, \tau) \) for \( f \) into (5.5.16) we obtain

\[ \psi(t_1 - \tau) E^0(p_2h, \tau) + \psi(t_1) E^0(p_2h, \tau) = m + \int_0^{t_1} \psi(t_1 - \xi) d\xi. \]

Hence (5.5.17) implies that

\[ |(\psi(t_1) + \psi(t_1))| \leq \psi(t_1) E^0(p_2h, \tau) + \psi(t_1) E^0(p_2h, \tau) \]

and (5.5.14) easily follows.

Now suppose that no \( \tau \in [0,t_1] \) exists satisfying (5.5.13). Then

\[ \text{sgn}(\psi(t_1) - \psi(t_1)) = \text{sgn}(\psi(t_1 - v) - \psi(t_1)^+) = \psi(1) = 1. \]

Since \( \psi(t_1) > 0 \) there is a positive \( v \) such that \( \psi(t_1) - (u + v)\psi(t_1) = 0 \).

In view of this one arrives, by substituting \( \tau = 0 \) into (5.5.15), at

\[ \psi(t_1) u(0) + f'(0) = -\psi(t_1) f(0) + f(t_1) - \int_0^{t_1} \psi(t_1 - \xi)p_2(D)\xi(\xi) d\xi. \]

Consequently,

\[ |ue(0) + f'(0)| \leq \frac{1}{\psi(t_1)} \left( \psi(t_1)m + \int_0^{t_1} \psi(t_1 - \xi) d\xi \right). \]
\[ M = \frac{1}{\varphi(T_{\lambda})} \left( \mu E_{\lambda}(p_2; h, 0) + E_{\lambda}^{\prime}(p_2; h, 0) \right), \]
and again (5.5.14) is established. This completely proves the theorem.

A sketch of \( \Gamma_m^+(p_2) \), where \( p_2(0) = D^2 - 1 \) and \( 0 < m < M_0 = 1 \), is given below.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{Figure 2.}
\end{figure}

In order to compare the sets \( \Gamma_m^+(D^2 - 1) \) and \( \Gamma_m^+(b^2 - 1) \) Figure 2 also contains the set \( \Gamma_m(b^2 - 1) \) (cf. Figure 1, p. 119). Obviously, \( \Gamma_m(D^2 - 1) \subset \Gamma_m(b^2 - 1) \).

Note that the part of \( \partial \Gamma_m^+(D^2 - 1) \) for which \( x > -m \) and \( y > 0 \) may be parameterized by \( (E(p_2; h, 1), E'(p_2; h, 1)) \) and that the boundaries of \( \Gamma_m(D^2 - 1) \) and \( \Gamma_m(b^2 - 1) \) coincide for \( \{(x, y)^T \mid x \geq 0, 0 \leq y \leq (m(2-m))^2\} \) and for \( \{(x, y)^T \mid x \leq 0, -(m-2-m)^2 \leq y \leq 0\} \).

Having dealt with \( 0 < m < M_0 \), we now consider the case when \( m \geq M_0 \). If the zeros \( \lambda_1 \) and \( \lambda_2 \) of \( p_2 \) have positive real parts then, according to Lemma 5.3.7, we have \( \Gamma_m^+(p_2) = \Gamma_m(p_2) \), and the results of Theorem 5.5.4 apply. With respect to the location of \( \lambda_1 \) and \( \lambda_2 \), two cases remain to be considered, viz. \( \lambda_1 < 0, \lambda_2 > 0 \), and \( \Re(\lambda_1) < 0, \Re(\lambda_2) < 0 \).

**Theorem 5.5.7.** Let the zeros \( \lambda_1 \) and \( \lambda_2 \) of the monic polynomial \( p_2 \) satisfy \( \lambda_1 < 0, \lambda_2 > 0 \) and let \( M_0 \leq m < m_0 \), where \( M_0 \) is given by (5.5.4). Then
\[ \Gamma_m^+(p_2) = \{(x, y)^T \mid |x| \leq m, |y - \lambda_1 x| \leq \lambda_2^{-1} \}. \]
PROOF. Let \( m \geq \lambda p_2 = (-\lambda_1,\lambda_2)^{-1} \) (cf. (5.5.4)). We first show that
\[
|f'(0) - \lambda_1 f(0)| \leq \lambda_2 \quad \text{for all } f \in \mathbb{P}_m(p_2; \mathbb{R}^n_0).
\]
The function \( g := f' - \lambda_1 f \) is bounded on \( \mathbb{R}^n_0 \) and satisfies the differential equation \( g' - \lambda_2 g = p_2(0) f \).
This equation has a unique bounded solution which is easily seen to be
\[
g(t) = m \int_0^t e^{\lambda_2 (t-\tau)} p_2(0) f(\tau) \, d\tau \quad (t \geq 0).
\]
Hence (cf. (5.1.4))
\[
|f'(0) - \lambda_1 f(0)| = |g(0)| \leq m \int_0^\infty e^{-\lambda_2 \tau} \, d\tau = \lambda_2^{-1} \quad (f \in \mathbb{P}_m(p_2; \mathbb{R}^n_0)).
\]
It follows that
\[
\mathbb{P}_m^+(p_2) \subset \{ (x,y)^T \mid |x| \leq m, |y - \lambda_1 x| \leq \lambda_2^{-1} \}.
\]
Now let \((x,y)\) satisfy \( y - \lambda_1 x = \lambda_2^{-1} \) and \( |x| \leq m \). We must prove that
\[
(x,y)^T \in \mathbb{P}_m^+(p_2).
\]
If \( x = (-\lambda_1,\lambda_2)^{-1} \) and \( y = 0 \) it is sufficient to observe that the constant function \( (-\lambda_1,\lambda_2)^{-1} \) belongs to \( \mathbb{P}_m(p_2; \mathbb{R}^n_0) \).
If \(-m \leq x < (-\lambda_1,\lambda_2)^{-1} \) then the function \( f_1 \) given by
\[
f_1(t) := (-\lambda_1,\lambda_2)^{-1} - (m - (-\lambda_1,\lambda_2)^{-1}) e^{\lambda_1 t}
\]
is an element of \( \mathbb{P}_m(p_2; \mathbb{R}^n_0) \) and therefore \( (f_1'(t),f_1(t))^T \in \mathbb{P}_m^+(p_2) \) (\( t \geq 0 \)).
Now \(-m \leq f_1(t) \leq (-\lambda_1,\lambda_2)^{-1} \) and \( f_1'(t) - \lambda_2 f_1(t) = \lambda_2^{-1} \).
If \( (-\lambda_1,\lambda_2)^{-1} \leq x \leq m \) then the function \( f_2 \) given by
\[
f_2(t) := (-\lambda_1,\lambda_2)^{-1} + (m - (-\lambda_1,\lambda_2)^{-1}) e^{\lambda_1 t}
\]
can be used to show that \( (x,y)^T \in \mathbb{P}_m^+(p_2) \).

A sketch of \( \mathbb{P}_m^+(p_2) \), where \( p_2(0) = D^2 - I \) and \( m \geq 1 \), is given below.
The only remaining case is Re(λ₁) < 0, Re(λ₂) < 0, and m ≥ M₀₂. Let g be the function satisfying \( p₂(0)g(t) = 1 \) (\( t > 0 \)), \( g(0) = m \) and \( g'(0) = 0 \). Hence, if \( \lambda_1, \lambda_2 = \alpha \pm \beta i \) with \( \alpha < 0 \) and \( \beta > 0 \) then

\[
g(t) = (\alpha^2 + \beta^2)^{-1} + (m - (\alpha^2 + \beta^2)^{-1})e^{\alpha t}(\cos(\beta t) - (\alpha i)^{-1} \sin(\beta t))
\]

The function g has a maximum at \( t = 0 \) and for some value \( t^* < 0 \) it increases from \(-m\) to \( m \) on [\( t^*, 0 \)]. The function \( \rho_m(p₂, t) \) is now defined as

\[
\rho_m(p₂, t) := \begin{cases} 
g(t-t^*) & 0 \leq t < t^* \\
m & t \geq t^*
\end{cases}
\]

Evidently, \( \rho_m(p₂, t) \in F_m(p₂, R₀^+) \). Moreover, \( \rho_m(p₂, t) \) has the following extremal property, the proof of which is omitted as it is similar to the proof of Theorem 5.5.7.

**THEOREM 5.5.9.** Let \( p₂ \in \mathcal{R} \) be a monic polynomial whose zeros \( \lambda₁ \) and \( \lambda₂ \) satisfy Re(\( λ₁ \)) < 0, Re(\( λ₂ \)) < 0 and let \( m \geq M₀₂ \), where \( M₀₂ \) is given by (5.5.4). Then for each \( u \in R \) one has

\[
\|\rho_m(\bar{p₂}, t) + \rho_m(\bar{p₂}, t)\|_{F_m(p₂, R₀^+)} = \max_{f \in F_m(p₂, R₀^+)} \|u + f\|_1.
\]
5.6. The Landau problem for third order differential operators

5.6.1. Introductory remarks

In the previous section we discussed in detail the Landau problem for general second order differential operators in the full-line case and in the half-line case. For general third order differential operators the analogous problems are much more complicated. However, a significant simplification occurs if the operators have the property $p_3(D) = p_3^*(D)$; this will be assumed throughout this section. This introduces a certain symmetry which makes the problem easier to handle. Consequently, only operators of the form $p_3(D) = D^3 + cD (c \in \mathbb{R})$ will be considered. Furthermore, we only consider the case $b^2 := c > 0$ ($b > 0$); the analysis and the results for the other two cases $c < 0$ and $c = 0$ are quite similar. We note that results for the case $c = 0$ may be found in Ter Horst (44).

In Subsection 5.6.2 it is shown that $\mathcal{Z}_h(p_3)$ can be parametrized by means of the perfect Euler $L$-splines $\mathcal{E}(p_3(h, \cdot ))$. According to Definition 3.2.12 the function $\mathcal{E}(p_3(h, \cdot ))$ is defined for those values of $h$ for which $p_3(-1) \neq 0$. Since (cf. (3.2.11))

$$\widetilde{p}_3(z) = (z-1)(2z-2e^{i\theta h}) (z \in \mathbb{C}),$$

one has $\widetilde{p}_3(-1) = -4(1 + \cos(\theta h))$ and therefore the restriction $h \not\in (2k + 1)\pi /8$ ($k \in \mathbb{N}$) must be made. It easily follows from (3.2.30) that for $0 \leq t < \pi /8$

$$\mathcal{E}(p_3(h, t)) = \frac{1}{\sqrt{2}} \left( \frac{h}{\sqrt{2}} - t \right) + \frac{1}{\sqrt{2}} \frac{1}{\cos(\theta h / 2)} \sin(\theta (t - \frac{h}{2})) .$$

(5.6.1)

We note that the fundamental function $\varphi$ corresponding to the operator $D^3 + b^2 D$ is given by (cf. Definition 1.3.3) $\varphi(t) = e^{it}(1 - \cos \theta t)$ and therefore (cf. Lemma 1.4.3) $D^3 + b^2 D$ is disconjugate on $(0, 2\pi /8)$. Since $\mathcal{E}'(p_3(h, \cdot ))$ is an Euler $L$-spline corresponding to the operator $D^3 + b^2 D$, it follows from Lemma 3.2.6 that the function $\mathcal{E}'(p_3(h, \cdot ))$ has precisely one zero in $(0, h)$, provided $0 < h < \pi /8$.

An easy computation shows that

$$\mathcal{E}(p_3(h, \cdot )) = \mathcal{E}(p_3(h, 0)) = \mathcal{E}(p_3(h, h)) =$$

$$= e^{-i(\tan(\theta h) - i\theta h)} (0 \leq h < \frac{\pi}{8}) .$$

(5.6.2)
In what follows the abbreviation $M(h) := 6^{-2}(\tan(8h/2) - 8h/2)$ will be used.

5.6.2. The full-line case ($I = \mathbb{R}$)

We recall (cf. Theorem 5.5.3) that the curve consisting of the points

\[ \left( E(p_3; r, \tau), E^*(p_3; h, \tau) \right)^T \quad (0 \leq \tau \leq 2h) \]

in $\Gamma_m(p_3)$, where $0 < m < \|p_3\|$. Now it will be shown that a similar phenomenon occurs for the third order operator $p_3(D) = D^3 + \beta^2 D$. However, the function $E(p_3; h_0, \tau)$ with $h_0 \in (0, \pi/8)$ satisfying $\|E(p_3; h_0, \tau)\| = m$ (cf. 5.6.2) generates a closed curve on the two-dimensional surface $\Gamma_m(p_3)$. Therefore we have to look for other extremal functions. It turns out that one also obtains extremal functions if appropriate constants are added to the perfect Euler $\pi$-splines $E(p_3; h_0, \tau)$, where $0 \leq h \leq h_0$. Details are given in the following.

**Theorem 5.6.1.** Let $m > 0$, let $E(p_3; h_0, \tau)$ be the perfect Euler $\pi$-spline corresponding to the operator $p_3(\alpha) = D^3 + \beta^2 D$ with $\beta > 0$ and, furthermore, let $h_0 \in (0, \pi/8)$ be such that $\|E(p_3; h_0, \tau)\| = m(h_0) = m$. Then $\Gamma_m(p_3)$ can be parametrized by

\[
(5.6.3) \quad \Gamma_m(p_3) = \left\{ \left( E(p_3; h, \tau) + c(m-M(h)), E^*(p_3; h, \tau), E^*(p_3; h, \tau) \right)^T \right\} \\
0 \leq h \leq h_0, \quad 0 \leq \tau \leq 2h, \quad c = \pm 1.
\]

**Proof.** Let the function $f$ be defined by

\[
(5.6.4) \quad f(t) := E(p_3; h, t) + c(m-M(h)) \quad (t \in \mathbb{R}).
\]

We first show that $(f(t), f'(t), f''(t))^T \in \Gamma_m(p_3)$ if $0 \leq h \leq h_0$, $0 \leq \tau \leq 2h$, and $c \in \{1, -1\}$. If $h = 0$, then $E(p_3; h, t) \equiv 0$ ($t \in \mathbb{R}$) and $M(h) = 0$, and thus we obtain the two points $f(0, 0)$ which obviously belong to $\Gamma_m(p_3)$. If $0 \leq h \leq h_0$, then $0 < M(h) < m$. Consequently, in view of the definition of $f$ and (5.6.2) we get

\[
(5.6.5) \quad \max_{0 \leq t \leq 2h} f(t) = f(h) = \begin{cases} 
\min \{n \} \quad (c = 1), \\
2k(h) - n \leq m \quad (c = -1),
\end{cases}
\]

\[
\min_{0 \leq t \leq 2h} f(t) = f(0) = \begin{cases} 
\min \{n - 2M(h) \} \quad (c = 1), \\
- m \quad (c = -1).
\end{cases}
\]
Hence \( \| f \| = m \) and \( \| p_3(0) f \| = \| p_3(0) E(p_3; h, \tau) \| = 1 \). Therefore \( f \in \Gamma_{m}(p_3; \mathbb{R}) \) and \( (f(t), f'(t), f''(t))^T \in \mathcal{M}_{m}(p_3) \). In order to prove that \( (f(t), f'(t), f''(t))^T \in \mathcal{M}_{m}(p_3) \) we assume that \( \tau \in [0, h] \), since a similar argument may be used if \( \tau \in [h, 2h] \). If \( (f(t), f'(t), f''(t))^T \not\in \mathcal{M}_{m}(p_3) \) then because of the convexity of \( \Gamma_{m}(p_3) \) and the fact that \( 0 \) is an interior point of \( \Gamma_{m}(p_3) \) (cf. Theorem 5.3.3), there exists a number \( \lambda > 1 \) such that \( \lambda (f(t), f'(t), f''(t))^T \not\in \mathcal{M}_{m}(p_3) \). Let \( q \in \mathcal{M}_{m}(p_3; \mathbb{R}) \) be a function satisfying \( q^{(i)}(\tau) = \lambda f^{(i)}(\tau) \), \( i = 0, 1, 2 \), then the function \( r \) given by

\[
r(t) := q(t) - \lambda f(t), \quad t \in \mathbb{R}
\]

satisfies the relations \( r^{(i)}(\tau) = 0 \), \( i = 0, 1, 2 \) and

\[
p_3(0) r(t) = p_3(0) q(t) - \lambda p_3(0) E(p_3; h, \tau) = p_3(0) q(t) + \lambda \geq 1
\]

for almost every \( t \in [0, h] \). Using Taylor's formula (cf. Lemma 1.4.4), we obtain

\[
(5.6.6) \quad r(t) = \int_{0}^{t} q(t - \xi) (p_3(0) q(\xi) + \lambda) d\xi \quad (0 \leq t \leq h).
\]

Here \( s(t) = \rho(t) - \lambda s(t) \), the fundamental function corresponding to the operator \( \rho^3 + \rho^2 \mathcal{M} \). It follows from (5.6.6) that \( r(0) = 0 \) and \( r(h) \neq 0 \). Hence \( q(h) \geq \lambda f(h) \) and \( q(0) \leq \lambda f(0) \). As a consequence of (5.6.5) we conclude \( q(h) > \lambda m > m \) or \( q(0) \leq -m \), contradicting \( q \in \mathcal{M}_{m}(p_3; \mathbb{R}) \). This contradiction establishes that \( (f(t), f'(t), f''(t))^T \not\in \mathcal{M}_{m}(p_3) \).

We proceed by showing that for all \( (x, y, z)^T \in \mathcal{M}_{m}(p_3) \) there exist numbers \( h^* \in [0, h_{0}], \tau^* \in [0, 2h_{0}], \) and \( \sigma^* \in (-1, 1) \) such that \( (x, y, z)^T = (f(\tau^*), f'(\tau^*), f''(\tau^*)) \). If \( y = z = 0 \) then \( x = \pm n \), and we may take \( h^* = \tau^* = 0, \sigma^* = \pm 1 \). It remains to consider the case \( (y, z) \neq (0, 0) \). We note that if \( q \in \mathcal{M}_{m}(p_3; \mathbb{R}) \) then the functions \( -q \) and \( t \rightarrow q(-t) \) also belong to \( \mathcal{M}_{m}(p_3; \mathbb{R}) \). Consequently, if \( (x, y, z)^T \in \mathcal{M}_{m}(p_3; \mathbb{R}) \) then also \( -(x, y, z)^T \in \mathcal{M}_{m}(p_3; \mathbb{R}) \) and \( (x, y, z)^T \in \mathcal{M}_{m}(p_3; \mathbb{R}) \). Because of this symmetry we may assume that \( y \geq 0 \) and \( z \leq 0 \). It will now be shown that numbers \( h^* = (C, h_{0}], \tau^* \in [0, h_{0}], \sigma^* = \pm 1 \) exist such that \( f(\tau^*) = x, f'(\tau^*) = y, \) and \( f''(\tau^*) = z \) if \( (x, y, z)^T \in \mathcal{M}_{m}(p_3; \mathbb{R}) \). Using (5.6.1) and (5.6.4) we obtain the following equations for \( h^*, \tau^*, \sigma^* \):
\[
(5.6.7) \begin{cases}
1) \quad E(p_3|h^*, r^*) + \sigma^*(m - N(h^*)) = \kappa , \\
2) \quad \cos(\beta r^* - \beta h^*) = \beta y + 1 , \\
3) \quad \sin(\beta r^* - \beta h^*) = \beta z .
\end{cases}
\]

Given \( y > 0 \) and \( z < 0 \) with \((y, z) \neq (0, 0)\) it easily follows that equations 1) and 2) of (5.6.7) uniquely determine \( h^* \in (0, \pi/8) \) and \( r^* \in [0, \pi/2] \).

Having determined these numbers \( h^* \) and \( r^* \) our next purpose is to find \( \sigma^* \in [-1, 1] \) such that \( E(p_3|h^*, r^*) + \sigma^*(m - N(h^*)) = \kappa \). Let \( m_1 \) be the smallest constant satisfying
\[
(5.6.8) \begin{cases}
1) \quad m_1 \geq N(h^*) , \\
2) \quad E(p_3|h^*, r^*) + \{m_1 - N(h^*)\} = \kappa \\
3) \quad E(p_3|h^*, r^*) - \{m_1 - N(h^*)\} = \kappa .
\end{cases}
\]

If \( m_1 \) is determined by the first equation of 1) of (5.6.8) then \( \sigma^* = 1 \)
else \( \sigma^* = -1 \).

Note that \( m_1 - m \) has to be proved. Let \( h_1 \in (0, \pi/8) \) satisfy \( N(h_1) = m_1 \). We conclude that \((x, y, z) = (f(r^*), f'(r^*), f''(r^*))\), where \( f \) is given by (5.6.4)
with \( m \) replaced by \( m_1 \), \( b \) by \( h^* \), and \( \sigma \) by \( \sigma^* \). Consequently, \((x, y, z) \in S^*_{\|p_3\|} \cap S^*_{\|p_3\|} \) |
with the first part of the proof. Since, by a simple reasoning, \( S^*_{\|p_3\|} \cap S^*_{\|p_3\|} \) |
for notation, and because of the assumption \((x, y, z) \in S^*_{\|p_3\|} \cap S^*_{\|p_3\|} \) |
we infer that \( n = m_1 \). Therefore \( h_1 = h_0 \) and \( \sigma^* \in [0, \pi/8] \). This completely proves the theorem.

\[\Box\]

**Remark.** Theorem 5.6.1 implies the following assertion. If \( W \) is a function |
defined on \( \mathbb{R}^3 \) such that \( |W| \) attains its maximum on \( \Gamma^*_n(p_3) \) |
attains the functional |
\[
\|W(f, f', f'')\| := \sup_{t \in \mathbb{R}} |W(f(t), f'(t), f''(t))|
\]
attains its maximum for a perfect Euler Spline plus an appropriate constant function. In particular, this will be the case when \( W \) is a linear function. This gives rise to the following
COROLLARY 5.6.1. Let $m > 0$, let $E(p_3; h_0, \cdot)$ be the perfect Euler $1$-spline corresponding to $p_3(D) = D^3 + s^2 D$ with $s > 0$ and, furthermore, let $h_0 < (2, \pi/3)$ be such that $E(p_3; h_0, \cdot) = m$. Let $p_2 \in \Pi_2$ be an arbitrary polynomial. Then there exist an $h \in [0, h_0)$ and a constant function $d$ such that

$$\max_{E(p_2(D))} \|p_2(D)\| = \|p_2(D) (E(p_2; h, \cdot) + d)\|.$$ 

REMARK 2. The corresponding results for the operator $p_3(D) = D^3$, which may also be found in Ter Morsche [44], can easily be obtained by letting $s$ tend to zero. A sketch of $\Gamma_m(D^3)$ is given in Figure 4. We observe that $\Gamma_m(D^3)$ is not strictly convex since its boundary contains straight line segments (for instance, the line segment PQ).
According to Remark 2 a parametrization of $\mathcal{F}_m(D^3)$ can be derived as follows. Letting $\beta$ tend to zero in $M(h) := \beta^{-3} (\tan(\beta h) - \beta h)$ (cf. p. 127) we obtain $M(h) = h^3/24$. Since $N(h, 0) = n$ (cf. Theorem 5.6.1) it follows that $h_0 = 2(3m)^{1/3}$. Moreover, the perfect Euler spline $E(D^3; h, \cdot)$ is obtained from (5.6.1) by letting $\beta$ tend to zero. In view of Theorem 5.6.1 we then have

$$\mathcal{F}_m(D^3) = \{(E(D^3; h, t_1), E(D^3; h, t_2), E^3(D^3; h, t)) \mid 0 \leq h \leq 2(3m)^{1/3}, 0 \leq t_1 < 2h, \sigma = \pm 1\},$$

where

$$E(D^3; h, t) = -\frac{1}{6} t^3 + \frac{1}{4} h t^2 - \frac{1}{24} h^3 \quad (0 \leq t \leq h).$$

On the boundary of $\mathcal{F}_m(D^3)$ we have drawn two closed curves denoted by 1 and 2, respectively. Curve 1 corresponds to the perfect Euler spline $E(D^3; h, \cdot)$ with $h = 2(3m)^{1/3}$. Curve 2 corresponds to $E(D^3; h, \cdot) + (m - \frac{1}{24} h^3)$ with $h = (12m)^{1/3}$; it obviously contains the point $B = (0, 0, h)$. With respect to the Landau problem it is evident from Figure 4 that $\|f''\| \leq (3m)^{1/3}$ and $\|f''\| \leq (3m)^{1/3}$ whenever $\|f''\| \leq 1$ and $\|f''\| \leq m$. 
6. ON THE LANDAU PROBLEM FOR PERIODIC FUNCTIONS

6.1. Introduction and summary

In Section 5.1 of the preceding chapter the Landau problem has been defined as the problem of determining the best possible upper bound for a functional of the form \( I = \| p(D) f \|_\infty \) on \( C_p(p, J) \). We recall (cf. Section 5.4) that for \( J = \mathbb{R} \) or \( J = \mathbb{R}_+^n \) this problem is equivalent to determining the maximum of a linear function on the convex sets \( \Gamma_p(p, J) \) and \( \Gamma^p_p(p, J) \), respectively.

In this chapter we shall deal with the Landau problem for periodic functions. In order to obtain an appropriate setting for the problems involved, a few definitions will be given first. In these definitions, and also throughout this chapter, \( T \) is a positive number denoting the period of the periodic functions. We recall (cf. p. 72) that periodic functions with period \( T \) are called \( T \)-periodic.

**DEFINITION 6.1.1.** The sets \( W^{(n)}(\mathbb{R}, T) \) and \( L^\infty(\mathbb{R}, T) \) are defined by

\[
W^{(n)}(\mathbb{R}, T) := \{ f \in W^{(n)}(\mathbb{R}) \mid f(t + T) = f(t), \ t \in \mathbb{R} \},
\]

\[
L^\infty(\mathbb{R}, T) := \{ f \in L^\infty(\mathbb{R}) \mid f(t + T) = f(t) \ (a.e., \ t \in \mathbb{R}) \},
\]

where \( W^{(n)}(\mathbb{R}) \) and \( L^\infty(\mathbb{R}) \) are given on p. 3 of Subsection 1.2.3.

**DEFINITION 6.1.2.** Given a differential operator \( p_n(D) \), the set \( F(p_n, T) \) is defined by

\[
F(p_n, T) := \{ f \in W^{(n)}(\mathbb{R}, T) \mid \| p_n(D) f \| \leq 1 \}.
\]

We proceed by defining the set \( \Gamma(p_n, T) \), which is related to \( F(p_n, T) \) in the same way as \( \Gamma_0(p_n, \mathbb{R}, 0) \) is related to \( F_0(p_n, \mathbb{R}) \) (cf. (5.1.2) and (5.1.3)).
DEFINITION 6.1.3.

\[(6.1.1) \quad \Gamma(p_n, T) := \{(f(0), f'(0), \ldots, f^{(n-1)}(0))^T \mid f \in F(p_n, T)\}.\]

Now the general Landau problem for T-periodic functions can be defined as the problem of determining the best possible upper bound for a functional of the form \( f \mapsto \|p_k(D)f\| \) on \( F(p_n, T) \); again (cf. Section 5.4) this problem is equivalent to determining the maximum of a linear functional on the set \( \Gamma(p_n, T) \). However, in contrast to \( \Gamma(p_n) \) (cf. Theorem 5.3.3), the set \( \Gamma(p_n, T) \) may be unbounded; this occurs when \( \text{Ker}(p_n) \) contains nontrivial T-periodic functions. In connection with this we define the set \( \text{Ker}(p_n, T) \) as follows.

DEFINITION 6.1.4.

\[ \text{Ker}(p_n, T) := \{ f \in \text{Ker}(p_n) \mid f(t+T) = f(t), t \in \mathbb{R} \}. \]

The set \( \text{Ker}(p_n, T) \) is spanned by the functions \( t \mapsto \sin(2\pi k t / T) \), \( t \mapsto \cos(2\pi k t / T) \), where \( k \) varies over the set of integers for which \( p_n(2\pi k / T) = 0 \). Consequently, as \( p_n \) has real coefficients it follows that

\[(6.1.2) \quad \text{Ker}(p_n, T) = \text{Ker}(p_n^*), \]

The main contents of this chapter are as follows. Section 6.2 investigates conditions on \( u \in L_p([0, T]) \) for the differential equation \( p_n(D)f = u \) to have a solution \( f \) in \( W^{n, 1}(\mathbb{R}, T) \). The uniqueness of such a solution is assured by restricting \( f \) to a set \( F(p_n, T) \subseteq F(p_n, T) \); this is done in Lemma 6.3.2. Corresponding to \( F(p_n, T) \) we then define \( \bar{\Gamma}(p_n, T) \) analogous to \( \Gamma(p_n, T) \) in (6.1.1).

In Section 6.3 we study the extremal functions for the Landau problem with respect to periodic functions. In this connection a few properties of the sets \( F(p_n, T) \) and \( \bar{\Gamma}(p_n, T) \) are derived. For instance, it is shown that \( \bar{\Gamma}(p_n, T) \) is a strictly convex and compact set having \( 0 \) as an interior point. The subject of Section 6.4 is connected with work done by Northcott [46] on the problem of maximizing \( \|f\|^p \) \( 0 \leq k \leq n-1 \) on \( \bar{\Gamma}(p_n, T) \). Our main result states that perfect ruler \( f \)-splines are extremal functions for the linear functional \( f \mapsto \|p_k(D) + b_{k+1}(D)\|f\| \), \( f \in F(p_n, T) \), under the following conditions: i) \( p_n \) has only real zeros with \( p_n(0) = 0 \); ii) the monic polynomial \( p_{k+1} \) is a divisor of \( p_n \) with the additional property
\[ p_{k+1}(t) = (t - a)p_k(t) \]. Finally, Section 6.5 contains a parametrization of the boundary of the specific set \( \Gamma (\mathbb{R}^3, 1) \).

### 6.2. A few preliminary lemmas and the sets \( \tilde{\Gamma}(p_n, T) \) and \( \Gamma(p_n, T) \)

As remarked earlier, the set \( \Gamma(p_n, T) \) is not necessarily bounded. The main purpose of this section is to introduce a specific bounded subset \( \tilde{\Gamma}(p_n, T) \) of \( \Gamma(p_n, T) \), which coincides with \( \Gamma(p_n, T) \) if this set is bounded itself. In order to achieve this we start out by characterizing the functions \( u \in L_\infty (\mathbb{R}, T) \) for which the differential equation

\[(6.2.1) \quad p_0(D)f(t) = u(t) \quad (a.e.) \quad (t \in \mathbb{R})\]

has a solution \( f \in W^{(n)}(\mathbb{R}, T) \). We note that \( f \in W^{(n)}(\mathbb{R}, T) \) implies that \( f \) is a \( T \)-periodic function with the property

\[(6.2.2) \quad f^{(i)}(0) = f^{(i)}(T) \quad (i = 0, 1, \ldots, n-1) . \]

**Lemma 6.2.1.** The differential equation (6.2.1) has a solution \( f \in W^{(n)}(\mathbb{R}, T) \) if and only if

\[(6.2.3) \quad \int_0^T g(r)p_n(D)f(r)\,dr = 0 \quad (g \in \text{Ker}(p_n, T)) . \]

**Proof.** Let \( f \in W^{(n)}(\mathbb{R}, T) \) satisfy (6.2.1) and let \( g \in \text{Ker}(p_n, T) \). As (cf. (6.1.2))) \( \text{Ker}(p_n, T) \approx \text{Ker}(p_n, T) \), integration by parts yields

\[ \int_0^T g(t)p_n(D)f(t)\,dt = (-1)^{n} \int_0^T f(t)p_n^*(D)g(t)\,dt = 0 . \]

Hence (6.2.3) is necessary. To prove that (6.2.3) is sufficient we write (6.2.1) as a system of first order linear differential equations by putting

\[ x_1 = f, x_2 = f', \ldots, x_n = f^{(n-1)} . \]

We then obtain

\[(6.2.4) \quad \dot{x} = Ax + Bu , \]

where \( x = (x_1, x_2, \ldots, x_n)^T, x = (0, 0, \ldots, 0, 1)^T \) and where \( A \) is the corresponding companion matrix.
It is well known that a solution of (6.2.4) can be written in the form

\[(6.2.5) \quad \chi(t) = e^{tA} \chi(0) + \int_0^t e^{(t-\tau)A} \mathbf{b} \psi(\tau) \, d\tau.\]

We have to prove that for a given \( u \in L_u(\mathbb{R}, T) \) satisfying (6.2.3), \( \chi(0) \) can be chosen in such a way that \( \chi(T) = \chi(0) \), i.e., such that

\[(6.2.6) \quad (I - e^{TA}) \chi(0) = \int_0^T e^{(T-\tau)A} \mathbf{b} \psi(\tau) \, d\tau.\]

Consequently, it must be shown that \( \int_0^T e^{(T-\tau)A} \mathbf{b} \psi(\tau) \, d\tau \) is in the range of \( (I - e^{TA}) \). This is equivalent to the assertion that

\[(6.2.7) \quad \chi^T \int_0^T e^{(T-\tau)A} \mathbf{b} \psi(\tau) \, d\tau = 0 \]

for all \( \chi \in \mathbb{R}^n \) satisfying \( \chi^T (I - e^{TA}) = 0 \); here the well-known fact is used that the range of a linear transformation is orthogonal to the kernel of its transpose. However, if \( \psi \in L_u(\mathbb{R}, T) \) satisfies \( \psi^T (I - e^{TA}) = 0 \) then the function \( g \) defined by \( \psi(t) := \psi^T e^{(T-t)A} \mathbf{b} \) belongs to \( \text{Ker}(\mathbb{P}_u^{*}, T) = \text{Ker}(\mathbb{P}_u, T) \). So (6.2.7) follows from (6.2.3) and the lemma is proved.

We note that a generalisation of Lemma 6.2.1 is contained in Brockett [7, pp. 50-53].

For convenience we define

\[(6.2.8) \quad \mathbb{P}_u := \left\{ u \in L_u(\mathbb{R}, T) \mid \int_0^T g(\tau) u(\tau) \, d\tau = 0, \ g \in \text{Ker}(\mathbb{P}_u, T) \right\}.

We note that \( f \in C^1(\mathbb{R}, T) \) implies that \( \mathbb{P}_u f \) belongs to \( \mathbb{P}_u \).

If \( \mathbb{P}_u f = u \in \mathbb{P}_u \) then also \( \mathbb{P}_u (f + g) = u \) for all \( g \in \text{Ker}(\mathbb{P}_u, T) \).

Therefore, in order to ensure a unique solution of (6.2.1) we impose additional conditions on the function \( f \). This is done in the following lemma.
Lemma 6.2.2. Given an arbitrary \( u \in \mathcal{U}(P_n, \mathbb{R}) \) the differential equation (6.2.1) has exactly one solution \( f \in W^{(n)}(\mathbb{R}, T) \) having the property

\[
\int_0^T g(t) f(t) \, dt = 0 \quad (g \in \text{Ker}(P_n, T)).
\]

**Proof.** It follows from Lemma 6.2.1 that a solution \( f_0 \) of (6.2.1) exists. Consequently, any solution \( f \) of (6.2.1) can be written in the form \( f = f_0 + h \), where \( h \in \text{Ker}(P_n, T) \). According to (6.2.9) we have to choose the function \( h \) such that

\[
\int_0^T g(t) h(t) \, dt = - \int_0^T g(t) f_0(t) \, dt \quad (g \in \text{Ker}(P_n, T)).
\]

This uniquely determines \( h \).

Lemma 6.2.2 motivates the introduction of the following subset of \( F(P_n, T) \).

**Definition 6.2.3.** The set \( \tilde{F}(P_n, T) \) is defined by

\[
\tilde{F}(P_n, T) := \left\{ f \in F(P_n, T) \mid \int_0^T g(t) f(t) \, dt = 0, \quad g \in \text{Ker}(P_n, T) \right\}.
\]

If \( \text{Ker}(P_n, T) \) contains only the null function then \( \tilde{F}(P_n, T) = F(P_n, T) \).

Similar to \( \Gamma(P_n, T) \) in Definition 6.1.3, the set \( \tilde{F}(P_n, T) \) associated with \( \tilde{F}(P_n, T) \) is defined as follows.

**Definition 6.2.4.**

\[
\tilde{F}(P_n, T) := \left\{ (f(0), f'(0), \ldots, f^{(n-1)}(0))^T \mid f \in \tilde{F}(P_n, T) \right\}.
\]

Properties of the sets \( \tilde{F}(P_n, T) \) and \( \tilde{F}(P_n, T) \) that are of importance for the Landau problem will be given in the following section.
6.3. Some properties of the sets $\tilde{\mathcal{F}}(p_n, T)$ and $\tilde{\mathcal{F}}(p_n, T)$

The main result of this section is that $\tilde{\mathcal{F}}(p_n, T)$ is a strictly convex and compact subset of $\mathbb{R}^n$ having 0 as an interior point. Preliminary to this it will be shown that the elements of $W^{(n)}(\mathbb{R}, T)$ satisfying (6.2.9) can be represented as convolution integrals.

**Lemma 6.3.1.** If $f \in W^{(n)}(\mathbb{R}, T)$ satisfies (6.2.9) then

\[(6.3.1) \quad f(t) = \frac{1}{T} \int_0^T p_n(t - \tau) p_n(\tau) f(\tau) d\tau \quad (t \in \mathbb{R}),\]

where $p_n$ is a $T$-periodic function, the Fourier series of which is given by

\[(6.3.2) \quad p_n(t) = \sum_{k \in \mathbb{Z}_n} \frac{e^{2\pi it/T}}{\hat{p}_n(k2\pi/T)} \quad (t \in \mathbb{R}),\]

$\mathbb{Z}_n$ being the set of integers $k$ for which $p_n(k2\pi/T) = 0$.

**Proof.** It follows from (6.3.2) that $p_n^{(n-2)}$ is continuous on $\mathbb{R}$. Moreover,

\[\int_0^T p_n(t) g(t) dt = 0 \quad (g \in \text{Ker}(p_n, T))\]

since Ker$(p_n, T)$ is spanned by $\cos(2k\pi t/T)$, $\sin(2k\pi t/T)$ with $k \in \mathbb{Z}_n$.

Substituting (6.3.2) in the right-hand side of (6.3.1), changing the order of summation and integration, and integrating by parts we obtain

\[
\frac{1}{T} \int \sum_{k \in \mathbb{Z}_n} \frac{e^{2\pi it/T}}{\hat{p}_n(k2\pi/T)} \int_0^T e^{-2\pi it/T} p_n(D) f(\tau) d\tau \cdot e^{2\pi it/T} \cdot p_n(D) f(\tau) d\tau -
\]

\[= \frac{1}{T} \int \sum_{k \in \mathbb{Z}_n} \frac{e^{2\pi it/T}}{\hat{p}_n(k2\pi/T)} \int_0^T f(t) \cdot p_n(-D) e^{-2\pi it/T} d\tau \cdot e^{2\pi it/T} \cdot p_n(D) f(\tau) d\tau =
\]

\[= \sum_{k \in \mathbb{Z}_n} \left( \frac{1}{T} \int \int_0^T f(t) e^{-2\pi it/T} d\tau \cdot e^{2\pi it/T} \cdot p_n(D) f(\tau) d\tau \right) e^{2\pi it/T} = f(t) \quad (t \in \mathbb{R}),
\]

as (cf. (6.2.9)) $\frac{1}{T} \int_0^T f(t) e^{-2\pi it/T} d\tau = 0$ for all $k \in \mathbb{Z}_n$. $\square$
It will be shown on p. 136 (cf. Remark 1) that if \( \mathbb{K}_n = \emptyset \) then \( P_n \) is an exponential \( z \)-spline corresponding to the operator \( P_n^{(0)} \), i.e., \( P_n \in \mathcal{S}(\mathbb{P}_n, T) \). However, \( P_n \) is an \( z \)-spline having double knots if \( \mathbb{K}_n = \{0\} \) (cf. Remark 2, p. 136).

**Lemma 5.3.2.** Let \( P_n \) be the function defined by (6.3.9). Then \( P_n \) satisfies the differential equation

\[
(6.3.3) \quad P_n^{(d)} (t) = \sum_{k \in \mathbb{K}_n} e^{2k\pi i t / T} (0 < t < T),
\]

where an empty sum is defined to be zero.

Furthermore, if \( Q_n \) denotes the analytic function coinciding with \( P_n \) on \((0,T)\) then

\[
(6.3.4) \quad Q_n(t + T) = Q_n(t) = c(t) \quad (t \in \mathbb{R}),
\]

where \( c \) is the fundamental function corresponding to \( P_n^{(0)} \) and \( c \) is given by

\[
(6.3.5) \quad c = \frac{P_n^{(n-1)}(T) - P_n^{(n-1)}(0)}{n}.
\]

**Proof.** A simple computation shows that the residue of the function

\[
(6.3.5) \quad z \mapsto \frac{Te^{tz}}{(e^{Tz} - 1)P_n(z)} \quad (t \in \mathbb{R})
\]

at the point \( z = 2k\pi i / T \) \( (k \in \mathbb{Z}, k \notin \mathbb{K}_n) \) equals \( e^{2k\pi i t / T} / P_n(2k\pi i / T) \). It can be shown that the sum of all residues of (6.3.5) equals zero, provided \( 0 \leq t < T \), as \( Q_n \) coincides with \( P_n \) on \((0,T)\). It follows from the residue theorem and (6.3.2) that

\[
(6.3.6) \quad Q_n(t) = \frac{T}{2\pi i} \oint_C \frac{e^{tz}}{(e^{Tz} - 1)P_n(z)} dz,
\]

where \( C \) is a contour in the complex plane surrounding the zeros of \( P_n \) and excluding the points \( z = 2k\pi i / T \) \( (k \notin \mathbb{K}_n) \).

Consequently,

\[
P_n^{(d)}(t) Q_n(t) = \frac{T}{2\pi i} \oint_C \frac{e^{tz}}{(e^{Tz} - 1)} dz
\]

and so (6.3.3) follows by the residue theorem.
Since $\Phi_n \in C^{(n-2)}(\mathbb{R})$ and $P_n$ is $T$-periodic it follows that $p_{n}^{(j)}(0) = p_{n}^{(j)}(T)$ ($j = 0,1,\ldots,n-2$) and therefore $\Phi_n^{(j)}(0) = \Phi_n^{(j)}(T)$ ($j = 0,1,\ldots,n-2$). Let the function $f$ be given by $f(t) := \Phi_n(t + T) - \Phi_n(t)$ ($t \in \mathbb{R}$). Then $f^{(j)}(0) = 0$ ($j = 0,1,\ldots,n-2$) and

$$p_{n}^{(j)}(0) = p_{n}^{(j)}(D)\Phi_n(t + T) - p_{n}^{(j)}(D)\Phi_n(t) = 0,$$

since the right-hand side of (6.3.3) is $T$-periodic. In view of the definition of $\Phi$ (cf. p. 6) we then have $f = \Phi_0$ for some constant $c \in \mathbb{R}$. Hence $\Phi_n(t + T) - \Phi_n(t) = c\Phi(t)$ and the constant $c$ equals

$$c = \Phi_n^{(n-1)}(T) - \Phi_n^{(n-1)}(0) = p_{n}^{(n-1)}(T) - p_{n}^{(n-1)}(0).$$

This proves the lemma. □

REMARK 1. We observe that, if $K_{nP} = \emptyset$ then formula (6.3.6) agrees with formula (3.2.9), with $\lambda$ and $\delta$ replaced by 1 and $T$, up to a multiplicative constant. Consequently, $\Phi_n(t) = a\Phi_1(p_n h, t)$ and $P_n(t) = a\Phi_1(p_n h, t)$ ($t \in \mathbb{R}$) for some constant $a \in \mathbb{R}$. Hence $P_n$ is an exponential $L$- spline corresponding to the operator $P_n(D)$ and with mesh distance $T$.

REMARK 2. If $K_{nP} = \{0\}$ then, according to (6.3.3), the function $P_n$ satisfies the differential equation $P_n(D)P_n(t) = -1$ ($0 < t < T$). Since $P_n \in C^{(n-2)}(\mathbb{R})$ and $Dp_{n}(D)p_{n}(t) = 0$ on each subinterval $[v_2, T] + [v_3, T]$ ($v \in \mathbb{R}$), the function $P_n$ is an $L$-spline corresponding to the operator $Dp_n(D)$ having double knots at the points $v_2$ (for $v \in \mathbb{R}$).

REMARK 3. If $p_{n}(D) = \delta^n$ and $T = 1$ then $-n! \Phi_n$ is the ordinary Bernoulli polynomial of degree $n$ (cf. Norlund [45], p. 65).

The next three results concern properties of the set $\mathcal{P}(p_n, T)$ as introduced in Definition 6.2.4.

THEOREM 6.3.1. The set $\mathcal{P}(p_n, T)$ is a convex and compact subset of $\mathbb{R}^n$ having $0$ as an interior point.

PROOF. Obviously, $\mathcal{P}(p_n, T)$ is a convex set, since $\mathcal{P}(p_n, T)$ is convex. With respect to the compactness property we proceed as follows. In view of (6.3.1) every $\mathcal{F} \in \mathcal{P}(p_n, T)$ satisfies (cf. Definition 6.2.3)
\[ f(t) = \frac{1}{T} \int_0^T P_n(t-\tau) f(\tau) d\tau \quad (t \in \mathbb{R}). \]

Hence
\[ |f(t)| \leq \frac{1}{T} \int_0^T |P_n(t-\tau)| d\tau = \frac{1}{T} \int_0^T |P_n(\tau)| d\tau, \]

and so \( \hat{\mathcal{T}}(p_n, T) \subseteq \mathcal{T}(p_n) \) with \( m = T^{-1} \int_0^T |P_n(\tau)| d\tau \) (cf. (5.3.2)).

Since \( \mathcal{T}(p_n) \) is a compact set (cf. Theorem 5.3.3), \( \mathcal{T}(p_n, T) \) is bounded and thus it suffices to prove that \( \hat{\mathcal{T}}(p_n, T) \) is closed. This assertion easily follows from the fact that the limit of a convergent sequence of \( T \)-periodic functions is again \( T \)-periodic.

Finally, we have to show that \( \mathcal{N} \) is an interior point of \( \hat{\mathcal{T}}(p_n, T) \). Since \( \hat{\mathcal{T}}(p_n, T) \) is a subset of a finite dimensional space it is sufficient to prove that for all \( \eta \in \mathbb{R}^0 \) with \( \eta \neq 0 \)
\[ \max \{ \eta^T x \mid x \in \hat{\mathcal{T}}(p_n, T) \} > 0. \]

This amounts to maximizing (cf. Lemma 6.3.1)
\[ (6.3.7) \int_0^{n^{-1}} G(t) p_n(D)f(t) dt, \]

over \( \hat{\mathcal{T}}(p_n, T) \), where \( G \) is given by
\[ (6.3.8) \quad G(t) := \frac{1}{T} \sum_{j=0}^{n^{-1}} \eta_j \varphi_j(t-T). \]

In view of Lemma 6.2.1 and the fact that \( \|p_n(D)f\| \leq 1 \) if \( f \in \hat{\mathcal{T}}(p_n, T) \), the maximum in (6.3.7) equals
\[ (6.3.9) \quad \max \left\{ \int_0^{n^{-1}} G(t) u(t) dt \mid u \in \mathfrak{P}_n, \|u\| \leq 1 \right\}. \]

Let the maximum in (6.3.9) be denoted by \( M_\eta \). If \( M_\eta = 0 \) then for all \( k \neq \mathfrak{P}_n \)
\[ (6.3.10) \quad \int_0^{n^{-1}} \sum_{j=0}^{n^{-1}} G(t) e^{-2k\pi it/T} dt = 0. \]
since \( \lambda \neq \lambda_0 \) implies that the functions \( t \mapsto \cos(2k\pi t/T) \) and \( t \mapsto \sin(2k\pi t/T) \) belong to \( \mathcal{U}_n \).

Using the Fourier series of \( \Phi_n \) and (6.3.9) we obtain for all \( \lambda \neq \lambda_0 \)

\[
\sum_{j=0}^{n-1} \frac{1}{p_j(2k\pi t/T)} = 0.
\]

However, (6.3.11) implies that \( p_0 = \eta_0 = \ldots = \eta_{n-1} = 0 \), which violates the assumption \( \lambda \neq \lambda_0 \). Consequently, \( \lambda \neq \lambda_0 \) and hence \( \Omega \) is an interior point of \( \mathcal{F}(\gamma_i) \).

We recall (cf. Remark 2, p. 139) that at the end of Chapter 5 it was observed that \( \mathcal{F}_n(D^1) \) contains straight line segments and therefore the set \( \mathcal{F}_n(D^1) \) is not strictly convex. This notion will again be encountered presently; a formal definition reads as follows.

**Definition 6.1.4.** A set \( \Lambda \subset \mathbb{R}^n \) is called strictly convex if for all \( x \in \Lambda \) and all \( y \in \Lambda \) the point \( \frac{1}{2}(x+y) \) is an interior point of \( \Lambda \).

It will now be shown that \( \mathcal{F}(\gamma_i) \) is strictly convex. To this end we need the following.

**Theorem 6.3.5.** Associated with each \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) \) \( \in \mathcal{F}(\gamma_i) \) there exists a unique \( \varepsilon \in \mathcal{F}(\gamma_i) \) such that \( \varepsilon^{(j)}(0) = \lambda_j \) \( \forall j \in \{0, 1, \ldots, n-1\} \).

Moreover, this unique \( \varepsilon \) is a perfect \( \varepsilon \)-spline corresponding to the operator \( p_n(D) \).

**Proof.** We note that the existence of \( \varepsilon \) is an immediate consequence of the compactness of \( \mathcal{F}(\gamma_i) \). So the uniqueness remains to be proved. Since \( \lambda \in \mathcal{F}(\gamma_i) \) there exists an \( \lambda \neq \lambda_0 \) such that

\[
\lambda^\top x = \max \{ \lambda^\top x \mid x \in \mathcal{F}(\gamma_i) \}.
\]

So, if \( \varepsilon \in \mathcal{F}(\gamma_i) \) is a function with \( \varepsilon^{(j)}(0) = \lambda_j \) \( \forall j \in \{0, 1, \ldots, n-1\} \), then in view of (6.3.7) and (6.3.9)

\[
(6.3.12) \quad \frac{1}{p_0} = \int_0^T G(t)p_n(D)f(t)dt = \max \left\{ \int_0^T G(t)u(t)dt \mid u \in \mathcal{U}_n, \|u\| \leq 1 \right\}.
\]
We recall that this problem has already been dealt with in Chapter 4 (cf. pp. 97, 98); there it is shown that the maximum in (6.3.12) equals

\[ \int_0^T |G(t) - g(t)| \, dt \]

for an appropriate function \( g \in \text{Ker}(p_n, T) \).

Hence, if the maximum value is attained at \( u_0 \in U_p \) with \( \|u_0\| \leq 1 \), then

\[ \int_0^T |G(t) - g(t)| \, dt = \int_0^T (G(t) - g(t))u_0(t) \, dt \leq \int_0^T |G(t) - g(t)| \, dt . \]

Therefore \( u_0(t) = \text{sgn}(G(t) - g(t)) \) (a.e.) on the set \( \{0 < t < T \mid G(t) \neq g(t)\} \).

Note that the Fourier coefficients in the Fourier series of \( G(t) - g(t) \) are not all zero. Consequently, the function \( G - g \) is a nontrivial function on \((0,T)\) and, because it is analytic on \((0,T)\), it has finitely many zeros in this interval. It follows that \( u_0 \) is essentially unique and jumps from \( -1 \) to \( +1 \) at the zeros of \( G - g \). Since \( p_n(t) f(t) = u_0(t) \) (a.e.) on \((0,T)\) and \( f \in \tilde{F}(p_n, T) \), the function \( f \) is uniquely determined. Finally, the assertion that \( f \) is a perfect \( z \)-spline corresponding to the operator \( p_n(t) \) is an immediate consequence of Definition 1.3.3 and the properties of \( u_0 \).

\[ \Box \]

Theorem 6.3.5 may now be used to prove

**Theorem 6.3.6.** \( \tilde{F}(p_n, T) \) is a strictly convex set.

**Proof.** It follows from the proof of Theorem 6.3.5 that for any \( \tilde{u} \neq \tilde{v} \) the maximum of the linear function \( x \mapsto \tilde{u} \cdot x \) on \( \tilde{F}(p_n, T) \) is attained at a unique point \( x \in \tilde{F}(p_n, T) \). Now let \( \tilde{u} \in \tilde{F}(p_n, T) \), \( \tilde{v} \in \tilde{F}(p_n, T) \) and assume that also \( \frac{1}{2} (\tilde{u} + \tilde{v}) \in \tilde{F}(p_n, T) \). There exists a linear function \( x \mapsto x_0 \cdot x \) attaining its maximum on \( \tilde{F}(p_n, T) \) at \( \frac{1}{2} (\tilde{u} + \tilde{v}) \). Since \( x_0 \cdot \frac{1}{2} (\tilde{u} + \tilde{v}) = \frac{1}{2} (x_0 \cdot \tilde{u} + x_0 \cdot \tilde{v}) \), this function also attains its maximum at \( \tilde{u} \) and \( \tilde{v} \). Because of the uniqueness one has \( \tilde{u} = \tilde{v} \), and in view of Definition 6.3.4 it follows that \( \tilde{F}(p_n, T) \) is strictly convex.

\[ \Box \]

From the proof of Lemma 6.2.2 we may conclude that for every \( f \in \tilde{F}(p_n, T) \) a function \( g \in \text{Ker}(p_n, T) \) exists such that \( f - g \in \tilde{F}(p_n, T) \). Consequently, if \( p_k(t) \) (\( 0 \leq k \leq n-1 \)) has the property that \( \text{Ker}(p_n, T) \subset \text{Ker}(p_k) \), then
maximizing \( \| p_k(D) f \| \) on \( F(p_n, T) \) is equivalent to maximizing \( \| p_k(D) f \| \) on \( F(p_n, T) \). It turns out that \( T \)-periodic perfect \( f \)-splines are extremal with respect to this problem. Detailed information is given in the following

**THEOREM 6.3.7.** Let \( p_n \in \mathbb{R}_n \) be a monic polynomial and let \( p_k(D) \) \((0 \leq k \leq n-1)\) be a linear differential operator of order \( k \) such that \( p_k(D)q(t) = 0 \) \((t \in \mathbb{R})\) for all \( q \in \ker(p_n, T) \). Then there exists a \( T \)-periodic perfect \( f \)-spline \( f \) corresponding to the operator \( p_n(D) \) such that

\[
(6.3.13) \quad \max_{f \in F(p_n, T)} \| p_k(D) f \| = \| p_k(D) s \|. 
\]

Moreover, if \( f_1 \in F(p_n, T) \) also yields the maximum in (6.3.13) then there exist numbers \( \xi \in [0, T), \sigma \in [-1, 1) \) and a function \( g_1 \in \ker(p_n, T) \) such that

\[
f_1(t) = \sigma (t - \xi) + g_1(t) \quad (t \in \mathbb{R}).
\]

**PROOF.** If \( f \in F(p_n, T) \) and if \( g \in \ker(p_n, T) \) is the function satisfying

\[
\int_0^T f(t) u(t) \, dt = \int_0^T g(t) u(t) \, dt \quad (u \in \mathbb{P}_n),
\]

then \( f - g \in F(p_n, T) \). By Lemma 6.3.1 one has

\[
f(t) = g(t) + \frac{1}{T} \int_0^T p_n(t - \tau) p_k(D) f(\tau) \, d\tau \quad (t \in \mathbb{R}).
\]

In view of this and taking into account the periodicity of \( f \), it follows that

\[
p_k(D) f(0) = \frac{1}{T} \int_0^T p_k(D) p_n(t - \tau) p_k(D) f(\tau) \, d\tau
\]

and therefore

\[
(6.3.14) \quad \max_{f \in F(p_n, T)} | p_k(D) f(0) | = \max_{f \in F(p_n, T)} | p_k(D) s(0) |.
\]

We observe that maximizing \( | p_k(D) f(0) | \) on \( F(p_n, T) \) is equivalent to maximizing a linear function on \( F(p_n, T) \), the maximum of which is attained at a
boundary point of \( \overline{F}(p_n(T)) \). By Theorem 6.3.5 there is a unique \( s \in \overline{F}(p_n(T)) \), \( s \) being a \( T \)-periodic perfect \( \ell \)-spline corresponding to the operator \( p_n(0) \), such that
\[
P_n(x) s(x) = \max_{f \in \overline{F}(p_n(T))} \|p_n(x) f(x)\|.
\]

Hence, by (6.3.16),
\[
P_n(x) s(x) = \max_{f \in \overline{F}(p_n(T))} \|p_n(x) f(x)\|.
\]

This establishes the first part of the theorem.

Now let \( f_1 \in \overline{F}(p_n(T)) \) yield the maximum in (6.3.13), let the numbers \( t \in [0,T), \sigma \in (-1,1) \) be such that
\[
\sigma p_n(x_1(t) = \max_{f \in \overline{F}(p_n(T))} \|p_n(x) f(x)\|,
\]
and let \( a_1 \in \text{Ker}(p_n(T)) \) be such that
\[
(t \mapsto cf_1(t) + a_1(t)) \in \overline{F}(p_n(T)).
\]

Then by the equality \( \sigma p_n(x_1(x)) = p_n(x) s(0) \) and the uniqueness of \( s \) one has \( s(t) = cf_1(t) + a_1(t) \quad (t \in \mathbb{R}) \), which is equivalent to the assertion in the second part of the theorem. \( \square \)

### 6.4. Perfect Euler \( \ell \)-splines as extremal functions

The problem of maximising \( \|B^{k}\| \quad (k = 0,1,\ldots,n-1) \) on \( \overline{F}(p_n(T)) \) has been investigated by Northcott [46]. He proved that perfect Euler splines are extremal with respect to this problem. A generalization of Northcott's result to the nonpolynomial case for some specific operators \( p_n(0) \) is due to Colombeau [19]. In this section we shall prove that perfect Euler \( \ell \)-splines are extremal for specific functionals of the form \( f \mapsto \|\sigma p_n(x) + p_{n+1}(x) f\| \), provided the polynomial \( p_n \) has a few particular properties to be specified in the following theorem.

**Theorem 6.4.1.** Let \( p_n \in \pi_n \quad (n \geq 2) \) be a monic polynomial having only real zeros and let \( p_n(0) = 0 \). Furthermore, let \( p_{n+1} \in \pi_{n+1} \quad (k \leq n-2) \) be any monic polynomial that is a divisor of \( p_n \), and let \( p_k \in \pi_k \) be such that
\( p_{k+1}(t) = (t - s)p_{k}(t) \quad (t \in \mathbb{R}) \). Then for any \((a, b) \in \mathbb{R}^2\) with \((a, b) \neq (0, 0)\) one has

\[
(6.4.1) \quad \max_{f \in \mathcal{F}(p_n, T)} \| (ap_{n}(D) + bp_{n+1}(D))f \| = \frac{1}{T} \int_0^T (ap_{n}(D) + bp_{n+1}(D))E(p_n, T) dT,
\]

where \( E(p_n, T) \) denotes the perfect Euler \( L \)-spline as defined in Definition 3.2.11. Moreover, if a function \( f_0 \in \mathcal{F}(p_n, T) \) yields the maximum in (6.4.1), then

\[
(6.4.2) \quad f_0(t) = aE(p_n, \frac{T}{2}, t - \xi) \quad (t \in \mathbb{R})
\]

for appropriate constants \( \xi \in [0, T] \) and \( a \in [-1, 1] \).

PROOF. According to the definition of \( X_{p_n} \) (cf. p. 137) and the assumption on the zeros of \( p_n \), we conclude that \( X_{p_n} = \{0\} \) and hence, by (6.3.3), \( p_n \) satisfies the differential equation

\[
(6.4.3) \quad p_n(t)p_n(t) = -1 \quad (0 \leq t < \pi).
\]

Let the function \( F \) be defined by

\[
(6.4.4) \quad F(t) := (ap_{n}(D) + bp_{n+1}(D))p_n(t) \quad (t \in \mathbb{R}).
\]

In view of (6.4.3) and the assumption on \( p_n \) and \( p_{n+1} \)

\[
(6.4.5) \quad p_{n-k}(t) = (aI + b(D - aI))p_n(t) = -a + ab,
\]

where \( p_{n-k}(t) := p_n(t)/p_n(t) \quad (t \in \mathbb{R}) \).

Furthermore, since \( F \in \mathcal{G}(n-k-2) \), one has

\[
(6.4.6) \quad F^{(j)}(0) = F^{(j)}(0) \quad (j = 0, 1, \ldots, n-k-3).
\]

It follows from Lemma 6.3.1 that for all \( f \in \mathcal{F}(p_n, T) \) and \( t \in \mathbb{R} \)

\[
(6.4.7) \quad (ap_{n}(D) + bp_{n+1}(D))f(t) = \frac{1}{T} \int_0^T F(t - \xi)p_n(D)f(t) d\xi.
\]

In view of this and taking into account Lemma 6.2.2, we conclude that the problem of maximizing \( \| (ap_{n}(D) + bp_{n+1}(D))f \| \) on \( \mathcal{F}(p_n, T) \) is equivalent to the problem of maximizing

\[
(6.4.8) \quad \frac{1}{T} \int_0^T F(T - t)u(t) d\tau.
\]
with respect to those functions \( u \in \mathcal{U}_p \), for which \( \|u\| \leq 1 \).

Since \( p_n \) has only real zeros with \( p_n(0) = 0 \) the set \( \text{Ker}(p_n, T) \) only contains constant functions and therefore (cf. (6.2.8))

\[
\mathcal{U}_p = \left\{ u \in L^1(0, T) \mid \int_0^T u(t) \, dt = 0 \right\} .
\]

Consequently,

\[
(6.4.9) \quad \sup_{x \in \mathcal{U}_p} \| (a_{k+1}(D) + b_{k+1}(D)) f \| = \sup_{x \in \mathcal{U}_p} \left\{ \frac{1}{T} \int_0^T F(T - t) u(t) \, dt \mid u \in L^1([0, T]), \int_0^T u(t) \, dt = 0 \right\} .
\]

If there exists a constant function \( \overline{\delta} \) such that

\[
(6.4.10) \quad \int_0^T \text{sgn}(F(T - t) - \overline{\delta}) \, dt = \int_0^T \text{sgn}(F(t) - \overline{\delta}) \, dt = 0 ,
\]

and \( \overline{\delta} \) coincides with \( F \) at at most finitely many points, then by Theorem 6.3.5 the supremum in (6.4.3) equals \( T^{-1} \int_0^T \text{sgn}(F(t) - \overline{\delta}) \, dt \).

In view of the remark on p. 98 such a constant function \( \overline{\delta} \) exists if one shows that for any constant function \( \delta, F - \delta \) has finitely many zeros. This can be proved as follows.

Writing \( H_d(t) = F(t) - \delta \) and using (6.4.5) one has

\[
(6.4.11) \quad p_{k+1}(D) H_d(t) = -a \cdot b - p_{n-k}(0) \delta \quad (0 < t < T)
\]

with (cf. (6.4.6))

\[
H^{(j)}_{\overline{\delta}}(0) = H^{(j)}_{\overline{\delta}}(T) \quad (j = 0, 1, \ldots, n-k-1) .
\]

Because of (6.4.11) \( H_d \) coincides on \((0, T)\) with an analytic function. If \( H_d \)

would have infinitely many zeros on \((0, T)\) then \( H_d \) would vanish identically on \((0, T)\). Hence \( F(t) = \delta \) \((0 < t < T)\) and therefore, for all \( f \in \tilde{F}(p_n, T) \), we would have (cf. (6.4.7))

\[
(a_{k+1}(D) + b_{k+1}(D)) f(t) = \frac{\delta}{T} \int_0^T p_n(D) f(t) \, dt = 0 ,
\]

since \( p_n(0) = 0 \). This, of course, is not true. Hence \( H_d \) has finitely many
zeros in \((C,T)\) and so a constant \(\tilde{c}\) can be found satisfying (6.4.10).

Our next purpose is to estimate the number of zeros of \(H_d\) in \((0,T)\) by using the Budan-Fourier theorem 1.4.14.

Now let \(\beta_{n-k}^\alpha(t) = (D^{-\lambda} \beta_{n-k}^\alpha(t))_0 \cdots (D^{-\lambda} \beta_{n-k}^\alpha(t))_{n-k-1} \cdot \beta_{n-k}^\alpha(t) = (D^{-\lambda} \beta_{n-k}^\alpha(t))_0 \cdots (D^{-\lambda} \beta_{n-k}^\alpha(t))_1 \cdot \beta_{n-k}^\alpha(t), \alpha \in \{1, 2, \ldots, n-k-3\}.\) One easily verifies that

\[
Z_d^\alpha H_d(t) = Z_d^\alpha H_d(t) (i = 1, 2, \ldots, n-k-3).
\]

If \(\alpha\) is (6.4.11) \(B_{n-k}^\alpha(t) \neq 0\) then in view of the Budan-Fourier theorem 1.4.14 and (6.4.12)

\[
Z(D_{n-k-1}^\alpha H_d(t), 0, T) = \sum_{\alpha} \langle h_d(t) \rangle_d^n \langle [1]_{d}^{\alpha} H_d(t) \rangle_d^{(n-k)}(t) \neq 0.
\]

If \(B_{n-k}^\alpha(t) = 0\) and \(\sum_{\alpha} \langle [1]_{d}^{\alpha} H_d(t) \rangle_d^{(n-k)}(t) \neq 0\), then in a similar way it can be shown that

\[
Z(D_{n-k-2}^\alpha H_d(t), 0, T) = 0.
\]

On the other hand, if \(B_{n-k}^\alpha(t) = 0\) and \(\sum_{\alpha} \langle [1]_{d}^{\alpha} H_d(t) \rangle_d^{(n-k)}(t) = 0\), then \(\sum_{\alpha} \langle [1]_{d}^{\alpha} H_d(t) \rangle_d^{(n-k)}(t) = 0\) \((0 < t < T)\). We assert that in this case \(\sum_{\alpha} \langle [n-k-2]_{d}^{\alpha} H_d(t) \rangle_d^{(n-k-2)}(t) \neq 0\). If not, then \(\sum_{\alpha} \langle [n-k-2]_{d}^{\alpha} H_d(t) \rangle_d^{(n-k-2)}(t) = 0\) \((0 < t < T)\) and so in view of (6.4.12) the function \(H_d\) would coincide on \((0,T)\) with a function in \(\text{Ker}(H_d)\). Consequently, \(H_d(t) = \gamma (0 < t < T)\) for some constant \(\gamma\). However, we have shown already that this cannot be the case. Hence \(\sum_{\alpha} \langle [n-k-2]_{d}^{\alpha} H_d(t) \rangle_d^{(n-k-2)}(t) \neq 0\). Using Theorem 1.4.14 again we then have from (6.4.12)

\[
Z(D_{n-k-2}^\alpha H_d(t), 0, T) = 0.
\]

We conclude that one of the following four possibilities occurs: \(H_d\) has no zeros in \((0,T)\), \(H_d\) has exactly one zero in \((0,T)\), \(H_d\) has exactly two distinct simple zeros in \((0,T)\), and \(H_d\) has exactly one zero of multiplicity two in \((0,T)\). However, since by (6.4.10) \(\int_0^T \text{gsn}(H_d(t)) \, dt = 0\) the function \(H_d\) has exactly one simple zero in \((0,T)\) located at \(t = T/2\) or two simple zeros in \((0,T)\) a distance \(T/2\) apart. Therefore, if \(\tilde{u}\) denotes the periodic extension of a function \(u\) that maximizes (6.4.9), then \(\tilde{u}(t + T/2) = -\tilde{u}(t)\). Consequently, the "\(p_n(D)\)-derivative" of a function \(\tilde{f}_1 \equiv \tilde{f}(p_n(T))\) satisfying
\( P_n(0)f_1(t) = \tilde{g}(t) \) (a.e.) jumps from \( \pm 1 \) at equally spaced points a distance \( T/2 \) apart. In view of the definition of a perfect Euler I-spline (cf. Definition 3.2.12), we conclude that \( f_1(t) = \sigma \xi (p_n(T/2,t - c)) \) for some constants \( \sigma \in [-1,1] \) and \( c \in [0,T] \). This proves the first part of the theorem.

The second part of the theorem follows from the observation that the periodic continuation of every function \( u \) that maximizes (6.4.7) jumps from \( \pm 1 \) at equally spaced knots a distance \( T/2 \) apart. So each function

\[ f_1(t) = \tilde{g}(p_n,T) \]

which yields the maximum in (6.4.1) and for which \( p_n(0)f_1(t) = u(t) \) \((0 < t < T)\) has the form (6.4.2). \( \square \)

**Remark 1.** In case \( p_n \) has only real zeros and \( p_n(0) \neq 0 \), then (6.4.7) has to be maximized without the restriction \( \int_0^T u(t)dt = 0 \). Hence \( u(t) = \text{sgn}(f(t)) \).

It is easy to construct an example such that \( f(t) > 0 \) \((t \in (0,T))\). The corresponding extremal function then satisfies \( p_n(0)f_1(t) = 1 \) and so \( f(t) = p_n^{-1}(0) \) \((t \in \mathbb{R})\); this function is not a perfect Euler I-spline.

**Remark 2.** If not all zeros of \( p_n \) are real then the problem as solved in Theorem 6.4.1 becomes more complicated. This is due to the fact that in this case an interval of disconjugacy for \( p_n(0) \) is finite, and the length of the interval \([0,T]\) is then of importance. If \( T \) is small enough one may expect that results similar to Theorem 6.4.1 hold.

### 6.5. A parametrization of the set \( \tilde{B}^2(0,1) \)

As an illustration of the preceding results a parametrization of the set \( \tilde{B}^2(0,1) \) will be given. In view of Lemma 6.3.1 a function \( f \in \tilde{B}^2(0,1) \) can be written as

\[
(6.5.1) \quad f(t) = \int_0^T P_2(t - \tau) \xi''(\tau) d\tau,
\]

where (cf. (6.3.2))

\[
P_2(t) = \sum_{k=0}^{2} \frac{2k\xi(t)}{(2k+1)} \quad (t \in \mathbb{R}).
\]

According to Remark 3 (cf. p. 139), apart from a multiplication factor \(-6\),
$P_3$ coincides on $(0,1)$ with the Bernoulli polynomial of degree three, i.e.,

$$P_3(t) = -\frac{1}{6} t^3 + \frac{1}{4} t^2 - \frac{1}{12} t \quad (0 \leq t \leq 1).$$

Hence, if $\eta_0$, $\eta_1$, and $\eta_2$ are real numbers with $(\eta_0, \eta_1, \eta_2) \neq (0,0,0)$ then (cf. (6.3.7) and (6.3.8))

$$\eta_0 f(0) + \eta_1 f'(0) + \eta_2 f''(0) = \int_0^1 f(1-t) \omega(t) \, dt,$$

where

$$\omega(t) = \eta_0 P_3(t) + \eta_1 P_3'(t) + \eta_2 P_3''(t) \quad (0 \leq t \leq 1).$$

Consequently $F \in \pi_3$ and $\int_0^1 f(t) \, dt = 0$. In order to maximize (6.5.2) on $F(D^3,1)$ one has to compute (cf. (6.4.9))

$$\max \left\{ \int_0^1 f(1-t) u(t) \, dt \mid u \in L^1_0(\mathbb{R}^2), \int_0^1 u(t) \, dt = 0, \|u\| \leq 1 \right\}.$$ 

We recall that this maximum equals $\int_0^1 |f(\tau) - \bar{a}| \, d\tau$, where $\bar{a}$ is a constant such that $\int_0^1 \text{sgn}(f(\tau) - \bar{a}) \, d\tau = 0$. Since $F - \bar{a} \in \pi_3$ it has at most three sign changes in $(0,1)$; evidently it has at least one sign change in $(0,1)$.

Defining $t^0_\tau := \frac{1}{2} (1 + \text{sgn}(t))$, we now may write

$$\text{sgn}(f(\tau) - \bar{a}) = \pm (1 - 2(t - \xi_1)_{+} + 2(t - \xi_2)_{+} - 2(t - \xi_3)_{+}).$$

with $0 \leq \xi_1 \leq \xi_2 \leq \xi_3 \leq 1$, $\xi_1 - \xi_2 + \xi_3 = 0$; the conditions on $\xi_1$, $\xi_2$, and $\xi_3$ follow from the fact that $\int_0^1 \text{sgn}(f(\tau) - \bar{a}) \, d\tau = 0$. The functions $f_0 \in F(D^3,1)$ for which $f_0''(t) = \pm \text{sgn}(f(1-t) - \bar{a}) (0 < t < 1)$ are then given by (cf. (6.5.11))

$$f_0(t) = \pm \int_0^1 P_3(t - \tau) (1 - 2(1 - t - \xi_1)_{+} + 2(1 - t - \xi_2)_{+} - 2(1 - t - \xi_3)_{+}) \, d\tau.$$ 

By a straightforward computation we get for $i = 0, 1, 2$

$$f^{(1)}_i(0) = \pm 2(P_{4-1}(\xi_1) - P_{4-1}(\xi_2) + P_{4-1}(\xi_3) - P_{4-1}(1)),$$

where (cf. Remark 3, p. 139) $P_2 = P_3'$, $P_3 = P_4'$, and $P_4$ is given by
\[ P_4(t) = -\frac{1}{24} t^4 + \frac{1}{12} t^3 - \frac{1}{34} t^2 + \frac{1}{720} \quad (0 \leq t \leq 1). \]

Since \( f_0 \) maximizes (6.5.2), the point \( (\xi_0^0, \xi_0^1, \xi_0^2)^T \in \tilde{F}(D^3, 1) \). On the other hand, for any \( \xi_1, \xi_2, \xi_3 \) with \( 0 \leq \xi_1 \leq \xi_2 \leq \xi_3 \leq 1 \) and \( \xi_1 = \xi_2 = \xi_3 = \frac{1}{3} \) the expressions in the right-hand side of (6.5.3) yield a point of \( \tilde{F}(D^3, 1) \), which assertion may be shown by choosing appropriate values for \( \eta_1, \eta_2 \) and \( \eta_3 \) in (6.5.2); we omit details. Therefore the right-hand side of (6.5.3) determines a parametrization of \( \tilde{F}(D^3, 1) \).

Taking \( \xi_3 = 1 \) and \( \xi_2 = \xi_1 + \frac{1}{3} \) and using (6.5.3) we obtain boundary points of the form

\[ (P_4(\xi_1) - P_4(\xi_1 + \frac{1}{3}), P_3(\xi_1) - P_3(\xi_1 + \frac{1}{3}), P_2(\xi_1) - P_2(\xi_1 + \frac{1}{3}))^T, \]

where \( 0 \leq \xi_3 \leq \frac{1}{3} \).

Now, by (6.5.4) and (1.2.20)

\[ 2(P_4(t) - P_4(t + \frac{1}{3})) = \frac{1}{6} t^3 - \frac{1}{9} t^2 + \frac{1}{192} = -E(D^3, \frac{t}{3}, t) \quad (0 \leq t \leq \frac{1}{3}). \]

Consequently, for all \( t \in \mathbb{R} \) the points \( (E(D^3, \frac{t}{3}, t), E'(D^3, \frac{t}{3}, t), E''(D^3, \frac{t}{3}, t))^T \) lie on the boundary of \( \tilde{F}(D^3, 1) \).
7. PERFECT $\ell$-SPLINES AND THE LANDAU PROBLEM ON THE HALF LINE

7.1. Introduction and summary

Landau problems are described in the introductory section of Chapter 5. Given a closed subinterval $J$ of $\mathbb{R}$ and a linear differential operator $p_n(D)$ of order $n$, the problem is to determine the best possible upper bound for $\|p_n(D)f\|_Y$ with respect to a specific subset $\mathcal{F}_n(p_n,J)$ of $W^{(n)}(J)$ (cf. (5.1.3)) where $p_n(D)$ is a given linear differential operator of order $k \leq n-1$. Functions for which the best possible upper bound is attained are called extremal functions (cf. p. 103). For second order differential operators $p_2(D)$ it is shown there that perfect Euler $\ell$-splines are extremal in the full-line case and one-sided perfect Euler $\ell$-splines are extremal in the half-line case.

In this chapter we shall deal exclusively with the half-line case, and a Landau problem then is an extremal problem with respect to the space $W^{(n)}_{\infty}([0,\infty))$ endowed with the sup norm. Of course, one of the main objectives in solving the half-line case Landau problem is to determine extremal functions; in view of this, we shall investigate how perfect $\ell$-splines are involved.

Throughout this chapter, the monic polynomial $p_n \in \mathcal{P}_n$ associated with the differential operator $p_n(D)$ is assumed to have only real zeros; this is done to avoid complications with respect to disconjugacy of $p_n(D)$.

We recall that Section 5.4 deals with the relation between the Landau problem on the half line and the set $\mathcal{F}_n(p_n)$ as defined on p. 110. It is shown there that maximizing $\|c_0 + c_1 f' + \ldots + c_{n-1} f^{(n-1)}\|_{\infty}$ on $\mathcal{F}_n(p_n,\mathbb{R}_0^+)$ is equivalent to maximizing a linear function on $\mathcal{F}_n^{+}(p_n)$. Therefore the problem of determining extremal functions is intimately connected with the problem of determining functions $f \in \mathcal{F}_n(p_n,\mathbb{R}_0^+)$ such that for some $x \in \mathbb{R}_0^+$ the point $(f(0), f'(0), \ldots, f^{(n-1)}(0)) \in \mathcal{F}_n^{+}(p_n)$, the boundary of $\mathcal{F}_n^{+}(p_n)$. This gives rise to the following interesting extremal problem; given $\hat{a} = (a_0, a_1, \ldots, a_{n-1})^T \in \mathbb{R}^n$, determine $\hat{f} \in \mathcal{F}_n(p_n,\mathbb{R}_0^+)$ with $\hat{f}(0) = a_0$ and $\hat{f}'(0) = a_1$ (i = 0, 1, \ldots, n-1) and such that
(7.1.1) \( \| p_n^{(0)} \|_{L^+} = \min \{ \| p_n^{(0)} \|_{L^+} \mid f \in P_n^{(0)} \} \),

\[ a_i, i = 0, 1, \ldots, n - 1 \].

Determining a function \( \tilde{f} \) satisfying (7.1.1) is one of the main purposes of this chapter.

A brief review of the contents of the various sections now follows. In Section 7.2 some preliminary material is collected. Starting from the set \( P_n^{(0)} \) of perfect \( L \)-splines having all knots in \( \mathbb{R}^+ \), we introduce the sets \( P_n^{(r)} \), \( P_n^{(0)} \), and \( P_n^{(e)} \). Properties of \( P_n^{(r)} \), \( P_n^{(0)} \), \( P_n^{(e)} \), and \( P_n^{(e)} \) due to Karlin [23] and Cavaretta [101], are given. Apart from this, we derive some compactness properties of \( P_n^{(e)} \) and \( P_n^{(e)} \). So-called representation theorems are dealt with in Section 7.3. A few fundamental results of this type, due to Karlin and Studden [25] and Karlin [23], are given for a set of functions spanned by a Chebyshev system, and for the set \( P_n^{(e)} \).

Our objective is to derive a representation theorem for the set \( P_n^{(e)} \); this is done via a representation theorem for \( \tilde{P}_n^{(e)} \), the set of \( \tilde{r} \)-approximate perfect \( L \)-splines. Section 7.4 is devoted to the problem of obtaining a function \( \tilde{f} \) satisfying (7.1.1). We determine \( \tilde{f} \) as the limit of a sequence of perfect \( L \)-splines; each perfect \( L \)-spline of the sequence is a solution of an extremal problem similar to (7.1.1), \( \mathbb{R}^+ \) then being replaced by an appropriate finite interval. These extremal \( L \)-splines converging to \( \tilde{f} \) exhibit a specific oscillation behaviour. In order to prove their existence we need the representation theorem of Section 7.3.

Finally, Section 7.5 contains, among other things, two examples. The first one concerns the operator \( D^3 \), and a complete description of the set \( \mathbb{R}_n^{(0)}(D^3) \) is given. The second example deals with the operator \( D^4 \). Even for this simple operator, it is difficult to determine an extremal function by analytical means. Instead, a sequence of perfect \( L \)-splines converging to an extremal function is obtained numerically.
7.2. Perfect $L$-splines and $\epsilon$-approximate perfect $L$-splines

We recall (cf. Definition 1.3.5) that perfect $L$-splines are introduced in Chapter 1. In connection with the Landau problem on the half line only perfect $L$-splines with knots in $\mathbb{R}_0^+$ are of interest. In this chapter a function $g \in \text{Ker}(p_n)$ will also be called a perfect $L$-spline corresponding to the operator $p_n(D)$; $g$ is said to be a perfect $L$-spline having $-1$ knots. For a given integer $r \geq -1$ the set $P_r(p_n)$ of perfect $L$-splines is now defined as follows.

**Definition 7.2.1.** Let $p_n \in \Pi_n$ be a monic polynomial having only real zeros and let $r+1 \in \mathbb{N}_0$. Then $P_r(p_n)$ is the set of perfect $L$-splines corresponding to the operator $p_n(D)$ and having at most $r$ knots, all located in $\mathbb{R}_0^+$.

It follows from the preceding definition that

$$\text{Ker}(p_n) = P_{-1}(p_n) \subset P_0(p_n) \subset P_1(p_n) \subset \ldots .$$

If $s \in P_r(p_n)$ has precisely $k$ knots $x_1, x_2, \ldots, x_k$ with $k \in \mathbb{N}_0$, then $s$ can be represented in the form

$$(7.2.1) \quad s(t) = g(t) + c \int_0^t \left( \varphi(t) + 2 \sum_{k=1}^{\infty} (-1)^k \varphi(t-x_k) \right) dt \quad (t \in \mathbb{R}_0),$$

where $g \in \text{Ker}(p_n)$, $c \in \mathbb{R}$ with $c \neq 0$, and where $\varphi$ is the fundamental function corresponding to $p_n(D)$.

In what follows, perfect $L$-splines in $P_r(p_n)$ will be needed for which the derivatives at 0, up to the order $n-1$, are prescribed. This leads to

**Definition 7.2.2.** Given $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1})^T \in \mathbb{R}^n$, the set $P_{r,a}(p_n)$ is defined by

$$P_{r,a}(p_n) := \{ s \in P_r(p_n) \mid s^{(i)}(0) = a_i, \quad i = 0, 1, \ldots, n-1 \} .$$

The purpose of this section is to derive some properties of $P_r(p_n)$ and $P_{r,a}(p_n)$ for later use.

We emphasize that throughout this chapter $p_n \in \Pi_n$ is a monic polynomial with real zeros only.

We observe that $P_r(p_n)$ is not a linear space of functions; it depends non-linearly on $n+r+1$ parameters. It is known (cf. Karlin [22]) that $P_r(p_n)$
is a univalent family of order \( n \) on \( \mathbb{R}_0^+ \) (cf. Definition 1.4.20), i.e., if \( f \in AC^{(n)}(\mathbb{R}_0^+) \) and if \( 0 \leq \xi_0 \leq \xi_1 \leq \ldots \leq \xi_{n+1} \) with \( \xi_i < \xi_{i+1} \) (\( i = 0, 1, \ldots, n \)) then a function \( s \in P_{\xi}(p_n) \) exists such that (cf. p. 4 for notation)

\[
(7.2.4) \quad s(\xi_0, \xi_1, \ldots, \xi_{n+1}) = \xi(\xi_0, \xi_1, \ldots, \xi_{n+1})
\]

An important property of perfect \( L \)-splines is contained in the following

**Theorem 7.2.3** (Karlin [22]). Let \( a > 0 \), let \( f \in W^{(n)}([0, a]) \), and let \( n + r + 1 \) points \( 0 \leq \xi_0 \leq \xi_1 \leq \ldots \leq \xi_{n+1} \) be given such that \( \xi_i < \xi_{i+1} \) (\( i = 0, 1, \ldots, n \)). Then a perfect \( L \)-spline \( s \in P_{\xi}(p_n) \) exists satisfying (7.2.2) and having its knots in \( (0, a) \). Moreover, any perfect \( L \)-spline \( s \in P_{\xi}(p_n) \) satisfying (7.2.2) has the property

\[
(7.2.3) \quad \|s_n(D)x\| \leq \|s_n(D)f\|_{[0, a]},
\]

The interpolating perfect \( L \)-spline of Theorem 7.2.3 is in general not uniquely determined. However, the following result holds.

**Lemma 7.2.4** (Cavaretta [10]). Let \( 0 \leq \xi_0 \leq \xi_1 \leq \ldots \leq \xi_{n+1} \) be \( n + r \) points with \( \xi_i < \xi_{i+1} \) (\( i = 0, 1, \ldots, n-1 \)). Then a nontrivial perfect \( L \)-spline \( s \in P_{\xi}(p_n) \) exists, uniquely determined up to a multiplicative constant, having precisely \( r \) knots and with the property \( s(\xi_0, \xi_1, \ldots, \xi_{n+1}) = 0 \). Moreover, the knots \( x_1 < x_2 < \ldots < x_r \) of any such \( s \) satisfy the inequalities

\[
(7.2.4) \quad \xi_i < x_i < \xi_{i+1} \quad (i = 0, 1, \ldots, r-1).
\]

In fact, in proving Lemma 7.2.4 Cavaretta dealt only with the polynomial case \( P_{\xi}(p_n) = D^n \). As the proof of Lemma 7.2.4 is quite similar to Cavaretta's, it will be omitted here.

The main difficulty in dealing with perfect \( L \)-splines arises from the fact that \( P_{\xi}(p_n) \) is not a univalent family (cf. Definition 1.4.20), i.e., a function \( s \in P_{\xi}(p_n) \) satisfying (7.2.2) is in general not uniquely determined. In order to overcome this difficulty, as a standard tool one uses the so-called "e-approximate" perfect \( L \)-splines. These functions will be defined by means of a specific positive linear operator.
If $f \in AC^{(n)}(\mathbb{R}^+_0)$ then by Taylor's formula (cf. Lemma 1.4.4) $f$ can be written in the form

$$f(t) = q(t) + \int_0^t e_t(t-r) p_n(D) f(t) \, dr \quad (t \geq 0),$$

where $q \in \text{Ker}(p_n)$ satisfies $q^{(i)}(0) = f^{(i)}(0) \quad (i = 0, 1, \ldots, n-1)$.

Let $\tilde{f}$ be defined by

$$\tilde{f}(t) := \begin{cases} f(t) & (t \geq 0), \\ q(t) & (t < 0). \end{cases}$$

Now for any $\epsilon > 0$ the "smoothing" operator $T_\epsilon$ is defined by

$$(T_\epsilon f)(t) := \frac{1}{\sqrt{\pi \epsilon}} \int_{-\infty}^{\infty} e^{-(t-\tau)^2/\epsilon} \tilde{f}(\tau) \, d\tau \quad (t \in \mathbb{R})$$

for functions $f \in AC^{(n)}_0(\mathbb{R}^+_0)$ for which the integral in (7.2.5) converges.

Note that if $p_n(D) f(t) = 0 \quad (t > 0)$, then $p_n(D) \tilde{f}(t) = 0$ for all $t \in \mathbb{R}$, and therefore $T_\epsilon f \in \text{Ker}(p_n)$.

We are now ready to define the sets $P_\epsilon(p_n)$ and $P_{\epsilon, 0}(p_n)$ of "$\epsilon$-approximate" perfect $L$-splines.

**Definition 7.2.5.** Let $\epsilon > 0$, let $P_{\epsilon}(p_n)$ and $P_{\epsilon, 0}(p_n)$ be the sets of perfect $L$-splines of Definitions 7.2.1 and 7.2.2. Furthermore, let the operator $T_\epsilon$ be given by (7.2.5). Then

$$P_\epsilon(p_n) := \{ T_\epsilon s \mid s \in P_{\epsilon}(p_n) \},$$

$$P_{\epsilon, 0}(p_n) := \{ T_\epsilon s \mid s \in P_{\epsilon, 0}(p_n) \}.$$

Just like $P_{\epsilon}(p_n)$ the set $P_\epsilon(p_n)$ depends nonlinearly on $n+r+1$ parameters. The important difference between the sets $P_{\epsilon}(p_n)$ and $P_{\epsilon, 0}(p_n)$ is that $P_{\epsilon}(p_n)$ is unisolvent (cf. p. 22), whereas $P_{\epsilon, 0}(p_n)$ is only a solvent family.

**Theorem 7.2.6 (Karlin [22]).** The set $P_\epsilon(p_n)$ is a unisolvent family of order $n+r+1$ on $\mathbb{R}^+_0$. Moreover, if $s_\epsilon \in P_\epsilon(p_n)$ satisfies
\( s_e(t_0, t_1, \ldots, t_{n+1}) = (y_{01} y_{11} \ldots y_{n+11})^T \), where \( 0 \leq t_0 \leq t_1 \leq \ldots \leq t_{n+1} \) and \( y_{i1} \in \mathbb{R} \) for \( i = 0, 1, \ldots, n+1 \) are given, then \( s_e \) depends continuously on the parameters \( t_0, t_1, \ldots, t_{n+1}, y_{01}, y_{11}, \ldots, y_{n+11} \).

In Karlin's proof of Theorem 7.2.6, it is rather complicated, it is assumed that \( p_n(D) = \mathbb{R}^n \). However, as in the case of Lemma 7.2.4 the proof can also be carried through for a general differential operator \( p_n(D) \), if \( p_n \) has only real zeros. We observe that a simpler proof of Theorem 7.2.6 can be given by making use of ideas of De Boor [4].

For later use we now derive some compactness properties of the sets \( P_\alpha^{(e)}(p_n) \) and \( P_\alpha^{(e)}(p_n) \).

**Lemma 7.2.7.** Given \( \varepsilon > 0 \), let \( (s_{e,m})_m \) be a sequence in \( P_\alpha^{(e)}(p_n) \). If there is an interval \([a,b] \in \mathbb{R}^1 \) with \( a < b \) and a positive \( M \) such that \( |s_{e,m}(t)| \leq M \) for all \( m \in \mathbb{N} \) and \( t \in [a,b] \), then a subsequence \( (s_{e,m_k})_k \) exists converging to \( s_e \in P_\alpha^{(e)}(p_n) \) such that for \( i = 0, 1, 2, \ldots \)

\[
\lim_{k \to \infty} s_{e,m_k}(t) = s^{(e)}_e(t),
\]

uniformly in \( t \) on every bounded interval of \( \mathbb{R} \).

**Proof.** Since \( s_{e,m} \in P_\alpha^{(e)}(p_n) \), we may write \( s_{e,m} = T^{\varepsilon}_{c_m} s_{e,m} \) with \( s_{e,m} \in P_\alpha^{(e)}(p_n) \).

Hence, according to (7.2.1),

\[
(7.2.6) \quad s_{e,m} = q_{e,m} + c_m h_m,
\]

where \( q_{e,m} \in \text{ker}(p_n), c_m \in \mathbb{R} \) and

\[
(7.2.7) \quad h_m(t) = \int_0^t \frac{1}{x(t) + 2} \left( \frac{1}{x(t) - x_{m,j}} \right) \delta^r(t - x_{m,j}) \, dt \quad (t > 0).
\]

Here \( x_{m,1} < x_{m,2} < \ldots < x_{m,q} \) are the knots of \( s_{e,m} \), if \( s_{e,m} \) has \( q \) knots with \( q < x \), then we take \( x_{m,q+1} = x_{m,q+2} = \ldots = x_{m,q} = x \) and \( s_{e,m}(t - x_{m,j}) = 0 \) \( (t \in \mathbb{R}, j = q+1, q+2, \ldots, r) \).

Without loss of generality it may be assumed that \( \lim_{m \to \infty} x_{m,j} = x_{j} \) \( (j = 1, 2, \ldots, r) \) with \( x_{r} \in \mathbb{R}^+ \) or \( x_{r} = \infty \), since there always is a subsequence of \( \{x_{m}\} \) having this property. We now define the function \( h \) by
(7.2.8) \( h(t) := \int_{0}^{\infty} \left( g(t) + \frac{2}{\sqrt{\pi}} \sum_{j=1}^{\infty} (-1)^{j} q_j (t-x_j^0) \right) dt \quad (t \in \mathbb{R}). \)

Since \( q_j (t-x_j^0) \) is \( t \)-bounded, uniformly in \( t \) on every bounded interval of \( \mathbb{R} \), we also have

(7.2.9) \( h(t) \rightarrow h(t) \quad (m \rightarrow \infty), \)

uniformly on every bounded interval of \( \mathbb{R}. \)

We proceed by proving that the sequence \( \{c_m\} \) is bounded. To this end we take \( n+1 \) points \( \xi_0 < \xi_1 < \cdots < \xi_n \in [a,b] \) and consider the divided difference \( s_{\xi_0, \xi_1, \ldots, \xi_n} \) (cf. Definition 1.4.25). In view of (7.2.6) and taking into account that \( T_{a} \gamma_m \subset \text{Ker}(p_n) \), it follows that

(7.2.10) \( \frac{1}{c_m} \frac{1}{n!} T_{n} h_{\xi_0, \xi_1, \ldots, \xi_n} = c_m T_{e_{\xi_0, \xi_1, \ldots, \xi_n}} \).

Because of (7.2.9) and the definition of \( T_{e_{\xi_0, \xi_1, \ldots, \xi_n}} \) (cf. (7.2.5)) one also has \( T_{e_{\xi_0, \xi_1, \ldots, \xi_n}} \rightarrow h(t) \) uniformly in \( t \) on every bounded interval. Hence

(7.2.11) \( T_{c_m} h_{\xi_0, \xi_1, \ldots, \xi_n} \rightarrow T_{e_{\xi_0, \xi_1, \ldots, \xi_n}} \) \( (m \rightarrow \infty), \)

since \( \{ e_{\xi_0, \xi_1, \ldots, \xi_n} \} \) is a bounded sequence the boundedness of \( \{c_m\} \) will follow from (7.2.10) if one shows that the points \( \xi_0, \xi_1, \ldots, \xi_n \) can be taken such that \( T_{e_{\xi_0, \xi_1, \ldots, \xi_n}} h_{\xi_0, \xi_1, \ldots, \xi_n} \neq 0 \).

By (7.2.8) and (7.2.5) one has for all \( t \in \mathbb{R} \)

(7.2.12) \( p_n(t) T_{e_{\xi_0, \xi_1, \ldots, \xi_n}} h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(t-x_j^0\right)^2/2} \left[ 1 + \frac{2}{\sqrt{\pi}} \sum_{j=1}^{\infty} \left(-1\right)^{j} q_j (t-x_j^0) \right] dt \),

where \( \left(t-x_j^0\right)^2 \equiv \left(1+\text{sgn}(t-x_j^0)\right)\).

So \( p_n(t) T_{e} h(t) \) is an analytic function and, moreover, it is nontrivial, since one easily verifies that

\( \lim_{c \rightarrow \infty} p_n(t) T_{e} h(t) \neq 0. \)

We conclude that \( p_n(t) T_{e} h(t) \) has finitely many zeros in each bounded interval. Consequently, points \( \xi_0, \xi_1, \ldots, \xi_n \) exist such that \( T_{e} h_{\xi_0, \xi_1, \ldots, \xi_n} \neq 0 \).

This proves the boundedness of \( \{c_m\} \). Therefore, a convergent subsequence of \( \{c_m\} \) exists and we may as well assume that \( c_m \rightarrow c \in \mathbb{R} \) \( (m \rightarrow \infty) \). It follows from the identity \( T_{e} g_m = T_{e} q_m + c_m T_{e} h_m \) \( (m \in \mathbb{N}) \) and the fact that
\((T \in \mathcal{M})\) and \(\{ T \mathbf{e}_m \} \) are bounded sequences on \([a,b]\) that \((T \mathbf{e}_m)\) is bounded on \([a,b]\). Since \(T \mathbf{e}_m \in \text{Ker}(p_n)\) and \(q, q', \ldots, q^{(n-1)}\) form a basis for \(\text{Ker}(p_n)\) we may write

\[
(T \mathbf{e}_m) \psi_n(t) = \sum_{k=0}^{n-1} a_{m,k} \psi^{(k)}(t) \quad (t \in \mathbb{R}).
\]

By taking \(n\) distinct points \(\zeta_1 < \zeta_2 < \ldots < \zeta_n\) in \([a,b]\) we have

\[
\begin{bmatrix}
0 & 0 & \cdots & \psi^{(n-1)} \\
\zeta_1 & \zeta_2 & \cdots & \zeta_n
\end{bmatrix} \neq 0.
\]

Since \(\psi, \psi', \ldots, \psi^{(n-1)}\) is a Chebyshev system on \([a,b]\) (cf. Example 1, p. 12). Consequently, the coefficients \(a_{m,k} (k = 0, 1, \ldots, n-1)\) in (7.2.12) are uniquely determined by the linear equations

\[
(T \mathbf{e}_m) (\zeta_i) = \sum_{k=0}^{n-1} a_{m,k} \psi^{(k)}(\zeta_i) \quad (i = 1, \ldots, n).
\]

Because of the boundedness of \((T \mathbf{e}_m)\) on \([a,b]\) we deduce that the sequences \(\{ a_{m,k} \}_{k=0}^{n-1} (i = 0, 1, \ldots, n-1)\) are bounded. Hence, there is a subsequence of \((T \mathbf{e}_m)\) converging to a function \(q \in \text{Ker}(p_n)\), uniformly on every bounded interval. We conclude that a subsequence \((T \mathbf{e}_{m_k})\) exists such that

\[
\lim_{k \to \infty} (T \mathbf{e}_{m_k}) (t) = \lim_{k \to \infty} \left( \sum_{k=0}^{n-1} a_{m_k,k} \psi^{(k)}(t) + c \mathbf{e}_m \mathbf{h}(t) \right) = \sum_{k=0}^{n-1} a_k \psi^{(k)}(t) + c \mathbf{e}_m \mathbf{h}(t),
\]

uniformly in \(t\) on every bounded interval, in (7.2.13) \(a_k := \lim_{k \to \infty} a_{m_k,k}\) and \(h\) is given by (7.2.8).

Since for every \(i \in \mathbb{N}\) one has \(c^i \mathbf{e}_m \mathbf{h}(t) = c^i \mathbf{e}_m \mathbf{h}(t) (n + m)\), uniformly in \(t\) on every bounded interval of \(\mathbb{R}\), the second part of the theorem easily follows from (7.2.13).

In a similar way one may prove the following

**Lemma 7.2.8.** Let \((\mathbf{e}_m)\) be a sequence in \(\mathcal{M}(p_n)\). If there is an interval \([a,b]\) with \(a < b\), and a positive \(N\) such that \(|e_m(t)| < M\) for all
If $n \in \mathbb{N}$ and $t \in [a,b]$, then a subsequence $(\sigma_k)_{k=1}^\infty$ exists converging to a function $s \in P_r(p_n)$ such that for $i = 0,1,\ldots,n-1$,

$$\lim_{k \to \infty} \sigma_k^{(1)}(t) = s^{(1)}(t),$$

uniformly in $t$ on every bounded interval of $\mathbb{R}$.

The next result is intimately connected with Lemma 7.2.7.

**Theorem 7.2.9.** Let $(\varepsilon_m)_{m=1}^\infty$ be a sequence of positive numbers with

$$\lim_{m \to \infty} \varepsilon_m = 0,$$

and let $(\nu_m)_{m=1}^\infty$ be the associated sequence of functions

$$\nu_m \in P_r(p_n).$$

If there is an interval $[a,b] = \mathbb{R}_0$ and a positive $M$ and a sequence $\{\nu_m(t)\}$ such that $|\nu_m(t)| \leq M$ for all $m \in \mathbb{N}$ and $t \in [a,b]$, then a subsequence $(\nu_k)_{k=1}^\infty$ exists converging to a function $s \in P_r(p_n)$ such that for $i = 0,1,\ldots,n-1$,

$$\lim_{k \to \infty} \nu_k^{(1)}(t) = s^{(1)}(t),$$

uniformly in $t$ on every bounded interval of $\mathbb{R}$.

**Proof.** The proof follows the same pattern as that of Lemma 7.2.7, the only difference is that we have to use the observation that for $j = 1,2,\ldots,n$,

$$\lim_{m \to \infty} T_{\varepsilon_m} \varphi_j(t - \varepsilon_m) = \varphi_j(t - \varepsilon_m),$$

uniformly in $t$ on every bounded interval of $\mathbb{R}$.

We recall that $P_r(p_n)$ is not a linear space, so if $s_1 \in P_r(p_n)$ and $s_2 \in P_r(p_n)$, then, in general, $(s_1 + s_2) \not\in P_r(p_n)$. However, for specific pairs $s_1, s_2$ the existence of an $s \in P_r(p_n)$ may be shown such that $s(s_1 + s_2)$ shares some properties with $s$. Details are given in the next lemma.

**Lemma 7.2.10.** Let $s_1 \in P_r(p_n)$ and $s_2 \in P_r(p_n)$ satisfy the condition

$$s_1(\ell_0, \ell_1, \ldots, \ell_{n+1-1}) = s_2(\ell_0, \ell_1, \ldots, \ell_{n+1-1})$$

for some sequence of points $0 = \ell_0 < \ell_1 < \ldots < \ell_{n+1-1}$. Then for every $t_0 \in \mathbb{R}$ with $s_1(t_0) \neq s_2(t_0)$ there exists a unique $s \in P_r(p_n)$ having the properties.
\[(7.2.14) \quad \begin{cases} 
\tau(t_0) = \frac{1}{2}(s_1(t_0) + s_2(t_0)), \\
\tau(t_0, \tau_1, \ldots, \tau_{n+1}) = \frac{1}{2}(s_1(t_0, \tau_1, \ldots, \tau_{n+1}) - \tau_{n+1}) \end{cases} \]
\[(7.2.15) \quad \min \{s_1(t), s_2(t)\} < s(t) < \max \{s_1(t), s_2(t)\} \]

for all \( t \in \mathbb{R}^+ \) with \( s_1(t) \neq s_2(t) \).

**Proof.** In view of Theorem 7.2.6 there exists a unique \( n \in \mathbb{Z} \setminus \{0\} \) satisfying \( \tau(t_0, \tau_1, \ldots, \tau_{n+1}) = \frac{1}{2}(s_1(t_0, \tau_1, \ldots, \tau_{n+1}) - \tau_{n+1}) \) and \( s(t_0) = \frac{1}{2}(s_1(t_0) + s_2(t_0)) \). So it remains to prove (7.2.15) for all \( t \neq \{t_0, \tau_1, \ldots, \tau_{n+1}\} \). Since \( s \neq s_1 \) and \( s \neq s_2 \), by Theorem 7.2.6, \( s \) coincides with \( s_1 \) and \( s_2 \) only at the \( n+1 \) points \( t_0, \tau_1, \ldots, \tau_{n+1} \). Further, it is no restriction to assume the situation as sketched in Figure 1.

![Figure 1](image)

Here \( t_1 \) and \( t_2 \) are two consecutive elements of \( \{t_0, \tau_1, \ldots, \tau_{n+1}\} \), \( t_1 \) being a zero of \( s_1 - s_2 \) of odd multiplicity and \( t_2 \) being a zero of \( s_1 - s_2 \) of even multiplicity. Since \( \tau(t_0, \tau_1, \ldots, \tau_{n+1}) = \frac{1}{2}(s_1(t_0, \tau_1, \ldots, \tau_{n+1}) - \tau_{n+1}) \), \( t_1 \) is a zero of \( s - s_1 \), \( s - s_2 \), and \( s_1 - s_2 \) of the same multiplicity. This assertion, obviously, also holds for \( t_2 \).

Consequently, as \( s_2(t_0) < s(t_0) < s_1(t_0) \) it follows that \( s_1(t) < s(t) < s_2(t) \) (\( t_1 < t < t_2 \)) and \( s_1(t) < s(t) < s_2(t) \) (\( t > t_2 \)). This ascertains (7.2.15) and the lemma is proved. \( \square \)
7.5. A representation theorem for the set $P_{n/2}(R_n)$

It is well known (cf. Karlin and Studden [25, p. 293]) that Chebyshev polynomials are "extremal" with respect to the problem of minimizing $\|p\|_{[-1,1]}$ on the set of polynomials $p$ in $\pi_n$ for which $\|p\|_{[-1,1]} \leq 1$. The Chebyshev polynomials are characterized by the fact that they oscillate a maximum number of times between $-\|p\|_{[-1,1]}$ and $\|p\|_{[-1,1]}$, i.e. points $-1 = r_1 < r_2 < \ldots < r_{n+1} = 1$ exist such that $\|p\|_{[-1,1]} = \|p(r_k)\|$ and $p(r_{k+1}) = -p(r_k)$ for $k = 1, 2, \ldots, n$. There are many problems in approximation theory for which the corresponding extremal functions of which are characterized by a specific oscillation behaviour (cf. Karlin and Studden [25, Chapter IX]).

In many such cases characterizations of extremal functions can be obtained by a suitable application of a so-called representation theorem. These representation theorems have in common that they ensure the existence of specific functions that oscillate a prescribed number of times between two given continuous functions. As an illustration we give a representation theorem for functions spanned by a Chebyshev system.

**Theorem 7.3.1** (Karlin and Studden [25, p. 72]). Let $\varphi_0, \varphi_1, \ldots, \varphi_n$ be a Chebyshev system on $[a,b]$ and let $f \in C([a,b])$, $g \in C([a,b])$ be such that a function $\nu(t)$ of the form $\nu(t) = \sum_{i=0}^{n} \varphi_i \nu_i$ exists satisfying $g(t) < \nu(t) < f(t)$ for $t \in [a,b]$. Then there exist unique functions $\check{\nu} = \sum_{i=0}^{n} \varphi_i \check{\nu}_i$ and $\bar{\nu} = \sum_{i=0}^{n} \varphi_i \bar{\nu}_i$ with the properties

\[
\begin{align*}
\text{i)} & \quad g(t) \leq \nu(t) \leq f(t) \quad (t \in [a,b]), \\
\text{ii)} & \quad \text{there exist } n+1 \text{ points } a = \tau_1 < \tau_2 < \ldots < \tau_{n+1} = b \text{ such that} \\
(7.3.1) & \quad \nu(\tau_k) = \begin{cases} g(\tau_k) & \text{(i odd)}, \\ f(\tau_k) & \text{(i even)}. \end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{i)} & \quad g(t) \leq \check{\nu}(t) \leq f(t) \quad (t \in [a,b]), \\
\text{ii)} & \quad \text{there exist } n+1 \text{ points } a = \check{\tau}_1 < \check{\tau}_2 < \ldots < \check{\tau}_{n+1} = b \text{ such that} \\
(7.3.2) & \quad \check{\nu}(\check{\tau}_k) = \begin{cases} g(\check{\tau}_k) & \text{(i even)}, \\ f(\check{\tau}_k) & \text{(i odd)}. \end{cases}
\end{align*}
\]
A representation theorem for functions spanned by a Chebyshev system and satisfying some appropriate boundary conditions is given by Pirkus [47].

Besides these results there is a representation theorem for perfect $L$-splines: before we go into this, it is convenient to have the following definitions available.

DEFINITION 7.3.2. Let $f$ and $g$ be two functions defined on an interval $J$ with the property $g(t) < f(t)$ $(t \in J)$. A function $h$ is said to oscillate $k$ times between $f$ and $g$ on $J$ if $g(t) \leq h(t) \leq f(t)$ $(t \in J)$ and if there exist at least $k+1$ points $\tau_1 < \tau_2 < \ldots < \tau_{k+1}$ in $J$ such that

$$\begin{cases}
  i) & h(\tau_1) = g(\tau_1) \quad (i \text{ odd}), \\
  & h(\tau_2) = f(\tau_2) \quad (i \text{ even}), \\
  \quad \text{or} \quad & \\
  ii) & h(\tau_1) = f(\tau_1) \quad (i \text{ even}), \\
  & h(\tau_2) = g(\tau_2) \quad (i \text{ odd}).
\end{cases}$$

(7.3.3)

It follows that if $h$ oscillates $k$ times it also oscillates $k-1$ times.

If $i)$ of (7.3.3) applies then the set $\{\tau_1, \tau_2, \ldots, \tau_{k+1}\}$ is called a set of $k+1$ oscillation points of $h$ of type I; similarly, if $ii)$ of (7.3.3) holds then $\{\tau_1, \tau_2, \ldots, \tau_{k+1}\}$ is called a set of $k+1$ oscillation points of $h$ of type II.

THEOREM 7.3.3 (Karlin [22]). Let $a > 0$ and let $f \in C([0,a])$, $g \in C([0,a])$ be such that a function $s \in P_1(p_n)$ exists satisfying $g(t) < s(t) < f(t)$ for $t \in [0,a]$. Then functions $g \in P_1(p_n)$ and $h \in P_1(p_n)$ exist oscillating $n+r$ times between $f$ and $g$ on $[0,a]$ and such that $g$ has a set of $n+r+1$ oscillation points of type I and $h$ has a set of $n+r+1$ oscillation points of type II.

The main purpose of this section is to derive a representation theorem for the set $P_1(p_n)$. This result will be used in Section 7.4 to show the existence of appropriate extremal functions. In order to achieve this we first deduce a representation theorem for the set $P_{1,n}(p_n)$ as introduced in Definition 7.2.5.
THEOREM 7.3.4. Let \( a \in \mathbb{R}^n \), \( a > 0 \), and let \( f \in C([0,a]) \), \( g \in C([0,a]) \) be such that \( s \in P_{[0,a]}(p_n) \) exist satisfying \( g(t) < s(t) < f(t) \) \((t \in [0,a])\). Then unique functions \( \phi \in P_{[0,a]}(p_n) \) and \( \psi \in P_{[0,a]}(p_n) \) exist oscillating \( n \) times between \( f \) and \( g \) on \([0,a]\) such that \( \phi \) has a set of \( r + 1 \) oscillation points of type \( X_\phi \).

PROOF. First we shall prove the existence of \( \phi \) and \( \psi \). Let \( s \in P_{[0,a]}(p_n) \) satisfy \( g(t) < s(t) < f(t) \) \((t \in [0,a])\). According to Theorem 7.2.6 for every sequence of \( n + r + 1 \) points \( 0 = \xi_0 = \xi_1 = \ldots = \xi_n \leq \xi_{n+1} \leq \xi_{n+2} \leq \ldots \leq \xi_{n+r} = a \) and for every \( \lambda \in \mathbb{R} \) there exists a unique \( v \in P_{[0,a]}(p_n) \) such that

\[
\begin{align*}
(7.3.4) \quad v &= v_{r+1}(\xi_0, \xi_1, \ldots, \xi_{n+r-1}) = s(\xi_0, \xi_1, \ldots, \xi_{n+r-1}) , \\
(7.3.5) \quad v &= v_{r+1}(\xi_0, \xi_1, \ldots, \xi_{n+r}) = \lambda , \\
& \text{where } j \epsilon \mathbb{N} \text{ is determined by } \xi_{n+j} < \xi_{n+j+1} = \ldots = \xi_{n+r} = a.
\end{align*}
\]

In view of the unsolvability of \( P_{[0,a]}(p_n) \) (cf. Theorem 7.2.6) it follows that \( v = s \) if \( \lambda = v_{r+1}(\xi_0, \xi_1, \ldots, \xi_{n+r}) = s(\xi_0, \xi_1, \ldots, \xi_{n+r}) \); the number \( v_{r+1}(\xi_0, \xi_1, \ldots, \xi_{n+r}) \) is denoted by \( \lambda_0 \). Note that \( \lambda \) depends on \( j \) and therefore on the sequence \( \xi_0, \xi_1, \ldots, \xi_{n+r} \). We define the vector \( \mathbf{h} = (h_0, h_1, \ldots, h_r)^T \in \mathbb{R}^{r+1} \) by

\[
(7.3.5) \quad h_i = \xi_{n+i} - \xi_{n+i-1} \quad (i = 0, 1, \ldots, r).
\]

Hence, \( \xi_0 \mathbf{h} = a \) and \( h_i \leq 0 \) \((i = 0, 1, \ldots, r)\). Let the simplex \( V \) be defined by

\[
V := \left\{ h \in \mathbb{R}^{r+1} \mid \sum_{i=0}^{r} h_i = a, \ h_i \geq 0, \ i = 0, 1, \ldots, r \right\}.
\]

The function \( v \) satisfying (7.3.4) will be denoted by \( v(\mathbf{h}, \lambda, \cdot) \).

In view of Theorem 7.2.6 it then follows that \( (\mathbf{h}, \lambda, \cdot) \rightarrow v(\mathbf{h}, \lambda, \cdot) \) is continuous. Since by assumption \( g(t) \leq v(\mathbf{h}, \lambda, \cdot)(t) < f(t) \) \((t \in [0,a])\) for every \( \mathbf{h} \in V \), \( \lambda_0 \) is an interior point of the set

\[
\Lambda_0 = \{ \lambda \in \mathbb{R} \mid g(t) < v(\mathbf{h}, \lambda, \cdot)(t) \leq f(t), \ t \in [0,a] \}.
\]

It follows from the compactness of \( P_{[0,a]}(p_n) \) (cf. Lemma 7.2.7) that \( \Lambda_0 \) is bounded. We next define
\[ \lambda^*(h) := \sup_{\lambda \in \mathcal{B}} \lambda \quad (3.1) \]

Our purpose is to show that for some \( h \in V \) either \( v(h, \lambda^*(h), t) = \emptyset \) or \( v(h, \lambda^*(h), t) \neq \emptyset \). To this end, we first prove that \( \lambda^*(h) \) depends continuously on \( h \). This will be done as follows. First we assert that for every \( h \in V \) and every \( \lambda \in [\lambda_0, \lambda^*(h)] \)

\[ (7.3.7) \quad g(t) < v(h, \lambda, t) < f(t) \quad (t \in [0, a]) \quad (*) \]

In order to establish \((7.3.7)\) we consider the function \( v(h, \lambda^*, t) \). Let \( \lambda_1 \neq \lambda_2 \). For all \( t \in (0, t_1, t_{n+1}, \ldots, t_{n+1}) \) it then follows that \( v(h, \lambda_1, t) \neq v(h, \lambda_2, t) \), since otherwise by \( i) \) of \((7.3.4)\) and Theorem 7.2.6 we would have \( v(h, \lambda_1, t) = v(h, \lambda_2, t) \), which in view of \( ii) \) of \((7.3.4)\) violates the assumption \( \lambda_1 \neq \lambda_2 \). Because of the continuity of \( v(h, \cdot^*, t) \), for every \( h \in V \) and every \( t \in (0, t_1, t_{n+1}, \ldots, t_{n+1}) \) the function \( v(h, \cdot^*, t) \) is strictly monotonic (cf. Figure 1, where \( \lambda_0 < \lambda_1 < \lambda_2 \)).

![Figure 1](image)

In view of the strict monotonicity of \( v(h, \cdot^*, t) \) it follows that for all \( t \in (0, t_1, t_{n+1}, \ldots, t_{n+1}) \) and all \( \lambda \in [\lambda_0, \lambda^*(h)] \)

\[ |v(h, \lambda, t) - s(t)| < |v(h, \lambda^*(h), t) - s(t)| \quad (t \in [0, a]) \quad (**) \]

From this and the inequality

\[ (7.3.8) \quad g(t) < v(h, \lambda^*(h), t) < f(t) \quad (t \in [0, a]) \quad (\star) \]

assertion \((7.3.7)\) easily follows.

Now let \( \{h_k\} \) be a sequence in \( V \) with \( \lim_{k \to \infty} h_k = h_0 \); it has to be shown that \( \lim_{k \to \infty} \lambda^*(h_k) = \lambda^*(h_0) \). As a consequence of Lemma 7.2.7 the sequence
(\lambda^{n}(h_{k})) is bounded, so it suffices to prove that every convergent sub-sequence of (\lambda^{n}(h_{k})) has limit \lambda^{n}(h_{0}). Consequently, we may assume, without loss of generality, that lim_{k \to \infty} \lambda^{n}(h_{k}) = \lambda^{n}_{\infty} \text{ for some } \lambda_{\infty} \in \mathbb{R}. \text{ In view of the continuity of } v(t, \cdot, t) \text{ it follows that}

\lim_{k \to \infty} v(h_{k}, \lambda^{n}(h_{k}), t) = v(h_{0}, \lambda_{\infty}, t) \quad \forall t \in [0, a] \quad (t \in [0, a])

\text{Hence } \lambda_{\infty} \in \lambda^{n}(h_{0}) \text{ and therefore (cf. (7.3.6)) } \lambda_{\infty} \in \lambda^{n}(h_{0}). \text{ If } \lambda_{\infty} \in \lambda^{n}(h_{0}) \text{ then by (7.3.7) one would have}

\forall t \in [0, a] \quad v(h_{0}, \lambda_{\infty}, t) < f(t)

\text{On the other hand, there exists a convergent sequence } (\mu_{k}) \text{ with } \mu_{k} \in (0, a) \text{ and having the property that either } v(h_{k}, \lambda^{n}(h_{k}), t_{k}) = g(t_{k}) \text{ or } v(h_{k}, \lambda^{n}(h_{k}), t_{k}) = f(t_{k}). \text{ This, however, contradicts (7.3.9) as is easily seen by letting } k \text{ tend to infinity. Hence } \lambda_{\infty} \in \lambda^{n}(h_{0}), \text{ proving the continuity of } \lambda^{n} \text{ as a function of } h.

\text{As a next step in the construction of } \mathfrak{A} \text{ and } \mathfrak{B}, \text{ a sequence } n_{0}(h), n_{1}(h), ..., n_{r}(h), \text{ corresponding to an arbitrary } h \in \mathfrak{V}, \text{ is defined by}

\begin{equation}
\begin{aligned}
(7.3.10) \quad m_{i}(h) := & \min \left( v(h, \lambda^{n}(h), t) - q(t) \mid t_{n_{i-1}} \leq t \leq t_{n_{i}} \right) \quad (i=1, ..., r) \\
& \min \left( v(h, \lambda^{n}(h), t) - v(h_{0}, \lambda_{\infty}, t) \mid t_{n_{i-1}} \leq t \leq t_{n_{i}} \right) \quad (i=1, ..., r-1)
\end{aligned}
\end{equation}

\text{We note that the nonnegative numbers } m_{i}(h) \text{ depend continuously on } h. \text{ The existence of the required function } g \text{ (2) will now be ensured if we prove that an } h \in \mathfrak{V} \text{ exists such that } m_{i}(h) = 0 \text{ for all } i = 0, 1, ..., r. \text{ Such an } h \in \mathfrak{V} \text{ will be obtained by showing that the assumption } \sum_{i=0}^{r} m_{i}(h) > 0 \text{ leads to a contradiction. So, let } \sum_{i=0}^{r} m_{i}(h) > 0 \forall h \in \mathfrak{V}. \text{ The vector-valued function } g \text{ defined by}

(7.3.11) \quad g(h) := \frac{1}{r} \left( m_{0}(h), m_{1}(h), ..., m_{r}(h) \right)^{T} \quad (h \in \mathfrak{V})

\text{then is a continuous function from } \mathfrak{V} \text{ into } \mathfrak{V}. \text{ According to Brouwer's fixed point theorem (cf. Kuga [29]), a point } h^{*} \text{ exists with } g(h^{*}) = h^{*}. \text{ Since for every } h \in \mathfrak{V} \text{ there is at least one } t_{0} \in (0, 1, ..., r) \text{ such that } m_{t_{0}}(h) = 0, \text{ one has } h_{t_{0}}^{*} = 0. \text{ Hence (cf. (7.3.5)) } \sum_{i=0}^{r} m_{i}(h^{*}) = \sum_{i=0}^{r} m_{i}(h) = 0. \text{ By (7.3.10) and because } v(h, \lambda^{n}(h), t_{n_{i+1}}) = s(t_{n_{i+1}}) \text{ it now follows that } m_{0}(h_{0}) > 0.

\text{This, however, contradicts the fact that}
C = k_0^a = \frac{a}{\sum_{i=0}^{r} n_i (k_0^a)}.

We conclude that an \( h \in V \) exists such that \( n_i(h) = 0 \) \((i = 0, 1, \ldots, r)\). The corresponding sequence \( \{ n_i(h) \} \) evidently satisfies \( n_i^{h, \lambda^r} > 0 \)
\((i = 0, 1, \ldots, r)\) and the corresponding function \( v(h, \lambda^r(h), \cdot) \) oscillates \( r \) times between \( f \) and \( g \) on [0, a]. Finally, if \( r \) is even (odd) \( v(h, \lambda^r(h), \cdot) \) has a set of \( r + 1 \) oscillation points of type III (II).

Consequently, we may take \( \bar{h} = v(h, \lambda^r(h), \cdot) \) if \( r \) is even and \( \bar{g} = v(h, \lambda^r(h), \cdot) \) if \( r \) is odd. If \( \lambda^r(h) \) is defined by \( \lambda^r(h) := \inf \lambda_r \) (cf. (7.3.6)), then in a similar way the existence of the remaining \( \bar{h} \) (or \( \bar{g} \)) can be established.

We proceed by showing the uniqueness of \( \bar{g} \), say. Let \( s_1 \in \varphi_{0, \lambda}^{(0)} \) be another function oscillating \( r \) times between \( f \) and \( g \), and having a set of \( r + 1 \) oscillation points \( \tau_1 < \tau_2 < \cdots < \tau_{r+1} \) of type I. In order to prove that \( s_1 = \bar{g} \) it suffices to show (cf. Theorem 7.2.6) that \( s_1 \) and \( \bar{g} \) coincide in at least \( r + 1 \) points in [0, a], counting multiplicities.

Let \( \bar{g} \) have a set of \( r + 1 \) oscillation points \( \tau_1 < \tau_2 < \cdots < \tau_{r+1} \) of type I.
Without loss of generality we may assume that for some \( i \) one has \( \tau_i < \tau_i' \), \( g(\tau_i) < g(\tau_i') \) and \( g(\tau_i') > g(\tau_i) \). Consequently, \( s_1 \) and \( \bar{g} \) coincide in at least one point in \( [\tau_i, \tau_i'] \). Furthermore, \( s_1 \) and \( \bar{g} \) coincide in at least one point in each of the intervals \([\tau_j, \tau_{j+1}']\) \((j = 1, 2, \ldots, r+1)\) and \([\tau_{r+1}', a]\). Hence, \( s_1 \) and \( \bar{g} \) coincide in at least \( r + 1 \) points in \([\tau_1, \tau_{r+1}]\) and they coincide in at least \( r + 1 \) points in \([\tau_1', \tau_{r+1}']\), counting multiplicities. We conclude that \( s_1 \) and \( \bar{g} \) coincide in at least \( (r + 1) + (r + 2) = 2r + 3 \) points in \([\tau_1, \tau_{r+1}]\) and therefore by Theorem 7.2.6 \( s_1 = \bar{g} \). This establishes the theorem.

From this representation theorem for the set \( V_{\varphi_{0, \lambda}^{(0)}}(\varphi_{0, \lambda}^{(0)}) \), we now easily obtain an analogous result for the set \( V_{\varphi_{0, \lambda}^{(0)}}(\varphi_{0, \lambda}^{(0)}) \) by letting \( c \) tend to zero.

**Theorem 7.2.6.** Let \( g \in \mathcal{E}_{\varphi}, a > 0 \), and let \( \varphi \in C([0, a]), g \in C([0, a]) \) be such that an \( e \in \mathcal{E}_{\varphi_{0, \lambda}} \) exists satisfying \( g(t) < e(t) < f(t) \) \((t \in [0, a])\). Then functions \( g \in \mathcal{E}_{\varphi_{0, \lambda}}(\mathcal{E}_{\varphi_{0, \lambda}}) \) and \( \bar{g} \in \mathcal{E}_{\varphi_{0, \lambda}}(\mathcal{E}_{\varphi_{0, \lambda}}) \) exist oscillating \( r \) times between \( f \) and \( g \) on [0, a], and such that \( \bar{g}(\bar{h}) \) has a set of \( r + 1 \) oscillation points of type I (II).

**Proof.** Since \( \tau_{r+1}^a + e(0) \) uniformly on every bounded interval of \( \mathbb{R}^+ \)
(cf. the proof of Theorem 7.2.9), it follows that for sufficiently small
\( c > 0 \) the function \( T_c \) satisfies the inequality \( g(t) < T_c g(t) < f(t) \) 
\((t \in (0, a))\). Let \( a_i(c) \) be defined by \( a_i(c) := \left( \frac{d}{dt} \right)^i g(t) \) at \( t = 0 \) 
\((i = 0, 1, \ldots, n - 1)\). According to Theorem 7.3.4 functions \( \phi_c \in \Phi_c(c) \) and 
\( \psi_c \in \Phi_c \) \((c > 0)\) exist oscillating \( r \) times between \( f \) and \( g \) on \((0, a)\), and such that \( \phi_c \) has a set of \( r+1 \) oscillation points of type I (II). By 
letting \( c \) tend to zero we obtain the desired functions \( \phi \) and \( \psi \).


7.4. An extremal property of perfect \( L \)-splines

Let \( a = (a_0, a_1, \ldots, a_{n-1})^T \in \mathbb{R}^n \), and let the set \( F_a \) \((a = \mathbb{R}_0^+ \) or \( J = (0, a) \) 
with \( a > 0)\) be defined by

\[
(7.4.1) \quad F_a(J) := \{ f \in W^1(J) | \| f \|_J \leq 1, f^{(i)}(0) = a_i, i = 0, 1, \ldots, n-1 \}.
\]

In this section we shall be concerned with the problem of minimizing 
\( \| p(J) \|_{\Phi} \) on \( F_a(\mathbb{R}_0^+) \), where \( p(J) \) is a given differential operator. It 
will be proved that a perfect \( L \)-spline \( \hat{p} \) exists, in general, infinitely many 
knots solves the problem. The extremal function will be obtained as the 
limit of a sequence of perfect \( L \)-splines \( p_{\epsilon} \) \((\epsilon > 0)\) and 
each \( p_{\epsilon} \) is an extremal function for the problem of minimizing 
\( \| p_{\epsilon}(J) \|_{\Phi} \) on 
\( F_a(J) \), where \( J \) is an appropriate finite interval depending on \( \epsilon \).

In order to ensure that \( F_a(\mathbb{R}_0^+) \) is nonempty the following lemma is of 
importance.

**Lemma 7.4.1.** The set \( F_a(\mathbb{R}_0^+) \) is nonempty if and only if the polynomial 
\( p_{n-1} \in \pi_{n-1} \), determined by the conditions \( p_{n-1}^{(i)}(0) = a_i \) \((i = 0, 1, \ldots, n-1)\), 
has the property that \( |p_{n-1}(t)| \leq 1 \) in a right-neighbourhood of 0.

**Proof.** If a function \( f \in F_a(\mathbb{R}_0^+) \) exists, then according to Taylor's formula 
one has \( p_{n-1}(t) = f(t) + O(t^n) \) \((t \to 0)\). Hence \( |p_{n-1}(t)| \leq 1 + O(t^n) \) \((t \to 0)\). 
Since \( p_{n-1} \in \pi_{n-1} \), we may conclude that \( |p_{n-1}(t)| \leq 1 \) in a right-neighbourhood 
of 0.

Conversely, let \( |p_{n-1}(t)| \leq 1 \) in \([0, \epsilon_0]\) for some \( \epsilon_0 > 0 \). If \( |p_{n-1}(t)| = 1 \) 
for all \( t \in [0, \epsilon_0] \), then \( p_{n-1}(t) = 1 \) \((t \in [0, \epsilon_0]) \) and we may choose 
\( f(t) = 1 \) or \( f(t) = -1 \) \((t \in \epsilon_0^+)\). If there exists a \( t_0 \in [0, \epsilon_0^+] \) such that 
\( |p_{n-1}(t)| < 1 \), then a function \( f \in F_a(\mathbb{R}_0^+) \) can be obtained as follows.

Let \( g_{2n-1} \in \pi_{2n-1} \) be the polynomial satisfying
\[ q_{2n-1}^{(0)}(0) = q_{2n-1}^{(1)}(0) = 0 \quad (i = 1, 2, \ldots, n-1), \]
\[ q_{2n-1}^{(0)}(1) = 0, \quad q_{2n-1}^{(1)}(1) = 0. \]

The function \( f_\lambda \) defined by

\[
f_\lambda(t) = \begin{cases} 
  P_{\lambda-1}(t) & (0 \leq t < \tau_1), \\
  P_{\lambda-1}(t)q_{2n-1}(\frac{t-\tau_1}{\tau_1}) & (\tau_1 \leq t < \tau_2 + \lambda), \\
  0 & (t \geq \tau_2 + \lambda),
\end{cases}
\]

has the properties that \( f_\lambda \in \mathcal{W}^n(\mathbb{R}_0^+) \), \( f_\lambda^{(1)}(0) = a_1 \) (\( i = 0, 1, \ldots, n-1 \)), and for sufficiently small \( \lambda > 0 \) one has \( \| f_\lambda \|_1 \leq 1 \), i.e., \( f_\lambda \in \mathcal{F}_1^* (\mathbb{R}_0^+) \). \( \square \)

If a function \( g \in \mathcal{W}^n(\mathbb{R}_0^+) \) exists such that \( p_\lambda \mathbf{g}(t) = 0 \) (\( t > 0 \)), then, of course, the minimum of \( \| p_\lambda \mathbf{g}(t) \|_1 \) on \( \mathcal{F}_1^* (\mathbb{R}_0^+) \) is zero and \( g \) is an extremal function. If such a function \( g \) does not exist then a sequence \( (g_{\lambda i}) \) of perfect \( \Lambda \)-splines \( g_{\lambda i} \in \mathcal{F}_\lambda^* (p_\lambda) \) will be determined, converging to an extremal function for the problem stated in the beginning of this section. As will be evident from what follows, this approach makes essentially use of the assumption that

\[
(7.4.2) \quad \mathcal{P}_\lambda^* (p_\lambda) \cap \mathcal{P}_\lambda^* (\mathbb{R}_0^+) = \mathcal{F}_1^* (\mathbb{R}_0^+) = \mathcal{G}, \quad (r \in \mathbb{N}_0^+).
\]

The condition \( p_\lambda(0) = 0 \) is sufficient to ensure that (7.4.2) holds, as then for every \( r \in \mathbb{N}_0^+ \) any function \( g \in \mathcal{F}_r^* (p_\lambda) \) (\( r \geq 0 \)) is unbounded on \( \mathbb{R}_0^+ \). For this reason we shall assume from now on that the following conditions are satisfied:

\[
(7.4.3) \quad \mathcal{P}_\lambda^* (\mathbb{R}_0^+) \neq \mathcal{G}, \quad \operatorname{Ker}(p_\lambda) \cap \mathcal{F}_\lambda^* (\mathbb{R}_0^+) = \mathcal{G}, \quad p_\lambda(0) = 0.
\]

We note that (7.4.3) implies that \( |a_0| \leq 1 \).

By the continuity of the perfect \( \Lambda \)-splines and by (7.4.3), it follows that for each \( s \in \mathcal{P}_\lambda^* (p_\lambda) \) with \( r \geq -1 \) there exists a positive number \( \rho_s \) defined by

\[
(7.4.4) \quad \rho_s := \max \{ t \mid |s(t)| = 1, 0 \leq t \leq s \}.
\]
DEFINITION 7.4.2. For \( r \in \mathbb{N}_0 \) we define

\[(7.4.5) \quad A_r := \sup \{ z \mid \exists s \in \mathcal{P}_r(\mathcal{A}_r) \} \cap \mathbb{N} \].

In what follows it will be shown that for each \( A_r \) the problem of determining a function \( f \in \mathcal{P}_r([0,A_r]) \) that minimizes \( \| p_r (D) f \|_{[0,A_r]} \) on \( \mathcal{P}_r([0,A_r]) \) is solved by an appropriate \( s \in \mathcal{P}_r(\mathcal{A}_r) \) exhibiting a specific oscillation behaviour.

THEOREM 7.4.3. Let \( r+1 \in \mathbb{N}_0 \) and let \( \alpha \in \mathbb{R}^n \) and the monic polynomial \( \mathcal{P}_r \in \mathbb{R}^n \) be such that \((7.4.3)\) is satisfied. Then there exists a perfect \( L \)-spline \( \mathcal{P}_r \in \mathcal{P}_r(\mathcal{A}_r) \) oscillating \( r+1 \) times between \(-1\) and \( 1 \) on \([0,A_r]\). Moreover, any \( \mathcal{P}_r \in \mathcal{P}_r(\mathcal{A}_r) \) having this oscillation property satisfies

\[(7.4.6) \quad \mathcal{P}_r (D) \mathcal{P}_r = \min \{ \| p_r (D) f \|_{[0,A_r]} \mid f \in \mathcal{P}_r([0,A_r]) \} \].

PROOF. Let \( \delta > 0 \). Since a function \( \mathcal{P}_r \in \mathcal{P}_r(\mathcal{A}_r) \) exists with \( |\mathcal{P}_r(t)| \leq 1 \) on \([0,A_r]\), it follows that \( |\mathcal{P}_r(t)| < 1 + \delta \) on \([0,A_r]\) for sufficiently small \( \delta > 0 \), where \( \mathcal{P}_r \) is the operator defined by \((7.2.5)\). Taking

\[ s_i (t) := \left \{ \begin{array}{ll} 0 & \text{if } t = 0, 1, \ldots, n-1 \end{array} \right. \]

and applying Theorem 7.3.4, we obtain two \( \varepsilon \)-approximate perfect \( L \)-splines,

\( s_i \in \mathcal{P}_r(\mathcal{A}_r) \) and \( \mathcal{P}_r \in \mathcal{P}_r(\mathcal{A}_r) \), oscillating \( r \) times between \(-1\) and \( 1 \) on \([0,A_r]\). Letting \( \delta \) and \( \varepsilon \) tend to zero we may assume, in view of Theorem 7.2.9, that \( s_i \) converges to \( s \in \mathcal{P}_r(\mathcal{A}_r) \) and \( \mathcal{P}_r \) converges to \( \mathcal{P}_r \in \mathcal{P}_r(\mathcal{A}_r) \), where both \( s \) and \( \mathcal{P}_r \) oscillate \( r \) times between \(-1\) and \( 1 \) on \([0,A_r]\). We assert that \( (s_i) = (s) \). If \( s_i \neq (s) \) then for sufficiently small \( \varepsilon > 0 \) one has that \( s_i \neq (s) \). In view of their oscillation properties (cf. Theorem 7.2.4) the functions \( s \) and \( s \) coincide at precisely \( n + r \) points, so Lemma 7.2.10 may be applied with \( t \) replaced by \( A_r \). The \( \varepsilon \)-approximate perfect \( L \)-spline obtained in this way is denoted by \( \mathcal{P}_r \).

Consequently, \( \mathcal{P}_r (A_r) = (s(A_r) \cap A_r) \) and \( |\mathcal{P}_r(t)| = 1 + \delta \) \((t \in [0,A_r])\). Letting again \( \delta \) and \( \varepsilon \) tend to zero we get a function \( \mathcal{P}_r \in \mathcal{P}_r(\mathcal{A}_r) \), with the properties

\[ |\mathcal{P}_r(t)| \leq 1 \quad (t \in [0,A_r]), \quad \mathcal{P}_r (A_r) = \{ (s(A_r) \cap A_r) \} \]

From the definition of \( A_r \) it follows that \( |\mathcal{P}_r(A_r)| = |s(A_r)\cap A_r| = |s(A_r)| = 1 \).
contradicting the assumption that \( z(\alpha_k) \neq z(\alpha_k) \). Hence, either \( g(\alpha_j) = g(\alpha_j) = 1 \) or \( g(\alpha_j) = g(\alpha_j) = -1 \). These equalities imply that either \( g \) or \( \bar{z} \) oscillates \( r+1 \) times between \(-1\) and \( 1 \) on \([0, \alpha_k]\). Consequently, \( P_{r+1}(p_n) \) contains at least one function \( s_k \) that oscillates \( r+1 \) times between \(-1\) and \( 1 \) on \([0, \alpha_k]\). At this juncture it is necessary to distinguish between the cases \( |a_0| < 1 \) and \( |a_0| = 1 \). If \( |a_0| < 1 \) then, evidently, the function \( s_k \) oscillates \( r+1 \) times on \([0, \alpha_k]\). If, however, \( |a_0| = 1 \) then we proceed as follows. Without loss of generality we assume that \( a_0 = 1 \). Since (7.4.3) holds one has \( (a_1, a_2, \ldots, a_{n-1}) \neq (0, 0, \ldots, 0) \), and if for some \( j \in \{1, 2, \ldots, n-1\} \) \( a_1 = a_2 = \ldots = a_{j-1} = 0 \) and \( a_j \neq 0 \) then \( a_j < 0 \). It follows from the reasoning above that for every \( \delta > 0 \) an \( s(\delta) \in P_{r+1}(p_n) \) exists oscillating \( r+1 \) times between \(-1-\delta\) and \( 1+\delta \) on an interval containing \([0, \alpha_k]\). Hence, in view of Lemma 7.2.8, there is a sequence \( (\varepsilon_k) \) of positive numbers with \( \lim_{k \to \infty} \varepsilon_k = 0 \) such that the corresponding sequence of perfect \( f \)-splines \( (s(\varepsilon_k)) \) converges to an \( s \in P_{r+1}(p_n) \), uniformly on \([0, \alpha_k]\). It may also be assumed that \( s(\varepsilon_k) \) has a set of \( r+2 \) oscillation points \( 0 < \tau_1, k < \tau_2, k < \ldots < \tau_{r+2}, k \in R_k \) such that \( \tau_{k+1} := \lim_{k \to \infty} \tau_k \text{ and } (\tau_{1+k}, \ldots, \tau_{r+2+k}) \in R_k \) exist and, moreover, that all sets of oscillation points are of the same type. If \( \tau_1 = 0 \) then, because of \( s(\varepsilon_k)(0) = a_0 = 1 \), the sets of oscillation points are of type II (cf. p. 162). Hence \( s(\varepsilon_k)(\tau_1, k) = \ldots \). Since \( s(\varepsilon_k)(0) = 1 \), \( s(\varepsilon_k)(0) = 0 \), \( (a_1, a_2, \ldots, a_{n-1}) \neq 0 \), it follows that \( \varepsilon_k(\varepsilon_k)(\tau_1, k) = \ldots \). As \( \delta_k > 0 \) we conclude that \( a_j > 0 \), contrary to our observation above and consequently \( \tau_1 > 0 \). It follows that also in the case \( |a_0| = 1 \) there exists an \( s \in P_{r+1}(p_n) \) oscillating \( r+1 \) times between \(-1\) and \( 1 \) on \([0, \alpha_k]\). To prove the second part of the theorem we note that every \( f \in P_{r+1}(0, \alpha_k) \) coincides with \( s_k \) in at least \( n+r+1 \) points, counting multiplicities. Applying Theorem 7.2.3 we then obtain (7.4.6). This completely proves the theorem.

As a simple illustration of Theorem 7.4.3 we consider two examples.
EXAMPLE 1. Let $r = 0$, $g = (1,0,-1)^T \in \mathbb{R}^3$ and $p_3(D) = D(D^2 - I)$.

A sketch of the extremal function $s_0 \in P_{0,B_0}^3(D)$ is given in Figure 1.

Taking into account (7.2.1) we have

$$s_0(t) = 2 - \cosh(t) + c \sinh(t) - t \quad (t \in \mathbb{R}),$$

where the constant $c$ is determined by the condition that $s_0$ has minimum value $-1$ for $t \geq 0$. It follows that $c = 1.044$ and $A_0 = 5.327$. Hence, if $f \in F_2(\mathcal{C}, A_0)$ then, according to (7.4.6),

$$\|f'' - f''[0,A_0]\| \geq \|s''_0 - s''_0\| = 1.044.$$

Figure 1 also contains a sketch of $s_{-1} \in P_{-1,B_0}^3(D)$, i.e., $s_{-1} \in \text{Ker}(p_3)$ satisfying the conditions $s_{-1}(0) = 1$, $s'_{-1}(0) = 0$ and $s''_{-1}(0) = -1$. Obviously, $s_{-1}(t) = 2 - \cosh(t) \quad (t \in \mathbb{R})$.

EXAMPLE 2. Let $r = 1$, $g = (0,1)^T \in \mathbb{R}^2$ and $p_2(D) = D^2$.

A sketch of the extremal function $s_0 \in P_{0,B_0}^3(D)$ is given in Figure 2.
The extremal functions \( s_{-1}, s_0, \) and \( s_1 \) (cf. Figure 2) are given by

\[
\begin{align*}
    s_{-1}(t) &= t, & s_0(t) &= t - \frac{1}{2} t^2, & s_1(t) &= t - \frac{1}{2} t^2 + \frac{1}{2} (t - \frac{1}{2})^2 \quad (t \in \mathbb{R}),
\end{align*}
\]

\( x_1 = 4 \) being the only knot of \( s_1 \). Obviously, \( \lambda_{-1} = 1, \lambda_0 = 2 + 2\sqrt{2} \) and \( \lambda_1 = 6 + 2\sqrt{2} \). The perfect spline \( s_1 \in \mathbb{P}_{1,2}(P_2) \) oscillates twice between \(-1\) and \(1\) on \([0, A_1]\) and therefore, by Theorem 7.4.1, \( \|s_1\| \) minimizes \( \|f\|_{[0, A_1]} \) on \( \mathbb{P}_{1,2}(0, A_1) \). Note that the function \( f_0 \) defined by

\[
f_0(t) := \begin{cases} 
    s_1(t) & (0 \leq t \leq \tau_1), \\
    1 & (t > \tau_1),
\end{cases}
\]

also minimizes \( \|f\|_{[0, A_1]} \) on \( \mathbb{P}_{1,2}(0, A_1) \), so \( s_1 \) is not uniquely determined by its extremal property. However, as a consequence of the following theorem, every \( f \in \mathbb{P}_{1,2}(0, A_1) \) satisfying (7.4.6) coincides with \( s_1 \) on \([0, \tau_1]\).

**THEOREM 7.4.4.** Let \( x \in \mathbb{N}_0 \), let \( \mathbf{a} = (a_0, a_1, ..., a_{n-1})^T \in \mathbb{R}^n \) (\( n \geq 2 \)), and let \( x : \mathbb{P}_{1,2}(0, n) \) satisfy the following two conditions:

1) There exist \( r + 1 \) points \( 0 < \tau_1 < \tau_2 < ... < \tau_{r+1} \) such that

\[
(7.4.7) \quad x(t) = (-1)^i \text{sgn}(P_n(t))x(0^+) \quad (i = 1, 2, ..., r+1);
\]

2) \( x \) has exactly \( r \) knots \( 0 < \kappa_1 < \kappa_2 < ... < \kappa_r \) satisfying the inequalities

\[
(7.4.8) \quad x_1 < \kappa_i < x_{i+1} \quad (i = 1, 2, ..., r+1),
\]

where \( x_1 := a \) in case \( j > n \); moreover, (7.4.8) is void if \( r = 0 \).

Then for all \( f : \mathbb{P}_{1,2}(0, \tau_{r+1}) \)

\[
(7.4.9) \quad \|x(f)\|_{[0, \tau_{r+1}]} \leq \|x(0)^{\tau_{r+1}}\|_{[0, \tau_{r+1}]};
\]

with equality if and only if \( f(t) = x(t) \) \( (t \in [0, \tau_{r+1}]) \).

**PROOF.** Using Taylor's formula and an auxiliary function \( H \), we first derive a relation valid for any function in \( \mathbb{A}_{1,2}(0, \tau_{r+1}) \). To this end, let \( \phi \) be the fundamental function corresponding to the differential operator \( P_n(0) \), and for \( t \in \mathbb{R} \) let \( H \) be defined by the determinant
\[(7.4.10) \quad H(t) := \begin{bmatrix}
\phi_1(t_1 - t) & \phi_2(t_2 - t) & \cdots & \phi_r(t_{r+1} - t) \\
\phi_1(t_1 - x_1) & \phi_2(t_2 - x_1) & \cdots & \phi_r(t_{r+1} - x_1) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1(t_1 - x_r) & \phi_2(t_2 - x_r) & \cdots & \phi_r(t_{r+1} - x_r)
\end{bmatrix}.
\]

Obviously, \( H \) can be written in the form

\[(7.4.11) \quad H(t) = \sum_{k=1}^{r+1} \delta_k \phi_k(t_k - t) \quad (t \in \mathbb{R}),
\]

where (cf. p. 4 for notation)

\[f_k = (-1)^{k-1}
\begin{bmatrix}
\phi_1(t_1 - t) & \cdots & \phi_1(t_{k-1} - t) & \phi_1(t_{k+1} - t) & \cdots & \phi_1(t_{r+1} - t) \\
x_1 & \cdots & x_{k-1} & x_k & \cdots & x_r
\end{bmatrix}.
\]

It follows from Lemma 1.4.19 and (7.4.8) that

\[(7.4.12) \quad \text{sgn}(f_k) = (-1)^{k-1} \quad (k = 1, 2, \ldots, r+1).
\]

Taking into account the definition of \( \phi_k \) (cf. p. 6) and (7.4.10) one easily observes that \( H^{(j)}(t_k) = 0 \) \((j = 0, 1, \ldots, n-2)\) and \( H(x_k) = 0 \) \((i = 1, 2, \ldots, r)\). We proceed by proving that \( H \) changes sign at the points \( x_k \). If \( t \in (x_k, x_{k+1}) \) \((k = 1, 2, \ldots, r-1)\) then

\[H(t) = (-1)^{k-1}
\begin{bmatrix}
\phi_1(t_1 - t) & \phi_1(t_2 - t) & \cdots & \phi_1(t_{r+1} - t) \\
x_1 & x_2 & \cdots & x_k & t & x_{k+1} & \cdots & x_r
\end{bmatrix}
\]

and Lemma 1.4.19 again yields

\[\text{sgn}(H(t)) = (-1)^k \quad (x_k < t < x_{k+1} ; k = 1, 2, \ldots, r-1).
\]

In a similar way it may be shown that

\[\text{sgn}(H(t)) =
\begin{cases}
1 & (0 < t < x_1), \\
(-1)^k & (x_k < t < x_{k+1}).
\end{cases}
\]

Now let \( f \in \mathcal{C}^{(n)}(0, \tau_{r+1}) \). Then by Taylor's formula (cf. Lemma 1.4.4) and (7.4.11)
\[
(7.4.13) \quad \int_0^{\tau + 1} H(n) p_n(D) f(n) dn = \int_0^{\tau + 1} \sum_{k=1}^{r+1} \theta_k (t_k - n) p_n(D) f(n) dn = \\
= \sum_{k=0}^{r+1} Y_k f(t_k) + \sum_{k=1}^{r+1} \theta_k f(t_k),
\]

where the coefficients \(\gamma_0, \gamma_1, \ldots, \gamma_{r+1}\) do not depend on \(f\).

Consequently, for every \(f \in \mathbb{F}_{[0, \tau + 1]}\),
\[
(7.4.14) \quad \sum_{k=0}^{r+1} \sum_{t=0}^{r+1} Y_k a_k \leq \int_0^{\tau + 1} |H(n)| |p_n(D)| f(n) f(n) [0, \tau + 1],
\]

where \(a_0, a_1, \ldots, a_{r+1}\) are the components of \(a\).

We now apply (7.4.13) to \(s \in \mathbb{F}_{[0, \tau + 1]}\). In view of the sign structure of \(H\) and \(\theta_k\) (cf. (7.4.12) and (7.4.17)), \(s\) satisfies the relation
\[
(7.4.15) \quad \sum_{k=0}^{r+1} \sum_{t=0}^{r+1} Y_k a_k = \int_0^{\tau + 1} |H(n)| |p_n(D)| f(n) f(n) [0, \tau + 1].
\]

Hence, combining (7.4.14) and (7.4.15), we conclude that
\[
|p_n(D)| f(n) f(n) [0, \tau + 1] = |p_n(D)| f(n) f(n) [0, \tau + 1],
\]

and, moreover, that \(|p_n(D)| f(n) f(n) [0, \tau + 1]\) if and only if \(f(t) = s(t)\) \((0 \leq t \leq \tau + 1)\). This proves the theorem.

We note that the existence of an \(s \in \mathbb{F}_{[0, \tau + 1]}\) satisfying conditions 1) and 2) of Theorem 7.4.4 is not warranted. With respect to this the following example is of interest.

**Example 3.** Let \(p_3(D) = p^3\) and let \(s_0\) be an element of \(\mathbb{F}_{[0, p_3]}\) (cf. Definition 7.2.1), the graph of which is given in Figure 3.
A perfect spline \( s_1 \in \mathcal{P}_1(p_3) \) is now obtained from \( s_0 \) by adding an appropriately situated knot \( x_1 \) such that \( s_1(t) = s_0(t) \) \((0 \leq t \leq x_1)\) and such that \( s_1 \) exhibits the oscillation behaviour as sketched in Figure 3. If we take \( \bar{a} = (s_0(0), s_0'(0), s_0''(0))^T \) it follows that \( s_1 \in \mathcal{P}_{1, \bar{a}}(p_3) \). In fact, we have (cf. Figure 3)
\[
s_1(t) = s_0(t) + \frac{1}{3} \| s_0'' \| (t-x_1)^3 \quad (t \in \mathbb{R}).
\]

Hence \( \| s_1'' \| = \| s_0'' \| \). Since \( x_1 > r_1 \), condition (7.4.8) does not hold and so Theorem 7.4.6, with \( \varepsilon \) replaced by \( s_1 \notin \mathcal{P}_{1, \bar{a}}(p_3) \), does not apply. However, in this specific case, if \( f \in \mathcal{P}_{1, \bar{a}}(C_0, A_1) \) satisfies \( \| f'' \| (0, A_1) = \| s_1'' \| = \| s_0'' \| \), then Theorem 7.4.6 may be applied with \( \varepsilon \) replaced by \( s_0 \in \mathcal{P}_{0, \bar{a}}(p_3) \), and it follows that \( f \) coincides with \( s_0 \) on \([0, r_1]\).

**Example 4.** Let \( x = 5, \bar{a} = (0, 0, 0, 6) \in \mathbb{R}^4 \) and \( p_4(\mathbb{R}) = \mathbb{R}^4 \). We note that \( \bar{a} \) and \( p_4 \) are chosen such that condition (7.4.3) is satisfied.
In order to obtain the perfect spline $s_6$ of Theorem 7.4.3, one has to solve a system of nonlinear equations of the form (cf. (7.2.1)))

\[
\begin{align*}
  s_5(t_1) &= t_1^3 - c_5 t_1^4 - 2c_5 \sum_{j=1}^{5} (-1)^j (t_1 - x_j)^4 = (-1)^{i+1} \\
  s_5(t_i) &= 3t_i^2 - 4c_5 t_i^3 - 8c_5 \sum_{j=1}^{5} (-1)^j (t_i - x_j)^3 = 0
\end{align*}
\]

in the twelve unknowns $t_1, t_2, \ldots, t_6, x_1, x_2, \ldots, x_5, c_5$, where

$c_5 = \|s_6^{(4)}\|/24$. This is done numerically by means of an iterative method. The numerical values of the knots $x_1, x_2, \ldots, x_5$ and the oscillation points $t_1, t_2, \ldots, t_6$ are given in Table 1; $c_5 = 0.47865$.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$t_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.15913</td>
<td>1.65864</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$t_2$</td>
</tr>
<tr>
<td>2.56369</td>
<td>2.32465</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$t_3$</td>
</tr>
<tr>
<td>4.13349</td>
<td>4.93522</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$t_4$</td>
</tr>
<tr>
<td>5.73972</td>
<td>6.54299</td>
</tr>
<tr>
<td>$x_5$</td>
<td>$t_5$</td>
</tr>
<tr>
<td>7.35339</td>
<td>8.14382</td>
</tr>
<tr>
<td>$t_6$</td>
<td>9.39218</td>
</tr>
</tbody>
</table>

Table 1

Note that the knots $x_1, x_2, \ldots, x_5$ and the oscillation points $t_1, t_2, \ldots, t_6$ satisfy the inequalities (7.4.8). Since $s_6^{(4)}(0) = -11.4876 < 0$, condition (7.4.7) is also satisfied and Theorem 7.4.4 may be applied. Consequently, if $\pi_0 \in F_5([0, t_6])$ minimizes $\|s_5^{(4)}\|_2([0, t_6])$ on $F_5([0, t_6])$ then $f_0(t) = s_5(t) \ (0 \leq t \leq t_6)$.

On the following pages we have shown that a perfect $\delta$-spline $s_6 \in F_5([P_n])$ oscillating $r + 1$ times between $-1$ and $1$ on $(0, \lambda_x)$ solves the problem of minimizing $\|P_n(D) f\|_2([0, \lambda_x])$ on $F_5([0, \lambda_x])$. Now we want to prove that a convergent subsequence of $(s_6)_{-1}$ solves the corresponding problem of minimizing $\|P_n(D) f\|_2$ on $F_5([\lambda_x])$. For this purpose we need the following property of the sequence $(\lambda_x)_{-1}$.

**Lemma 7.4.5.** Let $r + 1 \in \mathbb{N}$, and let $g \in \mathbb{R}^n \ (n \geq 2)$ and the monic polynomial $P_n \in \mathbb{R}[x]$ be such that (7.4.3) is satisfied. Then the sequence $(\lambda_x)_{-1}$ is nondecreasing and $\lim_{x \to -1} \lambda_x = \infty$. 
PROOF. It immediately follows from the definition of $A_r^r$ (cf. (7.4.5)) that $(h_{x-1}^{x-1})^r$ is nondecreasing. Hence $\lim_{x \to \infty} A_r = \alpha$ with $\alpha \leq \gamma$. Now let $(\{s_{x-1}^{x-1}\})$ be a sequence of perfect $L$-splines $s_{x-1}^{x-1} \in P_{\infty}(\mathcal{B})$ oscillating $r+1$ times between $-1$ and $1$ on $[0,A_r]$ and possessing property (7.4.6) (cf. Theorem 7.4.3). Since $s_{x-1}^{x-1} \in P_{\infty}(\mathcal{B})$, for all $x \geq 1$, the sequence $(\{s_{x-1}^{x-1}\})$ is nondecreasing. Moreover, it is bounded since $\|s_{x-1}^{x-1}\|_\infty \leq \|s_{x-1}^{x-1}\|_\infty$ for all $x \geq 1$. If $x = 1$, then the boundedness of $(\{s_{x-1}^{x-1}\})$ together with the inequality $|s_{x-1}^{x-1}(t)| \leq 1$ $(r \in \mathbb{N}_0, t \in [0,A_r])$ would yield that $(\{s_{x-1}^{x-1}\})$ is bounded. This, however, is incompatible with the number of oscillations of $s_{x-1}^{x-1}$ on $[0,1]$ tending to infinity as $x \to \infty$. Hence $\alpha = \gamma$. \]

**THEOREM 7.4.6.** Let $p \in \mathbb{R}^n (n \geq 1)$ and the monic polynomial $p_n \in \mathbb{P}_n$ be such that (7.4.3) is satisfied. Then there exists a perfect $L$-spline $s$, corresponding to the operator $p_n(D)$, with infinitely many knots and having the extremal property

$(7.4.16)$ \[ \|p_n(D)s\| = \min \{\|p_n(D)s\|_\infty \mid s \in P_{\infty}(\mathcal{B})\}. \]

**PROOF.** Let $s_{x-1}^{x-1} \in P_{\infty}(\mathcal{B})$ be a perfect $L$-spline oscillating $r+1$ times between $-1$ and $1$ on $[0,A_r]$ and satisfying (7.4.6). First we show that the total number of knots of $s_{x-1}^{x-1}$ in a bounded interval $[0,N]$ is bounded as $x \to \infty$. Let $N$ denote the number of knots of $s_{x-1}^{x-1}$ in $[0,N]$. Since by Lemma 7.4.5, $A_r \to \infty (r = \infty)$, the open intervals $(N,A_r)$ are nonempty for sufficiently large $r$. On $(N,A_r)$ the number of knots is bounded by $r-N$, and so by Lemma 7.2.4 $s_{x-1}^{x-1}$ cannot have more than $r-N$ zeros in $(N,A_r)$. Since $s_{x-1}^{x-1}$ oscillates $r+1$ times between $-1$ and $1$ on $(N,A_r)$, of which, say, $K_{x-1}^{x-1}$ times on $(N,N)$, one has $r-N \leq r-N + n$. Hence $N \leq n + K_{x-1}^{x-1}$. Now the sequence $(K_{x-1}^{x-1})$ is bounded, since otherwise the number of oscillations of $s_{x-1}^{x-1}$ on $(0,N)$ would tend to infinity as $x \to \infty$ and so $\|s_{x-1}^{x-1}\|_\infty = (r = \infty)$, which (cf. the proof of Lemma 7.4.5) cannot occur. Consequently, a number $q_N \leq \infty$ exists such that every $s_{x-1}^{x-1}$ coincides on $[0,N]$ with a function in $F_{\infty,q_N}(\mathcal{B})$. In view of the compactness property (cf. Lemma 7.2.0) there is a subsequence of $(s_{x-1}^{x-1})$ converging to an $s \in F_{\infty,q_N}(\mathcal{B})$, uniformly on $(0,N)$. By a standard diagonal procedure one obtains a subsequence $(s_{x_j}^{x_j})$ of $(s_{x-1}^{x-1})$ such that $s_{x_j}^{x_j} \to s (r = \infty)$, uniformly on every bounded interval of $(0,N)$. Moreover, $\|p_n(D)s_{x_j}^{x_j}\| = \|p_n(D)s\| (r = \infty)$, and $s$ is a perfect $L$-spline with infinitely many knots, but with finitely many in every bounded interval.
In order to prove (7.4.16), let \( f \in \mathcal{P}_n(\mathbb{M}_n^+ \setminus \mathbb{M}_0) \) be a function such that
\[
\|P_n(D)f\|_\omega < \|P_n(D)\|_\omega.
\]
Then for a sufficiently large \( j \) one would have
\[
\|P_n(D)\|_\omega < \|P_n(D)\|_{L^2}.
\]
However, this inequality contradicts the extremal property of \( \mathbb{M}_n^+ \) as given in (7.4.6). This proves the theorem. \( \square \)

7.5. A characterization of \( \mathcal{M}_n^+(p_n) \)

In the preceding section the problem of minimizing \( \|P_n(D)f\|_\omega \) on \( \mathcal{P}_n(\mathbb{M}_n^+ \setminus \mathbb{M}_0) \) is considered. Here we explain its relation to the set \( \mathcal{M}_n^+(p_n) \). To illustrate this, two examples will be given. We recall that \( p_n \in \mathcal{M}_n^+ \) has only real zeros and that \( p_n(0) = 0 \).

Let the function \( M \) be defined by (cf. (7.4.16))

\[
(7.5.1) \quad M(a) := \min \{ \|P_n(D)f\|_\omega \mid f \in \mathcal{P}_n(\mathbb{M}_n^+ \setminus \mathbb{M}_0) \} \quad (a \in \mathbb{R}^n);
\]

In case \( \mathcal{P}_n(\mathbb{M}_n^+ \setminus \mathbb{M}_0) = \emptyset \) we put \( M(a) := +\infty \).

Obviously, the function \( M \) is convex and therefore (cf. Roberts and Varberg [49, p. 93]) continuous on the open set

\[
(7.5.2) \quad \mathcal{V} := \{ a = (a_0, a_1, \ldots, a_{n-1})^T \in \mathbb{R}^n \mid |a_0| < 1 \}.
\]

In view of the definition of \( \mathcal{M}_n^+(p_n) \) (cf. (5.3.1) and (5.3.3)) a point \( a \in \mathbb{R}^n \) belongs to \( \mathcal{M}_n^+(p_n) \) if a function \( f \in \mathcal{P}_n(\mathbb{M}_n^+ \setminus \mathbb{M}_0) \) exists, i.e., a function \( f \in AC^n(\mathbb{M}_n^+ \setminus \mathbb{M}_0) \) for which \( |f|_{\omega} < n \) and \( |P_n(D)f|_{\omega} \leq 1 \), such that

\[
(7.5.3) \quad f_n^j(0) = a_j \quad (j = 0, 1, \ldots, n-1).
\]

If \( f \in \mathcal{P}_n(\mathbb{M}_n^+ \setminus \mathbb{M}_0) \) then, evidently,

\[
f / m \in \mathcal{P}_n(\mathbb{M}_n^+ \setminus \mathbb{M}_0)
\]

and it follows that \( M(f/m) \leq 1/m \). Hence, \( a \in \mathcal{M}_n^+(p_n) \) if and only if \( M(f/m) < 1/m \). The following lemma characterizes the boundary points \( a \) of \( \mathcal{M}_n^+(p_n) \) in terms of \( M(a/m) \).

**Lemma 7.5.1.** Let \( m > 0 \) and let \( a = (a_0, a_1, \ldots, a_{n-1})^T \in \mathbb{R}^n \) with \( |a_0| < 1 \). Then \( a \in \mathcal{M}_n^+(p_n) \) if and only if \( M(a/m) = 1/m \).

**Proof.** If \( a \in \mathcal{M}_n^+(p_n) \) then \( M(a/m) \leq 1/m \). Now let \( a \in \mathcal{M}_n^+(p_n) \) and assume that \( M(a/m) < 1/m \). Then by the continuity of \( M \) there is some \( \lambda > 1 \) such that \( |\lambda a_0| < 1 \) and \( M(\lambda a/m) < 1/m \). This implies that \( \lambda a \in \mathcal{M}_n^+(p_n) \), contradictory to the assumption that \( a \in \mathcal{M}_n^+(p_n) \). Consequently, \( M(a/m) = 1/m \).
Now let \( M(a/m) = 1/m \). Then, obviously, \( a \in \Gamma_m^+(\mathbb{R}) \). If \( a \notin \Gamma_m^+(\mathbb{R}) \) then some \( \lambda > 1 \) exists such that \( |\lambda a_0| < 1 \) and \( \lambda a \in \Gamma_m^+(\mathbb{R}) \). Hence \( M(a/m) \leq \lambda (\lambda a)^{-1} < m^{-1} \), yielding a contradiction.

Lemma 7.5.1 together with Theorem 7.4.6 gives

**Theorem 7.5.1.** Let \( n > 0 \), set \( \mathbf{a} = (a_0, a_1, \ldots, a_{n-1})^T \in \mathbb{R}^n \) with \( |a_0| < a_n \) and let \( p_n \in \mathbb{R}[x] \) be a mono polynomial having only real zeros with \( p_n(0) = 0 \). Then \( a \in \Gamma_m^+(\mathbb{P}_n) \) if and only if a perfect \( f \)-spline \( f \in \mathbb{E}_m(p_n, \mathbb{R}) \) exists with \( f^{(i)}(0) = a_i \) \( i = 0, 1, \ldots, n-1 \) and such that

\[
(7.5.3) \quad \|f_n(a)\|_{\infty} = \min \{ \|f_n(2)\|_{\infty} \mid \|f_n\|_{\infty} \leq a, f^{(i)}(0) = a_i, i = 0, 1, \ldots, n-1 \} = 1.
\]

**Remark.** In case \( |a_0| = a_n \) it is possible that the minimum in (7.5.3) is smaller than one; it may even be zero as is illustrated by the simple example: \( (1,0)^T \in \Gamma_m^+(\mathbb{R}^2) \) and \( M((1,0)^T) = 0 \).

Theorem 7.5.2 and Lemma 7.5.1 enable us to describe the boundary of \( \Gamma_m^+(\mathbb{P}^3) \) explicitly by determining various extremal functions (cf. Example 1 below). To do the same for \( \Gamma_m^+(\mathbb{P}^3) \) is much more difficult, as is evident from Example 2. For the construction of even one single point of \( \Gamma_m^+(\mathbb{P}^3) \) rather elaborate numerical computations are needed.

**Example 1.** Before describing the set \( \Gamma_m^+(\mathbb{P}^3) \) a preliminary remark is in order. If \( f \in \mathbb{E}_m(\mathbb{P}^3, \mathbb{R}) \) then \( t \mapsto m^{-1} f(m^{1/3} t) \) belongs to \( \mathbb{E}((\mathbb{P}^3, \mathbb{R}) \). Consequently, if \( a = (a_0, a_1, a_2)^T \in \Gamma_m^+(\mathbb{P}^3) \) then \( m^{-1} a_0, m^{-2/3} a_1, m^{-1/3} a_2 \in \Gamma_m^+(\mathbb{P}^3) \). Hence it suffices to describe \( \Gamma_m^+(\mathbb{P}^3) \).

On the interval \((0, h)\) the perfect Bézier spline \( E(\mathbb{P}^3, h, \cdot) \) is given by (cf. p. 131)

\[
(7.5.4) \quad E(\mathbb{P}^3, h, t) = -\frac{1}{6} t^3 + \frac{1}{2} h t^2 - \frac{1}{24} h^3.
\]

We recall (cf. (5.5.1)) that \( E(\mathbb{P}^3, h, t+h) = E(\mathbb{P}^3, h, t) \) \( (t \in \mathbb{R}) \) and that \( \|E(\mathbb{P}^3, h, \cdot)\| = h^2/24 \) (cf. p. 131). In what follows \( h \) is taken \( \sqrt{3} \) and accordingly \( \|E(\mathbb{P}^3, h, \cdot)\| = 1 \).
Now for any \( \gamma \in [-h,0] \) and \( r+1 \in \mathbb{N} \), a perfect \( r \)-spline \( s_{r+1,n} \) is constructed as follows. Let \( s_\gamma \in \mathcal{P}_r(D^3) \) be the perfect \( r \)-spline coinciding with \(-E(D^3,h,\gamma)\) on \([0,\gamma]\) and having the knots \( h,2h,\ldots,\gamma h \) (cf. Figure 1, where \( r = 3 \)).

![Figure 1](image)

Associated with \( s_\gamma \) we consider the perfect \( r \)-spline \( s_{r+1,n} \) with knots \( n,2n,\ldots,\gamma n \) and coinciding with \( s_\gamma \) for \( \tau \leq n \) (cf. Figure 1). Letting \( \gamma \in [-h,0] \) be such that \( s_{r+1,n}(\gamma) = -1 \), we define (cf. Figure 2) \( s_{r+1,n}(t) \) by the translation \( s_{r+1,n}(t) = s_{r+1}(t+\gamma) \) (\( t \in \mathbb{R} \)).

![Figure 2](image)

Taking into account Figure 2 it is evident that in general \( s_{r+1,n} \in \mathcal{P}_r(D^3) \), and \( s_{r+1,n} \) oscillates \( r+2 \) times between \(-1\) and \( 1 \) on \( (0,\gamma n] \). By Theorem 7.4.3 it follows that if \( \|s_{r+1,n}\|_\infty = 1 \) for any \( f \in \mathcal{F}_r((0,\gamma n]) \), where

\[
\mathcal{F}_r = \{s_{r+1,n}(0),s_{r+1,n}(h),s_{r+1,n}(2h),\ldots,s_{r+1,n}(\gamma n)\}^T = \{(s_{r+1,n}(0),s_{r+1,n}(h),s_{r+1,n}(2h),\ldots,s_{r+1,n}(\gamma n))\}^T.
\]

Letting \( n \) tend to infinity and using Theorem 7.5.2 we conclude that \( (s_{r+1,n}(0),s_{r+1,n}(h),s_{r+1,n}(2h),\ldots,s_{r+1,n}(\gamma n)) \in \mathcal{F}_r(D^3) \). Using similar arguments we also infer that

\[
(7.5.5) \quad (s_{r+1,n}(\tau),s_{r+1,n}(\tau+h),s_{r+1,n}(\tau+2h),\ldots,s_{r+1,n}(\tau+\gamma n)) \in \mathcal{F}_r(D^3) \quad (\tau \in [0,\gamma n-\gamma h]).
\]

Other boundary points of \( \Gamma_1^+(D^3) \) can be found by using the following
construction. Let \( s^* \) be the perfect spline with knots \( h, 2h, \ldots \) and coinciding with \( -1 = h^3/24 \cdot E(D^3, r, t) \) for \( t \geq 0 \) (cf. Figure 3); here \( h \in (0, 2V_3^2) \).

![Figure 3.](image)

In view of (7.5.4) \( s^* \) coincides on \([-h, h]\) with the polynomial \( q_3 \) given by

\[
q_3(t) = \frac{1}{6} t^3 - \frac{1}{4} h t^2 + \frac{1}{12} h^3 - 1 \quad (t \in \mathbb{R}) .
\]

One easily verifies that \( q_3 \) oscillates once between \(-1\) and 1 on any interval of the form \([r, \alpha]\), with \( \alpha \in [-h, h] \) and \( \alpha \) being determined by \( q_3(\alpha) = 1 \) (cf. Figure 3). Since \( q_3 \) can be interpreted as a perfect spline having zero knots, it follows from Theorem 7.4.4 that \( \|q_3''\|_{C([0, \alpha-\epsilon])} \geq 1 \) for all \( f \in P_2([0, \alpha-\epsilon]) \) with \( \alpha = (q_3(r), q_3'(r), q_3''(r))^T \). Hence \( \|f''\|_{C([0, \alpha])} \geq 1 \) for all \( f \in P_2([0, \alpha]) \). Since \( s^* \in P_2([0, \alpha]) \), Lemma 7.5.1 applies and so

\[
(7.5.6) \quad (q_3(t), q_3'(t), q_3''(t))^T \in \partial P_1(D^3) \quad (-h \leq t \leq h) .
\]

A sketch of \( \Gamma_1^*(D^3) \) in x-y-z space can now be given; this is done in Figure 4. Because of the symmetry of \( \Gamma_1^*(D^3) \) it suffices to consider the half-space \( y \geq 0 \) only.

The curves 1 and 2 in Figure 4 consist of points as given by (7.5.6) for two different values of \( h \in (0, 2V_3^2) \). Similarly, the curves 3 and 4 consist of points as given by (7.5.5) for two different values of \( n \in [-V_5, 0] \).
EXAMPLE 2. Let \( p_4(D) = D^4 \) and let \( q = (0,0,0,a_y)^T \in \mathbb{R}^4 \). The problem we pose is to determine \( a_y \) such that \( q \in \mathbb{R}^4 \). As pointed out, we may assume without loss of generality that \( m = 1 \). According to Lemma 7.5.1 the point \((0,0,0,a_y)^T \in \mathbb{R}^4 \) if and only if \( M(0,0,0,a_y) = 1 \). If \( f \in C((0,0,0,a_y)(\mathbb{R}^4) \) then \( t \mapsto f((6/a_y)^{4/3} t) \in C((0,0,0,6)(\mathbb{R}^4) \) and it follows that the equality \( M(0,0,0,a_y) = 1 \) is equivalent to \( M(0,0,0,6) = (6/a_y)^{4/3} \). So, in order to determine \( a_y \), it suffices to compute \( M(0,0,0,6) \). Similar to the proof of Theorem 7.4.6 this will be done by determining a sequence of perfect splines \( \{ s_n \}_{n=1}^\infty \) with \( s_n \in P_{2n}((0,6), a = (0,0,0,6)^T) \), and such that \( s_n \) oscillates \( r \) times between -1 and 1 on \((0,\tau)\). Hence, by setting (cf. Example 4, p. 173)

\[
\begin{align*}
    s_n(t) &= t^3 - c x^4 - 2c \sum_{j=1}^{r} \frac{(-1)^j}{j!} (t - x_j)^j + \left( t \in \mathbb{R}, \quad \tau < t < \tau_{r+1} \right),
\end{align*}
\]

we aim to compute oscillation points \( 0 < \tau_1 < \tau_2 < \ldots < \tau_{r+1} \tau \) such that
\[(7.5.8) \quad \eta_{x}(t_{i+1}) = (-1)^{i+1}, \quad \eta_{x}(t_{i}) = 0 \quad (i = 1, 2, \ldots, r+1).\]

In view of (7.5.6) the knots \(x_{1,x}, x_{2,x}, \ldots, x_{r+1,x}\) and the oscilla-
tion points \(x_{1',x}, x_{2',x}, \ldots, x_{r+1',x}\) and the constant \(c_{x} > 0\) are determined by a system of \(2r+2\) nonlinear equations. For \(r = 5, 6, \ldots, 19\) numerical values for \(x_{1,x}, x_{2,x}, \ldots, x_{r,x}, x_{1',x}, x_{2',x}, \ldots, x_{r+1',x}\) are obtained by means of an iterative method. Being interested in \(M(0,0,0,6)\) we only give here the numerical values of \(\|\eta_{x}^{(4)}\| (r = 5, 6, \ldots, 19)\).

\[
\begin{array}{cccccc}
 r & \|\eta_{x}^{(4)}\| & r & \|\eta_{x}^{(4)}\| & r & \|\eta_{x}^{(4)}\| \\
 5 & 11,48760 & 10 & 11,48773 & 15 & 11,48795 \\
 6 & 11,48764 & 11 & 11,48773 & 16 & 11,48787 \\
 7 & 11,48764 & 12 & 11,48774 & 17 & 11,48789 \\
 8 & 11,48769 & 13 & 11,48780 & 18 & 11,48789 \\
 9 & 11,48771 & 14 & 11,48783 & 19 & 11,48792 \\
\end{array}
\]

We observe that the sequence \(\{\|\eta_{x}^{(4)}\|\}_{r=5}^{19}\) is monotonically nondecreasing as it should be, since \(s_{x}\) minimizes \(\|f^{(4)}\|_{0,A_{x}}\) on \(F_{x}(0,A_{x})\), \(s_{x+1}\) minimizes \(\|f^{(4)}\|_{0,A_{x+1}}\) on \(F_{x}(0,A_{x+1})\) and (cf. (7.4.4) and (7.4.5)) \([0,A_{x}] \subset [0,A_{x+1}]\).

Taking into account the data of Table 1, one may reasonably expect that \(M(0,0,0,6) = 11.49\) rounded to two digits. Hence \(\delta_{0} = 0.9614\).

**Remark.** We note that an extremal function \(f \in F_{1}(D^{4}, R_{0})\) for which \((f(0), f'(0), f''(0), f'''(0)) = (0, 0, 0, 0, 0, 14)\) can easily be extended to a function \(g \in F_{1}(D^{4}, R)\) by means of

\[
g(t) = \begin{cases} 
   f(t) & (t \geq 0), \\
   -f(-t) & (t < 0).
\end{cases}
\]

Consequently, \(g = (0,0,0,0,0,14) \in \Phi_{1}(D^{4})\) and \(g\) is an extremal function for a related Landau problem on the full line.
REFERENCES

1. Ahlberg, J.H.; Nilson, E.N.; Walsh, J.L.
   The theory of splines and their applications.

2. Aitken, A.C.
   Determinants and matrices.

3. Boor, C. de
   On calculating with B-splines.

4. Boor, C. de
   A remark concerning perfect splines.

5. Boor, C. de
   A practical guide to splines.
   Applied Mathematical sciences 27.

6. Boor, C. de; Schoenberg, I.J.
   Cardinal interpolation and spline functions VIII.
   The Budan-Fourier theorem for splines and applications.
   in: Spline functions Karlsruhe 1975; ed. by K. Döhrner, G. Meinardus
   and W. Schenck, pp. 1-79.
   Lecture notes in mathematics 501.

7. Brockot, R.W.
   Finite dimensional linear systems.

8. Brujin, N.G. de
   Asymptotic methods in analysis.
9. Cavaretta, A.S.
   An elementary proof of Kolmogorov's theorem.

10. Cavaretta, A.S.
    Oscillatory and zero properties for perfect splines and monosplines.

11. Cavaretta, A.S.
    A refinement of Kolmogorov's inequality.
    J. Approximation Theory 27 (1979), 45-60.

12. Cheney, E.W.
    Introduction to approximation theory.

13. Chui, C.K.; Smith, P.W.
    A note on Landau's problem for bounded intervals.

14. Coppel, W.A.
    Disconjugacy.
    Lecture notes in mathematics 220.

15. Cox, M.G.
    The numerical evaluation of B-splines.

16. Davis, P.J.
    Interpolation and approximation.

17. Feller, W.
    An introduction to probability theory and its applications, Vol. II.

18. Pyke, D.J.
    Linear dependence relations connecting equal interval Nth degree splines and their derivatives.
19. Golomb, M.
Some extremal problems for differentiable periodic functions in $L_2(\mathbb{R})$.
WRC Technical Summary Report 1069.
Univ. of Wisconsin, Madison, 1970.

20. Jakimovski, A.; Russell, D.C.
On an interpolation problem.
in: General inequalities 2; ed. by E.F. Beckenbach, pp. 205-232.
International series of numerical mathematics 47.

21. Karlin, S.
Total positivity, Vol. I.

22. Karlin, S.
Interpolation properties of generalized perfect splines and the solution of certain extremal problems.

23. Karlin, S.
Oscillatory perfect splines and related extremal problems.

24. Karlin, S.; Michelli, C.A.
The fundamental theorem of algebra for monocplines satisfying boundary conditions.
Israel J. Math. 11 (1972), 405-461.

25. Karlin, S.; Studden, W. J.
Tchebycheff systems: with applications in analysis and statistics.

26. Koch, P.E.; Lyon, T.
Bounds for the error in trigonometric interpolation.
27. Kolmogorov, A.N.
On inequalities between the upper bounds of the successive derivatives of an arbitrary function on a finite interval.

28. Kroo, E.
Introductory functional analysis with applications.

29. Kuga, K.
Brouwer's fixed-point theorem: an alternative proof.

30. Landau, E.
Einige Ungleichungen für zweimal differentierbare Funktionen.

31. Luenberger, D.G.
Optimization by vector space methods.

32. Lyche, T.
A note on the condition numbers of the B-spline bases.

33. Marsden, K.
An identity for spline functions and its application to variation diminishing spline approximations.

34. Meinardus, G.
Bemerkungen zur Theorie der B-splines.

35. Neir, A.: Sharma, A.
Convergence of a class of interpolatory splines.
J. Approximation Theory 1 (1968), 243-250.
36. Meilman, A.A.
   The Sudan-Fourier theorem for splines.

37. Michelli, C.A.
   Cardinal $Z$-splines.
   in: Studies in spline functions and approximation theory; ed. by
   S. Karlin et al., pp. 203-250.

38. Michelli, C.A.
   on an optimal method for the numerical differentiation of smooth
   functions.
   J. Approximation theory 18 (1976), 189-204.

   A survey of optimal recovery.
   in: Optimal estimation in approximation theory; ed. by
   C.A. Micelli and T.J. Rivlin, pp. 1-54.

40. Morsche, H.G. ter
   On the existence and convergence of interpolating periodic spline
   functions of arbitrary degree.
   Meinardus und W. Schumpp, pp. 197-214.

41. Morsche, H.G. ter
   On the relations between finite differences and derivatives of
   cardinal spline functions.
   in: Spline functions Karlsruhe 1975; ed. by K. Böhm, G. Meinardus
   and W. Schumpp, pp. 210-219.
   Lecture notes in mathematics 501.

42. Morsche, H.G. ter
   The maximum suprenum norm of a $B$-spline.
   TH-Report 78-Mek 05.
43. Morsche, H.G. ter
   An extremum problem of the Landau type concerning the differential
   operators $D^2 + Y$ on the half-line.
   Memorandum 79-06.

44. Morsche, H.G. ter
   The Landau problem for the differential operator $D^3$
   Memorandum 79-07.

45. Nörlund, N.E.
   Vorlesungen über Differenzenrechnung.
   Springer-Verlag, Berlin, 1924.

46. Northcott, D.G.
   Some inequalities between periodic functions and their derivatives.

47. Pinkus, A.
   Applications of representation theorems to problems of Chebyshev
   approximation with constraints.
   in: Studies in spline functions and approximation theory; ed. by
   S. Karlin et al., pp. 83-111.

48. Polya, G.
   On the mean-value theorem corresponding to a given linear homo-
   geneous differential equation.

49. Roberts, A.; Varberg, D.
   Convex functions.

50. Schmidt, E.; Lancaster, P.; Watkins, D.
   Bases of splines associated with constant coefficient differential
   operators.
51. Schoenberg, I.J.
   Cardinal spline interpolation.
   Regional conference series in applied mathematics 12.

52. Schoenberg, I.J.
   The elementary cases of Landau’s problem of inequalities between derivatives.

53. Schoenberg, I.J.
   Notes on spline functions VI. Extremum problems of the Landau-type for the differential operators $B^k$.

54. Schoenberg, I.J.: Cavaretta, A.S.
   Solution of Landau’s problem concerning higher derivatives on the halfline.
   NMR Technical Summary Report 1052.
   Univ. of Wisconsin, Madison, 1970.

55. Schönhage, A.
   Approximationstheorie.

56. Schumaker, L.L.
   Spline functions: basic theory.

57. Schurer, F.
   A note on interpolating periodic quintic splines with equally spaced nodes.

58. Sharma, A.: Trigub, L.
   Landau-type inequalities for some linear differential operators.
59. Stečkin, S.B.
   Best approximation of linear operators.

60. Stečkin, S.B.
   On inequalities between the upper bounds of derivatives of a
   function on the halfline.

61. Subbotin, Yu.N.
   On the relations between finite differences and the corresponding
   derivatives.
   Trudy Mat. Inst. Steklov 76 (1965), 24-42.
   Extremal properties of polynomials; ed. by

62. Trokh, I.
   On the interval of disconjugacy of linear autonomous differential
   equations.
LIST OF SYMBOLS
symbols are defined on the pages listed.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_G^{(n)}(S)$</td>
<td>3</td>
</tr>
<tr>
<td>$A_T$</td>
<td>169</td>
</tr>
<tr>
<td>$B_{n,m}$</td>
<td>29</td>
</tr>
<tr>
<td>$C_{n,h}$</td>
<td>29</td>
</tr>
<tr>
<td>$C(3)$</td>
<td>3</td>
</tr>
<tr>
<td>$C(n)(3)$</td>
<td>3</td>
</tr>
<tr>
<td>$C(a_0,a_1,\ldots,a_{n-1})$</td>
<td>83</td>
</tr>
<tr>
<td>$D$</td>
<td>5</td>
</tr>
<tr>
<td>$D^x$</td>
<td>5</td>
</tr>
<tr>
<td>$D_1^{(1)}$</td>
<td>8</td>
</tr>
<tr>
<td>$D_1^{(l)}$</td>
<td>13</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>44</td>
</tr>
<tr>
<td>$E_n$</td>
<td>105</td>
</tr>
<tr>
<td>$E_0$</td>
<td>105</td>
</tr>
<tr>
<td>$E_0$</td>
<td>58</td>
</tr>
<tr>
<td>$E_0(p_n; h, *)$</td>
<td>66</td>
</tr>
<tr>
<td>$E_0(p_1; m, *)$</td>
<td>116</td>
</tr>
<tr>
<td>$F_0(p_2; h, *)$</td>
<td>120</td>
</tr>
<tr>
<td>$f(t_0, t_1, \ldots, t_{n-1})$</td>
<td>4</td>
</tr>
<tr>
<td>$f(x_0, x_1, \ldots, x_n)$</td>
<td>25</td>
</tr>
<tr>
<td>$f([p_n; x_0, x_1, \ldots, x_n])$</td>
<td>25</td>
</tr>
<tr>
<td>$F_{m}(p_n; a)$</td>
<td>103</td>
</tr>
<tr>
<td>$P(p_n; T)$</td>
<td>132</td>
</tr>
<tr>
<td>$P(p_n; T)$</td>
<td>136</td>
</tr>
<tr>
<td>$P_{m}(p_n; T)$</td>
<td>167</td>
</tr>
<tr>
<td>$P_{n}(p_n; T)$</td>
<td>110</td>
</tr>
<tr>
<td>$H_{n+1}$</td>
<td>30</td>
</tr>
<tr>
<td>$K_{n}$</td>
<td>46</td>
</tr>
<tr>
<td>$K_{x, y}$</td>
<td>97</td>
</tr>
<tr>
<td>$K_{x, a}$</td>
<td>95</td>
</tr>
<tr>
<td>$K_{x, a}$</td>
<td>96</td>
</tr>
<tr>
<td>$K_{x, a}$</td>
<td>90</td>
</tr>
<tr>
<td>$Ker(p_n)$</td>
<td>5</td>
</tr>
<tr>
<td>$Ker(p_n; D)$</td>
<td>5</td>
</tr>
<tr>
<td>$Ker(p_n; T)$</td>
<td>133</td>
</tr>
<tr>
<td>$K_{x_1, x_2, \ldots, x_{n+1}}$</td>
<td>45</td>
</tr>
<tr>
<td>$L(f, k)$</td>
<td>47</td>
</tr>
<tr>
<td>$L_0(m)$</td>
<td>3</td>
</tr>
<tr>
<td>$L_0(W, T)$</td>
<td>132</td>
</tr>
<tr>
<td>$M_{x_0}(f, a)$</td>
<td>12</td>
</tr>
<tr>
<td>$M((D_1)^{n}; f, a)$</td>
<td>13</td>
</tr>
<tr>
<td>$M(p_n; x_0, x_1, \ldots, x_n; *)$</td>
<td>20</td>
</tr>
<tr>
<td>$M_{x_1}(p_n; *)$</td>
<td>20</td>
</tr>
<tr>
<td>Symbol</td>
<td>Value</td>
</tr>
<tr>
<td>--------</td>
<td>-------</td>
</tr>
<tr>
<td>S^* {a_1, a_2, \ldots, a_n}</td>
<td>11</td>
</tr>
<tr>
<td>S^{-} (f, (a,b))</td>
<td>11</td>
</tr>
<tr>
<td>\gamma_{\mathbb{C}}</td>
<td>155</td>
</tr>
<tr>
<td>\omega(n, J)</td>
<td>135</td>
</tr>
<tr>
<td>\omega^*</td>
<td>3</td>
</tr>
<tr>
<td>\omega^\prime</td>
<td>14</td>
</tr>
<tr>
<td>\phi \omega</td>
<td>6</td>
</tr>
<tr>
<td>\psi^*</td>
<td>6</td>
</tr>
<tr>
<td>\psi_0 \psi_1 \cdots \psi_{m-1}</td>
<td>4</td>
</tr>
<tr>
<td>\psi_0 \psi_1 \cdots \psi_{m-1}</td>
<td>4</td>
</tr>
<tr>
<td>\Gamma^* (p_n, J, \xi)</td>
<td>104</td>
</tr>
<tr>
<td>\Gamma_\nu (p_n^*)</td>
<td>110</td>
</tr>
<tr>
<td>\Gamma_\nu^* (p_n)</td>
<td>110</td>
</tr>
<tr>
<td>\Gamma (p_n, J)</td>
<td>133</td>
</tr>
<tr>
<td>\Gamma (p_n, \nu)</td>
<td>136</td>
</tr>
<tr>
<td>\phi_n, \omega</td>
<td>38</td>
</tr>
<tr>
<td>\phi_{n+1, (n)}</td>
<td>43</td>
</tr>
<tr>
<td>\psi_1 (p_n, h, \nu)</td>
<td>59</td>
</tr>
<tr>
<td>\psi_\lambda</td>
<td>59</td>
</tr>
<tr>
<td>\psi_{n+1} (p_n, h, \nu)</td>
<td>67</td>
</tr>
<tr>
<td>\psi_{n+1} (p_n, h, \nu)</td>
<td>59</td>
</tr>
<tr>
<td>\psi_\lambda</td>
<td>59</td>
</tr>
</tbody>
</table>
**SUBJECT INDEX**

Numbers indicate the pages of the text on which the subjects are defined, treated, etc.

**B-splines**
- basic property of 31–32
- definition of 28
- Fourier transform of 38
- hyperbolic 29
- minimum supremum norm of polynomial 44–54
- normalized polynomial 45
- polynomial 29
- recurrence relations for 38–44
- total positivity property of 32
- trigonometric 29
- variation diminishing property of 33
- with equidistant knots 43

**Budan-Fourier theorem**
- classical 11
- for f-splines 24
- generalized 17

**Chebyshev system**
- definition of 19
- extended 19
- extended weak 19
- weak 19

**circulant matrix** 83

**coincident block** 4

**condition number of a normalized B-spline basis** 45

**differentiation formula** 74

**disconjugacy of a differential operator** 0
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>divided difference</td>
<td>25</td>
</tr>
<tr>
<td>error estimates</td>
<td>91 - 93</td>
</tr>
<tr>
<td>for cardinal $L$-spline interpolation</td>
<td>94, 99, 101</td>
</tr>
<tr>
<td>for periodic cardinal $L$-spline interpolation</td>
<td>146</td>
</tr>
<tr>
<td>extremal properties</td>
<td>167</td>
</tr>
<tr>
<td>of perfect $L$-splines</td>
<td>149</td>
</tr>
<tr>
<td>of T-periodic perfect Euler $L$-splines</td>
<td>146</td>
</tr>
<tr>
<td>half-line case</td>
<td>104</td>
</tr>
<tr>
<td>inequalities</td>
<td>102, 103</td>
</tr>
<tr>
<td>of Landau</td>
<td>102</td>
</tr>
<tr>
<td>of Kolmogorov</td>
<td>103</td>
</tr>
<tr>
<td>interpolation</td>
<td>75</td>
</tr>
<tr>
<td>cardinal $L$-spline</td>
<td>91 - 93</td>
</tr>
<tr>
<td>error estimates for cardinal $L$-spline</td>
<td>94, 99, 101</td>
</tr>
<tr>
<td>error estimates for periodic cardinal $L$-spline</td>
<td>78</td>
</tr>
<tr>
<td>fundamental solution of interpolation problem</td>
<td>4</td>
</tr>
<tr>
<td>Hermite</td>
<td>75</td>
</tr>
<tr>
<td>periodic $L$-spline</td>
<td>154</td>
</tr>
<tr>
<td>property of perfect $L$-splines</td>
<td>155</td>
</tr>
<tr>
<td>property of &quot;$c$-approximate&quot; $L$-splines</td>
<td>105</td>
</tr>
<tr>
<td>intrinsic error</td>
<td></td>
</tr>
<tr>
<td>knots</td>
<td></td>
</tr>
<tr>
<td>definition</td>
<td>5</td>
</tr>
<tr>
<td>multiple</td>
<td></td>
</tr>
<tr>
<td>multiplicity of</td>
<td>5</td>
</tr>
<tr>
<td>simple</td>
<td></td>
</tr>
</tbody>
</table>
$L$-splines

Budan-Fourier theorem for

- cardinal 7
- definition of 5
- degree of 5
- Euler 58
- exponential 57
- normalized exponential 59
- number of zeros of 23, 24
- order of 5
- periodic cardinal 75
- relation between finite differences and derivatives of cardinal 72

Landau problem

- for periodic functions
  - for second order differential operators (full-line case) 116-120
  - for second order differential operators (half-line case) 120-125
  - for third order differential operators (full-line case) 127-131
  - for third order differential operators (half-line case) 179-181

- mesh distance 7
- nodes 55

- oscillation points
  - definition of 162
  - of type I 162
  - of type II 162

- Peano's remainder formula 10

- perfect $L$-splines
  - compactness properties of 158
  - definition of 7
  - extremal properties of 167
  - interpolation property of 154
  - one-sided perfect Euler 120
  - perfect Euler 66
  - representation theorem for 162, 166
  - satisfying initial conditions 153
  - with knots in $k^+$ 153
  - "$L$-approximate" 155
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>polynomial splines</td>
<td></td>
</tr>
<tr>
<td>B-splines</td>
<td>27</td>
</tr>
<tr>
<td>cubic</td>
<td>55</td>
</tr>
<tr>
<td>definition of quintic</td>
<td>5</td>
</tr>
<tr>
<td>polynomials</td>
<td></td>
</tr>
<tr>
<td>Bernoulli</td>
<td>142</td>
</tr>
<tr>
<td>Euler</td>
<td>90</td>
</tr>
<tr>
<td>Euler-Frobenius</td>
<td>62</td>
</tr>
<tr>
<td>exponential</td>
<td>58</td>
</tr>
<tr>
<td>exponential f- monic</td>
<td>98</td>
</tr>
<tr>
<td>normalized exponential</td>
<td>59</td>
</tr>
<tr>
<td>normalized exponential f-</td>
<td>62, 59</td>
</tr>
<tr>
<td>power growth</td>
<td>77</td>
</tr>
<tr>
<td>representation theorems</td>
<td>161, 162, 163, 166</td>
</tr>
<tr>
<td>shift operator</td>
<td>44</td>
</tr>
<tr>
<td>smoothing operator</td>
<td>155</td>
</tr>
<tr>
<td>strict convexity</td>
<td>144</td>
</tr>
<tr>
<td>sign changes</td>
<td></td>
</tr>
<tr>
<td>strong</td>
<td>11</td>
</tr>
<tr>
<td>weak</td>
<td>11</td>
</tr>
<tr>
<td>strong zeros</td>
<td></td>
</tr>
<tr>
<td>definition of</td>
<td>12</td>
</tr>
<tr>
<td>multiplicity of</td>
<td>12, 13</td>
</tr>
<tr>
<td>Taylor’s formula</td>
<td>10</td>
</tr>
<tr>
<td>T-periodic function</td>
<td>75</td>
</tr>
<tr>
<td>truncated power basis</td>
<td>44</td>
</tr>
<tr>
<td>(uni) solvent family</td>
<td>22</td>
</tr>
</tbody>
</table>
**AUTHOR INDEX**

Numbers indicate the pages of the text on which the author’s work is referred to.

Ahlborg, J.B. 55, 81  
Aitken, A.C. 83, 84  
Boox, C. de 6, 11, 12, 17, 45, 46, 156  
Brocket, R.W. 135  
Brujn, N.G. de 48  
Cavaretta, A.S. 103, 120, 152, 154  
Cheney, E.W. 98  
Chui, K. 103  
Coppel, W.A. 9  
Cox, M.G. 38, 40  
Davis, P.G. 10, 25  
Feller, W. 31  
Pyfe, D.J. 55  
Golomb, N. 144  
Jakhimovski, A. 81  
Karlin, S. 6, 11, 17, 19, 21, 33, 152, 153, 154, 161, 162  
Koch, P.K. 40  
Kolmogorov, A. 102, 103  
Kreyszig, E. 107  
Kuga, K. 165  
Lancaster, P. 27  
Landau, E. 102  
Luenberger, D.G. 112  
Lyche, T. 40, 46, 54  
Madsen, M.J. 45  
Meinardus, G. 27, 44  
Meir, N. 75  
Melnkman, A.A. 11  
Michelli, C.A. 5, 7, 64, 65, 76, 77, 80, 105, 106, 107
<table>
<thead>
<tr>
<th>Name</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hama, K.</td>
<td>38, 44, 46, 52, 62, 65, 67, 75, 76, 126, 130</td>
</tr>
<tr>
<td>Nilsson, E.M.</td>
<td>55, 81</td>
</tr>
<tr>
<td>Nørlund, N.E.</td>
<td>58, 139</td>
</tr>
<tr>
<td>Northcott, D.G.</td>
<td>123, 144</td>
</tr>
<tr>
<td>Pinkus, A.</td>
<td>22, 162</td>
</tr>
<tr>
<td>Poly, Ch.</td>
<td>8</td>
</tr>
<tr>
<td>Rylkin, T.J.</td>
<td>98, 105</td>
</tr>
<tr>
<td>Roberts, A.W.</td>
<td>178</td>
</tr>
<tr>
<td>Russell, D.C.</td>
<td>81</td>
</tr>
<tr>
<td>Schmidt, E.</td>
<td>27</td>
</tr>
<tr>
<td>Schoenberg, J.J.</td>
<td>7, 11, 12, 17, 29, 30, 56, 62, 75, 77, 103, 119, 120</td>
</tr>
<tr>
<td>Schönboh, H.</td>
<td>98</td>
</tr>
<tr>
<td>Schumaker, L.</td>
<td>17</td>
</tr>
<tr>
<td>Schurer, F.</td>
<td>55, 72</td>
</tr>
<tr>
<td>Sharma, A.</td>
<td>75, 103, 119</td>
</tr>
<tr>
<td>Smith, F.W.</td>
<td>103</td>
</tr>
<tr>
<td>Steffin, S.B.</td>
<td>103, 105</td>
</tr>
<tr>
<td>Studden, W.J.</td>
<td>17, 19, 152, 161</td>
</tr>
<tr>
<td>Subbotin, Yu.N.</td>
<td>81</td>
</tr>
<tr>
<td>Troch, I.</td>
<td>9</td>
</tr>
<tr>
<td>Tsimbalario, J.</td>
<td>103, 119</td>
</tr>
<tr>
<td>Varberg, D.E.</td>
<td>178</td>
</tr>
<tr>
<td>Walsh, J.L.</td>
<td>55, 81</td>
</tr>
<tr>
<td>Watkins, D.</td>
<td>27</td>
</tr>
</tbody>
</table>
SAMENVATTING

In hun eenvoudigste vorm zijn spline-functies (kortgevat: splines) functies die bestaan uit (stuksgewijze) polynomen, meestal van laag graad, die in de zogenoemde knooppunten met een zekere mate van gladheid aansluiten. Systematisch onderzoek naar het gedrag van splines dateert van 1946, toen Schoenberg een aantal belangrijke artikelen over deze functies publiceerde. Het blijkt dat splines, mede vanwege hun flexibiliteit, interessante interpolatie- en extremale eigenschappen hebben, die hen ook uitermate geschikt maken voor het numeriek benaderen van functies. Zowel de theorie en de toepassing van splines uitgegroeid tot een belangrijk onderdeel van de approximatietheorie. Allereerst hebben hun introductie gedaan. Zo zijn de polynomen vervangen door bijvoorbeeld exponentiële functies, of algemener door functies die in de nullruimte liggen van een lineaire differentiaaloperator $L$: de bijbehorende splines worden daarom $L$-splines genoemd.

In dit proefschrift komen $L$-splines aan de orde waarbij de differentiaaloperator constante reële coëfficiënten heeft, en dus geschreven kan worden als $p_n(D)$ waarbij $D$ de gewone eerste orde differentiaaloperator en $p_n$ het karakteristieke polynoom van $L$ is. Het eerste deel van het proefschrift gaat over interpolatie eigenschappen van zogenoemde "cardinal" $L$-splines, d.w.z. splines waarvan de opeenvolgende knooppunten onderling dezelfde afstand $h$ hebben. Uitgaande van een reëelwaardige functie $f$ op $E$, een lineaire differentiaaloperator $L$, en een vaste knooppuntstand $h$, wordt gevraagd een $L$-spline met knooppunten $0, ch, 2h, ...$ aan te geven die $f$ in de punten $0, ch, 2h, ...$ interpolerend, hierbij is $c$ een willekeurig gegeven getal in $(0, h)$. De theorie van dit soort interpolatie voor polynomiale splines, d.w.z. splines die behoren bij de operator $p_n(D) = D^n$, is uitvoerig behandeld door Schoenberg voor de gevallen $c = h$ en $c = 2h$. Een uitbreiding van deze theorie tot willekeurige $c$ in $(0, h)$ is door Ter Morsche gegeven, in eerste instantie voor periodieke cardinal polynomiale spline interpolatie. In dit verband dient eveneens te worden gewezen op het werk van Michelli,
die cardinal $\mathcal{C}$-spline interpolatie heeft onderzoek voor algemene operatoren $p_n$, waarbij $p_n$ alleen reëlle nulpunten heeft. In dit proefschrift wordt cardinal $\mathcal{C}$-spline interpolatie bestudeerd zonder deze eis voor $p_n$. Een belangrijk hulpmiddel daarbij zijn relaties tussen differenties en afgeleiden van cardinal $\mathcal{C}$-splines. Deze relaties laten zich op een overzichtelijke manier weergeven met behulp van zogenoemde exponentiële cardinal $\mathcal{C}$-splines; dit zijn splines die, analoog aan de exponentiële functie, de eigenschap hebben dat $s(x+h) = \lambda s(x)$, voor zekere $\lambda \in \mathbb{R}$ en alle $x \in \mathbb{R}$. De genoemde relaties tussen differenties en afgeleiden worden ook toegepast bij het onderzoek van periodieke $\mathcal{C}$-spline interpolatie. Existentie- en eenduidigheidssystemen en ook foutschatten worden afgeleid, zowel voor cardinal $\mathcal{C}$-spline interpolatie als voor periodieke cardinal $\mathcal{C}$-spline interpolatie.

Bij de foutschatten gaat het om het bepalen van sup $|f(x) - s(x)|$ over alle (periodieke) functies $f$ met $|p_n(f)| \leq 1$. Hier en in het vervolg wordt met $\|\cdot\|_a$ steeds de supremumnorm op $J \subseteq \mathbb{R}$ bedoeld; als $J = \mathbb{R}$ dan laten we de index $J$ weg, als $J = \mathbb{R}^+_0 := [0,\infty)$ dan noteren we $\|\cdot\|_{J_+}$. Het blijkt dat bij deze foutschatten de zogenoemde perfecte Euler $\mathcal{C}$-splines een belangrijke rol spelen. Dit zijn exponentiële cardinal $\mathcal{C}$-splines $\mathcal{E}$ waarvoor geldt $f(x+h) = \lambda f(x)$ voor alle $x \in \mathbb{R}$ en $|p_n(D)f(x)| = 1$ voor alle $x \in \mathbb{R}$ behalve in de knooppunten.

Bij de uitbreiding van de theorie van cardinal $\mathcal{C}$-spline interpolatie tot operator $p_n$, waarbij $p_n$ ook niet-reële nulpunten mag hebben, is de belangrijkste moeilijkheid dat een basis in de nulruimte van $p_n$ niet op elk interval een Chebyshevvoorsysteem vormt. Op intervallen waar dit wel het geval is kan, naar een klassiek resultaat van Pólya, de operator $p_n$ onbundeld worden in eerste orde differentiaaloperatoren van de vorm $D_i = w_i \frac{\partial}{\partial x_i}$, waarbij $w_i$ ($i = 1, 2, \ldots, n$) op het betreffende interval positieve functies zijn; er geldt dan $p_n(D) = D_n \cdots D_1$. Een dergelijke onbinding is van belang voor het tellen van nulpunten van $\mathcal{C}$-splines; de stelling van Rolle en de stelling van Budan-Fourier als vervanging daarvan kunnen eveneens geformeerd worden voor $D_1, D_2, \ldots, D_n$.

Het tweede deel van het proefschrift gaat over extremale eigenschappen van perfecte $\mathcal{C}$-splines met betrekking tot zogenoemde Landauproblemen. In 1913 bewees Landau voor tweemaal differentieerbare functies $f$ op $\mathbb{R}$ de ongelijkheid $\|f\|_2 \leq \sqrt{2} \|\mathcal{E} f\|_2$; voor functies $f$ gedefinieerd op de halfrechte $\mathbb{R}_0^+$ bewees hij dat geldt $\|f\|_4 \leq 2 \|\mathcal{E} f\|_4$. Deze ongelijkheden kunnen niet verscherpt worden, d.w.z. de constanten $\sqrt{2}$ en 2 kunnen niet worden vervangen.
door kleinere getallen. Landau's eerste ongelijkheid werd in 1939 door Kolmogorov gegeneraliseerd tot hogere afgeleiden. Op $H_0$ is een dergelijke generalisatie van Landau's tweede ongelijkheid heel wat geoppercevierd. Uiteindelijk is dit probleem opgelost door Schoenberg en Caversatto in die zin, dat zij zogenaamde extremale functies bepaald hebben: die zijn functies waarbij de betreffende ongelijkheden in gelijkheden overgaan. Bevinden zij zich bijzondere gevallen van het algemene Landauprobleem dat we nu beschrijven.

Laat $J$ een willekeurig gesloten deelinterval van $\mathbb{R}$ zijn, en $p_n(D)$ en $p_n^<(D)$ lineaire differentiaaloperatoren van de orde $n$ en $k = n-1$, bepaald dan bij gegeven $n > 0$ het supremum van $\|p_n(D)f\|_J$ over alle functies $f$ die $(n-1)$ keer differentieerbaar zijn op $J$, een absoluut continue $(n-1)$-ste afgeleide hebben en waarvoor geldt dat $\|f\|_J \leq m$ en $\|p_n(D)f\|_J \leq 1$. Deze klasse functies geven we aan met $\Gamma_n(p_n^oJ)$. Een functie $f$ waarvoor het supremum van $\|p_n(D)f\|_J$ wordt bereikt, noemen we een extremale functie genoemd. Als dus geformuleerd blijkt in de gevallen $J = \mathbb{R}$ en $J = \mathbb{R}^+$ het Landauprobleem equivalent te zijn met het probleem het maximum te bepalen van een lineaire functie over de verzameling van alle punten $(f(0), f'(0), \ldots, f^{(n-1)}(0))^T$, met $f \in p_n(p_n^oJ)$. We besteden daarom in het bijzonder aandacht aan de gevallen $J = \mathbb{R}$ en $J = \mathbb{R}^+$; de zojuist genoemde verzamelingen worden dan respectievelijk met $\Gamma_n(p_n^o\mathbb{R})$ en $\Gamma_n(p_n^o\mathbb{R}^+)$ aangeduid. Een aantal speciale gevallen van het Landau probleem wordt opgelost door middel van een parametrisering van de verzamelingen $\Gamma_n(p_n^o\mathbb{R})$ en $\Gamma_n(p_n^o\mathbb{R}^+)$. Wij besluiten met een korte opsomming van de inhoud van de verschillende hoofdstukken.

Hoofdstuk 1 heeft een inleidend karakter. Het bevat de definities van de verschillende soorten splines die in het proefschrift een rol spelen. Verder komen o.a. aan de orde: gedisc conventueerdheid van een differentiaaloperator, verschillende soorten Chebyshev-systemen en gedeelde differenties. Dit alles is standaardmateriaal in de approximatie-theorie. We nemen een generalisatie van de stelling van Budan-Fourier voor functies met stuksgewijze continue afgeleiden.

In Hoofdstuk 2 worden 8-splines behandeld, behorende bij willekeurige differentiaaloperatoren $p_n(D)$. Erst besteden we aandacht aan een aantal belangrijke eigenschappen van deze functies. Vervolgens wordt met behulp van de stelling van Budan-Fourier de zogenaamde "totale positiviteit" van een rij van opvolgende 8-splines bewezen. Via de Fourier-getransformeerden

In het begin van Hoofdstuk 5 wordt de betekenis van het Landsauprobleem aangegeven voor de bepaling van optimale differentiatie-algoritmen. Verder worden algemene eigenschappen van de verzamelingen $\Gamma_n^k(p_n)$ en $\Gamma_n^k(p_0)$ afgeleid. In het bijzonder wordt aandacht gegeven aan de gevallen $n = 2$ en $n = 3$. We tonen aan dat bij de parametrizering van de rand van $\Gamma_2^k(p_n)$ en $\Gamma_2^k(p_n)$ perfecte Euler f-splines een fundamentele rol spelen.

Het Landsauprobleem voor periodieke functies is het onderwerp van Hoofdstuk 6. Hier wordt o.a. een verregaande generalisatie gegeven van een resultaat van Northcott, dat betrekking heeft op het maximaliseren van $\|f^{(k)}\|_{\infty}$ voor periodieke functies $f$ met $\|f^{(n)}\|_{\infty} < 1$.

In Hoofdstuk 7, tenslotte, wordt het Landsauprobleem op de halfrechte $R^+_0$ beschouwd voor algemene $p_n(x)$, waarbij echter t.a.v. het polynoom $p_n$ wordt verondersteld dat het uitsluitend reële nulpunten heeft en dat bovendien geldt dat $p_n(0) = 0$. Met behulp van een representatiestelling voor een geschikte klasse van periodieke f-splines tonen we aan dat een extremale functie wordt verkregen als de limiet van een rij van geschikte perfecte f-splines, die een specifiek oscillatiedrag vertonen. Op grond van de bereikte resultaten kan de rand van de verzameling $\Gamma_n^k(p_n)$ in detail worden beschreven. Bij een parameterisering van de rand van de verzameling $\Gamma_1^k(p_0)$ doet zich echter reeds aanzienlijke analytische moeilijkheden voor. Een punt van deze rand wordt numeriek bepaald.
CURRICULUM VITAE


Sinds 1968 is hij verbonden aan de Onderafdeling der Wiskunde en Informatica van de Technische Hogeschool Eindhoven, waar hij als wetenschappelijk hoofdmedewerker deel uitmaakt van de Groep Basisonderwijs. Tevens is hij lid van de vakgroep Toegespaste Analysé, Matematische Fysica, Mechanica en Numerieke Wiskunde, waar hij zijn werkzaamheden verricht in een overeenstemmingsrelatie met Prof.Dr.Ir. F. Schurer.
STELLINGEN
bij het proefschrift

INTERPOLATIONAL AND EXTREMAL PROPERTIES
OF $L$-SPLINE FUNCTIONS

van

R.G. ter Morsche

1

De Euler $L$-spline $\Psi_{-1} (p_n; h, \cdot)$ en de perfecte Euler $L$-spline $E (p_n; h, \cdot)$ hebben respectievelijk de eigenschappen

$$\lim_{h \to 0} h^{-n+1} \Psi_{-1} (p_n; h, h t) = \Psi_{-1} (D^n; 1, t), \quad (t \in \mathbb{R}).$$

$$\lim_{h \to 0} h^{-n} E (p_n; h, h t) = E (D^n; 1, t).$$

Subsecties 3.2.3 en 3.2.7 van dit proefschrift.

2

Zij $f \in C^{(n-1)} (\mathbb{R})$ met absoluut continue $(n-1)$-ste afgeleide. De interpolerende cardinal $L$-spline $s_x$ in Theorem 4.4.2 van dit proefschrift heeft de eigenschap dat voor alle $x \in \mathbb{R}$

$$|f(x) - s_x(x)| = O (h^n) \quad (h \to 0).$$

Sectie 4.4 van dit proefschrift.

3

Zij $p_n$ een monisch polynoom van de graad $n$ met uitsluitend reële nulpunten en zij $E (p_n; h, \cdot)$ de bijbehorende perfecte Euler $L$-spline. Dan is $\sup_{t \in \mathbb{R}} |E (p_n; h, t)|$ een continu en monotoon stijgende functie van $h$ op $[0, \infty)$, dit kan eenvoudig worden aangetoond met behulp van een resultaat van Sharma en Tribhulario.

Subsecties 3.2.7 en 5.5.2 van dit proefschrift.

Zij \( f : \mathbb{R} \rightarrow \mathbb{R}^n \) (n \( \geq 2 \)) een differentieerbare functie met absoluut continue afgeleide \( f' \). Zij verder \( |f(t)| := \sqrt{f_1^2(t) + \ldots + f_n^2(t)} \), waarbij \( f_1, \ldots, f_n \) de componenten van \( f \) zijn.

Als \( |f(t)| \leq 1 \) en \( |f'(t)| \leq 1 \) voor (bijna) alle \( t \in \mathbb{R} \), dan geldt

\[
|f'(t)|^2 \leq 1 + \sqrt{1 - |f(t)|^2}^2 \quad (t \in \mathbb{R})
\]

en deze ongelijkheid kan niet verscherpt worden.


Gegeven zijn \( 2n+4 \) zuivere getallen \( y_0, \ldots, y_n, y_0', \ldots, y_n', y_0'', \ldots, y_n'' \).

De polynomiale vijfde-graad spline \( s \in C_5([-1], 1]) \) met knooppunten van
ciliciëriteit t.w.t. in \( 1, \frac{1}{n}, \ldots, \frac{n-1}{n} \) die voldoet aan de interpolatievoorwaarden:

\[
s\left(\frac{1}{n}\right) = y_0, \quad s\left(\frac{1}{n}\right) = y_0' \quad (i = 0, 1, \ldots, n),
\]

\[
s''(0) = y_0'', \quad s''(1) = y_n''
\]

can op eenvoudige wijze worden geconstrueerd door gebruik te maken van
voortbrengende functies.

Morcha, R.G. ter, On the construction of a \( (0,2) \)-interpolating deficient quintic spline function. TH-Report 74-WSR-02, Eindhoven University of

Zij \( K \) de klasse van functies \( f \) gedefinieerd op \( (0, \infty) \) en met de eigenschappen:

1) \( f \) is integeerbaar op ieder interval \([a, b] \subset (0, \infty) \),

2) er bestaan positieve constanten \( a \) en \( b \) zodanig dat

\[
f(t) = O(t^{-a}) \quad (t \to 0), \quad f(t) = O(t^b) \quad (t \to \infty)
\]

Zij verder \( A_n \) de rij van positieve lineaire operatoren gedefinieerd door
\[ A_n f(x) := \frac{n^n}{(n-1)!} \int_0^\infty \frac{t^{n-1}}{e^t} f(x) dt \quad (x > 0). \]

De saturatieklasse van \((A_n)\) met betrekking tot \(M\) bestaat dan uit functies \(f\) waarvan de afgeleide locaal Lipschitz-continu is, en de saturatieorde be- draagt \(1/n\), d.w.z. op elke compact interval van \((0,\infty)\) is

\[ |A_n f(x) - f(x)| = O\left(\frac{1}{n}\right) \quad (n \to \infty) \]

dan en slechts dan als \(f\) tot de genoemde deelklasse van \(M\) behoort.


Laat \(Y_1, Y_2, \ldots\) onafhankelijke en identiek verdeeld stochastische variabelen zijn met een verdelingsfunctie \(F_x\), die afhangt van een parameter \(x \in (a,b)\). Zij \(\int_a^b x f_x(t) dt = 1\), zij \(E_x\) de verwachtingsoperator met betrekking tot \(F_x\) en zij \(E_x Y_n = x \cdot (x \in (a,b); n \in \mathbb{N})\). Als de rij \((L_n)\) van positieve lineaire operatoren gedefinieerd is door

\[ L_n f(x) = E_x f(x_n) \]

met \(x_n := (Y_1 + Y_2 + \cdots + Y_n)/n\) en \(f \in \mathcal{C}([a,b])\), dan is de saturatieklasse van \((L_n)\) de verzameling functies op \([a,b]\) met een Lipschitz-continu afgeleide, en de saturatieorde is \(1/n \cdot (Y_n - x)^2\).


Zij \(n \in \mathbb{N}\) en zij \((a_n)\) een rij complexe getallen met de eigenschap dat voor alle \(n \in \mathbb{Z}\)

\[ |a_n| \leq 2a^2, \quad |a_n^2 - 2a_{n+1} + a_n| \leq 1. \]

Dan geldt

\[ |a_{n+1} - a_n| \leq 2m \quad (n \in \mathbb{Z}), \]

en deze ongelijkheid kan niet verscherpt worden.
De door Sierpinski gegeven oplossing van het probleem:

bewijs dat voor m = 1 en m = 2 de vergelijking \( x^3 + y^3 + z^3 = mxyz \) geen oplossingen x,y,z in \( \mathbb{N} \) heeft, en bepaal alle x,y,z in \( \mathbb{N} \) waarvoor \( x^3 + y^3 + z^3 = 3xyz \).

kan aanzienlijk vereenvoudigd worden als men gebruik maakt van de identiteit

\[ x^3 + y^3 + z^3 - 3xyz = (x+y+z)((x-y)^2 + (y-z)^2 + (z-x)^2). \]

Sierpinski, W., 250 problems in elementary number theory (problem 156).

In onderwijskundige verhandelingen over hoorcolleges wordt te weinig aandacht geschonken aan de medeverantwoordelijkheid van studenten voor de kwaliteit van deze onderwijsvorm.


Een aantal belangrijke beleidsohnemens met betrekking tot de financiering en de organisatie van het universitaire onderzoek in Nederland, zoals geformuleerd in de beleidsnota universitair onderzoek, staat op gespannen voet met het in dezelfde nota geformuleerde uitgangspunt, dat de universiteiten en hogescholen een zelfstandig beleid moeten kunnen voeren.


Het opzettelijk en bij herhaling veroorzaakt van een buitenspelsituatie bij het voetbalspel is een vorm van spelbodrijf en zou als zodanig door de scheidsrechter bestraft dienen te worden.

H.G. ter Morsche
16 april 1982.